ON THE CAUCHY PROBLEM FOR THE MODIFIED KORTEWEG–DE VRIES EQUATION WITH STEPLIKE FINITE-GAP INITIAL DATA

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ABSTRACT. We solve the Cauchy problem for the modified Korteweg–de Vries equation with steplike quasi-periodic, finite-gap initial conditions under the assumption that the perturbations have a given number of derivatives and moments finite.

1. Introduction

The purpose of the present paper is to investigate the Cauchy problem for the modified Korteweg–de Vries (mKdV) equation

$$v_t(x,t) = -v_{xxx}(x,t) + 6v(x,t)^2 v_x(x,t), \qquad v(x,0) = v(x), \tag{1.1}$$

(where subscripts denote partial derivatives as usual) for the case of steplike initial conditions $v(x)$. More precisely, we will assume that $v(x)$ is asymptotically close to (in general) different real-valued, quasi-periodic, finite-gap potentials $u_\pm(x)$ in the sense that

$$\pm\int_0^{\pm\infty} \left|\frac{d^n}{dx^n}\big(v(x) - u_\pm(x)\big)\right| (1+|x|^{m_0})dx < \infty, \quad 0 \le n \le n_0 + 1, \tag{1.2}$$

for some positive integers m_0, n_0. Here by quasi-periodic, finite-gap potentials we mean algebro-geometric, quasi-periodic, finite-gap potentials which arise naturally as the stationary solutions of the mKdV hierarchy as discussed in [8]. If (1.2) holds for all m_0, n_0 we will call it a Schwartz-type perturbation.

If $u_\pm = 0$ this problem is of course well understood, but for non-decaying initial conditions the only result we are aware of is the one by Kappeler, Perry, Shubin and, Topalov [13]. In order to solve the Cauchy problem for the mKdV equation (1.1) with initial data satisfying (1.2) for suitable m_0, n_0, our main ingredient will be the corresponding result for the KdV equation [3], [5] combined with the Miura transform.

Next, let us state our main result. Denote by $C^n(\mathbb{R})$ the set of functions $x \in \mathbb{R} \mapsto q(x) \in \mathbb{R}$ which have n continuous derivatives with respect to x and by $C^n_k(\mathbb{R}^2)$ the set of functions $(x,t) \in \mathbb{R}^2 \mapsto q(x,t) \in \mathbb{R}$ which have n continuous derivatives with respect to x and k continuous derivatives with respect to t.

Theorem 1.1. *Let $u_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the mKdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $u_\pm(x) = u_\pm(x,0)$. Let $m_0 \ge 8$ and $n_0 \ge m_0 + 5$ be fixed natural numbers.*

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Suppose, that $v(x) \in C^{n_0+1}(\mathbb{R})$ is a real-valued function such that (1.2) *holds. Then there exists a unique classical solution $v(x,t) \in C_1^{n_0-m_0-1}(\mathbb{R}^2)$ of the initial-value problem for the mKdV equation* (1.1) *satisfying*

$$\pm\int_0^{\pm\infty} \left|\frac{\partial^n}{\partial x^n}\big(v(x,t) - u_\pm(x,t)\big)\right| (1+|x|^{\lfloor \frac{m_0}{2}\rfloor -4})dx < \infty, \quad n \le n_0 - m_0 - 1, \tag{1.3}$$

for all $t \in \mathbb{R}$.

In particular, this theorem shows that the mKdV equation has a solution within the class of steplike Schwartz-type perturbations of finite-gap potentials:

Corollary 1.2. *Let $u_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the mKdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $u_\pm(x) = u_\pm(x,0)$. In addition, suppose, that $v(x)$ is a steplike Schwartz-type perturbations of $u_\pm(x)$. Then the solution $v(x,t)$ of the initial-value problem for the mKdV equation* (1.1) *is a steplike Schwartz-type perturbations of $u_\pm(x,t)$ for all $t \in \mathbb{R}$.*

For a unique continuation result within this class of solutions we refer to [4].

2. The KdV equation with steplike finite-gap initial data

As a preparation we recall some basic facts on the Cauchy problem for the KdV equation

$$q_t(x,t) = -q_{xxx}(x,t) + 6q(x,t)q_x(x,t), \qquad q(x,0) = q(x), \tag{2.1}$$

for the case of steplike initial conditions $q(x)$ from [3], [5]. More precisely, we will assume that $q(x)$ is asymptotically close to (in general) different quasi-periodic, finite-gap potentials $p_\pm(x)$ in the sense that

$$\pm\int_0^{\pm\infty} \left|\frac{d^n}{dx^n}\big(q(x) - p_\pm(x)\big)\right| (1+|x|^{m_0})dx < \infty, \quad 0 \le n \le n_0, \tag{2.2}$$

for some positive integers m_0, n_0. The main result reads

Theorem 2.1 ([3])**.** *Let $p_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $p_\pm(x) = p_\pm(x,0)$. Let $m_0 \ge 8$ and $n_0 \ge m_0 + 5$ be fixed natural numbers.*

Suppose, that $q(x) \in C^{n_0}(\mathbb{R})$ is a real-valued function such that (2.2) *holds. Then there exists a unique classical solution $q(x,t) \in C_1^{n_0-m_0-2}(\mathbb{R}^2)$ of the initial-value problem for the KdV equation* (2.1) *satisfying*

$$\pm\int_0^{\pm\infty} \left|\frac{\partial^n}{\partial x^n}\big(q(x,t) - p_\pm(x,t)\big)\right| (1+|x|^{\lfloor \frac{m_0}{2}\rfloor -2})dx < \infty, \quad n \le n_0 - m_0 - 2, \tag{2.3}$$

and

$$\pm\int_0^{\pm\infty} \left|\frac{\partial}{\partial t}\big(q(x,t) - p_\pm(x,t)\big)\right| (1+|x|^{\lfloor \frac{m_0}{2}\rfloor -2})dx < \infty, \tag{2.4}$$

for all $t \in \mathbb{R}$.

In order to invert the Miura transform we will also need the solutions of the associated Lax system.

Introduce the Lax operators corresponding to the finite-gap solutions $p_\pm(x,t)$,

$$\begin{aligned} L_\pm(t) &= -\partial_x^2 + p_\pm(x,t),\\ P_\pm(t) &= -4\partial_x^3 + 6p_\pm(x,t)\partial_x + 3\partial_x p_\pm(x,t). \end{aligned} \tag{2.5}$$

Then the time dependent Baker–Akhiezer functions $\hat\psi_\pm(\lambda,x,t)$ are the unique solutions of the Lax system ([1], [8])

$$\begin{aligned} L_\pm(t)\hat\psi_\pm &= \lambda\hat\psi_\pm,\\ \frac{\partial\hat\psi_\pm}{\partial t} &= P_\pm(t)\hat\psi_\pm, \end{aligned} \tag{2.6}$$

which satisfy $\hat\psi_\pm(\lambda,.,t)\in L^2(0,\pm\infty)$ and are normalized according to $\hat\psi_\pm(\lambda,0,0)=1$. We will denote by $\breve\psi_\pm(\lambda,.,t)$ the other branch which satisfies $\breve\psi_\pm(\lambda,.,t)\in L^2(0,\mp\infty)$.

Similarly, for a solution $q(x,t)$ of the KdV equation as in Theorem 2.1 define the Lax operators $L(t)$ and $P(t)$ as in (2.5) but with $q(x,t)$ in place of $p_\pm(x,t)$.

Lemma 2.2. *Let $q(x,t)$ be a solution of the KdV equation as in Theorem 2.1. Then there exist unique solutions of the Lax system*

$$\begin{aligned} L(t)\hat\phi_\pm &= \lambda\hat\phi_\pm,\\ \frac{\partial\hat\phi_\pm}{\partial t} &= P(t)\hat\phi_\pm, \end{aligned} \tag{2.7}$$

which satisfy $\hat\phi_\pm(\lambda,.,t)\in L^2(0,\pm\infty)$ and are normalized according to

$$\hat\phi_\pm(\lambda,x,t) = \hat\phi_\pm(\lambda,x,t)(1+o(1)) \qquad \text{as} \quad x\to\infty. \tag{2.8}$$

Moreover, we have

$$\hat\phi_\pm(\lambda,x,t) > 0 \qquad \text{for} \quad \lambda\le\inf\sigma(L(t)), \tag{2.9}$$

where $\sigma(L(t))=\sigma(L(0))$ denotes the spectrum of the operator $L(t)$ in $L^2(\mathbb{R})$.

Proof. The first part follows from [5, Lemma 5.1]. To see (2.9) recall that the Weyl solutions of $L(t)\phi=\lambda\phi$ have no zeros for $\lambda<\inf\sigma(L(t))$ and thus $\hat\phi_\pm(\lambda,x,t)>0$ for $\lambda<\inf\sigma(L(t))$ since the same is true for $\hat\psi_\pm(\lambda,x,t)$. Moreover, by continuity we obtain $\hat\phi_\pm(\lambda,x,t)\ge 0$ for $\lambda\le\inf\sigma(L(t))$ and since (nonzero) solutions of a second order equation can only have first order zeros, we obtain (2.9). □

The solutions $\hat\phi_\pm(\lambda,x,t)$ can also be represented with the help of the transformation operators as

$$\hat\phi_\pm(\lambda,x,t) = \hat\psi_\pm(\lambda,x,t) \pm \int_x^{\pm\infty} K_\pm(x,y,t)\hat\psi_\pm(\lambda,y,t)dy, \tag{2.10}$$

where $K_\pm(x,y,t)$ are real-valued functions that satisfy

$$K_\pm(x,x,t) = \pm\frac{1}{2}\int_x^{\pm\infty}(q(y,t)-p_\pm(y,t))dy. \tag{2.11}$$

Moreover, as a consequence of [2, (A.15)], the following estimate is valid

$$\begin{aligned} \left|\frac{\partial^{n+l}}{\partial x^n \partial y^l} K_\pm(x,y,t)\right| \leq & C_\pm(x,t)\Big(Q_\pm(x+y,t) \\ & + \sum_{j=0}^{n+l-1} \left|\frac{\partial^j}{\partial x^j}\big(q(\frac{x+y}{2},t) - p_\pm(\frac{x+y}{2},t)\big)\right|\Big), \end{aligned} \tag{2.12}$$

for $\pm y > \pm x$, where $C_\pm(x,t) = C_{n,l,\pm}(x,t)$ are continuous positive functions decaying as $x \to \pm\infty$ and

$$Q_\pm(x,t) := \pm \int_{\frac{x}{2}}^{\pm\infty} \big|q(y,t) - p_\pm(y,t)\big| dy. \tag{2.13}$$

Finally we recall, that for $\lambda \leq \inf \sigma(L(t))$ the equation $L(t)\phi = \lambda\phi$ has two minimal positive (also known as principal or recessive) solutions which are uniquely determined (up to a multiple) by the requirement

$$\pm \int_0^{\pm\infty} \frac{dx}{\phi_\pm(\lambda,x)^2} = \infty.$$

For $\lambda = \inf \sigma(L(t))$ the two minimal positive solutions could be linearly dependent and the $L(t) - \lambda$ is called critical in this case (and subcritical otherwise). And positive solution can be written as a linear combination of the two minimal positive solutions and in the critical case there is only one positive solution up to multiples. We refer to (e.g.) [12] for further details.

In particular, Lemma 2.2 implies that for $\lambda \leq \inf \sigma(L(t))$ the solutions $\hat{\phi}_\pm(\lambda,x,t)$ are the two minimal positive solutions of $L(t)\phi = \lambda\phi$ and thus any positive solution of this equation is a multiple of

$$\hat{\phi}_\sigma(\lambda,x,t) = \frac{1+\sigma}{2}\hat{\phi}_+(\lambda,x,t) + \frac{1-\sigma}{2}\hat{\phi}_-(\lambda,x,t), \qquad \sigma \in [-1,1]. \tag{2.14}$$

Finally, we also recall the following uniqueness result.

Theorem 2.3 ([3])**.** *Let $p_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $p_\pm(x) = p_\pm(x,0)$. Suppose $q(x,t)$ is a solution of the KdV Cauchy problem satisfying*

$$\pm \int_0^{\pm\infty} \left(|q(x,t) - p_\pm(x,t)| + \left|\frac{\partial}{\partial t}\big(q(x,t) - p_\pm(x,t)\big)\right| \right)(1+x^2)dx < \infty, \tag{2.15}$$

then $q(x,t)$ is unique within this class of solutions.

3. The Miura transformation

Our key ingredient will be the Miura transform [14] and its inversion (see also [6], [9], [10], [11] and the references therein). Let $v(x,t)$ be a (classical) solution of the mKdV equation

$$v_t(x,t) = -v_{xxx}(x,t) + 6v(x,t)^2 v_x(x,t). \tag{3.1}$$

More precisely we will assume that

$$v_t, v_x, \dots, v_{xxxx}, \quad \text{and} \quad v_{xt} \tag{3.2}$$

exist and are continuous.

Then

$$q_j(x,t) = v(x,t)^2 + (-1)^j v_x(x,t), \qquad j = 0, 1, \tag{3.3}$$

are classical solutions of the KdV equation. Moreover,

$$\phi_j(x,t) = \exp\left((-1)^j \int_0^x v(y,t)dy + (-1)^j \int_0^t (2v(0,s)^3 - v_{xx}(0,s)ds\right) \tag{3.4}$$

is a positive solution of

$$-\frac{\partial^2}{\partial x^2}\phi_j(x,t) + q_j(x,t)\phi_j(x,t) = 0, \tag{3.5}$$

$$\frac{\partial}{\partial t}\phi_j(x,t) + \big(\mp 2q_j(x,t)v(x,t) + q_{j,x}(x,t)\big)\phi_j(x,t) = 0. \tag{3.6}$$

In other words, $\phi_j(x,t)$ solves the Lax system

$$L_j(t)\phi_j = 0, \qquad \frac{\partial}{\partial t}\phi_j = P_j(t)\phi_j, \tag{3.7}$$

where the operators $L_j(t)$ and $P_j(t)$ are defined as in (2.5) but with $q_j(x,t)$, $j = 0, 1$, in place of $p_\pm(x,t)$. All claims are straightforward to check.

Conversely, let $q_j(x,t)$ be a solution of the KdV equation and let $\phi_j(x,t)$ be a positive solution of (3.7), then one sees after a quick calculation that

$$v(x,t) = (-1)^j \frac{\partial}{\partial x}\log\phi_j(x,t) \tag{3.8}$$

is a solution of the mKdV equation.

4. Finite-gap solutions of the mKdV equation

In this section we want to briefly look at quasi-periodic, finite-gap solutions of the mKdV equation and their relation to the quasi-periodic, finite-gap solutions of the KdV equation (see also [7], [8]).

Let $u_\pm(x,t)$ be quasi-periodic, finite-gap solutions of the mKdV equation. Fix a number $j = 0$ or $j = 1$ for the Miura transformation. Then

$$p_{\pm,j}(x,t) = u_\pm(x,t)^2 + (-1)^j u_{\pm,x}(x,t) \tag{4.1}$$

are quasi-periodic, finite-gap solutions of the KdV equation. Moreover, it is well-known (see, for example, [9]), that $\inf\sigma(L_{\pm,j}(t)) \geq 0$, where $L_{\pm,j}(t)$ is defined by (2.5). Therefore, a positive solution $\psi_{\pm,j}(x,t)$ defined as in (3.4) with $u_\pm$ instead of v, must be a convex combination of the two branches of the Baker–Akhiezer function $\hat\psi_{\pm,j}(0,x,t)$ and $\check\psi_{\pm,j}(0,x,t)$ corresponding to $p_{\pm,j}(x,t)$, that is,

$$\psi_{\pm,j}(x,t) = (1 - \alpha_{\pm,j}(t))\hat\psi_{\pm,j}(0,x,t) + \alpha_{\pm,j}(t)\check\psi_{\pm,j}(0,x,t). \tag{4.2}$$

Moreover, either 0 is the lowest band edge of $\sigma(L_{\pm,j})$, in which case $\hat\psi_{\pm,j}(0,x,t) = \check\psi_{\pm,j}(0,x,t)$ and $\alpha_{\pm,j}(t)$ drops out, or 0 is below the spectrum $\sigma(L_{\pm,j})$, in which case we must have $\alpha_{\pm,j}(t) = 0$ or $\alpha_{\pm,j}(t) = 1$ (since otherwise 0 would be an eigenvalue of operator, corresponding to the potential $u_\pm(x,t)^2 - (-1)^j u_{\pm,x}(x,t)$).

Since the converse is also true, all quasi-periodic, finite-gap solutions of the mKdV equation arise in this way from quasi-periodic, finite-gap solutions of the KdV equation.

Moreover, by virtue of Theorem 2.3 we can already show the following result which proves the uniqueness part of Theorem 1.1.

Theorem 4.1. *Let $u_\pm(x,t)$ be quasi-periodic, finite-gap solutions of the mKdV equation and $v(x,t)$ a solution of the Cauchy problem for the mKdV equation as above such that $q_0(x,t)$ (or $q_1(x,t)$) satisfies* (2.15). *Then $v(x,t)$ is unique within this class.*

Proof. Let $v(x,t)$ and $\tilde{v}(x,t)$ be two solutions corresponding to the same initial condition $v(x,0) = \tilde{v}(x,0) = v(x)$. Then, by uniqueness for KdV, $q_0(x,t) = \tilde{v}(x,t)^2 + \tilde{v}_x(x,t)$. Moreover, $\phi_0(x,t)$ and $\tilde{\phi}_0(x,t)$ defined by (3.4) both solves (2.7) and coincide for $t=0$. Hence they are equal by [5, Lem. 2.4] and so are $v(x,t)$ and $\tilde{v}(x,t)$. □

5. Proof of the main theorem

Let $u_\pm(x,t)$ be two quasi-periodic, finite-gap solutions of the mKdV equation and suppose $v(x,t)$ is a (classical) solution of the mKdV equation. Then

$$q(x,t) = v(x,t)^2 + (-1)^j\, v_x(x,t) \tag{5.1}$$

is a classical solution of the KdV equation and $p_{\pm,j}(x,t)$, defined by (4.1) are quasi-periodic, finite-gap solutions of the KdV equation. Choose numbers $j_\pm \in \{0,1\}$ for the Miura transform such that (compare (3.4))

$$\begin{aligned} \psi_\pm(x,t) &= \hat{\psi}_{\pm,j_\pm}(0,x,t) \\ &= \exp\left((-1)^{j_\pm}\int_0^x u_\pm(y,t)dy + (-1)^{j_\pm}\int_0^t (2u_\pm(0,s)^3 - u_{\pm,xx}(0,s)ds\right) \end{aligned} \tag{5.2}$$

and thus

$$\frac{\partial}{\partial x}\psi_\pm(x,t) = (-1)^{j_\pm} u_\pm(x,t)\psi_\pm(x,t), \tag{5.3}$$

which is possible by the considerations from the last section.

Lemma 5.1. *Let $u_+(x,t)$ and $v(x,t)$ be as introduced above such that*

$$\int_0^\infty \big(|v(x,t)-u_+(x,t)| + |v_t(x,t)-u_{+,t}(x,t)|\big)dx < \infty. \tag{5.4}$$

Then

$$\phi_+(x,t) := \psi_+(x,t)\exp\left((-1)^{j_+ +1}\int_x^\infty (v(y,t)-u_+(y,t))dy\right) \tag{5.5}$$

is a minimal positive solutions of $(-\partial_x^2 + q_{j_+}(x,t))\phi = 0$. Moreover,

$$\frac{\partial}{\partial x}\phi_+(x,t) = v(x,t)\phi_+(x,t), \tag{5.6}$$

$$\frac{\partial}{\partial t}\phi_+(x,t) = \big((-1)^{j_+} 2q_{j_+}(x,t)v(x,t) - q_{j_+,x}(x,t)\big)\phi_+(x,t). \tag{5.7}$$

Proof. First of all note that $\psi_+(x,t) = \hat{\psi}_{+,j_+}(0,x,t)$ is the minimal positive solutions of $L_{+,j_+}\psi = 0$ and by our choice of j_+ we have (5.3) from which (5.6) is immediate. Similarly, (5.7) follows after a straightforward computation. □

Now we are ready to proof our main theorem: We begin with the initial condition $v(x)$ and define

$$q(x) = v(x)^2 + (-1)^{j_+} v_x(x). \tag{5.8}$$

By our assumptions (1.2) we infer that $q(x)$ satisfies (2.2). Hence, by Theorem 2.1 there is a corresponding solution $q(x,t)$ of the KdV equation and by Lemma 2.2 associated solution $\hat{\phi}_+(\lambda,x,t) := \hat{\phi}_{+,j_+}(\lambda,x,t)$.

Recall (5.2) and define $\phi_+(x)$ by

$$\phi_+(x) := \psi_+(x,0)\exp\left((-1)^{j_++1}\int_x^\infty (v(y)-u_+(y,0))dy\right) \tag{5.9}$$

which, by Lemma 5.1 is a minimal positive solution of $L(0)$. Moreover, since

$$\phi_+(x) = \psi_+(x,0)(1+o(1)) \quad \text{as} \quad x\to\infty \tag{5.10}$$

we conclude

$$\phi_+(x) = \hat{\phi}_+(0,x,0). \tag{5.11}$$

Consequently

$$v(x,t) = \frac{\partial}{\partial x}\log\hat{\phi}_+(0,x,t) \tag{5.12}$$

is a solution of the mKdV equation which satisfies the initial condition

$$v(x,0) = \frac{\partial}{\partial x}\log\hat{\phi}_+(0,x,0) = \frac{\partial}{\partial x}\log\phi_+(x) = v(x) \tag{5.13}$$

as required.

To see (1.3) set $\phi_+(x,t) := \hat{\phi}_+(0,x,t)$ and observe that from (2.10)

$$\frac{\phi_+(x,t)}{\psi_+(x,t)} = 1 + \int_x^\infty K_+(x,y,t)\frac{\psi_+(y,t)}{\psi_+(x,t)}dy, \tag{5.14}$$

and thus

$$1/2 < \frac{\phi_+(x,t)}{\psi_+(x,t)} < 2$$

for $x > x_0(t)$. Moreover, differentiating (5.14) we obtain

$$\begin{aligned} v(x,t) - u_\pm(x,t) &= \frac{\partial}{\partial x}\log\frac{\phi_+(x,t)}{\psi_+(x,t)} \\ &= \frac{\psi_+(x,t)}{\phi_+(x,t)}\Bigg(- K_+(x,x,t) \\ &\qquad + \int_x^\infty \big(K_{+,x}(x,y,t) - u_+(x,t)K(x,y,t)\big)\frac{\psi_+(y,t)}{\psi_+(x,t)}dy\Bigg) \end{aligned} \tag{5.15}$$

which implies

$$|v(x,t) - u_\pm(x,t)| \le C(t)\left(Q(2x,t) + \int_x^\infty Q(x+y,t)dy\right). \tag{5.16}$$

The higher derivatives then follow in a similar fashion using

$$\frac{\partial}{\partial x}\big(v(x,t)-u_\pm(x,t)\big) = q(x,t) - p_+(x,t) - \left(\frac{\phi_{+,x}(x,t)}{\phi_+(x,t)}\right)^2 + \left(\frac{\psi_{+,x}(x,t)}{\psi_+(x,t)}\right)^2.$$

This shows (1.3) for the plus sign. To see it for the minus sign, repeat the argument with j_-.

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