# WEYL–TITCHMARSH THEORY FOR SCHRÖDINGER OPERATORS WITH STRONGLY SINGULAR POTENTIALS

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Abstract. We develop Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials such as perturbed spherical Schrödinger operators (also known as Bessel operators). It is known that in such situations one can still define a corresponding singular Weyl $m$-function and it was recently shown that there is also an associated spectral transformation. Here we will give a general criterion when the singular Weyl function can be analytically extended to the upper half plane. We will derive an integral representation for this singular Weyl function and give a criterion when it is a generalized Nevanlinna function. Moreover, we will show how essential supports for the Lebesgue decomposition of the spectral measure can be obtained from the boundary behavior of the singular Weyl function. Finally, we will prove a local Borg–Marchenko type uniqueness result. Our criteria will in particular cover the aforementioned case of perturbed spherical Schrödinger operators.

## 1. Introduction

The present paper is concerned with spectral theory for one-dimensional Schrödinger operators

$$H = -\frac{d^2}{dx^2} + q(x), \qquad x \in (a,b), \tag{1.1}$$

on the Hilbert space $L^2(a,b)$ with a real-valued potential $q \in L^1_{\mathrm{loc}}(a,b)$. If we assume one endpoint to be regular it is well-known that one can associate a single function $m(z)$, the Weyl–Titchmarsh (or Weyl) $m$-function, with $H$, such that $m(z)$ contains all the information about $H$. In particular, a corresponding spectral measure $d\mu$ such that $H$ is unitarily equivalent to multiplication by the identity function in $L^2(\mathbb{R}, d\mu)$ can be obtained by considering the vague limit of $\pi^{-1}\mathrm{Im}(m(\lambda + \mathrm{i}\varepsilon))d\lambda$ as $\varepsilon \downarrow 0$. Moreover, by the celebrated Borg–Marchenko theorem $H$ is uniquely determined by $m(z)$.

If neither endpoint is regular, similar results hold by considering some point $c \in (a,b)$ at the price that one now has to deal with two by two Weyl matrices. This is clearly unavoidable if the spectral multiplicity is two (for necessary and sufficient conditions for simplicity of the spectrum we refer to [18], see also the references therein), but it naturally raises the question if there are cases where a single function is sufficient and a *singular* Weyl $m$-function $M(z)$ can be introduced. One such a case is if one endpoint is in the limit circle case (cf. [13]). Moreover, another example, the perturbed spherical Schrödinger (or Bessel) operator, got much attention and singular $m$-functions were introduced by Gesztesy and Zinchenko [17] and

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Fulton and Langer [14], [15] recently (see also [28]). In this respect note that the literature on spectral theory for these operators is extensive, see e.g. [2], [6], [21], [25], [33] and the references therein.

The key ingredient for defining a Weyl $m$-function is an entire system of linearly independent solutions $\phi(z,x)$, $\theta(z,x)$ of the underlying differential equation $Hu = zu$, $z \in \mathbb{C}$, normalized such that the Wronskian $W(\theta(z),\phi(z))$ equals one. To make the connection with $H$, one solution, say $\phi(z,x)$, has to be chosen such that it lies in the domain of $H$ near the endpoint $a$ (i.e., $\phi(z,.)$ is square integrable near $a$ and satisfies the boundary condition at $a$ if $H$ is limit circle at $a$). If $a$ is regular, this can be easily done by specifying the initial conditions at $x = a$. Once $\phi(z,x)$ and $\theta(z,x)$ are given, the Weyl $m$-function $M(z)$ can be defined by the requirement that the solution

$$\psi(z,x) = \theta(z,x) + M(z)\phi(z,x) \tag{1.2}$$

is in the domain of $H$ near $b$.

While this prescription sounds straightforward, it has turned out to be rather subtle! The definition of $\phi(z,x)$ is unproblematic and, as demonstrated in Gesztesy and Zinchenko [17], a necessary and sufficient condition for the existence of $\phi(z,x)$ is the assumption that $H$ restricted to $(a,c) \subset (a,b)$ has purely discrete spectrum. This is also confirmed by the fact that in the special case of spherical Schrödinger operators the solution $\phi(z,x)$ (together with detailed growth estimates) can be obtained by the usual iteration schemes (see e.g. [12, Sect. 14], [21], [25], [32, Sect. 12.2], [33]). On the other hand, a similar construction for the second (singular) solution $\theta(z,x)$ is not known to the best of our knowledge! More precisely, a construction was given in [21] and later on used and generalized by several authors, but unfortunately this construction is wrong (cf. [25, Rem. 2.7] for a counterexample). Consequently, this creates a gap in the proofs of all papers relying on this construction. The only case of spherical Schrödinger operators, where existence of $\theta(z,x)$ is known, is when the potential is analytic of Fuchs type near $a = 0$. In this case a second entire solution comes for free from the Frobenius method. Naturally, this special case has attracted much attention recently [14], [15], [28].

This clearly raises the question when such a second solution $\theta(z,x)$ exists or (essentially) equivalent, when the singular Weyl $m$-function is analytic in the entire upper (and hence lower) half plane. This question was circumvented in [17] by observing that for the definition of an associated spectral measure, all one needs is a definition of $\theta(z,x)$ for $z$ in a vicinity of the real line. Such a $\theta(z,x)$ is easy to obtain and existence of an associated spectral transformation was established in [17]. However, analyticity of the Weyl $m$-function in the upper half plane is not only interesting from an academic point of view, but also from a practical perspective! In fact, the Borg–Marchenko uniqueness theorem involves the asymptotic expansion of the Weyl $m$-function as $|z| \to \infty$ along nonreal rays. Hence analyticity in the upper half plane is a vital prerequisite for such kind of results.

It is the purpose of the present paper to give a positive answer to the above raised questions: As our first result we will show that a second (singular) entire solution $\theta(z,x)$ always exists in Section 2. This will be done using an explicit construction based on the Mittag–Leffler theorem. Next we will reproduce the main result concerning existence of a spectral transformation from [17] in Section 3. In fact, we will provide some generalizations and a simplified proof. Moreover, we will show

how essential supports for the Lebesgue decomposition of the spectral measure can be obtained from the boundary behavior of the singular Weyl function.

In Section 4 we will derive an integral representation for singular Weyl functions which generalizes the usual Herglotz representation. Moreover, it will allow us to show that in all cases of practical interest there is a choice of $\phi(z,x)$ and $\theta(z,x)$ such that the singular Weyl function is in the generalized Nevanlinna class $N_\kappa^\infty$ (cf. Appendix C for a definition of this class). In fact, in the special case of perturbed spherical Schrödinger operators quite some effort has been put into showing that the singular Weyl function is a generalized Nevanlinna function [10], [15], [28], but so far only the case of analytic perturbations could be treated. Our approach is both more general and much simpler.

In Section 5 the explicitly solvable example of the (unperturbed) Bessel operator will be investigated to illustrate our findings.

Then we will investigate exponential growth properties of solutions as a preparation for our next section. Existence of solutions with the necessary growth properties will require an additional (mild) hypothesis on the Dirichlet and Neumann eigenvalues of the operator restricted to $(a,c) \subset (a,b)$. Our approach will be non-constructive and based on a version of the Corona Theorem for rings of entire functions due to Hörmander [22]. After these preparations we will prove a Borg–Marchenko-type uniqueness result in its local version due to Gesztesy and Simon [16], [35] (our proof is modeled after the simple proof from Bennewitz [4]).

Finally we will show in Section 8 how our results can be applied to perturbed spherical Schrödinger operators.

Appendix A shows how our considerations simplify in the special case where the endpoint is limit circle and shows that the singular $m$-function is a Herglotz–Nevanlinna function in this case. Appendix B investigates another explicitly solvable example. Appendix C collects some facts about generalized Nevanlinna functions.

In a follow-up paper [24] we will further investigate the special example of perturbed spherical Schrödinger operators

$$H = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \qquad x \in (0,\infty). \tag{1.3}$$

There we will precisely identify the value of $\kappa$ from the generalized Nevanlinna class $N_\kappa^\infty$ to which the singular $m$-function belongs and connect our results with the theory of super singular perturbations [5], [7], [8], [34]. Let us mention that this connection was first observed by Djiksma and Shondin [10] and Kurasov and Luger [28] in the particular cases of unperturbed Bessel operators and the Bessel operators perturbed by a Coulomb term, respectively. In the case where the perturbation is analytic similar results have been obtained in [15] with the help of the Frobenius method as pointed out earlier.

## 2. The singular Weyl $m$-function

To set the stage, we will consider one-dimensional Schrödinger operators on $L^2(a,b)$ with $-\infty \le a < b \le \infty$ of the form

$$\tau = -\frac{d^2}{dx^2} + q, \tag{2.1}$$

where the potential $q$ is real-valued satisfying

$$q \in L^1_{loc}(a,b). \tag{2.2}$$

We will use $\tau$ to denote the formal differential expression and $H$ to denote a corresponding self-adjoint operator given by $\tau$ with separated boundary conditions at $a$ and/or $b$.

If $a$ (resp. $b$) is finite and $q$ is in addition integrable near $a$ (resp. $b$), we will say $a$ (resp. $b$) is a *regular* endpoint. We will say $\tau$, respectively $H$, is *regular* if both $a$ and $b$ are regular.

We will choose a point $c \in (a,b)$ and also consider the operators $H^D_{(a,c)}$, $H^D_{(c,b)}$ which are obtained by restricting $H$ to $(a,c)$, $(c,b)$ with a Dirichlet boundary condition at $c$, respectively. The corresponding operators with a Neumann boundary condition will be denoted by $H^N_{(a,c)}$ and $H^N_{(c,b)}$.

Moreover, let $c(z,x)$, $s(z,x)$ be the solutions of $\tau u = z\,u$ corresponding to the initial conditions $c(z,c)=1$, $c'(z,c)=0$ and $s(z,c)=0$, $s'(z,c)=1$.

Then we can define the Weyl solutions

$$u_\pm(z,x) = c(z,x) \pm m_\pm(z) s(z,x), \tag{2.3}$$

where $m_\pm(z)$ are the Weyl $m$-functions corresponding to the base point $c$. For further background we refer to [36, Chap. 9] or [37].

To define an analogous singular Weyl $m$-function at the, in general singular, endpoint $a$ we will first need the analog of the system of solutions $c(z,x)$ and $s(z,x)$. Hence our first goal is to find a system of entire solutions $\theta(z,x)$ and $\phi(z,x)$ such that $\phi(z,x)$ lies in the domain of $H$ near $a$ and such that the Wronskian $W(\theta(z),\phi(z)) = 1$. To this end we start with a hypothesis which will turn out necessary and sufficient for such a system of solutions to exist.

**Hypothesis 2.1.** *Suppose that the spectrum of $H^D_{(a,c)}$ is purely discrete for one (and hence for all) $c \in (a,b)$.*

Note that this hypothesis is for example satisfied if $q(x) \to +\infty$ as $x \to a$ (cf. Problem 9.7 in [36]). Further examples will be given in Sections 5 and 8 (cf. also Example 3.13 in [17]).

**Lemma 2.2.** *Hypothesis 2.1 is necessary and sufficient to have a (nontrivial) solution $\phi(z,x)$ of $\tau u = zu$ which is in the domain of $H$ near $a$ and which is entire with respect to $z$. Moreover, one can choose $\phi(z,x)$ such that*

$$\phi(z,x) = \alpha(z) c(z,x) + \beta(z) s(z,x), \tag{2.4}$$

*where $\alpha(z)$ and $\beta(z)$ are real entire functions with no common zeros.*

*Proof.* Suppose Hypothesis 2.1 holds. Then $m_-(z)$ is meromorphic with simple poles at the points in $\sigma(H^D_{(a,c)})$ and by the Weierstraß product theorem ([31, Thm. II.10.1]) there is a (real) entire function $\alpha(z)$ which has simple zeros (and no other zeros) at the poles of $m_-(z)$. Hence we can set $\phi(z,x) = \alpha(z) u_-(z,x)$.

Conversely, let $\phi(z,x) = \alpha(z)c(z,x) + \beta(z)s(z,x)$ be entire. Then $\alpha(z) = \phi(z,c)$ is also entire. Moreover,

$$\beta(z) = \frac{\phi(z,x) - \alpha(z)c(z,x)}{s(z,x)}$$

is entire since the possible poles on the real line are removable (use that the left-hand side of this formula is independent of $x$ and the possible poles on the right-hand side vary as $x$ varies). Note that

$$\alpha(z) = \phi(z,c), \qquad \beta(z) = \phi'(z,c) \tag{2.5}$$

Finally, recalling (2.3), we see that

$$m_-(z) = -\frac{\phi'(z,x)}{\phi(z,x)} = -\frac{\beta(z)}{\alpha(z)} \tag{2.6}$$

is meromorphic and thus Hypothesis 2.1 holds. □

**Corollary 2.3.** *The function $\phi(z,x)$ is uniquely defined up to a real entire function without zeros. That is, if $\tilde\phi(z,x)$ is another real entire solution which is nontrivial for all $z \in \mathbb{C}$ and in the domain of $H$ near $a$, we have*

$$\tilde\alpha(z) = \mathrm{e}^{g(z)}\alpha(z), \qquad \tilde\beta(z) = \mathrm{e}^{g(z)}\beta(z) \tag{2.7}$$

*for some real entire function $g(z)$.*

It remains to find a second solution:

**Lemma 2.4.** *Suppose Hypothesis 2.1 holds. Then there is a second solution*

$$\theta(z,x) = \gamma(z)c(z,x) + \delta(z)s(z,x), \tag{2.8}$$

*where $\gamma(z)$ and $\delta(z)$ are real entire functions with no common zeros, such that*

$$W(\theta(z),\phi(z)) = \gamma(z)\beta(z) - \alpha(z)\delta(z) = 1. \tag{2.9}$$

*Here $W(u,v) = u(x)v'(x) - u'(x)v(x)$ is the usual Wronski determinant.*

*Proof.* We will make the following ansatz

$$\gamma(z) = \frac{\beta(z)}{\alpha(z)^2+\beta(z)^2} - \eta(z)\alpha(z), \qquad \delta(z) = \frac{-\alpha(z)}{\alpha(z)^2+\beta(z)^2} - \eta(z)\beta(z),$$

where $\eta(z)$ is a meromorphic function with poles at the the zeros of $\alpha(z)^2+\beta(z)^2$ to be determined next.

Denote by $\{z_j\}$ the zeros of the entire function $\alpha(z)^2+\beta(z)^2$ and suppose that $z_j$ is a zero of order $n_j \in \mathbb{N}$. Then at each $z_j$ we have

$$\beta^{(k)}(z_j) = \sigma_j \mathrm{i}\alpha^{(k)}(z_j), \qquad 0 \le k < n_j,$$

with some $\sigma_j \in \{\pm 1\}$. Now choose $\eta(z)$ such that the principal part of $\eta(z)$ near $z_j$ matches the one from $\sigma_j\mathrm{i}(\alpha(z)^2+\beta(z)^2)^{-1}$, that is,

$$\frac{\sigma_j \mathrm{i}}{\alpha(z)^2+\beta(z)^2} = \eta(z) + O(z-z_j)^0.$$

Such a function exists by the Mittag–Leffler theorem [31, Thm. II.10.10]. Then one computes

$$\begin{aligned}\gamma(z) &= \frac{\beta(z)}{\alpha(z)^2+\beta(z)^2} - \eta(z)\alpha(z)\\ &= \frac{\sigma_j\mathrm{i}\alpha(z) + O(z-z_j)^{n_j}}{\alpha(z)^2+\beta(z)^2} - \frac{\sigma_j\mathrm{i}\alpha(z)}{\alpha(z)^2+\beta(z)^2} + O(z-z_j)^0 = O(z-z_j)^0\end{aligned}$$

and thus all poles of $\gamma(z)$ are removable. Similarly, all poles of $\delta(z)$ are removable

$$\begin{aligned}\delta(z) &= \frac{-\alpha(z)}{\alpha(z)^2+\beta(z)^2} - \eta(z)\beta(z)\\ &= \frac{-\alpha(z)}{\alpha(z)^2+\beta(z)^2} - \frac{(\sigma_j \mathrm{i})^2\alpha(z) + O(z-z_j)^{n_j}}{\alpha(z)^2+\beta(z)^2} + O(z-z_j)^0 = O(z-z_j)^0.\end{aligned}$$

Finally, $W(\theta(z),\phi(z)) = 1$ is immediate (at least away from the zeros) by our ansatz. If $\gamma(z)$ or $\delta(z)$ are not real, replace them by $\frac{1}{2}(\gamma(z)+\gamma(z^*)^*)$ and $\frac{1}{2}(\delta(z)+\delta(z^*)^*)$, respectively. □

Note again

$$\gamma(z) = \theta(z,c), \qquad \delta(z) = \theta'(z,c). \tag{2.10}$$

**Corollary 2.5.** *Given $\phi(z,x)$ and $\theta(z,x)$, any other real entire solution $\tilde{\theta}(z,x)$ satisfying $W(\tilde{\theta}(z),\phi(z)) = 1$ is given by*

$$\tilde{\theta}(z,x) = \theta(z,x) - f(z)\phi(z,x) \tag{2.11}$$

*for some real entire function $f(z)$.*

Given a system of real entire solutions $\phi(z,x)$ and $\theta(z,x)$ as in the above lemma we can define the singular Weyl $m$-function

$$M(z) = -\frac{W(\theta(z),u_+(z))}{W(\phi(z),u_+(z))} = -\frac{\gamma(z)m_+(z)-\delta(z)}{\alpha(z)m_+(z)-\beta(z)} \tag{2.12}$$

such that the solution which is in the domain of $H$ near $b$ (cf. (2.3)) is given by

$$u_+(z,x) = a(z)\big(\theta(z,x) + M(z)\phi(z,x)\big), \tag{2.13}$$

where $a(z) = -W(\phi(z),u_+(z)) = \beta(z) - m_+(z)\alpha(z)$. By construction we obtain

**Lemma 2.6.** *The singular Weyl $m$-function $M(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies $M(z) = M(z^*)^*$.*

*Proof.* It is is clear from (2.12) that $M(z)$ is meromorphic in $\mathbb{C}\backslash\mathbb{R}$ since $m_+(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and all other functions are entire. Moreover, $W(\phi(z),u_+(z))$ cannot vanish for $z \in \mathbb{C}\backslash\mathbb{R}$ since every such zero would correspond to a complex eigenvalue of $H$. Hence the claim follows. □

Rather than $u_+(z,x)$ we will use

$$\psi(z,x) = \theta(z,x) + M(z)\phi(z,x). \tag{2.14}$$

**Remark 2.7.** *By Corollary 2.3 and Corollary 2.5 any system of real entire solutions $\tilde{\theta}(z,x)$ and $\tilde{\phi}(z,x)$ satisfying $W(\tilde{\theta}(z),\tilde{\phi}(z)) = 1$ is related to the ones constructed above via*

$$\tilde{\theta}(z,x) = \mathrm{e}^{-g(z)}\theta(z,x) - f(z)\phi(z,x), \qquad \tilde{\phi}(z,x) = \mathrm{e}^{g(z)}\phi(z,x),$$

*where $g(z)$, $f(z)$ are real entire function. The singular Weyl $m$-functions are related via*

$$\tilde{M}(z) = \mathrm{e}^{-2g(z)}M(z) + \mathrm{e}^{-g(z)}f(z).$$

*In particular, the maximal domain of holomorphy or the structure of poles and singularities do not change.*

## 3. Spectral transformations

Denote by $f \in L^2_c(a,b)$ the subset of square integrable functions with compact support. Set

$$\hat{f}(z) = \int_a^b \phi(z,x) f(x) dx, \qquad f \in L^2_c(a,b) \tag{3.1}$$

and note that $\hat{f}(z)$ is entire. Note

$$\hat{f}(z^*)^* = \int_a^b \phi(z,x) f(x)^* dx. \tag{3.2}$$

Moreover, recall that for every $f \in L^2(a,b)$ there is an associated spectral measure $\mu_f$ whose Borel transform is given by

$$m_f(z) := \langle f, (H-z)^{-1} f\rangle = \int_{\mathbb{R}} \frac{d\mu_f(\lambda)}{\lambda - z}. \tag{3.3}$$

Here $\langle f, g\rangle = \int_a^b f(x)^* g(x) dx$ denotes the scalar product in $L^2(a,b)$.

Moreover, recall

$$(H-z)^{-1} f(x) = \int_a^b G(z,x,y) f(y) dy, \tag{3.4}$$

where

$$G(z,x,y) = \begin{cases} \phi(z,x)\psi(z,y), & y \ge x, \\ \phi(z,y)\psi(z,x), & y \le x, \end{cases} \tag{3.5}$$

is the Green function of $H$.

**Lemma 3.1.** *For every $f \in L^2_c(a,b)$ we have*

$$m_f(z) = E_f(z) + \hat{f}(z)\hat{f}(z^*)^* M(z), \tag{3.6}$$

*where $\hat{f}(z)$ is given by* (3.1) *and $E_f(z)$ is entire and satisfies $E_f(z^*)^* = E_f(z)$.*

*Proof.* A straightforward calculation using

$$\begin{aligned}\langle f, (H-z)^{-1} f\rangle &= \int_a^b f(x)^* \int_a^b G(z,x,y) f(y) dy\, dx \\ &= \int_a^b f(x)^* \left( \psi(z,x) \int_a^x \phi(z,y) f(y) dy + \phi(z,x) \int_x^b \psi(z,y) f(y) dy \right) dx\end{aligned}$$

and (2.14) proves the claim with

$$E_f(z) = \int_a^b f(x)^* \left( \theta(z,x) \int_a^x \phi(z,y) f(y) dy + \phi(z,x) \int_x^b \theta(z,y) f(y) dy \right) dx.$$

□

For the proof of our main result we will need the following form of the Stieltjes inversion formula:

**Lemma 3.2** (Stieltjes inversion formula)**.** *Suppose $m(z)$ is the Borel transform of a finite measure $d\mu$,*

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}, \tag{3.7}$$

*then*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} F(\lambda) \mathrm{Im}\big(m(\lambda + \mathrm{i}\varepsilon)\big) d\lambda = \int_{\lambda_0}^{\lambda_1} F(\lambda) d\mu(\lambda) \tag{3.8}$$

*for every $F \in C[\lambda_0, \lambda_1]$, where*

$$\int_{\lambda_0}^{\lambda_1} F\, d\mu = \frac{1}{2}\left(\int_{(\lambda_0,\lambda_1)} F\, d\mu + \int_{[\lambda_0,\lambda_1]} F\, d\mu\right). \tag{3.9}$$

Upon choosing $f(x) = \chi_{[c,d]}(x)\phi(\lambda_0, x)$ for some fixed $\lambda_0 \in \mathbb{R}$ we have $\hat{f}(\lambda_0) > 0$ and obtain

$$M(z) = \frac{-E_f(z) + \langle f, (H - z)^{-1} f\rangle}{\hat{f}(z)^2} \tag{3.10}$$

for $z$ in a vicinity of $\lambda_0$. Thus $M(z)$ shares many properties of the Herglotz–Nevanlinna function $\langle f, (H - z)^{-1} f\rangle$. In particular, we can use this to associate a measure with $M(z)$ by virtue of the Stieltjes–Livšić inversion formula ([23, Lem. 2.1]).

**Lemma 3.3.** *There is a unique Borel measure $d\rho$ defined via*

$$\frac{1}{2}\Big(\rho\big((\lambda_0, \lambda_1)\big) + \rho\big([\lambda_0, \lambda_1]\big)\Big) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} \mathrm{Im}\big(M(\lambda + \mathrm{i}\varepsilon)\big) d\lambda \tag{3.11}$$

*such that*

$$d\mu_f = |\hat{f}|^2 d\rho, \qquad f \in L^2_c(a, b), \tag{3.12}$$

*where $d\mu_f$ is the spectral measure of $f$ associated with the self-adjoint operator $H$.*

*Proof.* Fix $\lambda_0 < \lambda_1$ and $f \in L^2_c(a, b)$ such that $\hat{f}(\lambda) \neq 0$ for $\lambda \in [\lambda_0, \lambda_1]$. Then (3.10) implies

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} F(\lambda) \mathrm{Im}\big(M(\lambda + \mathrm{i}\varepsilon)\big) d\lambda &= \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} F(\lambda) \mathrm{Im}\left(\frac{m_f(\lambda + \mathrm{i}\varepsilon)}{\hat{f}(\lambda + \mathrm{i}\varepsilon)\hat{f}(\lambda - \mathrm{i}\varepsilon)^*}\right) d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} \frac{F(\lambda)}{|\hat{f}(\lambda)|^2} \mathrm{Im}\big(m_f(\lambda + \mathrm{i}\varepsilon)\big) d\lambda \\
&= \int_{\lambda_0}^{\lambda_1} \frac{F(\lambda)}{|\hat{f}(\lambda)|^2} d\mu_f(\lambda)
\end{aligned}$$

by Lemma 3.2. Hence we can choose $F(\lambda) = 1$ to obtain (3.11) (split the interval into smaller subintervals and use different $f$'s for different subintervals if necessary).

Moreover, replacing $F(\lambda)$ by $|\hat{f}(\lambda)|^2 F(\lambda)$ we obtain

$$\int_{\mathbb{R}} F |\hat{f}|^2 d\rho = \int_{\mathbb{R}} F d\mu_f$$

for every continuous function with compact support away from the real zeros of $\hat{f}(\lambda)$ (note that the zeros are discrete since $\hat{f}(z)$ is entire). Next observe that at every real zero $\lambda_0$ of $\hat{f}$ we have $\mu_f(\{\lambda_0\}) = 0$ (if $\mu_f(\{\lambda_0\}) > 0$, then $\lambda_0$ must be an eigenvalue of $H$ with corresponding eigenfunction $\phi(\lambda_0, .)$ and $\mu_f(\{\lambda_0\}) = |\langle \phi(\lambda_0), f\rangle|^2/\|\phi(\lambda_0)\|^2 = |\hat{f}(\lambda_0)|^2/\|\phi(\lambda_0)\|^2$ contradicting $\hat{f}(\lambda_0) = 0$) and hence we can remove this restriction implying (3.12). □

Now we come to our main result in this section which summarizes the main result from [17].

**Theorem 3.4** ([17]). *Define*

$$\hat{f}(\lambda) = \lim_{c\uparrow b} \int_a^c \phi(\lambda, x) f(x) dx, \tag{3.13}$$

*where the right-hand-side is to be understood as a limit in $L^2(\mathbb{R}, d\rho)$. Then the map*

$$U : L^2(a, b) \to L^2(\mathbb{R}, d\rho), \qquad f \mapsto \hat{f}, \tag{3.14}$$

*is unitary and its inverse is given by*

$$f(x) = \lim_{r\to\infty} \int_{-r}^{r} \phi(\lambda, x) \hat{f}(\lambda) d\rho(\lambda), \tag{3.15}$$

*where again the right-hand-side is to be understood as a limit in $L^2(a, b)$. Moreover, $U$ maps $H$ to multiplication by $\lambda$.*

*Proof.* Equation (3.12) implies

$$\|f\|^2 = \int_{\mathbb{R}} d\mu_f = \int_{\mathbb{R}} |\hat{f}|^2 d\rho = \|\hat{f}\|^2_\rho, \quad f \in L^2_c(a, b),$$

and thus the unique extension of this map to $L^2(a, b)$ is isometric. Here we have used $\|.\|_\rho$, $\langle ., .\rangle_\rho$ to distinguish the norm, scalar product in $L^2(\mathbb{R}, d\rho)$ from the one in $L^2(a, b)$. Moreover,

$$\langle f, F(H)f\rangle = \int_{\mathbb{R}} F d\mu_f = \int_{\mathbb{R}} F|\hat{f}|^2 d\rho = \langle \hat{f}, F\hat{f}\rangle_\rho, \qquad f \in L^2_c(a, b),$$

for every bounded Borel function $F$. By polarization and approximation we even get

$$\langle f, F(H)g\rangle = \langle \hat{f}, F\hat{g}\rangle_\rho, \qquad f, g \in L^2(a, b),$$

for every bounded Borel function $F$.

Now consider $f, g \in L^2(a, b)$ and two bounded Borel functions $F, G$. Set $h = F(H)g$. Then the above equation (with $F \to GF$) implies

$$\int_{\mathbb{R}} G\hat{f}^* (\hat{h} - F\hat{g}) d\rho = 0, \qquad h = F(H)g,$$

for every bounded Borel function $G$ and thus $\hat{f}(\lambda)^*(\hat{h}(\lambda) - F\hat{g}(\lambda)) = 0$ for $\rho$-a.e. $\lambda$. Furthermore, since for every $\lambda_0$ we can find an $f$ such that $\hat{f}(\lambda_0) \neq 0$, we even get $\hat{h} = F\hat{g}$. This shows that $\mathrm{ran}(U)$ contains (e.g.) all characteristic functions of intervals and thus $\mathrm{ran}(U) = L^2(\mathbb{R}, d\rho)$. □

Moreover, the spectral types can be read off from the boundary behavior of the singular Weyl function in the usual way.

**Corollary 3.5.** *The following sets*

$$\begin{aligned} \Sigma_{ac} &= \{\lambda | 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon)) < \infty\},\\ \Sigma_{s} &= \{\lambda | \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon)) = \infty\}, \qquad (3.16)\\ \Sigma_{p} &= \{\lambda | \lim_{\varepsilon\downarrow 0} \varepsilon\mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon)) > 0\},\end{aligned}$$

$$\Sigma = \Sigma_{ac}\cup\Sigma_s = \{\lambda | 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))\} \qquad (3.17)$$

*are minimal supports for $\rho_{ac}$, $\rho_s$, $\rho_{pp}$, and $\rho$, respectively. In fact, we could even restrict ourselves to values of $\lambda$, where the* lim sup *is a* lim *(finite or infinite).*

*Moreover, the spectrum of $H$ is given by the closure of $\Sigma$,*

$$\sigma(H) = \overline{\Sigma}, \qquad (3.18)$$

*the set of eigenvalues is given by*

$$\sigma_p(H) = M_p, \qquad (3.19)$$

*and the absolutely continuous spectrum of $\mu$ is given by the essential closure of $\Sigma_{ac}$,*

$$\sigma(H_{ac}) = \overline{\Sigma}_{ac}^{ess}. \qquad (3.20)$$

*Recall $\overline{\Omega}^{ess} = \{\lambda\in\mathbb{R} | |(\lambda-\varepsilon,\lambda+\varepsilon)\cap\Omega| > 0$ for all $\varepsilon>0\}$, where $|\Omega|$ denotes the Lebesgue measure of a Borel set $\Omega$.*

*Proof.* It suffices to show this result restricted to sufficiently small intervals $[\lambda_0,\lambda_1]$. Now choose $f\in L^2_c(a,b)$ such that $\hat{f}(\lambda)\neq 0$ for $\lambda\in[\lambda_0,\lambda_1]$ as in the previous lemma.

Then, by Lemma 3.1, the above sets (restricted to $[\lambda_0,\lambda_1]$) remain unchanged if we replace $M(z)$ by the Herglotz–Nevanlinna function $m_f(z)$. Moreover, the measures $\mu_f$ and $\rho$ are mutually absolutely continuous on $[\lambda_0,\lambda_1]$ and thus the claim follows from standard results (cf., e.g., [36, Sect. 3.2]). □

We conclude this section with one more simple but useful observation.

**Lemma 3.6.** *Recall the Green function $G(z,x,y)$ of $H$ defined in* (3.5)*. Then*

$$(UG(z,x,.))(\lambda) = \frac{\phi(\lambda,x)}{\lambda-z} \quad \text{and} \quad (U\partial_x G(z,x,.))(\lambda) = \frac{\phi'(\lambda,x)}{\lambda-z} \qquad (3.21)$$

*for every $x\in(a,b)$ and every $z\in\mathbb{C}\setminus\sigma(H)$.*

*Proof.* First of all note that $G(z,x,.)\in L^2(a,b)$ for every $x\in(a,b)$ and $z\in\mathbb{C}\setminus\sigma(H)$. Moreover, from $(H-z)^{-1}f = U^{-1}\frac{1}{\lambda-z}Uf$ we have

$$\int_a^b G(z,x,y)f(y)\,dy = \int_{\mathbb{R}} \frac{\phi(\lambda,x)\hat{f}(\lambda)}{\lambda-z} d\rho(\lambda). \qquad (3.22)$$

Here equality is to be understood in $L^2$, that is, for a.e. $x$ (cf. (3.15)). However, the left-hand side is continuous with respect to $x$ and so is the right-hand side, at least if $\hat{f}$ has compact support. Since this set is dense, the first equality in (3.21) follows. Similarly, the second follows after differentiating (3.22) with respect to $x$. □

Differentiating with respect to $z$ we even obtain

**Corollary 3.7.** *We have*

$$(U\partial_z^k G(z,x,.))(\lambda) = \frac{(-1)^k k!\phi(\lambda,x)}{(\lambda-z)^{k+1}} \text{ and } (U\partial_z^k\partial_x G(z,x,.))(\lambda) = \frac{(-1)^k k!\phi'(\lambda,x)}{(\lambda-z)^{k+1}} \tag{3.23}$$

*for every $x \in (a,b)$, $k \in \mathbb{N}_0$, and every $z \in \mathbb{C}\setminus\sigma(H)$.*

**Remark 3.8.** *We have seen in Remark 2.7 that $M(z)$ is not unique. However, given $\tilde{M}(z)$ as in Remark 2.7, the spectral measures are related by*

$$d\tilde{\rho}(\lambda) = \mathrm{e}^{-2g(\lambda)} d\rho(\lambda).$$

*Hence the measures are mutually absolutely continuous and the associated spectral transformation just differ by a simple rescaling with the positive function $\mathrm{e}^{-2g(\lambda)}$.*

## 4. An integral representation for singular $m$-functions

One of the most important properties of Herglotz–Nevanlinna functions is the existence of an integral representation. Our first lemma gives such an integral representation for our singular $m$-function.

**Lemma 4.1.** *Suppose $\hat{g}(z)$ is an entire function such that $\hat{g}(\lambda) > 0$ for $\lambda \in \mathbb{R}$ and $(1+\lambda^2)^{-1}\hat{g}(\lambda)^{-1} \in L^1(\mathbb{R}, d\rho)$. Then*

$$M(z) = E(z) + \hat{g}(z)\int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right)\frac{d\rho(\lambda)}{\hat{g}(\lambda)}, \tag{4.1}$$

*where $E(z)$ is a real entire function.*

*Proof.* Abbreviate $\hat{g}$ times the integral on the right-hand side of (4.1) by $\tilde{M}(z)$. Since $\tilde{M}(z)$ is holomorphic for $z \in \mathbb{C}\setminus\mathbb{R}$ it suffices to show that $M(z) - \tilde{M}(z)$ is holomorphic near any point $\lambda_0 \in \mathbb{R}$. So fix $\lambda_0 \in \mathbb{R}$ and choose some real-valued $f \in L^2_c(a,b)$ such that $\hat{f}(z)$ defined in (3.1) does not vanish at $\lambda_0$. Then by virtue of Lemma 3.1 we obtain

$$\begin{aligned}
M(z) - \tilde{M}(z) &= -\frac{E_f(z)}{\hat{f}(z)^2} + \frac{m_f(z)}{\hat{f}(z)^2} - \tilde{M}(z)\\
&= -\frac{E_f(z)}{\hat{f}(z)^2} + \frac{1}{\hat{f}(z)^2}\int_{\mathbb{R}\setminus I}\frac{d\mu_f(\lambda)}{\lambda - z} - \hat{g}(z)\int_{\mathbb{R}\setminus I}\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right)\frac{d\rho(\lambda)}{\hat{g}(\lambda)}\\
&\quad - \hat{g}(z)\int_I \frac{\lambda}{1+\lambda^2}\frac{d\rho(\lambda)}{\hat{g}(\lambda)} + \int_I \frac{1}{\lambda - z}\left(\frac{\hat{f}(\lambda)^2}{\hat{f}(z)^2} - \frac{\hat{g}(z)}{\hat{g}(\lambda)}\right)d\rho(\lambda),
\end{aligned}$$

where $I$ is some small interval containing $\lambda_0$ such that $\hat{f}(z)$ does not vanish in a neighborhood of $I$. Now observe that all terms in the above representation are holomorphic near $\lambda_0$. In fact, for the first four terms this is clear and concerning the last one note that the integrand is holomorphic as a function of both variables in a neighborhood of $(\lambda_0, \lambda_0)$. □

In order to find a corresponding $\hat{g}(z)$ note that $\phi(\lambda,x)^2 + \phi'(\lambda,x)^2$ is positive for $\lambda \in \mathbb{R}$ and in $L^1(\mathbb{R}, (1+\lambda^2)^{-1}d\rho)$ by Lemma 3.6.

As a consequence we get a criterion when our singular Weyl $m$-function is a generalized Nevanlinna function with no nonreal poles and the only generalized pole of nonpositive type at $\infty$. We will denote the set of all such generalized Nevanlinna functions by $N_\kappa^\infty$ and refer to Appendix C for further information.

**Theorem 4.2.** *Fix the solution $\phi(z,x)$. Then there is a corresponding solution $\theta(z,x)$ such that $M(z)\in N_\kappa^\infty$ if and only if $(1+\lambda^2)^{-k-1}\in L^1(\mathbb{R},d\rho)$.*

*Proof.* If $(1+\lambda^2)^{-k-1}\in L^1(\mathbb{R},d\rho)$ we can choose $\hat{g}(z)=(1+z^2)^k$ and by the above lemma we have

$$M(z)=f(z)+(1+z^2)^k\int_{\mathbb{R}}\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)\frac{d\rho(\lambda)}{(1+\lambda^2)^k},$$

where $f(z)$ is entire. Hence choosing $\tilde{\theta}(z,x)=\theta(z,x)-f(z)\phi(z,x)$ (cf. Remark 2.7) the corresponding Weyl $m$-function

$$\tilde{M}(z)=(1+z^2)^k\int_{\mathbb{R}}\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)\frac{d\rho(\lambda)}{(1+\lambda^2)^k}\tag{4.2}$$

is of the required type.

Conversely, a generalized Nevanlinna function from $N_\kappa^\infty$ admits the integral representation (C.4)–(C.5), where the measure $d\rho$ coincides with the one from Lemma 3.3. □

Note that the condition $(1+\lambda^2)^{-k-1}\in L^1(\mathbb{R},d\rho)$ is related to the growth of $M(z)$ along the imaginary axis (cf. Lemma C.2). In order to identify possible values of $k$ one can try to bound $\lambda^{-k}$ by a linear combination of $\phi(\lambda,x)^2$ and $\phi'(\lambda,x)^2$ which are in $L^1(\mathbb{R},(1+\lambda^2)^{-1}d\rho)$ as pointed out before.

## 5. An example

Our prototypical example will be the spherical Schrödinger equation given by

$$H_l=-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2},\qquad x\in(0,b),\ l\ge-\frac{1}{2},\tag{5.1}$$

with the usual boundary condition at $x=0$ (for $l\in[-\frac{1}{2},\frac{1}{2})$)

$$\lim_{x\to0}x^l((l+1)f(x)-xf'(x))=0.\tag{5.2}$$

Two linearly independent solutions of

$$-u''(x)+\frac{l(l+1)}{x^2}u(x)=zu(x)\tag{5.3}$$

are given by

$$\phi_l(z,x)=z^{-\frac{2l+1}{4}}\sqrt{\frac{\pi x}{2}}J_{l+\frac{1}{2}}(\sqrt{z}x),\tag{5.4}$$

$$\theta_l(z,x)=-z^{\frac{2l+1}{4}}\sqrt{\frac{\pi x}{2}}\begin{cases}\frac{-1}{\sin((l+\frac{1}{2})\pi)}J_{-l-\frac{1}{2}}(\sqrt{z}x), & l+\frac{1}{2}\in\mathbb{R}_+\setminus\mathbb{N}_0,\\ Y_{l+\frac{1}{2}}(\sqrt{z}x)-\frac{1}{\pi}\log(z)J_{l+\frac{1}{2}}(\sqrt{z}x), & l+\frac{1}{2}\in\mathbb{N}_0,\end{cases}\tag{5.5}$$

where $J_{l+\frac{1}{2}}$ and $Y_{l+\frac{1}{2}}$ are the usual Bessel and Neumann functions [1]. All branch cuts are chosen along the negative real axis unless explicitly stated otherwise. If $l$ is an integer they of course reduce to spherical Bessel and Neumann functions and can be expressed in terms of trigonometric functions (cf. e.g. [36, Sect. 10.4])

In particular, both functions are entire and their Wronskian is given by

$$W(\theta_l(z),\phi_l(z))=1.\tag{5.6}$$

Moreover, on $(0,\infty)$ and $l \in \mathbb{N}_0$ we have

$$\begin{aligned} u_+(z,x) &= \theta_l(z,x) + M_l(z)\phi_l(z,x) \\ &= -(\mathrm{i}\sqrt{-z})^{l+\frac{1}{2}} \sqrt{\frac{\pi x}{2}} \Big( Y_{l+\frac{1}{2}}(\mathrm{i}\sqrt{-z}x) - \mathrm{i} J_{l+\frac{1}{2}}(\mathrm{i}\sqrt{-z}x) \Big) \end{aligned} \tag{5.7}$$

with

$$M_l(z) = \begin{cases} \frac{-1}{\sin((l+\frac{1}{2})\pi)} (-z)^{l+\frac{1}{2}}, & l+\frac{1}{2} \in \mathbb{R}_+ \setminus \mathbb{N}_0, \\ \frac{-1}{\pi} z^{l+\frac{1}{2}} \log(-z), & l+\frac{1}{2} \in \mathbb{N}_0, \end{cases} \tag{5.8}$$

where all branch cuts are chosen along the negative real axis. The associated spectral measure is given by

$$d\rho_l(\lambda) = \chi_{[0,\infty)}(\lambda) \lambda^{l+\frac{1}{2}} \frac{d\lambda}{\pi}, \qquad l \geq -\frac{1}{2}, \tag{5.9}$$

and the associated spectral transformation is just the usual Hankel transform. Furthermore, one infers that $M_l(z)$ is in the generalized Nevanlinna class $N_\kappa^\infty$ with $\kappa = \lfloor l/2 + 3/4 \rfloor$.

For more information we refer to Section 4 of [17], to [11], where the limit circle case $l \in [-1/2, 1/2)$ is considered, and to Section 5 of [15], where the case $q(x) = l(l+1)/x^2 - a/x$ is worked out (see also [10], [28]).

## 6. Exponential growth rates

While the system $\phi(z,x)$ and $\theta(z,x)$ found in Section 2 was good enough to define a singular Weyl $m$-function and an associated spectral measure, it does not suffice for the proof of our local Borg–Marchenko theorem. For this we will need information on the exponential growth rate of the solutions $\phi(z,x)$ and $\theta(z,x)$. To this end we introduce the following additional assumption.

**Hypothesis 6.1.** *Suppose that the spectrum of $H^X_{(a,c)}$, $X \in \{D, N\}$ is purely discrete*

$$\sigma(H^D_{(a,c)}) = \sigma_d(H^D_{(a,c)}) = \{\mu_j(c)\}_{j\in\mathbb{N}}, \quad \sigma(H^N_{(a,c)}) = \sigma_d(H^N_{(a,c)}) = \{\nu_j(c)\}_{j\in\mathbb{N}_0} \tag{6.1}$$

*and bounded below. In addition, suppose one of the following conditions:*

(i) *For all values $s > \frac{1}{2}$ we have*

$$\sum_{j\in\mathbb{N}} \frac{1}{1 + |\mu_j(c)|^s} < \infty. \tag{6.2}$$

(i') *Assume*

$$\mu_j(c) = \left( \frac{\Delta(c)}{\pi} j \right)^2 + o(j^2). \tag{6.3}$$

(ii) *For sufficiently large $j$ we have*

$$\mu_j(c) - \nu_{j-1}(c) \geq \frac{1}{\nu_{j-1}(c)^r} + \frac{1}{\mu_j(c)^r}, \qquad \nu_j(c) - \mu_j(c) \geq \frac{1}{\nu_j(c)^r} + \frac{1}{\mu_j(c)^r}, \tag{6.4}$$

*for some $r > 0$.*

**Remark 6.2.** *Clearly (i) implies that the resolvent $(H^X_{(a,c)}-z)^{-1}$ is trace class and (i') implies (i).*

*The condition (ii) requires that the Dirichlet and Neumann spectra don't get too close to each other. Recall that both spectra are interlacing:*

$$\nu_0(x) < \mu_1(x) < \nu_1(x) < \mu_2(x) < \cdots . \tag{6.5}$$

*Condition (ii) is also reasonable, since by virtue of Krein's theorem* [30, Thm. 27.2.1] *we have*

$$m_-(z) = C(z-\nu_0)\prod_{j\in\mathbb{N}} \frac{1-\frac{z}{\nu_j(c)}}{1-\frac{z}{\mu_j(c)}} = -C'\prod_{j\in\mathbb{N}} \frac{1-\frac{z}{\nu_{j-1}(c)}}{1-\frac{z}{\mu_j(c)}}, \quad C, C' > 0. \tag{6.6}$$

*However, if $\nu_j(c)$ and $\mu_j(c)$ or $\nu_{j-1}(c)$ and $\mu_j(c)$ are eventually too close, this would contradict the well-known asymptotics*

$$m_-(z) = -\sqrt{-z} + O(1), \tag{6.7}$$

*where the branch cut is chosen along the negative real axis.*

*Finally, we note that the assumption that $H$ is bounded from below could be dropped in principle. However, for negative potentials the convergence exponent (i.e., the infimum over all $s$ for which (6.2) holds) can be larger than $\frac{1}{2}$ (see e.g.* [3]*) and hence one has to work with higher order Hadamard products below. In this respect one should also mention a result by Krein* [26] *which states that if $\tau$ is regular at $a=0$ and limit circle at $b=\infty$, then the growth order of $\phi(z,x)$ is in* $[1/2, 1]$. *Moreover, if it is equal to $1/2$ (resp. $1$), then $\phi(z,x)$ is of maximal (resp. minimal) type.*

To this end it will be convenient to look at the eigenvalue counting function

$$N_{H^X_{(a,c)}}(R) = \#\{\lambda\in\sigma(H^X_{(a,c)}) | \lambda \le R\} = \dim\operatorname{ran} P_{H^X_{(a,c)}}((-\infty,R]), \tag{6.8}$$

where $P_{H^X_{(a,c)}}(\Omega)$ is the family of spectral projections associated with the self-adjoint operator $H^X_{(a,c)}$. The connection with our hypothesis is given by the fact that the convergence exponent of $\mu_j(c)$ (i.e., the infimum over all $s$ for which (6.2) holds) is equal to the order of the counting function ([30, Lem. 3.2.2]):

$$\inf\Big\{s\Big| \sum_{j\in\mathbb{N}} \frac{1}{1+|\mu_j(c)|^s} < \infty\Big\} = \limsup_{R\to\infty} \frac{\log(N_{H^D_{(a,c)}}(R))}{\log(R)}. \tag{6.9}$$

Condition (i') of Hypothesis (6.1) is equivalent to

$$N_{H^D_{(a,c)}}(R) = \frac{\Delta(c)}{\pi} R^{1/2} + o(R^{1/2}). \tag{6.10}$$

Moreover, since changing the boundary conditions (at one endpoint) amounts to a rank-one resolvent perturbation, the corresponding eigenvalue counting functions satisfy $|N_{H^D_{(a,c)}}(R) - N_{H^N_{(a,c)}}(R)| \le 1$ (see e.g. [19, Corollary II.2.1]). Hence since we will eventually only be interested in the growth rates of these functions it will suffice to look at $H^D_{(a,c)}$.

Finally, recall that if $H^D_{(a,c)}$ is regular we have the well-known Weyl asymptotics

$$N_{H^D_{(a,c)}}(R) = \frac{c-a}{\pi} R^{1/2} + o(R^{1/2}), \tag{6.11}$$

that is, (6.10) holds with $\Delta(c) = c-a$.

**Lemma 6.3.** *The first condition (i) in Hypothesis 6.1 is independent of the value of $c$ and holds for $H^D_{(a,c)}$ if and only it holds for $H^N_{(a,c)}$. Moreover, condition (i) cannot hold for any $s \le \frac{1}{2}$.*

*Similarly, condition (i') in Hypothesis 6.1 is independent of the value of $c$ and holds for $H^D_{(a,c)}$ if and only it holds for $H^N_{(a,c)}$ (with the same $\Delta(c)$). Moreover,*

$$\Delta(x) = \Delta_0 + x - a. \tag{6.12}$$

*Proof.* As already pointed out, the boundary condition is irrelevant. To see that (i) is independent of $c$ it suffices to consider the operator $\tilde{H}^D_{(a,c+\delta)} := H^D_{(a,c)} \oplus H^D_{(c,c+\delta)}$ which differs from $H^D_{(a,c+\delta)}$ by a rank one resolvent perturbation. Hence

$$N_{\tilde{H}^D_{(a,c+\delta)}}(R) = N_{H^D_{(a,c)}}(R) + N_{H^D_{(c,c+\delta)}}(R) = N_{H^D_{(a,c)}}(R) + \frac{\delta}{\pi} R^{1/2} + o(R^{1/2})$$

by (6.11) since the points $c$ and $c+\delta$ are regular.

Similarly, the last claim of the first part follows using $\tilde{H}^D_{(a,c)} := H^D_{(a,a+\varepsilon)} \oplus H^D_{(a+\varepsilon,c)}$ and

$$N_{\tilde{H}^D_{(a,c)}}(R) \ge N_{H^D_{(a+\varepsilon,c)}}(R) = \frac{c-a-\varepsilon}{\pi} R^{1/2} + o(R^{1/2})$$

since if (6.2) holds for $s = \frac{1}{2}$, then $\int_1^\infty R^{-3/2} N_{H^D_{(a,c)}}(R) dR < \infty$ by [30, Lem. 3.2.1].

The second part follows analogously. □

Now we are ready to draw some first conclusions for $\phi(z,x)$.

**Lemma 6.4.** *Suppose Hypothesis 6.1 (i) holds. Then there is a real entire solution $\phi(z,x)$ of growth order $\frac{1}{2}$. This solution is unique up to a constant and given by*

$$\phi(z,x) = \phi(0,x) \prod_{j\in\mathbb{N}} \Big(1 - \frac{z}{\mu_j(x)}\Big) \tag{6.13}$$

*(if $\mu_{j_0}(x) = 0$ one has to replace $\phi(0,x)(1 - z/\mu_{j_0}(x))$ by $\dot{\phi}(0,x)z$, where the dot indicates a derivative with respect to $z$). Furthermore,*

$$\phi(z,x) = \phi(z,x_0) \mathrm{e}^{-(x-x_0)\sqrt{-z}} (1 + O(1/\sqrt{-z})), \qquad x_0, x \in (a,b), \tag{6.14}$$

*as $|z| \to \infty$ along any nonreal ray.*

*Proof.* By Lemma 2.2 we already know that there is an entire solution $\phi(z,x)$. Moreover, choosing

$$\alpha(z) = \prod_{j\in\mathbb{N}} \Big(1 - \frac{z}{\mu_j(c)}\Big), \qquad \beta(z) = -C' \prod_{j\in\mathbb{N}_0} \Big(1 - \frac{z}{\nu_j(c)}\Big) \tag{6.15}$$

in (2.4), where $C'$ is chosen such that (6.6) holds, we see that there is an entire solution of growth order at most $1/2$ since $\alpha(z)$, $\beta(z)$ are of growth order $1/2$ by Borel's theorem ([30, Thm. 4.3.3]) and the same is true for $c(z,x)$, $s(z,x)$ by [36, Lem. 9.18]. Moreover, any solution of growth order at most $1/2$ is uniquely determined by its zeros and its value (respectively its derivative, if $\phi(0,x) = 0$) at 0 by the Hadamard product theorem [30, Thm. 4.2.1]. But the zeros of $\phi(z,x)$ are precisely the eigenvalues $\mu_j(x)$ of $H^D_{(a,x)}$ and all of them are simple since the Herglotz–Nevanlinna function $m_-(z,x) = \phi'(z,x)/\phi(z,x)$ has a first order pole at every eigenvalue. Thus any entire solution $\phi(z,x)$ of growth order at most $1/2$ is given by (6.13). By Lemma (6.3) and Borel's theorem the growth order of (6.13) is equal to $1/2$.

Finally,

$$\phi(z,x) = \phi(z,c)(c(z,x) - m_-(z)s(z,x))$$

and using the well-known asymptotics of $c(z,x)$, $s(z,x)$, and $m_-(z)$ (cf. [36, Lem. 9.18 and Lem. 9.19]) we see (6.14) for $x_0 = c$ and $x > x_0$. The case $x_0 < x$ follows after reversing the roles of $x_0$ and $x$. Since $c$ is arbitrary the proof is complete. □

**Lemma 6.5.** *Suppose Hypothesis 6.1 (i') holds. Then*

$$\phi(z,x) = \mathrm{e}^{-(\Delta_0 + x - a)\sqrt{-z} + o(z/\mathrm{Im}(\sqrt{-z}))} \tag{6.16}$$

*as $|z| \to \infty$ along any ray except the positive real axis.*

*Proof.* The claim follows from (6.10) and [30, Thm. 12.1.1]. □

Next, we establish a practical condition which ensures that the first part of our hypothesis holds.

**Lemma 6.6.** *Suppose $a \in \mathbb{R}$ and $q(x)$ satisfies*

$$\min(q(x), 0) \ge -\frac{C}{(x-a)^2} - Q(x) \tag{6.17}$$

*with $C < \frac{1}{4}$ and $(x-a)Q(x) \in L^1(a,c)$ or $C = \frac{1}{4}$ and $(x-a)(1 - \log(x-a))Q(x) \in L^1(a,c)$.*

*Then Hypothesis 6.1 (i') holds with $\Delta(x) = x - a$.*

*Proof.* Denote the right-hand side of (6.17) by $\hat{q}(x) := -C(x-a)^2 - Q(x)$. First of all note that the operator $\hat{H}$ associated with $\hat{q}$ has purely discrete spectrum and is bounded from below, by [25, Thm. 2.4] (see also Section 8 below). In particular, the associated differential equation $\hat{\tau} - \lambda$ is nonoscillatory for every $\lambda \in \mathbb{R}$. Hence the same is true for $\tau - \lambda$ by Sturm's comparison theorem and thus $H$ has no essential spectrum and is bounded from below (cf. [37, Thm. 14.9]).

Next, we can again replace $H^D_{(a,c)}$ by $\tilde{H}^D_{(a,c)} := H^D_{(a,a+\varepsilon)} \oplus H^D_{(a+\varepsilon,c)}$ for any $\varepsilon \in (0, c-a)$ since both operators differ by a rank one resolvent perturbation. Denote by $\tilde{N}_{(a,c)}(R)$, $N_{(a,a+\varepsilon)}(R)$, and $N_{(a+\varepsilon,c)}(R)$ the corresponding eigenvalue counting functions. Then,

$$\tilde{N}_{(a,c)}(R) = N_{(a,a+\varepsilon)}(R) + N_{(a+\varepsilon,c)}(R) = \frac{c-a}{\pi} R^{1/2} + N_\varepsilon(R) + o(R^{1/2})$$

with

$$N_\varepsilon(R) = N_{(a,a+\varepsilon)}(R) - \frac{\varepsilon}{\pi} R^{1/2}$$

by (6.11) since the points $a + \varepsilon$ and $c$ are regular. Next, note that the eigenvalue counting function associated with $\hat{H}^D_{(a,a+\varepsilon)}$ satisfies

$$\hat{N}_{(a,a+\varepsilon)}(R) = \frac{\varepsilon}{\pi} R^{1/2} + o(R^{1/2})$$

by [25, Thm. 2.5] (see also Section 8). Since $q(x) \ge \hat{q}(x)$ by assumption, the min-max principle implies

$$N_{(a,a+\varepsilon)}(R) \le \hat{N}_{(a,a+\varepsilon)}(R)$$

and we obtain

$$-\frac{\varepsilon}{\pi} R^{1/2} \le N_\varepsilon(R) \le o(R^{1/2})$$

for any $\varepsilon \in (0, c-a)$. Hence $N_\varepsilon(R) = o(R^{1/2})$ and (6.10) holds with $\Delta(x) = x - a$. This is equivalent to (i') as noted earlier. □

**Remark 6.7.** *This condition is not far from being optimal since any potential $q(x)$ satisfying*

$$q(x) \le -\frac{C}{(x-a)^2}$$

*for some $C > \frac{1}{4}$ fails to be bounded from below. To see this observe that the potential $-C(x-a)^{-2}$ is oscillatory near $a$ and thus the same is true for any potential below it by Sturm's comparison theorem.*

In order to prove our next lemma we will need the following version of the corona theorem for entire functions:

**Theorem 6.8** ([22])**.** *Let $R_s(\mathbb{C})$, $s > 0$, be the ring of all entire functions $f(z)$ for which there are constants $A, B > 0$ such that*

$$|f(z)| \le B\mathrm{e}^{A|z|^s}. \tag{6.18}$$

*Then $f_j(z)$, $j = 1, \dots, n$ generate $R_s(\mathbb{C})$ if and only if*

$$|f_1(z)| + \cdots + |f_n(z)| \ge a\mathrm{e}^{-b|z|^s} \tag{6.19}$$

*for some constants $a, b > 0$.*

*Proof.* This follows from Theorem 1 of [22] upon choosing $\Omega = \mathbb{C}$, $p(z) = |z|^s$, $K_1 = K_2 = 0$, $K_2 = K_4 = \max(1, 2^{s-1})$. □

Moreover, we will need the following technical estimate for Hadamard products.

**Lemma 6.9.** *Let $\{\zeta_j\}_{j=1}^\infty \subseteq \mathbb{C}\backslash\{0\}$ be a sequence with no finite accumulation point and with growth exponent $\rho < 1$. Let $r > 0$. Then for every $s$ with $\rho < s < 1$ there are constants $A$ and $B$ (depending on $s$ and $r$) such that the Hadamard product satisfies*

$$\Big| \prod_{j=1}^\infty \Big(1 - \frac{z}{\zeta_j}\Big) \Big| \ge B \exp(-A|z|^s), \tag{6.20}$$

*except possibly when $z$ belongs to one of the discs $|z - \zeta_j| < |\zeta_j|^{-r}$.*

Arguments like this are well-known (see also [30, Sect. 12]) but we could not find a reference for the particular version required here and hence we provide a proof for the sake of completeness.

*Proof.* We split

$$\prod_{j=1}^\infty \Big(1 - \frac{z}{\zeta_j}\Big) = \prod_{|\zeta_j| > 2|z|} \Big(1 - \frac{z}{\zeta_j}\Big) \prod_{|\zeta_j| \le 2|z|} \Big(1 - \frac{z}{\zeta_j}\Big)$$

By virtue of $|1 - z| \ge \mathrm{e}^{-c|z|}$ for $2|z| \le 1$ (with $c = 2\log(2)$), the first product can be estimated by

$$\Bigg| \prod_{|\zeta_j| > 2|z|} \Big(1 - \frac{z}{\zeta_j}\Big) \Bigg| \ge \prod_{|\zeta_j| > 2|z|} \mathrm{e}^{-c|z/\zeta_j|} \ge \mathrm{e}^{-c 2^{s-1} |z|^s \sum_j |\zeta_j|^{-s}}$$

as required. For the second product we use our requirement $|z-\zeta_j| \geq |\zeta_j|^{-r}$ for all $j$ to obtain

$$\left| \prod_{|\zeta_j|\leq 2|z|} \Big(1-\frac{z}{\zeta_j}\Big) \right| = \prod_{|\zeta_j|\leq 2|z|} \Big|\frac{z-\zeta_j}{\zeta_j}\Big| \geq \prod_{|\zeta_j|\leq 2|z|} |\zeta_j|^{-r-1}.$$

Finally, we have

$$(r+1) \sum_{|\zeta_j|\leq 2|z|} \log|\zeta_j| \leq (r+1) n(2|z|)\log(2|z|) \leq C|z|^s \log|z|$$

for $|z| \geq 2$, where $n(R) = \#\{\zeta_j | |\zeta_j| \leq R\}$ is the counting function for zeros. Hence the claim holds for any $s' > s$ and we are done. □

Now we are ready to show

**Lemma 6.10.** *Suppose Hypothesis 6.1 (i) and (ii) hold. Then for every $s > \frac{1}{2}$ there is a second solution as in Lemma 2.4 with $\gamma(z), \delta(z) \in R_s(\mathbb{C})$. In particular, we have $\theta(., x) \in R_s(\mathbb{C})$ in this case.*

*Proof.* We will fix some $s > \frac{1}{2}$ such that $\alpha(z), \beta(z) \in R_s(\mathbb{C})$. Using (2.8) the Wronski condition implies that we need to find $\gamma(z), \delta(z) \in R_s(\mathbb{C})$ such that

$$\gamma(z)\beta(z) - \delta(z)\alpha(z) = 1. \tag{6.21}$$

This follows from Theorem 6.8 once we show

$$|\alpha(z)| + |\beta(z)| \geq B\exp(-A|z|^s) \tag{6.22}$$

for some constants $A, B > 0$, which depend only on $s$ and $r$.

By (6.5) and Hypothesis 6.1 there is an $N \in \mathbb{N}$ such that the disks $|z-\mu_j| < |\mu_j|^{-r}$ and $|z-\nu_i| < |\nu_i|^{-r}$ are disjoint for all $j, i \geq N$. Hence Lemma 6.9 implies that we have (6.20) for either $\alpha(z)$ or $\beta(z)$ and hence in particular for the sum of both which proves the desired inequality (6.22). If $\gamma(z)$, $\delta(z)$ are not real, $\frac{1}{2}(\gamma(z)+\gamma(z^*)^*)$, $\frac{1}{2}(\delta(z)+\delta(z^*)^*)$ will be. □

**Remark 6.11.** *While under Hypothesis 6.1 (i) a solution $\phi(z,x)$ of growth order at most $\frac{1}{2}$ is unique up to a constant, $\theta(z,x)$ is only unique up to $f(z)\phi(z,x)$, where $f(z)$ is an entire function of growth order at most $\frac{1}{2}$.*

## 7. A local Borg–Marchenko uniqueness result

As in the previous sections we assume that we have an entire system of solutions $\theta(z,x)$ and $\phi(z,x)$ satisfying $W(\theta(z),\phi(z)) = 1$.

**Lemma 7.1.** *The singular Weyl $m$-function $M(z)$ and the solution $\psi(z,x)$ defined in (2.14) have the following asymptotics*

$$M(z) = -\frac{\theta(z,x)}{\phi(z,x)} + O\Big(\frac{1}{\sqrt{-z}\,\phi(z,x)^2}\Big), \tag{7.1}$$

$$\psi(z,x) = \frac{1}{2\sqrt{-z}\,\phi(z,x)}\left(1 + O\Big(\frac{1}{\sqrt{-z}}\Big)\right) \tag{7.2}$$

*as $|z| \to \infty$ in any sector $|\mathrm{Re}(z)| \leq C|\mathrm{Im}(z)|$.*

*Proof.* This follows by solving the well-known asymptotical formula (e.g., combine Lem. 9.19 and (9.101), (9.103) from [36]) for the diagonal of the Green's function

$$G(z,x,x) = \phi(z,x)\psi(z,x) = \frac{1}{2\sqrt{-z}} + O(z^{-1})$$

for $\psi(z,x)$ and $M(z)$. □

In particular, (7.1) shows that asymptotics of $M(z)$ immediately follow once one has corresponding asymptotics for $\theta(z,x)$ and $\phi(z,x)$. Moreover, the leading asymptotics up to $O(z^{-1/2}\phi(z,x)^{-2})$ depends only on the values of $q(x)$ for $x \in (a,c)$ (and on the choice of $\phi(z,x)$ and $\theta(z,x)$). The converse is the content of the following Borg–Marchenko type uniqueness result.

**Theorem 7.2.** *Let $q_0$ and $q_1$ be two potentials satisfying Hypothesis 6.1 (i) and (ii). Let $c \in (a,b)$ and $\phi_j(z,x), \theta_j(z,x) \in R_s(\mathbb{C})$, $j = 0,1$, for some $s \ge \frac{1}{2}$ be given.*

*Then, if $\frac{\phi_1(z,x)}{\phi_0(z,x)} = 1 + o(1)$ for one (and hence by (6.14) for all) $x \in (a,c)$ and $M_1(z) - M_0(z) = O(\phi_0(z,c-\varepsilon)^{-2})$ for every $\varepsilon > 0$ as $z \to \infty$ along some nonreal rays dissecting the complex plane into sectors of length less than $\pi/s$, we have $q_0(x) = q_1(x)$ for $x \in (a,c)$.*

*Proof.* First note that the entire function

$$\begin{aligned}\phi_1(z,x)\theta_0(z,x) - \phi_0(z,x)\theta_1(z,x) =& \phi_1(z,x)\psi_0(z,x) - \phi_0(z,x)\psi_1(z,x)\\ &+ (M_1(z) - M_0(z))\phi_0(z,x)\phi_1(z,x)\end{aligned}$$

vanishes as $z \to \infty$ along our nonreal ray for fixed $x \in (a,c)$. For the first two terms this follows from (7.2) together with our hypothesis that $\phi_0(z,x)$ and $\phi_1(z,x)$ have the same asymptotics. For the last term this follows by our assumption on $M_1(z) - M_0(z)$. Moreover, by our hypothesis this function has an order of growth $s$ and thus we can apply the Phragmén–Lindelöf theorem (e.g., [30, Sect. 6.1]) in the angles bounded by our rays. This shows that the function is bounded on all of $\mathbb{C}$. By Liouville's theorem it must be constant and since it vanishes along a ray, it must be zero; that is, $\phi_1(z,x)\theta_0(z,x) = \phi_0(z,x)\theta_1(z,x)$ for all $z \in \mathbb{C}$ and $x \in (a,c)$. Dividing both sides of this identity by $\phi_0\phi_1$, differentiating with respect to $x$, and using $W(\theta_j(z),\phi_j(z)) = 1$ shows $\phi_1(z,x)^2 = \phi_0(z,x)^2$. Taking the logarithmic derivative further gives $\phi_1'(z,x)/\phi_1(z,x) = \phi_0'(z,x)/\phi_0(z,x)$ and differentiating once more shows $\phi_1''(z,x)/\phi_1(z,x) = \phi_0''(z,x)/\phi_0(z,x)$. This finishes the proof since $q_j(x) = z + \phi_j''(z,x)/\phi_j(z,x)$. □

Note that while at first sight it might look like the condition $\frac{\phi_1(z,x)}{\phi_0(z,x)} = 1 + o(1)$ requires knowledge of $\phi_j(z,x)$, this is not the case, since the high energy asymptotics will only involve some qualitative information on the kind of the singularity at $a$ as we will show in the next section. Moreover, one could even allow different values of $b_0$ and $b_1$ for $q_0$ and $q_1$, respectively.

Finally, assuming Hypothesis 6.1 (i'), then by Lemma 6.5, one can write the condition $M_1(z)-M_0(z) = O(\phi(z,c-\varepsilon)^{-2})$ in the more usual form $M_1(z)-M_0(z) = O(\mathrm{e}^{-2(\Delta_0+c-a-\varepsilon)\sqrt{-z}})$ for every $\varepsilon > 0$. In this respect recall that Hypothesis 6.1 (i') holds under the assumptions of Lemma 6.6 with $\Delta_0 = 0$.

## 8. Applications to perturbed Bessel potentials

Here we want to continue our example from Section 5 and consider perturbations

$$H = H_l + q, \qquad x \in (0, b), \tag{8.1}$$

(taking the same boundary conditions as for $H_l$ at $x = 0$), where $q$ satisfies

**Hypothesis 8.1.** *Let $l \in [-\frac{1}{2}, \infty)$. Set*

$$\tilde{q}(x) = \begin{cases} |q(x)|, & l > -\frac{1}{2}, \\ (1 - \log(x))|q(x)|, & l = -\frac{1}{2}, \end{cases} \tag{8.2}$$

*and suppose $q \in L^1_{loc}(0, \infty)$ is real-valued such that*

$$x\tilde{q}(x) \in L^1(0, 1). \tag{8.3}$$

By [25, Lem 2.2] there is an entire solution $\phi(z, x)$ of growth order $\frac{1}{2}$ which is square integrable near 0. In particular, it is shown that this solution satisfies

$$\phi(z, x) = z^{-\frac{l+1}{2}} \left( \sin\big(\sqrt{z}x - \frac{l\pi}{2}\big) + o\big(\mathrm{e}^{x|\mathrm{Im}(\sqrt{z})|}\big) \right), \tag{8.4}$$

$$\phi'(z, x) = z^{-\frac{l}{2}} \left( \cos\big(\sqrt{z}x - \frac{l\pi}{2}\big) + o\big(\mathrm{e}^{x|\mathrm{Im}(\sqrt{z})|}\big) \right), \tag{8.5}$$

as $|z| \to \infty$.

**Lemma 8.2.** *Suppose Hypothesis 8.1. Then $H$ satisfies Hypothesis 6.1 (i') and (ii) for any $r \geq 0$.*

*Proof.* Without loss of generality choose $c = 1$ in Hypothesis 6.1. Then the claim is immediate from the eigenvalue asymptotics [25, Thm. 2.5]

$$\mu_j = \left( \pi\Big(j + \frac{l}{2}\Big) + \varepsilon_j \right)^2, \tag{8.6}$$

$$\nu_j = \left( \pi\Big(j + \frac{l+1}{2}\Big) + \tilde{\varepsilon}_j \right)^2, \tag{8.7}$$

where $\varepsilon_j, \tilde{\varepsilon}_j \to 0$. □

Moreover, using the detailed asymptotics from [25, Lem 2.2] we can even strengthen the content of Lemma 6.10 in this special case.

**Lemma 8.3.** *Suppose Hypothesis 8.1. Then there is a system of real entire solutions $\phi(z, x)$ and $\theta(z, x)$ with $W(\theta(z), \phi(z)) = 1$ such that $\phi(., x), \theta(., x) \in R_{1/2}(\mathbb{C})$ for every $x > 0$.*

*Proof.* Let us take $c = 1$ for notational simplicity. For the function $\phi(z, x)$ we can take the one from [25, Lem 2.2] (it is unique up to a constant by Remark 6.11). Then the estimates (8.4) and (8.5) imply

$$\begin{aligned} \frac{1}{\sqrt{z}}\beta(z) - \mathrm{i}\alpha(z) &= \frac{1}{\sqrt{z}}\phi'(z, 1) - \mathrm{i}\phi(z, 1) \\ &= z^{-\frac{l+1}{2}} \left( \mathrm{e}^{-\mathrm{i}(\sqrt{z} - \frac{l\pi}{2})} + o\big(\mathrm{e}^{|\mathrm{Im}(\sqrt{z})|}\big) \right). \end{aligned}$$

Hence, for $\mathrm{Im}(z) \geq 0$ (implying $\mathrm{Im}(\sqrt{z}) \geq 0$) we obtain

$$\frac{1}{\sqrt{z}}\beta(z) - \mathrm{i}\alpha(z) = z^{-\frac{l+1}{2}} \mathrm{e}^{\mathrm{Im}(\sqrt{z})} \left( \mathrm{e}^{-\mathrm{i}(\mathrm{Re}(\sqrt{z}) - \frac{l\pi}{2})} + o(1) \right).$$

Since $\alpha(z)$ and $\beta(z)$ have no common zeros this shows

$$|\alpha(z)| + |\beta(z)| \geq B(1+|z|)^{-\frac{l+1}{2}} \mathrm{e}^{|\mathrm{Im}(\sqrt{z})|} \geq B\mathrm{e}^{-|\sqrt{z}|}$$

for $\mathrm{Im}(z) \geq 0$. But since both sides of the inequality do not depend on the sign of $\mathrm{Im}(z)$, the inequality holds for all $z \in \mathbb{C}$. The claim now follows from this estimate as in the proof of Lemma 6.10. □

In particular, the results of the previous sections apply to this example. For instance, Theorem 7.2 now reads as follows.

**Theorem 8.4.** *Let $q_0$ and $q_1$ be two potentials satisfying Hypothesis 8.1 with the same $l$ and let $c \in (0, b)$. Choose $\phi_j(z,x)$ as in (8.4) and $\theta_j(.,x) \in R_{1/2}(\mathbb{C})$ according to Lemma 8.3.*

*Then, if $M_1(z) - M_0(z) = O\big(\mathrm{e}^{-2(c-\varepsilon)|\mathrm{Im}(\sqrt{z})|}\big)$ for every $\varepsilon > 0$ as $z \to \infty$ along any nonreal ray, we have $q_0(x) = q_1(x)$ for $x \in (0, c)$.*

Moreover, our asymptotics for $\phi(z,x)$ imply that $M(z)$ is a generalized Nevanlinna function (cf. Appendix C).

**Theorem 8.5.** *Choose $\phi(z,x)$ as in (8.4). There is a choice for $\theta(z,x)$ such that*

$$M(z) = (1+z^2)^k \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) \frac{d\rho(\lambda)}{(1+\lambda^2)^k}, \quad k = \lceil \frac{l+1}{2} \rceil. \tag{8.8}$$

*In particular, $M(z) \in N_\kappa^\infty$ for some $\kappa \leq k$.*

*Proof.* Just note that by (8.4) and (8.5) we have $\phi(\lambda,x)^2 + \lambda^{-1}\phi'(\lambda,x)^2 = \lambda^{-l-1}(1+o(1))$. Since this function is in $L^1(\mathbb{R}, (1+\lambda^2)^{-1}d\rho)$ the result follows from Theorem 4.2. □

In the case of analytic perturbations this result was first obtained by [15] and [28] (the latter only considered a Coulomb term $q(x) = q_1/x$). Moreover, the correct value of $\kappa = \lfloor \frac{l}{2} + \frac{3}{4} \rfloor$ (cf. Section 5) was determined, whereas the above result gives only an estimate from above. On the other hand, our result does not require analyticity of the perturbation! Determining the correct value of $\kappa$ will require a much more detailed approach and will be given in a follow-up paper [24]. Moreover, our result does not show how to obtain $\theta(z,x)$. This will also be addressed in [24].

## Appendix A. The limit circle case

In this appendix we want to show that the singular Weyl $m$-function is a Herglotz–Nevanlinna function if one has the limit circle case at $a$. We will use the same abbreviations as in Section 2. The results should be compared with [13].

**Hypothesis A.1.** *Let $\phi_0(x)$, $\theta_0(x)$ be two real-valued solutions of $\tau u = \lambda_0 u$ for some fixed $\lambda_0 \in \mathbb{R}$ satisfying $W(\theta_0, \phi_0) = 1$.*

*Assume that the limits*

$$\lim_{x\to a} W_x(\phi_0, u(z)), \qquad \lim_{x\to a} W_x(\theta_0, u(z)) \tag{A.1}$$

*exist for every solution $u(z)$ of $\tau u = zu$.*

**Remark A.2.** *Using the Plücker identity*

$$W_x(f_1,f_2)W_x(f_3,f_4) + W_x(f_1,f_3)W_x(f_4,f_2) + W_x(f_1,f_4)W_x(f_2,f_3) = 0 \tag{A.2}$$

*it is not hard to see, that Hypothesis A.1 is independent of $\lambda_0$.*

It is not hard to see that our hypothesis hold if $\tau$ is limit circle (l.c.) at $a$.

**Lemma A.3.** *Suppose $\tau$ is l.c. at $a$. Then Hypothesis A.1 holds. Moreover, the limits* (A.1) *are holomorphic with respect to $z$ whenever $u(z,x)$ is.*

*Proof.* Given two solutions $u(x)$, $v(x)$ of $\tau u = zu$, $\tau v = \hat{z} v$ it is straightforward to check

$$(z-\hat{z})\int_c^x u(y)v(y)dy = W_x(u,v) - W_c(u,v) \tag{A.3}$$

(clearly it is true for $x = c$, now differentiate with respect to $x$). Hence the limit exists whenever $u$ and $v$ are square integrable near $a$ and the first claim follows.

Moreover, this also shows that

$$\lim_{x\to a} W_x(\phi_0, u(z)) = W_c(\phi_0, u(z)) + (\lambda_0 - z)\int_x^c \phi_0(y)u(z,y)dy$$

and to see the second claim it remains to show that the integral on the right-hand side is if $u(z,x)$ is holomorphic in a neighborhood of $z_0$. Then for all $z$ in a bounded neighborhood of $z_0$ we can estimate

$$|u(z,x)| \le C_1|c(z_0,x)| + C_1|s(z_0,x)|, \quad |u'(z,x)| \le C_3|c'(z_0,x)| + C_4|s'(z_0,x)|.$$

This follows (e.g.) by inspecting the proof of [36, Thm. 9.9]. Hence the limits (A.1) are holomorphic in the same domain as $u(z,x)$. □

It will be shown below in Corollary A.5 that Hypothesis A.1 is in fact equivalent to $\tau$ being l.c. at $x = a$.

So let $\tau$ satisfy Hypothesis A.1 and introduce

$$\begin{aligned} \phi(z,x) &= W_a(c(z),\phi_0)s(z,x) - W_a(s(z),\phi_0)c(z,x),\\ \theta(z,x) &= W_a(c(z),\theta_0)s(z,x) - W_a(s(z),\theta_0)c(z,x). \end{aligned} \tag{A.4}$$

Here the solutions $c(z,x)$ and $s(z,x)$ are defined as in Section 2. Clearly $\phi(z,x)^* = \phi(z^*,x)$ and $\theta(z,x)^* = \theta(z^*,x)$. Moreover, $\phi(\lambda_0,x) = \phi_0(x)$ and $\theta(\lambda_0,x) = \theta_0(x)$.

**Lemma A.4.** *Suppose Hypothesis A.1. Then $\theta(z,x)$, $\phi(z,x)$ defined in* (A.4) *satisfy*

$$W(\theta(z),\phi(z)) = 1 \tag{A.5}$$

*and*

$$\begin{aligned} &W_a(\theta(z),\phi(\hat{z})) = 1,\\ &W_a(\phi(\hat{z}),\phi(z)) = W_a(\theta(\hat{z}),\theta(z)) = 0. \end{aligned} \tag{A.6}$$

*Proof.* It suffices to show the last two identities.

$$\begin{aligned} W_a(\theta,\hat{\phi}) =& - W_a(\hat{c},\phi_0)\big(W_a(c,\theta_0)W_a(\hat{s},s) - W_a(s,\theta_0)W_a(\hat{s},c)\big)\\ &+ W_a(\hat{s},\phi_0)\big(W_a(c,\theta_0)W_a(\hat{c},s) + W_a(s,\theta_0)W_a(\hat{c},c)\big)\\ =& - W_a(\hat{c},\phi_0)W_a(\hat{s},\theta_0) + W_a(\hat{s},\phi_0)W_a(\hat{c},\theta_0)\\ =& W(\hat{c},\hat{s})W(\theta_0,\phi_0) = 1, \end{aligned} \tag{A.7}$$

where we have twice used the Plücker identity (A.2) which remains valid in the limit $x \to a$. Similarly,

$$\begin{aligned} W_a(\hat{\phi},\phi) =& W_a(\hat{c},\phi_0)\big(W_a(c,\phi_0)W_a(\hat{s},s) - W_a(s,\phi_0)W_a(\hat{s},c)\big) \\ &- W_a(\hat{s},\phi_0)\big(W_a(c,\phi_0)W_a(\hat{c},s) - W_a(s,\phi_0)W_a(\hat{c},c)\big) \\ =& W_a(\hat{c},\phi_0)W_a(\hat{s},\phi_0) - W_a(\hat{s},\phi_0)W_a(\hat{c},\phi_0) = 0 \end{aligned} \tag{A.8}$$

and likewise for $W_a(\hat{\theta},\theta)$. □

**Corollary A.5.** *Suppose Hypothesis A.1 then $\theta(z,x)$, $\phi(z,x)$ defined in* (A.4) *satisfy*

$$\begin{aligned} W_c(\phi(z)^*,\phi(z)) &= 2\mathrm{Im}(z)\int_a^c |\phi(z,x)|^2 dx, \\ W_c(\theta(z)^*,\theta(z)) &= 2\mathrm{Im}(z)\int_a^c |\theta(z,x)|^2 dx. \end{aligned} \tag{A.9}$$

*In particular $\tau$ is l.c. at $a$ and $\theta(z,x)$, $\phi(z,x)$ are entire with respect to $z$.*

*Proof.* Choose $u = \phi(z)$ and $v = \phi(z)^*$ in (A.3), let $x \to a$ and use Lemma A.4 for the first formula. For the second replace $\phi(z)$ by $\theta(z)$. □

**Lemma A.6.** *Suppose Hypothesis A.1 and let $H$ be some self-adjoint operator associated with $\tau$ and the boundary condition induced by $\phi_0$ at $a$.*

*Then $\phi(z,x)$ defined in* (A.4) *lies in the domain of $H$ near $a$, that is, $W_a(\phi(z),\phi_0) = 0$. Moreover,*

$$m_-(z) = \frac{W_a(\phi_0,c(z))}{W_a(\phi_0,s(z))}. \tag{A.10}$$

*Proof.* First of all it is easy to check $W_a(\phi(z),\phi_0) = 0$. Moreover, using $0 = W_a(\phi_0,u_-(z)) = W_a(\phi_0,c(z)) - m_-(z)W_a(\phi_0,s(z))$ shows (A.10). □

Now we can introduce the singular Weyl $m$-function $M(z)$ as in Section 2 such that

$$\psi(z,x) = \theta(z,x) + M(z)\phi(z,x) \in L^2(c,b) \tag{A.11}$$

and $\psi(z,x)$ satisfies the boundary condition of $H$ at $b$ if $\tau$ is l.c. at $b$.

**Theorem A.7.** *Suppose Hypothesis A.1 and let $H$ be some self-adjoint operator associated with $\tau$ and the boundary condition induced by $\phi_0$ at $a$.*

*The singular Weyl $m$-function defined in* (A.11) *is a Herglotz–Nevanlinna function and satisfies*

$$\mathrm{Im}(M(z)) = \mathrm{Im}(z)\int_a^b |\psi(z,x)|^2 dx. \tag{A.12}$$

*Proof.* Now choose $c = a$, $u(x) = \psi(z,x)$ and $v(x) = \psi(z,x)^*$ in (A.3),

$$-2\mathrm{Im}(z)\int_a^x |\psi(z,y)|^2 dy = -\mathrm{i}W_x(\psi(z),\psi(z)^*) - 2\mathrm{Im}(M(z)),$$

and observe that $W_x(\psi(z),\psi(z)^*)$ vanishes as $x \uparrow b$ since both functions are in the domain of $H$ near $b$. □

Moreover, we can also strengthen Lemma 3.6.

**Lemma A.8.** *Suppose Hypothesis A.1 and let $H$ be some self-adjoint operator associated with $\tau$ and the boundary condition induced by $\phi_0$ at $a$. Let $U$ be the associated spectral transformation as in Section 3.*

*Then*

$$(U\psi(z,.))(\lambda) = \frac{1}{\lambda - z} \tag{A.13}$$

*for every $z \in \mathbb{C} \setminus \sigma(H)$. Differentiating with respect to $z$ we even obtain*

$$(U\partial_z^k \psi(z,.))(\lambda) = \frac{(-1)^k k!}{(\lambda - z)^{k+1}}. \tag{A.14}$$

*Proof.* From Lemma 3.6 we obtain

$$(U\tilde{\psi}(z,x,.))(\lambda) = \frac{W_x(\theta(z), \phi(\lambda))}{\lambda - z},$$

where

$$\tilde{\psi}(z,x,y) = \begin{cases} \psi(z,y), & y > x, \\ M(z)\phi(z,y), & y < x. \end{cases}$$

Now the claim follows by letting $x \to a$ using Lemma A.4. □

**Remark A.9.** *If $\tau$ is regular at $a$ we can choose $c = a$ and the approach presented here reduces to the usual one (cf.* [36, Sect. 9.3]*).*

## Appendix B. Another example

Our next example is connected with a class of explicit soliton-type solutions of $\tau u = zu$, and we again set $(a,b) = (0,\infty)$. The constructions are similar to the ones in Theorem 1.2 and Proposition 1.3 of [20], the main difference being that here we will be interested in solutions with singularities at $x = 0$.

For simplicity we consider the scalar case though formulas (B.2), (B.5), and the first equality in (B.8) below are written in the standard form which holds also for a general construction, where $A$ and $S$ are matrices; see [20].

Let a vector $\begin{bmatrix} \upsilon_1 & \upsilon_2 \end{bmatrix} \in \mathbb{C}^2$ be given such that

$$0 = \begin{bmatrix} \upsilon_1 & \upsilon_2 \end{bmatrix} J \begin{bmatrix} \upsilon_1 & \upsilon_2 \end{bmatrix}^*, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{B.1}$$

that is,

$$\upsilon_1 \upsilon_2^* = \upsilon_2 \upsilon_1^*.$$

Fix $A \in \mathbb{C} \setminus \{0\}$ and define the vector function $\Lambda(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}$ as the solution of the linear equations with constant coefficients

$$\begin{cases} \Lambda_1'(x) = A\Lambda_2(x), \\ \Lambda_2'(x) = -\Lambda_1(x), \end{cases} \tag{B.2}$$

which satisfies the following initial condition:

$$\Lambda(0) = \begin{bmatrix} \upsilon_1 & \upsilon_2 \end{bmatrix}. \tag{B.3}$$

Define also the function

$$S(x) = \int_0^x |\Lambda_2(t)|^2 dt. \tag{B.4}$$

It follows from (B.1)–(B.4) that the identity

$$AS(x)-S(x)A^*=\Lambda(x)J\Lambda(x)^*, \qquad x\geq 0, \tag{B.5}$$

holds. Furthermore, it is straightforward to check that

$$\Lambda_1(x)=\upsilon_1\cos(\sqrt{A}x)+\upsilon_2\sqrt{A}\sin(\sqrt{A}x), \tag{B.6}$$

$$\Lambda_2(x)=\upsilon_2\cos(\sqrt{A}x)-\frac{\upsilon_1}{\sqrt{A}}\sin(\sqrt{A}x). \tag{B.7}$$

For $x>0$ introduce the transfer matrix function $w_A$ in L. Sakhnovich form

$$w_A(z,x)=I_2+J\Lambda(x)^*S(x)^{-1}(z-A)^{-1}\Lambda(x)=\begin{bmatrix}1+\frac{\Lambda_2(x)^*\Lambda_1(x)}{S(x)(z-A)} & \frac{|\Lambda_2(x)|^2}{S(x)(z-A)}\\ -\frac{|\Lambda_1(x)|^2}{S(x)(z-A)} & 1-\frac{\Lambda_1(x)^*\Lambda_2(x)}{S(x)(z-A)}\end{bmatrix}, \tag{B.8}$$

where $I_2$ is the $2\times 2$ identity matrix. Clearly, $w_A$ is well-defined since $S(x)$ is positive for all $x>0$. Then the potential $q$ corresponding to our triplet $\{\Lambda_1(x),\Lambda_2(x),S(x)\}$ is given by the formula

$$q(x)=2\Big(\big(S(x)^{-1}|\Lambda_2(x)|^2\big)^2+S(x)^{-1}\big(\Lambda_1(x)^*\Lambda_2(x)+\Lambda_2(x)^*\Lambda_1(x)\big)\Big). \tag{B.9}$$

Introduce

$$T(z,x):=T_1(z)\begin{bmatrix}e^{i\sqrt{z}x} & 0\\ 0 & e^{-i\sqrt{z}x}\end{bmatrix}T_1(z)^{-1}, \quad T_1(z)=\begin{bmatrix}1 & 1\\ i\sqrt{z} & -i\sqrt{z}\end{bmatrix}. \tag{B.10}$$

Using (B.2), (B.5), (B.8), and (B.9) one can show [20] that the vector function

$$\begin{aligned} y(z,x)&=\begin{bmatrix}1 & 0\end{bmatrix}w_A(z,x)T(z,x)\\ &=\begin{bmatrix}1+\frac{\Lambda_2(x)^*\Lambda_1(x)}{S(x)(z-A)} & \frac{|\Lambda_2(x)|^2}{S(x)(z-A)}\end{bmatrix}T(z,x),\\ &=\begin{bmatrix}1 & 0\end{bmatrix}T(z,x)+\frac{\Lambda_2(x)^*}{S(x)(z-A)}[\Lambda_1(x)\ \ \Lambda_2(x)]T(z,x)\end{aligned} \tag{B.11}$$

satisfies $\tau u=zu$. Note that

$$T(z,x)=\begin{bmatrix}\cos(\sqrt{z}x) & \sin(\sqrt{z}x)/\sqrt{z}\\ -\sqrt{z}\sin(\sqrt{z}x) & \cos(\sqrt{z}x)\end{bmatrix} \tag{B.12}$$

is an entire matrix function of $z$. Hence, $y(z,x)$ gives a system of linearly independent solutions, which are meromorphic in $z$ with only possible pole at $z=A$. Moreover, it is true that

$$y'(z,x)=\begin{bmatrix}-S(x)^{-1}|\Lambda_2(x)|^2 & 1\end{bmatrix}w_A(z,x)T(z,x). \tag{B.13}$$

It follows from (B.9) that $q$ may have a singularity at zero (it is shown below that $q$ indeed has a singularity). Moreover, according to (B.2), (B.3), and (B.9) $q$ does not change if we replace the initial condition $\Lambda(0)=\begin{bmatrix}\upsilon_1 & \upsilon_2\end{bmatrix}$ by $\Lambda(0)=h\begin{bmatrix}\upsilon_1 & \upsilon_2\end{bmatrix}$ ($h\neq 0$). Thus, without loss of generality assume that

$$\upsilon_1=\upsilon_1^*, \qquad \upsilon_2=1. \tag{B.14}$$

Now, consider the behavior of the functions $\Lambda_k$ and $S$ in a neighborhood of $x=0$. By (B.6) and (B.7) we get

$$\Lambda_1(x)=\upsilon_1+O(x), \quad \Lambda_2(x)=1+O(x), \quad S(x)=x(1+O(x)). \tag{B.15}$$

Therefore, by (B.9), we obtain

$$q(x) = \frac{2}{x^2}(1 + O(x)). \tag{B.16}$$

So, we deal with a system with singularity.

Since $T(z,x) = I_2 + O(x)$, it follows from (B.11) and (B.15) that

$$\tilde{\phi}(z,x) := y(z,x)\begin{bmatrix}1\\ -\upsilon_1\end{bmatrix} \in L^2(0,c). \tag{B.17}$$

Moreover, $\tilde{\phi}$ satisfies $\tau u = zu$ since $y$ does. Using the following equality

$$T(z,x) = \cos(\sqrt{z}x)I_2 + \frac{\sin(\sqrt{z}x)}{\sqrt{z}}\begin{bmatrix}0 & 1\\ -z & 0\end{bmatrix},$$

it is straightforward to check that

$$\begin{aligned}\widetilde{\phi}(z,x) = \frac{1}{S(x)(z-A)}\Big(&\big(S(x)(z-A) + \Lambda_2(x)^*\Lambda_1(x)\big)\big(\cos(\sqrt{z}x) - \upsilon_1\frac{\sin(\sqrt{z}x)}{\sqrt{z}}\big)\\ &- |\Lambda_2(x)|^2\big(\upsilon_1\cos(\sqrt{z}x) + \sqrt{z}\sin(\sqrt{z}x)\big)\Big).\end{aligned} \tag{B.18}$$

Clearly, $\widetilde{\phi}(z,x)$ is meromorphic with unique possible pole at $z = A$. However, it is immediate from (B.6), (B.7), and (B.14) that

$$\cos(\sqrt{z}x) - \upsilon_1\frac{\sin(\sqrt{z}x)}{\sqrt{z}} = \Lambda_2(x) + O(z-A), \quad z \to A; \tag{B.19}$$

$$\upsilon_1\cos(\sqrt{z}x) + \sqrt{z}\sin(\sqrt{z}x) = \Lambda_1(x) + O(z-A), \quad z \to A. \tag{B.20}$$

Formulas (B.18)–(B.20) imply that $\widetilde{\phi}(z,x)$ is entire in $z$.

Let us show that

$$\phi(z,x) := (z-A^*)^{-1}\tilde{\phi}(z,x) = (z-A^*)^{-1}y(z,x)\begin{bmatrix}1\\ -\upsilon_1\end{bmatrix} \tag{B.21}$$

is real-valued on $\mathbb{R}$. To prove the claim it suffices to show that the function

$$F(z) := S(x)(z-A) + \Lambda_1(x)\Lambda_2(x)^* \tag{B.22}$$

is real-valued on $\mathbb{R}$. The latter immediately follows from the identity (B.5).

Next, consider the solution

$$\theta(z,x) = -(z-A)y(z,x)\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{B.23}$$

Straightforward calculations give

$$\theta(z,x) = -\frac{\sin(\sqrt{z}x)}{S(x)\sqrt{z}}\Big(S(x)(z-A) + \Lambda_2(x)^*\Lambda_1(x)\Big) - \frac{\cos(\sqrt{z}x)}{S(x)}|\Lambda_2(x)|^2. \tag{B.24}$$

Using the fact that the function $F(z)$ defined by (B.22) is real for $z \in \mathbb{R}$, we conclude that the solution $\theta(z,x)$ is entire in $z$ and real for $z \in \mathbb{R}$.

Now, let us show that $W(\theta,\phi) = 1$. Indeed, according to (B.11)–(B.13) we have

$$\begin{aligned}W(\theta,\phi) &= \frac{A-z}{z-A^*}\det\left(w_A(z,x)T(z,x)\begin{bmatrix}0 & 1\\ 1 & -\upsilon_1\end{bmatrix}\right)\\ &= (z-A)(z-A^*)^{-1}\det w_A(z,x).\end{aligned} \tag{B.25}$$

By (B.8), using (B.5), we get

$$\det w_A(z,x) = (z-A)^{-1}(z-A^*), \tag{B.26}$$

and hence we derive $W(\theta,\phi)=1$.

The behavior of the discrete spectrum of the operators $H^X_{(0,c)}$ $X\in\{D,N\}$, in this example follows from Theorem 2.5 in [25] and from Lemma 6.3. Solutions $\phi$ and $\theta$, which were constructed above, give an example of the entire solutions that were discussed in Lemma 2.4. Finally, using these solutions we can construct explicitly a singular Weyl $m$-function $M$. Namely, observe that

$$T(z,x)\begin{bmatrix}1\\ \mathrm{i}\sqrt{z}\end{bmatrix}=\begin{bmatrix}\mathrm{e}^{\mathrm{i}\sqrt{z}x}\\ \mathrm{i}\sqrt{z}\mathrm{e}^{\mathrm{i}\sqrt{z}x}\end{bmatrix}\in L^2(c,\infty)$$

for $\mathrm{Im}(\sqrt{z})>0$. Here the branch cut in $\sqrt{z}$ is chosen along the negative real axis. Since the transfer function $w_A(z,x)$ is bounded as $x\to+\infty$, we get

$$\widetilde{\psi}(z,x)=y(x,z)\begin{bmatrix}1\\ \mathrm{i}\sqrt{z}\end{bmatrix}=\begin{bmatrix}1 & 0\end{bmatrix}w_A(z,x)\begin{bmatrix}\mathrm{e}^{\mathrm{i}\sqrt{z}x}\\ \mathrm{i}\sqrt{z}\mathrm{e}^{\mathrm{i}\sqrt{z}x}\end{bmatrix}\in L^2(c,\infty). \tag{B.27}$$

Therefore, by (B.21), (B.23), and (B.27) we get

$$\psi(z,x)=\theta(z,x)+M(z)\phi(z,x)\in L^2(c,\infty),\quad M(z):=-\frac{(z-A)(z-A^*)}{\mathrm{i}\sqrt{z}+\upsilon_1}, \tag{B.28}$$

that is, $M(z)$ above is a singular Weyl $m$-function of our system. Thus, we proved the following lemma.

**Lemma B.1.** *Let the parameters $A\in\mathbb{C}\setminus\{0\}$ and $\upsilon_1\in\mathbb{R}$ be fixed and set $\upsilon_2=1$. Then the relations* (B.9), (B.6), (B.7), *and* (B.4) *explicitly determine a singular potential $q$ satisfying* (B.16). *The corresponding entire solutions $\phi(z,x)$ and $\theta(z,x)$, such that $\phi(z,x)$ is nonsingular at $x=0$ and $W(\theta(z),\phi(z))=1$, are given by* (B.21), (B.18), *and* (B.24), *respectively.*

*Moreover, the singular $m$-function corresponding to this problem is given by* (B.28).

## Appendix C. Generalized Nevanlinna functions

In this appendix we collect some information on the classes $N_\kappa$ of generalized Nevanlinna functions [27]. By $N_\kappa$, $\kappa\in\mathbb{N}_0$, we denote the set of all functions $M(z)$ which are meromorphic in $\mathbb{C}_+\cup\mathbb{C}_-$, satisfy the symmetry condition

$$M(z)=M(z^*)^* \tag{C.1}$$

for all $z$ from the domain $\mathcal{D}_M$ of holomorphy of $M(z)$, and for which the Nevanlinna kernel

$$\mathcal{N}_M(z,\zeta)=\frac{M(z)-M(\zeta)^*}{z-\zeta^*},\quad z,\zeta\in\mathcal{D}_M,\quad z\neq\zeta^*, \tag{C.2}$$

has $\kappa$ negative squares. That is, for any choice of finitely many points $\{z_j\}_{j=1}^n\subset\mathcal{D}_M$ the matrix

$$\{\mathcal{N}_M(z_j,z_k)\}_{1\le j,k\le n} \tag{C.3}$$

has at most $\kappa$ negative eigenvalues and exactly $\kappa$ negative eigenvalues for some choice of $\{z_j\}_{j=1}^n$. Note that $N_0$ coincides with the class of Herglotz–Nevanlinna functions.

Let $M \in N_\kappa$, $\kappa \geq 1$. A point $\lambda_0 \in \mathbb{R}$ is said to be a generalized pole of non-positive type of $M$ if either

$$\limsup_{\varepsilon \downarrow 0} \varepsilon |M(\lambda_0 + \mathrm{i}\varepsilon)| = \infty$$

or the limit

$$\lim_{\varepsilon \downarrow 0} (-\mathrm{i}\varepsilon) M(\lambda_0 + \mathrm{i}\varepsilon)$$

exists and is finite and negative. The point $\lambda_0 = \infty$ is said to be a generalized pole of non-positive type of $M$ if either

$$\limsup_{y \uparrow \infty} \frac{|M(\mathrm{i}y)|}{\mathrm{i}y} = \infty$$

or

$$\lim_{y \uparrow \infty} \frac{M(\mathrm{i}y)}{\mathrm{i}y}$$

exists and is finite and negative. All limits can be replaced by non-tangential limits.

We are interested in the special subclass $N_\kappa^\infty \subset N_\kappa$ of generalized Nevanlinna function with no nonreal poles and the only generalized pole of nonpositive type at $\infty$. It follows from Theorem 3.1 (and its proof) and Lemma 3.3 of [27] that

**Theorem C.1.** *A function $M \in N_\kappa^\infty$ admits the representation*

$$M(z) = (1+z^2)^k \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \frac{d\rho(\lambda)}{(1+\lambda^2)^k} + \sum_{j=0}^{l} a_j z^j, \tag{C.4}$$

*where $k \leq \kappa$, $l \leq 2\kappa + 1$,*

$$a_j \in \mathbb{R}, \quad \textit{and} \quad \int_{\mathbb{R}} (1+\lambda^2)^{-k-1} d\rho(\lambda) < \infty. \tag{C.5}$$

*The measure $\rho$ is given by the Stieltjes–Livšić inversion formula*

$$\frac{1}{2}\big(\rho\big((\lambda_0, \lambda_1)\big) + \rho\big([\lambda_0, \lambda_1]\big)\big) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} \mathrm{Im}\big(M(\lambda + \mathrm{i}\varepsilon)\big) d\lambda. \tag{C.6}$$

*The representation* (C.4) *is called irreducible if $k$ is chosen minimal, that is, either $k = 0$ or $\int_{\mathbb{R}} (1+\lambda^2)^{-k} d\rho(\lambda) = \infty$.*

*Conversely, if* (C.5) *holds, then $M(z)$ defined via* (C.4) *is in $N_\kappa^\infty$ for some $\kappa$. If $k$ is minimal, $\kappa$ is given by:*

$$\kappa = \begin{cases} k, & l \leq 2k, \\ \lfloor \frac{l}{2} \rfloor, & l \geq 2k+1, \ l \textit{ even, or, } l \textit{ odd and } a_l > 0, \\ \lfloor \frac{l}{2} \rfloor + 1, & l \geq 2k+1, \ l \textit{ odd and, } a_l < 0. \end{cases} \tag{C.7}$$

For additional equivalent conditions we refer to Definition 2.5 in [9].

Given a generalized Nevanlinna function in $N_\kappa^\infty$, the corresponding $\kappa$ is given by the multiplicity of the generalized pole at $\infty$ which is determined by the facts that the following limits exist and take values as indicated:

$$\lim_{y \uparrow \infty} -\frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa - 1}} \in (0, \infty], \qquad \lim_{y \uparrow \infty} \frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa + 1}} \in [0, \infty).$$

Again the limits can be replaced by non-tangential ones. This follows from Theorem 3.2 in [29]. To this end note that if $M(z) \in N_\kappa$, then $-M(z)^{-1}$, $-M(1/z)$, and

$1/M(1/z)$ also belong to $N_\kappa$. Moreover, generalized zeros of $M(z)$ are generalized poles of $-M(z)^{-1}$ of the same multiplicity.

**Lemma C.2.** *Let $M(z)$ be a generalized Nevanlinna function given by* (C.4)–(C.5) *with $l < 2k+1$. Then, for every $0 < \gamma < 2$, we have*

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{1+|\lambda|^{2k+\gamma}} < \infty \quad \Longleftrightarrow \quad \int_1^\infty \frac{(-1)^k \mathrm{Im}(M(\mathrm{i}y))}{y^{2k+\gamma}} dy < \infty. \tag{C.8}$$

*Concerning the case $\gamma = 0$ we have*

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{(1+\lambda^2)^k} = \lim_{y\to\infty} \frac{(-1)^k \mathrm{Im}(M(\mathrm{i}y))}{y^{2k-1}}, \tag{C.9}$$

*where the two sides are either both finite and equal or both infinite.*

*Proof.* The first part follows directly from [23, §3.5] (see also [36, Lem. 9.20]). The second part follows by evaluating the limit on the right-hand side using the integral representation plus monotone convergence (see e.g. [23, §4]). □

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