Reducibility of skew-product systems with multidimensional Brjuno base flows

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Abstract

We develop a renormalization method that applies to the problem of the local reducibility of analytic skew-product flows on $\mathbb{T}^d \times \mathrm{SL}(2,\mathbb{R})$. We apply the method to give a proof of a reducibility theorem for these flows with Brjuno base frequency vectors.

1 Introduction

We consider real analytic vector fields on $\mathcal{M} = \mathbb{T}^d \times \mathrm{G}$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ is a Lie group with the Lie algebra $\mathfrak{g}$. Vector fields of the form,

$$X(x,y) = (\omega, f(x)y) \tag{1.1}$$

where $\omega \in \mathbb{R}^d$ and $f : \mathbb{T}^d \to \mathfrak{g}$, generate the following system of differential equations

$$\frac{dx}{dt} = \omega\,, \qquad \frac{dy}{dt} = f(x)y, \tag{1.2}$$

where $(x,y) \in \mathcal{M}$.

We can assume, without restriction, that ω is incommensurate with respect to integers, i.e. $\omega \cdot \nu \neq 0$ for all $\nu \in \mathbb{Z}^d \backslash \{0\}$. The linear flow on the base $\mathbb{T}^d$, given by $x \mapsto x + \omega t$, for $t \geq 0$, is in this case quasiperiodic with frequency vector ω.

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The flow of (1.2), referred to as the skew-product flow, for every $t \in \mathbb{R}$, is the mapping on $\mathcal{M}$,

$$(x, y) \mapsto (x + \omega t, \Phi^t(x)y), \tag{1.3}$$

where $\Phi^t(x)$ is the matrix solution of (1.2) that satisfies $\Phi^0(x) = \mathrm{I}$, with I the identity in G. $\Phi^t(x)$ satisfy the *cocycle property*

$$\Phi^{t+s}(x) = \Phi^t(x + s\omega)\Phi^s(x), \qquad \forall t, s \in \mathbb{R}\,, \quad \forall x \in \mathbb{T}^d. \tag{1.4}$$

On $\mathcal{M}$, one can perform coordinate changes of the form

$$\psi(x, y) = (x, \Psi(x)^{-1}y), \qquad x \in \mathbb{T}^d\,, \quad y \in \mathrm{G}, \tag{1.5}$$

with $\Psi : \mathbb{T}^d \to \mathrm{G}$, that preserve the class of vector fields that generate skew-product flows. Under such a diffeomorphism ψ, a vector field X of the form (1.1), transforms as $X \mapsto \psi^* X := (D\psi)^{-1} X \circ \psi$, where the pullback of the vector field X under a coordinate change ψ is given by

$$\psi^* X(x, y) = (\omega, \Psi^\star f(x)y)\,, \qquad \Psi^\star f = \Psi^{-1}[f - D_\omega]\Psi. \tag{1.6}$$

Here D_ω is the directional derivative on $\mathbb{T}^d$ in the direction of ω.

The problem of reducibility of the system (1.2) concerns the existence of a coordinate change $z = \Psi(x)^{-1}y$ that transforms the system into

$$\frac{dx}{dt} = \omega\,, \qquad \frac{dz}{dt} = uz, \tag{1.7}$$

with a constant function $x \mapsto u \in \mathfrak{g}$.

Definition 1.1 We say that X is (C^ω) *conjugated* to Y if there exists a (C^ω) coordinate change ψ on $\mathcal{M}$, such that $\psi^* X = Y$. We call X of the form (1.1) *reducible* if it is conjugated to K, where $K(x, y) = (\omega, uy)$, with some constant element $u \in \mathfrak{g}$.

The system of equations (1.2) is equivalent to a linear system of ODE's with time dependent coefficients. In the case $d = 1$, the coefficients are periodic in time and the results of Floquet and Lyapunov show that the system is always reducible.

We are interested in the case $d > 1$ and the following reducibility problem: given a constant matrix $u \in \mathfrak{g}$ and the associated system in constant coefficients (1.7), we are interested in the reducibility of systems "nearby" to the one with constant coefficients. This is an old and well-studied problem in dynamics. The problem typically involves small divisors, and further conditions are needed in order to achieve reducibility.

Classical results [4, 8] and the later extensions [9, 16, 22, 23, 30] have already shown that this is possible in the case of Diophantine frequency vectors ω, i.e. those for which there exist a constant $\kappa > 0$ and $\beta \geq 0$, such that

$$|\omega \cdot \nu| \geq \kappa \|\nu\|^{-(d-1+\beta)}\,, \qquad \forall \nu \in \mathbb{Z}^d \backslash \{0\}. \tag{1.8}$$

In the case of $\mathrm{G} = \mathrm{SL}(2, \mathbb{R})$, motivated by the study of one-dimensional Schrödinger equation with a quasi-periodic potential, reducibility results have been obtained in the perturbative regime by KAM methods by Dinaburg and Sinai and Eliasson [8, 9], following the seminal ideas of Moser [29]. Eliasson has obtained, in the continuous-time case, the reducibility of analytic skew-product systems, close to a constant coefficient system of the form (1.7), in the case of Diophantine ω and fibered rotation number (associated to eigenvalues of u) either Diophantine or rational with respect to ω. The closeness condition does not depend on the fibered rotation number. A discrete-time version of this result was obtained in [1]. In the case of compact groups G, local reducibility results have been obtained in [22, 23].

A non-perturbative version of the Eliasson result [9] (the closeness to a constant does not depend on the frequency) in the single frequency case can be found in [3]. In the multifrequency case, it was shown by Bourgain that the results fail to extend to the non-perturbative regime [5].

Another approach to the reducibility problems is based on renormalization methods. For discrete-time cocycles on G, i.e. pairs $(\alpha, A) : (x, y) \mapsto (x + \alpha, A(x)y)$ on $\mathbb{T}^d \times \mathrm{G}$, over rotations by an irrational $\alpha \in \mathbb{T}^1$, and for $\mathrm{G} = \mathrm{SU}(2)$, a renormalization method was introduced by Rychlik [31]. The method was later improved by Krikorian who obtained several global results on reducibility of cocycles [24, 25]. Similar methods have been applied in [15] to prove that, for almost all α, a measurable conjugacy of an analytic quasi-periodic cocycle on $\mathrm{SO}(n, \mathbb{R})$ to a constant, implies analytical reducibility. Avila and Krikorian have proved a global reducibility result for analytic cocycles on $\mathrm{SL}(2, \mathbb{R})$ [2]: for almost every α, if such a cocycle is L^2-conjugated to some constant cocycle in $\mathrm{SO}(2, \mathbb{R})$, then it is analytically reducible to it.

Different renormalization methods have been applied to the problem of reducibility of skew-product flows on $\mathbb{T}^d \times \mathrm{SL}(2, \mathbb{R})$ in [20, 27]. These methods are similar to the ones previously used in the context of invariant tori for Hamiltonian flows [21] and more general vector fields on $\mathbb{T}^d \times \mathbb{R}^m$ [18]. For skew-product flows, the $d = 2$, analytic, Brjuno case was considered by Lopes Dias [27], while Koch and Lopes Dias considered the Diophantine multi-frequency case [20], in the C^k setting, with k finite and sufficiently large. The former approach relies on the estimates coming from the one-dimensional continued fraction algorithm and is consequently restricted to $d = 2$. In the latter, the results have been extended to $d > 2$, in the Diophantine case, by using estimates from a multidimensional continued fraction algorithm [17].

In this paper, we develop a renormalization method that applies to the problem of reducibility of analytic skew-product flows with multidimensional Brjuno base rotation. It extends the results of [27] to dimensions $d > 2$, and the results of [20] (in the analytic case) to Brjuno frequency vectors.

Definition 1.2 A vector $\omega \in \mathbb{R}^d$ is *Brjuno* if the following condition is satisfied [6, 7]

$$\sum_{n=1}^{\infty} 2^{-n} \ln(1/\Omega_n) < \infty\,, \qquad \Omega_n := \min_{\nu \in \mathbb{Z}^d, 0<|\nu|\le 2^n} |\omega \cdot \nu|\,. \tag{1.9}$$

The set of Brjuno vectors is of full Lebesgue measure. In particular, it contains all Diophantine vectors, for all $\kappa > 0$ and all $\beta \geq 0$. Conversely, there are vectors that are Brjuno and are not Diophantine for any κ and β.

A condition less restrictive than Diophantine (that agrees with Brjuno in the $d = 2$ case) has also been recently considered in [28], in the contexts of linearizability of torus flows. The multidimensional Brjuno ω case has also been recently considered in the skew-product case by Gentile [13], using a different renormalization method based on resummation of perturbation series as in quantum field theory (see e.g. [11, 12]). In fact, Gentile's results are valid under a Brjuno condition on the fibered rotation number with respect to ω.

We will consider vector fields close to an integrable vector field $K(x, y) = (\omega, uy)$, with $u \in \mathfrak{sl}(2, \mathbb{R})$ Diophantine with respect to ω. We will assume that u has purely imaginary eigenvalues.

Definition 1.3 A matrix $u \in \mathfrak{sl}(2, \mathbb{R})$ with eigenvalues $\pm i\rho$ is *Diophantine with respect to ω* if there exists constants $\mathcal{C}, \tau > 0$ such that

$$|\omega \cdot \nu - 2\rho| > \frac{\mathcal{C}}{|\nu|^{\tau}}\,, \qquad \forall \nu \in \mathbb{Z}^d \backslash \{0\}. \tag{1.10}$$

As a corollary of our main result (Theorem 1.5 below), we will obtain the following.

Theorem 1.4 *Let $\omega \in \mathbb{R}^d$ be a Brjuno vector and K a vector field of the form $K(x, y) = (\omega, uy)$, with $u \in \mathfrak{sl}(2, \mathbb{R})$ Diophantine with respect to ω. If X is a real analytic vector field on $\mathbb{T}^d \times \mathrm{SL}(2, \mathbb{R})$ of the form (1.1) with the base frequency vector ω, C^{ω}-sufficiently close to K, then there exists $v \in \mathfrak{sl}(2, \mathbb{R})$ such that X_v, where $X_v(x, y) = X(x, y) + (0, vy)$, for $(x, y) \in \mathbb{T}^d \times \mathrm{SL}(2, \mathbb{R})$, is C^{ω}-reducible to K. The map $X \mapsto v$ is analytic.*

This theorem is often referred to as a normal form theorem [23]. We will not use this terminology here, in order to avoid the confusion with a different theorem that is an essential part of the methods used here. We note that this result can be used to improve the arithmetic condition on ω in the main result of [8]. Following Chapter 4 of [23], by looking at the dependence of v on u, and using Whitney extension, one can show the existence of a positive measure set of parameters (energy) for which the one-dimensional quasi-periodic Schrödinger equation is reducible.

The renormalization approach that we pursue here provides an alternative method to KAM theory to deal with small divisors. By scaling the torus, one turns some of them into "large divisors", that can be eliminated by a coordinate change. Construction of such a coordinate change (see Section 6) is an important part of our renormalization method that differs it from renormalization methods used for discrete time cocycles (see e.g. [2]). These are the basic steps of a one step renormalization operator $\mathcal{R}$, defined between spaces of vector fields of the form (1.1). We will construct a sequence of renormalization operators, with an integrable fixed set, and show that there is a "stable manifold" associated to them. We will show that the vector fields on the stable manifold are reducible. The result of Theorem 1.4 is obtained simply by showing that the family of vector fields parameterized by v intersects the stable manifold. The proof is given at the end of Section 5.

The scaling of $\mathbb{T}^d$ is achieved by a sequence of non-integer matrices $T_n \in \mathrm{GL}(d,\mathbb{R})$. This is in contrast to the earlier approaches where a sequence of scaling matrices in $\mathrm{SL}(d,\mathbb{Z})$ is generated from a continued fraction algorithm. The flow on a homogeneous space $\mathrm{SL}(d,\mathbb{Z})\backslash\mathrm{SL}(d,\mathbb{R})$ can provide a way to construct a sequence of matrices $P_n \in \mathrm{SL}(d,\mathbb{Z})$ and the corresponding matrices $T_n = P_{n-1}^{-1}P_n$ [17, 26]. The flow is generated by the right action on a matrix M_ω associated to $\omega \in \mathbb{R}^d$, of a one-parameter subgroup E^t, with $t \geq 0$, of $\mathrm{SL}(d,\mathbb{R})$, that has the property to expand the direction of ω and contract perpendicular directions. Given a sequence of stopping times $n \mapsto t_n$, $n \in \mathbb{N}$, one can generate a sequence of matrices $P_n \in \mathrm{SL}(d,\mathbb{Z})$, whose left action brings the resulting matrix back to the fundamental domain for the left action of $\mathrm{SL}(d,\mathbb{Z})$ on $\mathrm{SL}(d,\mathbb{R})$. In order for these matrices to be useful for renormalization, one needs to have a good control of the matrices P_n. In [17], the appropriate bounds have been obtained for the case of Diophantine frequency vectors. In the approach pursued here, applicable to the larger set of Brjuno frequency vectors, we perform the scaling of $\mathbb{T}^d$ using non-integer matrices, that expand the direction of ω and contract all perpendicular directions. The scaling produces a deformation of the Fourier lattice $\mathcal{V}$ and we control the deformation produced by repeated scaling.

Let $\mathbb{I}$ be the identity operator and $\mathbb{E}$ be the torus averaging projection operator on spaces of analytic functions from $\mathbb{T}^d$ into $\mathfrak{g}$ (see Section 2). We will use the same symbols for the corresponding operators on spaces of vector fields on $\mathbb{T}^d \times \mathrm{G}$ of the form (1.1). In particular, $\mathbb{E}X(x,y) = (\omega, \mathbb{E}f(x)y)$.

Theorem 1.5 *Let $\omega \in \mathbb{R}^d$ be a Brjuno vector and K a vector field of the form $K(x,y) = (\omega, uy)$, with $u \in \mathfrak{sl}(2,\mathbb{R})$ Diophantine with respect to ω. There exists an open neighborhood $\mathcal{D}_0$ of K of real analytic vector fields on $\mathbb{T}^d \times \mathrm{SL}(2,\mathbb{R})$, of the form* (1.1)*, and an analytic function $W : (\mathbb{I} - \mathbb{E})\mathcal{D}_0 \to \mathbb{E}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. Every vector field X on the graph $\mathcal{W}$ of W is C^ω reducible to K via a conjugacy ψ_X of the form* (1.5)*. Moreover, the map $X \mapsto \psi_X$ is analytic.*

This theorem follows from the results of Section 4 (Theorem 4.7) and Section 5 (Theorem 5.4). It immediately implies Theorem 1.4.

We remark that the frequency vector ω is fixed, and an open neighborhood of K means

$\{\omega\} \times B$, where B is an open neighborhood of u in the space of analytic functions from $\mathbb{T}^d$ into $\mathfrak{sl}(2,\mathbb{R})$. Our second remark is that our estimates are uniform within classes of Brjuno vectors ω and $u \in \mathfrak{sl}(2,\mathbb{R})$ Diophantine with respect to ω, described at the end of Section 4.1.

The paper is organized as follows. In Section 2 we introduce the spaces of analytic vector fields and maps that we consider. In Section 3 we construct a one step renormalization operator. In Section 4 we construct a sequence of renormalization operators with a "trivial" limit set and show that there exists a stable manifold $\mathcal{W}$ of vector fields associated to this set. Section 5 deals with the reducibility of vector fields on the stable manifold. Section 6 is independent, and contains a proof of a normal form theorem for skew-product flows on $\mathbb{T}^d \times \mathrm{G}$, that has been used in the construction of a single renormalization step.

2 Preliminaries

2.1 Spaces of vector fields with skew-product flows

On $\mathbb{R}^n$ and $\mathbb{C}^n$ we use the following norms: $|v| = \sum_{j=1}^n |v_j|$ and $\|v\| = \max_j |v_j|$. Let $\varrho > 0$. Denote by D_ϱ the set of all points $x \in \mathbb{C}^d$ characterized by $\|\operatorname{Im} x\| < \varrho$.

Let $\{e_1, \ldots, e_d\}$ be a basis and $\mathcal{Z} = \{\sum_{i=1}^d z_i e_i | z_i \in \mathbb{Z}\}$ be a lattice in $\mathbb{R}^d$. Let also $\mathcal{V}$ be its dual lattice, i.e. the set of points $v \in \mathbb{R}^d$ satisfying $\exp(iv \cdot z) = 1$ for all $z \in \mathcal{Z}$.

Let $\mathrm{F}^{n\times n}$ be the space of $n \times n$ matrices with entries in F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We will consider functions defined from the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$ into $\mathbb{R}^{n\times n}$, that can be regarded as maps from $\mathbb{R}^d$ into $\mathbb{R}^{n\times n}$, by lifting to the universal cover. We will make use of analyticity to extend them to the complex domain D_ϱ. Let $\mathcal{F}_\varrho(\mathcal{V})$ be the Banach space of all $\mathcal{Z}$-periodic analytic maps $f : D_\varrho \to \mathbb{C}^{n\times n}$, that can be expanded in Fourier series as

$$f(x) = \sum_{v\in\mathcal{V}} f_v e^{iv\cdot x}\,, \tag{2.1}$$

where $v \cdot x = \sum_j v_j x_j$, and for which the following norm is finite

$$\|f\|_\varrho = \sum_{v\in\mathcal{V}} \|f_v\| e^{\varrho|v|}. \tag{2.2}$$

Here and in what follows, for matrices M with matrix elements M_{ij}, we use the operator norm corresponding to the $|\cdot|$ norm of vectors, that is the matrix norm $\|M\| = \max_j \sum_i |M_{ij}|$. The space $\mathcal{F}_\varrho(\mathcal{V})$, with the norm $\|\cdot\|_\varrho$ is a Banach algebra.

The torus averaging projection operator $\mathbb{E}$ is defined on functions $f \in \mathcal{F}_\varrho(\mathcal{V})$, with the expansion (2.1), as $\mathbb{E}f = f_0$. The identity operator acts as $\mathbb{I}f = f$.

We define $\mathcal{F}'_\varrho(\mathcal{V})$ to be the subspace of all maps in $f \in \mathcal{F}_\varrho(\mathcal{V})$ for which the norm

$$\|f\|'_\varrho = \|D_\omega f\|_\varrho + \|f\|_\varrho \tag{2.3}$$

is finite. $D_\omega f = \sum_{i=1}^{d} \omega_i \cdot \frac{\partial}{\partial x_i} f$ is the directional derivative of f in the direction of $\omega \in \mathbb{R}^d$.

Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{C})$ be a Lie group and $\mathfrak{g}$ its corresponding Lie algebra. The subspace of maps in $\mathcal{F}_\varrho(\mathcal{V})$ that take values in $\mathfrak{g}$ is denoted by $\mathcal{F}_\varrho(\mathfrak{g}, \mathcal{V})$. Similarly, the subset of maps in $\mathcal{F}_\varrho(\mathcal{V})$ that take values in G is denoted by $\mathcal{F}_\varrho(\mathrm{G}, \mathcal{V})$. Analogously, we can define $\mathcal{F}'_\varrho(\mathfrak{g}, \mathcal{V})$ and $\mathcal{F}'_\varrho(\mathrm{G}, \mathcal{V})$.

The space of vector fields on $D_\varrho \times \mathrm{G}$ that we consider, $\mathcal{A}_\varrho(\mathcal{V})$, is an affine space of vector fields X, of the form $X(x, y) = (\omega, f(x)y)$, for $(x, y) \in D_\varrho \times \mathrm{G}$, where $f \in \mathcal{F}_\varrho(\mathfrak{g}, \mathcal{V})$, with the norm $\|X\|_\varrho = \|\omega\| + \|f\|_\varrho$. The space $\mathcal{A}_\varrho(\mathcal{V})$ corresponding to $\omega = 0$ is a Banach spaces that will be denoted by $\mathcal{A}^0_\varrho(\mathcal{V})$. The norm of $Z \in \mathcal{A}^0_\varrho(\mathcal{V})$ with $Z(x, y) = (0, f(x)y)$ is thus $\|Z\|_\varrho = \|f\|_\varrho$. For these spaces, we will also use the notation $\mathcal{A}_\varrho(\mathfrak{g}, \mathcal{V})$ and $\mathcal{A}^0_\varrho(\mathfrak{g}, \mathcal{V})$, respectively, when we would like to emphasis which group G we consider. The subspace of vector fields $Z \in \mathcal{A}^0_\varrho(\mathcal{V})$ for which the norm $\|Z\|'_\varrho = \|f\|'_\varrho$ is finite will be denoted by $\mathcal{A}^{0'}_\varrho(\mathcal{V})$.

We define $\mathcal{B}_\varrho(\mathcal{V})$ as the space of maps ψ on $D_\varrho \times \mathrm{G}$, of the form $\psi(x, y) = (x, \Psi^{-1}(x)y)$, where $\Psi \in \mathcal{F}_\varrho(\mathrm{G}, \mathcal{V})$. The distance from the identity map $\mathrm{Id} \in \mathcal{B}_\varrho(\mathcal{V})$ is measured by the norm $\|\psi - \mathrm{Id}\|_\varrho = \|\Psi^{-1} - \mathrm{I}\|_\varrho$. The subspace of maps $\psi \in \mathcal{B}_\varrho(\mathcal{V})$ for which the norm $\|\psi - \mathrm{Id}\|'_\varrho = \|\Psi^{-1} - \mathrm{I}\|'_\varrho$ is finite, will be denoted by $\mathcal{B}'_\varrho(\mathcal{V})$.

For the maps between Banach spaces, we will use the following notion of analyticity [14]. Such a map will be called analytic if it is Fréchet differentiable. Thus, sums, products and compositions of analytic maps are analytic. We also have that the limit of a uniformly convergent sequence of such analytic maps is analytic.

A map $X \mapsto \Psi$, from $\{\omega\} \times B \subset \mathcal{A}_\varrho(\mathcal{V})$ into $\mathcal{F}_\delta(\mathcal{V})$, will be called analytic if the map $f \mapsto \Psi$, from $B \subset \mathcal{F}_\varrho(\mathfrak{g}, \mathcal{V})$ into $\mathcal{F}_\delta(\mathcal{V})$, is analytic. The map $X \mapsto \psi$ of the form $\psi(x, y) = (x, \Psi^{-1}(x)y)$, from $\{\omega\} \times B \subset \mathcal{A}_\varrho(\mathcal{V})$ into $\mathcal{B}_\delta(\mathcal{V})$, will be called analytic if the map $X \mapsto \Psi$ is analytic.

In the following, when the lattice $\mathcal{V}$ is fixed and no ambiguity can arise, we will simplify the notation by writing $\mathcal{A}_\varrho$ instead of $\mathcal{A}_\varrho(\mathcal{V})$, and analogously for other spaces.

2.2 Lie algebra $\mathfrak{sl}(2, \mathbb{R})$

The focus of this paper will be on the Lie group $\mathrm{G} = \mathrm{SL}(2, \mathbb{R})$ and its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. The group $\mathrm{G} = \mathrm{SL}(2, \mathbb{R})$ is a non-compact Lie group of real 2×2 matrices with determinant 1. The corresponding Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ consists of 2×2 real traceless matrices. The Lie bracket is given by the usual matrix commutator. A basis in $\mathfrak{sl}(2, \mathbb{R})$ is given by $\{H, E_+, E_-\}$, where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.4}$$

These matrices satisfy the following commutation relations

$$\mathrm{ad}_H E_+ = 2E_+, \qquad \mathrm{ad}_H E_- = -2E_-, \qquad \mathrm{ad}_{E_+} E_- = H. \tag{2.5}$$

The adjoint operator ad_u on $\mathfrak{g}$, for a given $u \in \mathfrak{g}$, is defined by the action $\mathrm{ad}_u v = uv - vu$, on an arbitrary $v \in \mathfrak{g}$.

Given $u \in \mathfrak{sl}(2, \mathbb{R})$ with imaginary eigenvalues, there exists a unique ρ, such that

$$u = \rho M \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M^{-1}, \tag{2.6}$$

for some $M \in \mathrm{SL}(2, \mathbb{R})$.

The spectrum of the adjoint operator ad_u, with $u \in \mathfrak{sl}(2, \mathbb{R})$, is $\{0, 2i\rho, -2i\rho\}$, where ρ is the unique number from the above decomposition.

3 Single renormalization step

Let $\varrho > 0$ and let $\omega \in \mathbb{R}^d$ be a fixed vector of unit length. Let K be a fixed vector field of the form $K(x, y) = (\omega, uy)$, where $u \in \mathfrak{g}$, with eigenvalues $\pm i\rho$, and $\rho > 0$.

3.1 Renormalization operator

On a subset of $\mathcal{A}_\varrho$ the one-step renormalization operator is defined (formally) as

$$\mathcal{R}(X) = \eta^{-1} \mathcal{T}^* \mathcal{U}_X^*(X), \tag{3.1}$$

where $\mathcal{T}^*$ is the pullback of the map $\mathcal{T} : (x, y) \mapsto (Tx, y)$, with $Tx = \eta^{-1} x_\parallel + \beta x_\perp$. Here $x = x_\parallel + x_\perp$ is the decomposition of $x \in \mathbb{R}^d$ into components parallel and perpendicular to ω. More precisely, if $X(x, y) = (\omega, f(x)y)$, then, using $T\omega = \eta^{-1}\omega$, we find

$$\mathcal{T}^*(X)(x, y) = (\eta\omega, f \circ T(x)y). \tag{3.2}$$

We will assume that the renormalization parameters η and β are chosen such that they are positive numbers smaller than 1.

The map $\mathcal{U}_X \in \mathcal{B}_\varrho$ is a change of coordinates chosen in such a way that $\mathcal{U}_X^* X$ is in a normal form, referred to as *resonant*. The precise definition of resonant is given below. It will follow from the definition that the vector field $K(x, y) = (\omega, uy)$, with $u \in \mathfrak{g}$, is resonant and $\mathcal{U}_K$ is the identity. An explicit calculation shows that $\mathcal{R}(K)(x, y) = (\omega, \eta^{-1} uy)$.

The renormalization parameters η and β and the renormalization operator $\mathcal{R}$ depend on the renormalization step giving rise to a sequence of renormalization operators $\mathcal{R}_n$, $n \in \mathbb{N}$, and a sequence of vector fields $K_n = \mathcal{R}_n(K_{n-1})$, with $K_0 = K$.

3.2 Resonant versus non-resonant modes

Definition 3.1 Let $\tilde{\tau} > 0$ be given. We define the resonant index set as $I^+ = \{v \in \mathcal{V} : \|Tv\| \leq \tilde{\tau}|v|\}$. The non-resonant index set I^- is defined as its complement in $\mathcal{V}$. When necessary, we will explicitly state the dependence of I^- or I^+ on $\mathcal{V}$, as $I^-(\mathcal{V})$ or $I^+(\mathcal{V})$, respectively. The terms in the expansion (2.1) of an $f \in \mathcal{F}_\varrho(\mathcal{V})$ will be called modes; those with $v \in I^+$ will be called resonant, those with $v \in I^-$ will be called non-resonant. The projection operator onto the sets of resonant and non-resonant modes in $\mathcal{F}_\varrho(\mathcal{V})$ will be denoted by $\mathbb{I}^+$ and $\mathbb{I}^-$, respectively.

We will use the same symbols for the corresponding projection operators in $\mathcal{A}_\varrho$, defined by $\mathbb{I}^- X(x,y) = (0, \mathbb{I}^- f(x)y)$, $\mathbb{I}^+ X(x,y) = (\omega, \mathbb{I}^+ f(x)y)$, on an arbitrary X of the form (1.1).

We assume that the components of ω are rationally independent with respect to $\mathcal{V}$, in the sense that the first component $v_\parallel$ of any nonzero vector $v \in \mathcal{V}$ is nonzero. Then, given any $L \geq 1$, we can find $\ell > 0$ such that

$$|v_\perp| > L \quad \text{or} \quad |v_\parallel| \geq \ell\,, \qquad \forall v \in \mathcal{V} \setminus \{0\}\,. \tag{3.3}$$

In other words, all points in $\mathcal{V}$, except for the origin, lie outside the disc $|v_\perp| \leq L$ and $|v_\parallel| < \ell$.

3.3 Elimination of non-resonant modes

In this section, given a vector field $X \in \mathcal{A}_\varrho(\mathfrak{sl}(2,\mathbb{R}))$, close to $K(x,y) = (\omega, uy)$, we construct a change of coordinates $\mathcal{U}_X \in \mathcal{B}_\varrho$ such that $\mathbb{I}^- \mathcal{U}_X^* X = 0$. The explicit construction is carried out in Section 6, where a normal form theorem is proved that applies to skew-product flows on $\mathbb{T}^d \times \mathrm{G}$. A similar theorem was proved in [18] in the context of vector fields on $\mathbb{T}^d \times \mathbb{R}^m$, but that theorem does not apply in the present setting, since the main assumption (Assumption 5.1 of Chapter 5 of [18]) is not satisfied.

Given $X \in \mathcal{A}_\varrho$, define the operator $\hat{X}$ as $\hat{X}Z := [X, Z]$, for an arbitrary Z in $\mathcal{A}_\varrho$ or $\mathcal{A}_\varrho^0$. Here, the right hand side of the equality denotes the usual commutator of the vector fields, i.e. $[X, Z] = (DZ)X - (DX)Z$.

The action of the operator DX, with $X(x,y) = (\omega, f(x)y)$, on an arbitrary vector field Y of the form $Y(x,y) = (\omega, g(x)y)$, where $f, g \in \mathcal{F}_\varrho$, is given by

$$[(DX)Y](x,y) = (0, D_\omega f(x) + f(x)g(x)y). \tag{3.4}$$

The action of DX on a vector field Z of the form $Z(x,y) = (0, g(x)y)$ is given by

$$[(DX)Z](x,y) = (0, f(x)g(x)y). \tag{3.5}$$

Note that for $X \in \mathcal{A}_\varrho$, the operator $\hat{X}$ is well defined on vector fields $Z \in \mathcal{A}_\varrho^0$, of the above form, with $g \in \mathcal{F}'_\varrho(\mathfrak{g})$, and acts as

$$(\hat{X}Z)(x,y) = (0, \hat{f}g(x)y), \qquad \hat{f}g(x) = D_\omega g(x) - \mathrm{ad}_{f(x)} g(x). \tag{3.6}$$

In particular, for $K(x,y) = (\omega, uy)$, the operator $\hat{K}$ takes the form

$$\hat{K}Z(x,y) = (0, \hat{u}g(x)y), \qquad \hat{u} = D_\omega - \mathrm{ad}_u. \tag{3.7}$$

The following proposition will be used repeatedly.

Proposition 3.2 *If $f \in \mathcal{F}_\varrho$, and $X \in \mathcal{A}_\varrho$, we have*

(i) $\|f(x)\| \le \|f\|_\varrho$

(ii) $\|\mathrm{ad}_f\|_\varrho \le 2\|f\|_\varrho$

(iii) $\|DX|_{\mathcal{A}_\varrho^0}\|_\varrho \le \|X\|_\varrho$

Proof. The property (i) follows from the definition of the norms. Inequality (ii) follows directly from the Banach algebra property of $\mathcal{F}_\varrho$, i.e. the fact that for all $f, g \in \mathcal{F}_\varrho$, we have $\|fg\|_\varrho \le \|f\|_\varrho \|g\|_\varrho$. The same property also implies (iii), using the identity (3.5). **QED**

We assume that the renormalization parameters $\eta, \beta, \ell, L, \tilde{\tau}, \sigma$ are positive and satisfy

$$\eta < 1/2\,, \qquad 2\sigma L < \ell\,, \qquad \sigma = \frac{1}{2}(1-\beta)\eta, \qquad \tilde{\tau} = \frac{1}{2}(1+\beta)\,. \tag{3.8}$$

In the following, let $\gamma > 0$ be such that

$$\gamma^{-1} = \min_{v \in I^-, |v| < 8\rho/\sigma} \left\{ 6^{-1}, \frac{|v_\parallel - 2\rho|}{2 + |v_\parallel|} \sigma^{-1} \right\}. \tag{3.9}$$

Proposition 3.3 *Let $0 < \sigma < 1/4$, $\varrho > 0$, let $\omega \in \mathbb{R}^d$ be a unit vector, and let $u \in \mathfrak{sl}(2,\mathbb{C})$ with eigenvalues $\pm i\rho$ be given. There exist $\gamma > 6$ such that the operator $\hat{u}$: $\mathbb{I}^- \mathcal{F}'_\varrho(\mathfrak{sl}(2,\mathbb{C})) \to \mathbb{I}^- \mathcal{F}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$ has an inverse and for every $g \in \mathbb{I}^- \mathcal{F}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$,*

$$\|\hat{u}^{-1} g\|'_\varrho \le \frac{\gamma}{\sigma} \|g\|_\varrho\,. \tag{3.10}$$

Proof. If v belongs to I^-, then we have $|Tv| > \tilde{\tau}|v|$, and thus, $\eta^{-1}|v_\parallel| + \beta|v_\perp| > \tilde{\tau}|v_\parallel| + \tilde{\tau}|v_\perp|$. This immediately implies that $|v_\parallel| > \sigma|v_\perp|$. This inequality, together with (3.3) and (3.8), also implies that $|v_\parallel| > \sigma$.

Let $f_v(x) = c_v e^{ix\cdot v} \in \mathbb{I}^- \mathcal{F}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$ and $g_v = \hat{u} f_v$. Then, we have $g_v(x) = (iv_\| - \mathrm{ad}_u) f_v(x)$,

$$\|f_v\|_\varrho \le \max\left\{\frac{1}{|v_\||}, \frac{1}{|v_\| \pm 2\rho|}\right\} \|g_v\|_\varrho \tag{3.11}$$

and

$$\|D_\omega f_v\|_\varrho \le \max\left\{1, \frac{|v_\||}{|v_\| \pm 2\rho|}\right\} \|g_v\|_\varrho\,. \tag{3.12}$$

If $v \in I^-$ and $|v| > 8\rho/\sigma$, then

$$\frac{|v_\| \pm 2\rho|}{2 + |v_\||} \ge \frac{|v_\|| - 2\rho}{2 + |v_\||} > \sigma/6 \tag{3.13}$$

Here we have also used that $\sigma < 1/4$. The number of modes indexed by $v \in I^-$ with $|v| < 8\rho/\sigma$ is finite. This gives $\|f_v\|'_\varrho \le (\gamma/\sigma)\|g_v\|_\varrho$.

Taking into account that the image of $\mathbb{I}^- \mathcal{F}'_\varrho(\mathfrak{sl}(2,\mathbb{C}))$ under $\hat{u}$ is dense in $\mathbb{I}^- \mathcal{F}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$, the above bounds show that if $f \in \mathbb{I}^- \mathcal{F}'_\varrho(\mathfrak{sl}(2,\mathbb{C}))$ and $g = \hat{u} f$, then

$$\|f\|'_\varrho \le \frac{\gamma}{\sigma} \|g\|_\varrho\,. \tag{3.14}$$

This proves the claim. **QED**

Proposition 3.3 verifies the main assumption of the normal form theorem proved in Section 6, with $\zeta = \sigma/\gamma$. The following lemma follows directly from the main result of that section (Theorem 6.4).

Lemma 3.4 *Let $\varrho > 0$, $0 < \sigma < 1/4$, let $\omega \in \mathbb{R}^d$ be a unit vector, and let $u \in \mathfrak{sl}(2,\mathbb{C})$ with eigenvalues $\pm i\rho$, and $\rho > 0$, be given. Let γ be chosen as in* (3.9)*. There exist universal constants $C_1, C_2 > 0$, such that the following holds. If X is any vector field in $\mathcal{A}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$ of the form* (1.1) *satisfying*

$$\|X - K\|_\varrho \le C_1(\sigma/\gamma)\,, \qquad \|\mathbb{I}^- X\|_\varrho \le C_1(\sigma/\gamma)^2\,, \tag{3.15}$$

where $K(x,y) = (\omega, uy)$, then there exists a vector field $Z \in \mathbb{I}^- \mathcal{A}^{0\prime}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$ solving the equation $X + [Z,X] = 0$ and a change of coordinates $\mathcal{U}_X : D_\varrho \times \mathrm{SL}(2,\mathbb{C}) \to D_\varrho \times \mathrm{SL}(2,\mathbb{C})$, of the form $\mathcal{U}_X(x,y) = (x, U_f^{-1}(x)y)$, with $U_f \in \mathcal{F}'_\varrho(\mathrm{SL}(2,\mathbb{C}))$, solving the equations $\mathbb{I}^- \mathcal{U}_X^* X = 0$. The vector field $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_\varrho(\mathfrak{sl}(2,\mathbb{C}))$, and

$$\begin{aligned}
\|Z\|'_\varrho &\le C_2(\gamma/\sigma)\|\mathbb{I}^- X\|_\varrho\,,\\
\|\mathcal{U}_X - \mathrm{Id}\|'_\varrho &\le C_2(\gamma/\sigma)\|\mathbb{I}^- X\|_\varrho\,,\\
\|\mathcal{U}_X^* X - X\|_\varrho &\le C_2(\gamma/\sigma)\|\mathbb{I}^- X\|_\varrho\,,\\
\|\mathcal{U}_X^* X - X - [Z,X]\|_\varrho &\le C_2(\gamma/\sigma)^3\|\mathbb{I}^- X\|_\varrho^2\,.
\end{aligned} \tag{3.16}$$

The maps $f \mapsto U_f$ and $X \mapsto \mathcal{U}_X$ are continuous in the region defined by (3.15)*, and analytic in its interior. Moreover, if $X : \mathbb{R}^d \times \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}^d \times \mathrm{SL}(2,\mathbb{R})$, then $U_f : \mathbb{R}^d \to \mathrm{SL}(2,\mathbb{R})$.*

3.4 Scaling of resonant modes

We now consider the action of $\mathcal{T}^*$ on resonant vector fields, i.e. those consisting of only resonant modes. In particular, the following lemma assures contraction of $v \neq 0$ resonant modes.

Lemma 3.5 *$\mathcal{T}^*$ is a bounded operator from $\mathbb{I}^+\mathcal{A}_\varrho(\mathcal{V})$ to $\mathcal{A}_\varrho(T\mathcal{V})$, where $\varrho > 0$, with the property that*

$$\begin{aligned} \|\mathcal{T}^*\mathbb{E}X\|_\varrho &\leq \|\mathbb{E}X\|_\varrho\,, \\ \|\mathcal{T}^*\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_\varrho &\leq e^{-\varrho\frac{1-\beta}{2}L}\|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_\varrho\,. \end{aligned} \tag{3.17}$$

Proof. Consider first a vector field $Y_0(x,y) = (\omega, c_0 y)$, with $c_0 \in \mathfrak{g}$. Then, $\mathcal{T}^*(Y)(x,y) = (\eta\omega, c_0 y)$, and the first bound in (3.17) follows.

To obtain the second bound, because of our choice of the norm, it suffices to verify it for a single non-zero $v \in I^+$. It follows essentially from the definitions that if $Y_v(x,y) = (0, c_v e^{ix\cdot v} y)$ then

$$\|\mathcal{T}^*Y_v\|_\varrho \leq e^A\|Y_v\|_\varrho\,, \qquad A = \varrho|Tv| - \varrho|v|\,. \tag{3.18}$$

Assume now that v belongs to I^+, and that $v \neq 0$. Thus $|Tv| \leq \tilde{\tau}|v|$. It leads to $|v_\parallel| < 2\sigma|v_\perp|$, if we use that $\eta\tilde{\tau} < 1/2$. This inequality excludes frequencies v that satisfy $|v_\perp| \leq L$ and $|v_\parallel| \geq \ell$, due to the condition (3.8). Thus, we must have $|v_\perp| > L$ by condition (3.3). Consequently,

$$A \leq -\varrho(1-\tilde{\tau})|v| \leq -\varrho\frac{1-\beta}{2}L\,, \tag{3.19}$$

and the second bound in (3.17) follows. **QED**

3.5 Estimates for a single renormalization step

Combining the two preceding lemmas, we obtain the following theorem. Notice that the subspace $\mathbb{E}\mathcal{A}_\varrho$ is invariant under $\mathcal{R}$, and the restriction of $\mathcal{R}$ to that subspace defines a linear operator that will be denoted by $\mathcal{L}$, by the action $\mathcal{R}(Y)(x,y) = (\omega, \mathcal{L}(c)y)$, on an arbitrary $Y \in \mathbb{E}\mathcal{A}_\varrho$ with $Y(x,y) = (\omega, cy)$, for all $(x,y) \in D_\varrho \times \mathrm{G}$.

Theorem 3.6 *Let $\mathrm{G} = \mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. There exist universal constants $C, R > 0$, such that the following holds, under the previous assumptions on L, ℓ, η, β, and γ. Let*

B be the open ball in $\mathcal{A}_\varrho(\mathcal{V})$ of radius $R(\sigma/\gamma)^2$, centered at K. Then $\mathcal{R}$ is a bounded analytic map from B to $\mathcal{A}_\varrho(T\mathcal{V})$, satisfying $\|\mathcal{L}^{-1}\| \le 1$ and

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &\le C\eta^{-1}(\gamma/\sigma)e^{-\varrho^{\frac{1-\beta}{2}}L}\|(\mathbb{I}-\mathbb{E})X\|_\varrho\,, \\ \|\mathbb{E}\mathcal{R}(X)-\mathcal{R}(\mathbb{E}X)\|_\varrho &\le C\eta^{-1}(\gamma/\sigma)^3\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,. \end{aligned} \tag{3.20}$$

Moreover, if X takes values in $\{\omega\}\times \mathrm{SL}(2,\mathbb{R})$ when restricted to $\mathbb{R}^d\times\mathrm{SL}(2,\mathbb{R})$, so does $\mathcal{R}(X)$.

Proof. There exists a universal constant $R>0$, such that the conditions (3.15) in Lemma 3.4 hold, whenever X belongs to the domain B, defined by $\|X-K\|_\varrho < R(\sigma/\gamma)^2$.

By Lemma 3.5, we have

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &= \eta^{-1}\|\mathcal{T}^*(\mathbb{I}-\mathbb{E})\mathcal{U}_X^*X\|_\varrho \\ &\le \eta^{-1}e^{-\varrho^{\frac{1-\beta}{2}}L}\big[\|(\mathbb{I}-\mathbb{E})X\|_\varrho + \|\mathcal{U}_X^*X-X\|_\varrho\big]\,. \end{aligned} \tag{3.21}$$

Using the bound in (3.16) on the norm of $\mathcal{U}_X^*X-X$, together with the fact that $\mathbb{I}^-\mathbb{E}=0$, we obtain the first inequality in (3.20).

By Lemma 3.5, we also have a bound

$$\|\mathbb{E}\mathcal{R}(X)-\mathcal{R}(\mathbb{E}X)\|_\varrho = \eta^{-1}\|\mathcal{T}^*\mathbb{E}(\mathcal{U}_X^*X-X)\|_\varrho \le \eta^{-1}\|\mathbb{E}(\mathcal{U}_X^*X-X)\|_\varrho\,. \tag{3.22}$$

Using Lemma 3.4, the norm on the right hand side of (3.22) can be estimated as follows:

$$\|\mathbb{E}(\mathcal{U}_X^*X-X)\|_\varrho \le C_2(\gamma/\sigma)^3\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2 + \|\mathbb{E}[Z,X]\|_\varrho\,, \tag{3.23}$$

where $Z=\mathbb{I}^-Z$ is the vector field described in Lemma 3.4. Since $\mathbb{E}Z=0$, we have $\mathbb{E}[Z,\mathbb{E}X]=0$. As a result,

$$\|\mathbb{E}[Z,X]\|_\varrho = \|\mathbb{E}[Z,(\mathbb{I}-\mathbb{E})X]\|_\varrho \le C_3\|Z\|_\varrho\|(\mathbb{I}-\mathbb{E})X\|_\varrho \le C_4(\gamma/\sigma)\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,. \tag{3.24}$$

Here, we have used Proposition 3.2 and the bound on $\|Z\|_\varrho$ from Lemma 3.4. Combining the last three equations yields the second inequality in (3.20).

In order to bound the inverse of $\mathcal{L}$, let Y be a vector field in $\mathbb{E}\mathcal{A}_\varrho$. Then Y can be written as $Y(x,y)=(\omega,cy)$, for some $c\in\mathfrak{g}$, and the bound on $\|\mathcal{L}^{-1}\|$ now follows from the fact that

$$\mathcal{R}(Y)(x,y) = \eta^{-1}(T^{-1}\omega, cy) = (\omega, \eta^{-1}cy)\,, \tag{3.25}$$

and thus $\mathcal{L}(c)=\eta^{-1}c$. This completes the proof. **QED**

4 A sequence of renormalization operators

Let $\omega \in \mathbb{R}^d$ be a given vector that satisfies the Brjuno condition (1.9). For the sake of convenience, we will perform a linear transformation of our coordinate system (a rotation and a scaling) such that ω takes the form $\omega = (1, 0, \ldots, 0)$. The lattice obtained from $(2\pi\mathbb{Z})^d$ under this transformation will be denoted by $\mathcal{Z}_0$, and its dual lattice by $\mathcal{V}_0$. Notice that, after that transformation, the Brjuno condition (1.9) on ω and the Diophantine with respect to ω condition (1.10) on u can be restated in the same form with $\omega = (1, 0, \ldots, 0)$ and $\nu \in \mathcal{V}_0$ (with not necessarily the same constant $\mathcal{C} > 0$). This ω that satisfies the Brjuno condition with frequencies $\nu \in \mathcal{V}_0$ will be called $\mathcal{V}_0$-*Brjuno*, and u that is Diophantine with respect to this ω and frequencies $\nu \in \mathcal{V}_0$ will be called $\mathcal{V}_0$-*Diophantine* with respect to ω.

Starting with a lattice $\mathcal{V}_0$ in $\mathbb{R}^d$ and a $\mathcal{V}_0$-Brjuno frequency vector ω, the objective of this section is to construct a sequence of matrices T_n, a sequence of lattices $\mathcal{V}_n = T_n \mathcal{V}_{n-1}$, and a sequence of renormalization operators $\mathcal{R}_n : \mathcal{A}_\varrho(\mathcal{V}_{n-1}) \to \mathcal{A}_\varrho(\mathcal{V}_n)$ with a fixed set $K_n \in \mathbb{E}\mathcal{A}_\varrho$, $n \in \mathbb{N}$, and a stable manifold $\mathcal{W}$ associated to this set.

4.1 Construction of the renormalization transformations

We begin by restating the Brjuno condition (1.9) in another form. Let now $\mathcal{V}_0$ be a lattice in $\mathbb{R}^d$, and let ω be $\mathcal{V}_0$-Brjuno frequency vector. Using the same set of frequencies, define

$$a_n = \sum_{k=n}^{\infty} 2^{n-k} \left[2^{-k-\kappa} \ln(1/\Omega'_{k+\kappa}) + (k+\kappa')^{-2} \right], \qquad \Omega'_n = \min_{0<|\nu_\perp|<2^n} |\nu_\parallel| \,, \tag{4.1}$$

for all positive integers n. Here $\kappa, \kappa' > 2$ are two integer constants that will be determined later on. Then the Brjuno condition (1.9) is equivalent to the condition that the resulting sequence a_n is summable. The terms involving $(k+\kappa')^{-2}$ are clearly summable, and do not influence the summability of other terms. Parameters κ and κ' produce "shifts" of the terms in the resulting series.

Define $\lambda_0 = 1$ and

$$\lambda_n = 2^{-n-\kappa} e^{-2^{n+\kappa} a_n} \,, \quad \eta_n = \frac{\lambda_n}{\lambda_{n-1}} \,, \quad A_n = \sum_{k=n}^{\infty} a_k \,, \quad \beta_n = \frac{A_{n+1}}{A_n} \,, \tag{4.2}$$

for all positive integers n. Consider the corresponding scaling transformations

$$P_n(x) = \lambda_n^{-1} x_\parallel + A_1^{-1} A_{n+1} x_\perp \,, \qquad T_n(x) = \eta_n^{-1} x_\parallel + \beta_n x_\perp \,. \tag{4.3}$$

Notice that $P_n = T_1 T_2 \ldots T_n$ by equation (4.2). These quantities will now be used to define the n-th step renormalization transformation $\mathcal{R}_n$. We need to verify that the renormalization parameters satisfy the assumptions made in the previous section at each renormalization step. Clearly, β_n is positive and less than one, since $n \mapsto A_n$ is a decreasing sequence. The condition $\eta_n < 1/2$ follows from the fact that $a_n > a_{n-1}/2$ for $n > 1$, and that $\lambda_1 < 1/2$.

The geometric data $\mathcal{V}$, L and ℓ used in step n are

$$\mathcal{V}_{n-1} = P_{n-1}\mathcal{V}_0\,, \qquad L_{n-1} = A_1^{-1}A_n 2^{n+\kappa}\,, \qquad \ell_{n-1} = 2^{n+\kappa}\eta_n\,. \tag{4.4}$$

These definitions immediately imply bounds (3.8). The following proposition shows that the condition (3.3) holds for all $v \in \mathcal{V}_{n-1}$.

Proposition 4.1 *If $v \in \mathcal{V}_{n-1}$ is nonzero, then either $|v_\parallel| \geq \ell_{n-1}$ or $|v_\perp| > L_{n-1}$.*

Proof. Assume that $v \in \mathcal{V}_{n-1}$ satisfies $0 < |v_\perp| \leq L_{n-1}$. Then the corresponding lattice point $\nu = P_{n-1}^{-1}v$ in $\mathcal{V}_0$ satisfies $|\nu_\perp| \leq A_1 A_n^{-1} L_{n-1} = 2^{n+\kappa}$, and thus $|\nu_\parallel| \geq \Omega'_{n+\kappa}$ by (4.1). Since we have $\lambda_n < 2^{-n-\kappa}\Omega'_{n+\kappa}$, this yields

$$|v_\parallel| = \lambda_{n-1}^{-1}|\nu_\parallel| \geq \eta_n \lambda_n^{-1}\Omega'_{n+\kappa} > \eta_n 2^{n+\kappa} = \ell_{n-1}\,, \tag{4.5}$$

as claimed. **QED**

Definition 4.2 Let us define $\rho_{n-1} = \lambda_{n-1}^{-1}\rho$ and $\sigma_n = \frac{1}{2}(1-\beta_n)\eta_n$ for $n \in \mathbb{N}$. We also define

$$\gamma_n = \max_{v \in I^-(\mathcal{V}_{n-1}),\, |v| < 8\rho_{n-1}/\sigma_n} \left\{ 6, \frac{2+|v_\parallel|}{|v_\parallel - 2\rho_{n-1}|}\sigma_n \right\}. \tag{4.6}$$

Proposition 4.3 *If $u \in \mathfrak{sl}(2,\mathbb{R})$ with eigenvalues $\pm i\rho$ is $\mathcal{V}_0$-Diophantine with respect to ω, then there exists a constant ξ that may depend only on ρ, $\mathcal{C}$ and τ, such that for all $n \in \mathbb{N}$,*

$$\gamma_n < \xi(\lambda_{n-1}(\prod_{i=1}^{n-1}\beta_i)\sigma_n)^{-\tau}. \tag{4.7}$$

Proof. Since

$$\gamma_n < \max_{v \in I^-(\mathcal{V}_{n-1}),\, |v| < 8\rho_{n-1}/\sigma_n} \left\{ 6, \frac{2\sigma_n + 8\rho_{n-1}}{|v_\parallel - 2\rho_{n-1}|} \right\}, \tag{4.8}$$

writing $\nu = P_{n-1}^{-1}v$, we have

$$\gamma_n < \max_{v \in I^-(\mathcal{V}_{n-1}),\, |v| < 8\rho_{n-1}/\sigma_n} \left\{ 6, \frac{2\sigma_n + 8\rho_{n-1}}{\lambda_{n-1}^{-1}|\nu_\parallel - 2\rho|} \right\} < \max_{v \in I^-(\mathcal{V}_{n-1}),\, |v| < 8\rho_{n-1}/\sigma_n} \left\{ 6, \frac{2+8\rho}{|\nu_\parallel - 2\rho|} \right\}. \tag{4.9}$$

If u is Diophantine with respect to ω then there exists a universal constant $\mathcal{C} > 0$ such that

$$\gamma_n < \max_{v \in I^-(\mathcal{V}_{n-1}),\, |v| < 8\rho_{n-1}/\sigma_n} \left\{ 6, \frac{2+8\rho}{\mathcal{C}}|\nu|^\tau \right\}. \tag{4.10}$$

Since

$$|v| = |v_{\|}| + |v_{\perp}| = \lambda_{n-1}^{-1}|\nu_{\|}| + (\prod_{i=1}^{n-1}\beta_i)|\nu_{\perp}| \geq (\prod_{i=1}^{n-1}\beta_i)|\nu|, \tag{4.11}$$

we have

$$\gamma_n < \max_{v\in I^-(\mathcal{V}_{n-1}),|v|<8\rho_{n-1}/\sigma_n}\left\{6, \frac{2+8\rho}{\mathcal{C}}(\prod_{i=1}^{n-1}\beta_i)^{-\tau}|v|^{\tau}\right\}. \tag{4.12}$$

Thus,

$$\gamma_n < \xi(\lambda_{n-1}(\prod_{i=1}^{n-1}\beta_i)\sigma_n)^{-\tau}, \tag{4.13}$$

where ξ is a constant that may depend only on ρ, $\mathcal{C}$ and τ. **QED**

For technical convenience we introduce the following quantities.

Definition 4.4 Let

$$\mu_k = \exp\Big\{-\varrho\frac{1-\beta_k}{2}L_{k-1}\Big\} = \exp\Big\{-\frac{\varrho}{2A_1}2^{k+\kappa}a_k\Big\}, \qquad k \geq 1\,. \tag{4.14}$$

The following proposition provides important estimates for the construction of the renormalization scheme.

Proposition 4.5 *For all $k \geq 1$, $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$. Furthermore, given $C, N > 0$, if κ' and then κ are chosen sufficiently large, then for all $k \geq 1$,*

$$\mu_k \leq Ce^{-N2^{k+\kappa}a_k}, \qquad \mu_k \leq C\eta_k^N, \qquad \mu_k \leq C(1-\beta_k)^N, \qquad \mu_k \leq C\gamma_k^{-N} \tag{4.15}$$

Proof. The inequality $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$ follows from the fact that $a_{k+1}/2 < a_k < 2a_{k+1}$. Let now $c = \varrho/2$. By choosing κ and κ' sufficiently large, we have $c/A_1 \geq (2+3\tau)N$. Keeping κ' fixed, and increasing κ further, if necessary, we obtain the first two bounds in (4.15) by using that $2^{k+\kappa}a_k \geq 2^{k+\kappa}(k+\kappa')^{-1} \geq c'2^{\kappa}k$, for some constant $c' > 0$. The same inequality, together with $1-\beta_k = a_k/A_k > (k+\kappa')^{-2}(2+3\tau)N/c$, implies the third bound in (4.15). These inequalities, together with Proposition 4.3, also imply the last bound. **QED**

In particular, the first bound shows that the μ_k decrease at least exponentially with k.

Note that in the definition of Ω_n' (4.1), we can replace min with a "lower bound". Our estimates are then uniform within classes of Brjuno vectors ω of unit length that admit the same sequence of lower bounds $n \mapsto \Omega_n'$ and matrices $u \in \mathfrak{sl}(2,\mathbb{R})$ Diophantine with respect to ω with the same $\mathcal{C}$, ρ and τ.

4.2 Stable manifold

Let $K_0 = K$ and let K_n be such that $K_n(x,y) = (\omega, \lambda_n^{-1} uy)$, for $(x,y) \in D_\varrho \times \mathrm{G}$. Having verified all of the assumptions made in the previous section, we can now apply Theorem 3.6 to the n-th step renormalization transformation $\mathcal{R}_n$, defined by the parameters introduced above. Denote by $\mathcal{L}_n$ the corresponding linear operator $\mathcal{L}$ from $\mathbb{E}\mathcal{F}_\varrho(\mathfrak{g}, \mathcal{V}_{n-1})$ to $\mathbb{E}\mathcal{F}_\varrho(\mathfrak{g}, \mathcal{V}_n)$.

Define $\mathcal{A}_{\varrho,k} = \mathcal{A}_\varrho(\mathcal{V}_k)$ and $\mathcal{A}^0_{\varrho,k} = \mathcal{A}^0_\varrho(\mathcal{V}_k)$, for all non-negative integers k. To simplify notation, the norms in $\mathcal{A}_{\varrho,k}$ and $\mathcal{A}^0_{\varrho,k}$ will not be given indices. From Theorem 3.6 we immediately obtain the following theorem.

Theorem 4.6 *There exist universal constants $r, C > 0$ such that the following holds, for every positive integer n. Let B_{n-1} be the open ball in $\mathcal{A}_{\varrho,n-1}(\mathfrak{sl}(2,\mathbb{C}))$ of radius $r\sigma_n^2/\gamma_n^2$, centered at K_{n-1}. Then $\mathcal{R}_n$ is a bounded analytic map from B_{n-1} to $\mathcal{A}_{\varrho,n}(\mathfrak{sl}(2,\mathbb{C}))$, satisfying $\|\mathcal{L}_n^{-1}\| \le 1$ and*

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}_n(X)\|_\varrho &\le C\gamma_n\sigma_n^{-2}\mu_n \|(\mathbb{I}-\mathbb{E})X\|_\varrho \,, \\ \|\mathbb{E}\mathcal{R}_n(X) - \mathcal{R}_n(\mathbb{E}X)\|_\varrho &\le C\gamma_n^3\sigma_n^{-4} \|(\mathbb{I}-\mathbb{E})X\|_\varrho^2 \,. \end{aligned} \tag{4.16}$$

Moreover, if X takes values in $\{\omega\} \times \mathrm{SL}(2,\mathbb{R})$ when restricted to $\mathbb{R}^d \times \mathrm{SL}(2,\mathbb{R})$, so does $\mathcal{R}_n(X)$.

In what follows, a domain $\mathcal{D}_{n-1}$ for $\mathcal{R}_n$ is a subset of the ball B_{n-1} described in Theorem 4.6, that is open in $\mathcal{A}_{\varrho,n-1}$ and contains the vector field K_{n-1}. Given a domain $\mathcal{D}_{n-1}$ for each $\mathcal{R}_n$, the domain $\widetilde{\mathcal{D}}_n$ of the combined renormalization operator $\widetilde{\mathcal{R}}_{n+1} = \mathcal{R}_{n+1} \circ \mathcal{R}_n \circ \ldots \circ \mathcal{R}_1$ is defined recursively as the set of all vector fields in the domain of $\widetilde{\mathcal{R}}_n$ that are mapped under $\widetilde{\mathcal{R}}_n$ into the domain $\mathcal{D}_n$ of $\mathcal{R}_{n+1}$. By Theorem 4.6, these domains are open and non-empty, and the transformations $\widetilde{\mathcal{R}}_n$ are analytic.

The following theorem, valid for the case of $\mathrm{G} = \mathrm{SL}(2,\mathbb{C})$, is the corollary of the stable manifold theorem of [18] (Section 6, Theorem 6.1), for sequences of maps between Banach spaces.

Theorem 4.7 *If κ' and then κ are chosen sufficiently large, then there exist a sequence of domains $\mathcal{D}_{n-1}$, for the renormalization transformations $\mathcal{R}_n$, $n \in \mathbb{N}$, such that the set $\mathcal{W} = \cap_{n=0}^\infty \widetilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I}-\mathbb{E})\mathcal{D}_0 \to \mathbb{E}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. For every $X \in \mathcal{W}$ and every $n \in \mathbb{N}$, if $\chi_n = \prod_{i=1}^n \mu_i$, then*

$$\begin{aligned} \big\|\widetilde{\mathcal{R}}_n(X) - K_n\big\|_\varrho &\le \chi_n^{1/2} \|(\mathbb{I}-\mathbb{E})X\|_\varrho \,, \\ \big\|\mathbb{E}[\widetilde{\mathcal{R}}_n(X) - K_n]\big\|_\varrho &\le \chi_n \|(\mathbb{I}-\mathbb{E})X\|_\varrho^2 \,, \\ \big\|(\mathbb{I}-\mathbb{E})\widetilde{\mathcal{R}}_n(X)\big\|_\varrho &\le \chi_n^{1/2} \|(\mathbb{I}-\mathbb{E})X\|_\varrho \,. \end{aligned} \tag{4.17}$$

Proof. Let us first rescale our transformations $\mathcal{R}_n$. Let $r_n = r_{n-1}\sigma_{n+1}^2/\gamma_{n+1}^2$ for every positive integer n, with $r_0 > 0$ smaller than half the constant r from Theorem 4.6.

Consider the transformations $R_n : \mathcal{A}^0_{\varrho,n-1} \to \mathcal{A}^0_{\varrho,n}$, with $n \in \mathbb{N}$, given by

$$R_n(Z) = r_n^{-1}\big[\mathcal{R}_n(K_{n-1} + r_{n-1}Z) - K_n\big] . \tag{4.18}$$

In order to show explicitly that the assumptions of the stable manifold theorem of [18] are verified, we begin by introducing the projection operator $\mathbb{P} = \mathbb{E}$. The restriction $R_n\mathbb{P}$ defines a linear map from $\mathbb{P}\mathcal{A}^0_{\varrho,n-1}$ to $\mathbb{P}\mathcal{A}^0_{\varrho,n}$, which will be denoted by L_n. By Theorem 4.6, R_n is analytic and bounded on the ball $\|Z\|_\varrho < 2$, and satisfies

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})R_n(Z)\|_\varrho &\le \varepsilon_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho , \\ \|(\mathbb{I}-\mathbb{P})R_n(Z)\|_\varrho &\le \vartheta_n\|(\mathbb{I}-\mathbb{P})Z\|_\varrho , \\ \|\mathbb{P}R_n(Z) - R_n(\mathbb{P}Z)\|_\varrho &\le \varphi_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho^2 , \end{aligned} \tag{4.19}$$

where

$$\varepsilon_n = \vartheta_n = C\sigma_n^{-2}\gamma_n\sigma_{n+1}^{-2}\gamma_{n+1}^2\mu_n , \quad \varphi_n = C\sigma_n^{-4}\gamma_n^3\sigma_{n+1}^{-2}\gamma_{n+1}^2 . \tag{4.20}$$

Here, $C \ge 1$ is a universal constant. In addition, we have $\|L_n^{-1}\| < 1/4$. We will restrict R_n to the domain $D_{n-1} \subset \mathcal{A}^0_{\varrho,n-1}$, defined by

$$\|\mathbb{P}Z\|_\varrho < 1 , \qquad \|(\mathbb{I}-\mathbb{P})Z\|_\varrho < 1 , \qquad \|(\mathbb{I}-\mathbb{E})Z\|_\varrho < \delta_{n-1} , \tag{4.21}$$

where $\delta_{n-1} = (6\varphi_n)^{-1}$. By Proposition 4.5, if κ' and κ are chosen sufficiently large (such that the bounds of the same proposition are valid with N large enough), then $C\sigma_n^{-2}\gamma_n\sigma_{n+1}^{-2}\gamma_{n+1}^2\mu_n^{1/2} \le 1/6$, for all positive integers n. This inequality implies

$$\varepsilon_n = \vartheta_n \le \mu_n^{1/2}/6 , \qquad \varepsilon_n\delta_{n-1} \le \delta_n , \tag{4.22}$$

for all $n \ge 1$. The hypotheses of the stable manifold theorem (see Theorem 6.1 of Section 6 in [18]) have now been verified, with $\varepsilon = 1/6$ and $\vartheta = 1/3$, and the conclusions of this theorem imply the statements in Theorem 4.7. **QED**

5 Reducibility of skew-product systems

Let $K_0 = K$, where $K(x,y) = (\omega, uy)$, and let $K_n = \mathcal{R}_n(K_{n-1})$, i.e. $K_n(x,y) = (\omega, \lambda_n^{-1}uy)$, as before. Let $X_0 = X \in \mathcal{W}$, and let $X_n = \mathcal{R}_n(X_{n-1})$ be the renormalization orbit of X. For the case of $\mathrm{G} = \mathrm{SL}(2,\mathbb{C})$, that we consider in this section, the stable manifold $\mathcal{W}$ has been constructed in Section 4.2. We will show that $X \in \mathcal{W}$ is analytically reducible to K.

Recall that a vector field $X \in \mathcal{A}_\varrho$ of the form (1.1) is (analytically) reducible to K, if there is an (analytic) map $\psi : \mathbb{T}^d \times \mathrm{G} \to \mathbb{T}^d \times \mathrm{G}$, of the form $\psi(x,y) = (x, \Psi(x)^{-1}y)$, such that $\psi^*(X) = K$. Modulo smoothness assumptions, this is equivalent to

$$\Psi(x+\omega t)e^{ut}\Psi(x)^{-1} = \Phi^t(x), \qquad t \in \mathbb{R}\,, \quad x \in \mathbb{T}^d, \tag{5.1}$$

where $\Phi^t(x)$ is the associated matrix solution of (1.2), with $\Phi^0(x)$ the identity, as in (1.3). $\Phi(x)$ satisfies the equation

$$\Phi^t(x) = \mathrm{I} + \int_0^t f(x+\omega s)\Phi^s(x)\,ds. \tag{5.2}$$

This equation can be used to construct $\Phi(x)$, using the contraction mapping principle.

Now, observe that if X_n is reducible to K_n, then X_{n-1} is reducible to K_{n-1}. Since $X_n = \eta_n^{-1}\mathcal{T}_n^*\mathcal{U}_{n-1}^*(X_{n-1})$, where $\mathcal{U}_{n-1} = \mathcal{U}_{X_{n-1}}$, and $K_n = \eta_n^{-1}\mathcal{T}_n^*(K_{n-1})$, if there exists a map ψ_n such that $\psi_n^*(X_n) = K_n$, then we have $\psi_{n-1}^*(X_{n-1}) = K_{n-1}$. The maps ψ_{n-1} and ψ_n are related, by writing $\mathcal{U}_{n-1}(x,y) = (x, U_{n-1}^{-1}(x)y)$, via

$$\Psi_{n-1} = U_{n-1}\Psi_n \circ T_n^{-1}\,, \qquad n \in \mathbb{N}. \tag{5.3}$$

The proof of reducibility of the vector fields $X \in \mathcal{W}$ that follows is based on this relationship.

Let $\mathcal{F}_0(\mathcal{V}_n)$ and $\mathcal{F}_0'(\mathcal{V}_n)$ be the spaces of continuous functions $f : \mathbb{T}^d \to \mathbb{R}^{2\times 2}$, where $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}_n$, lifted to $\mathbb{R}^d$, with frequencies in $\mathcal{V}_n$, equipped respectively with the norms

$$\|f\|_0 = \sum_{v\in\mathcal{V}_n} \|f_v\|\,, \tag{5.4}$$

and

$$\|f\|_n' = r_n^{-1}\|f\|_0 = r_n^{-1}\sum_{v\in\mathcal{V}_n} \|f_v\|\,, \tag{5.5}$$

where $r_n = \chi_n^{1/3}$, for $n \in \mathbb{N}$, and $r_0 = 1$.

Let $\mathcal{B}_0(\mathcal{V}_n)$ be the set of maps $F : \mathbb{T}^d \to \mathrm{G}$, where $\mathrm{G} \subset \mathrm{GL}(2,\mathbb{R})$, of the form $F = \mathrm{I} + f$ with $f \in \mathcal{F}_0(\mathcal{V}_n)$. Let $\mathcal{B}_n$ be the set of maps $F : \mathbb{T}^d \to \mathrm{G}$ of the form $F = \mathrm{I} + f$ with $f \in \mathcal{F}_0'(\mathcal{V}_n)$. Let B_n be the unit open ball in $\mathcal{B}_n$, centered at the identity I.

Define (formally) the map M_n from B_n into $\mathcal{B}_{n-1}$, as

$$M_n(F) = U_{n-1}F \circ T_n^{-1}. \tag{5.6}$$

Proposition 5.1 *If κ' and κ are sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \geq 1$, the map M_n is well defined and analytic, as a function from B_n to $\mathcal{B}_{n-1}$. Furthermore, M_n takes values in the ball of radius $1/2$ centered at the identity in B_{n-1}, and $\|DM_n(F)\| \leq \mu_n^{1/4}$, for all $F \in B_n$.*

Proof. M_n is well-defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$M_n(F) = \mathrm{I} + g + (U_{n-1} - \mathrm{I})(\mathrm{I} + g)\,, \qquad g = f \circ T_n^{-1}\,, \tag{5.7}$$

where $F = \mathrm{I} + f$. In order to estimate $U_{n-1} - \mathrm{I}$, we can apply Lemma 3.4. We will use Proposition 4.5 and assume that κ' and then κ have been chosen sufficiently large, without always mentioning it. By Lemma 3.4 and Theorem 4.7, there exist a constant $C > 0$, such that

$$\begin{aligned} \|U_{n-1} - \mathrm{I}\|_\varrho &\leq C\gamma_n\sigma_n^{-1}\|\mathbb{I}^- X_{n-1}\|_\varrho \leq C\gamma_n\sigma_n^{-1}\chi_{n-1}^{1/2}\|(\mathbb{I} - \mathbb{E})X\|_\varrho \\ &\leq \chi_{n-1}^{1/3}\|(\mathbb{I} - \mathbb{E})X\|_\varrho \leq \chi_n^{1/12}\,, \end{aligned} \tag{5.8}$$

for all $n > 1$, and for all $X \in \mathcal{W} \cap B$, provided that the neighborhood B of K has been chosen sufficiently small (depending on κ' and κ). The first inequality in (5.8) and the final bound also hold for $n = 1$.

We further have,

$$\|(U_{n-1} - \mathrm{I})(\mathrm{I} + g)\|_0 \leq \|U_{n-1} - \mathrm{I}\|_\varrho(1 + \|g\|_0), \tag{5.9}$$

and $\|g\|_0 = \|f\|_0 \leq r_n\|f\|_n' < r_n$. Here, and in what follows, we assume that $F \in B_n$. By using that $r_n/r_{n-1} = \mu_n^{1/3}$, we obtain $\|g\|_{n-1}' \leq \mu_n^{1/3}$. When combined with (5.8) and (5.9), this shows that M_{n-1} maps B_n into $B_{n-1}/2$. Using the fact that the inclusion map from B_n into B_{n-1} is bounded in norm by $\mu_n^{1/3}$, together with the property $\|f_1 f_2\|_{n-1}' \leq r_{n-1}\|f_1\|_{n-1}'\|f_2\|_{n-1}'$, we obtain that $\|DM_n(F)\| \leq \mu_n^{1/4}$, for all $n \geq 1$, and for all $F \in B_n$. **QED**

Proposition 5.2 *Under the same assumptions as in Proposition 5.1, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that the following holds. Given any $X \in \mathcal{W} \cap B$, and any sequence of functions $F_k \in B_k$, $k \in \mathbb{N}$, define*

$$\Psi_{n,k} = M_{n+1} \circ \ldots \circ M_k(F_k)\,, \qquad 0 \leq n < k\,. \tag{5.10}$$

Then the limits $\Psi_n = \lim_{k\to\infty}\Psi_{n,k}$ exist in $\mathcal{B}_n$, are independent of the choice of the maps $F_0, F_1, \ldots$, and satisfy the identities (5.3)*. The map $X \mapsto \Psi_0$ is analytic and bounded on $\mathcal{W} \cap B$.*

Proof. By Proposition 5.1, the maps $M_n : B_n \to B_{n-1}/2$ contract distances by a factor of at least 1/2. Thus, if $1 \leq n < k < k'$, then the difference $\Psi_{n,k'} - \Psi_{n,k}$ is bounded in norm by 2^{n-k+1}. This shows that the sequence $k \mapsto \Psi_{n,k}$ converges uniformly in $\mathcal{B}_n$ to a limit Ψ_n, which is independent of the choice of the functions F_k. By choosing $F_k = \Psi_k$ for all k, we obtain the identities (5.3). The analyticity of $X \mapsto \Psi_0$ follows via the chain rule from the analyticity of the maps used in our construction, and from uniform convergence. **QED**

In what follows, the maps M_1 and $\mathcal{R}_1$ associated to a vector field $X \in \mathcal{W}$ or the form $X(x,y) = (\omega, f(x)y)$ will be denoted by M_X and $\mathcal{R}_X$, respectively, in order to show explicitly the dependence on the vector field X. The map Ψ_0 associated with the vector field $X \in \mathcal{W}$ will be denoted by Ψ_X.

Proposition 5.3 *Let $\varrho > \delta > 0$. Under the same assumptions as in Proposition 5.1, there exists an open neighborhood B of K in $\mathcal{A}_\varrho(\mathcal{V}_0)$, such that Ψ_X has an analytic continuation to $\|\mathrm{Im}\, x\| < \delta$, for each $X \in B \cap \mathcal{W}$. With this continuation, $X \mapsto \Psi_X$ defines a bounded analytic map from $B \cap \mathcal{W}$ to $\mathcal{F}_\delta(\mathrm{SL}(2,\mathbb{C}), \mathcal{V}_0)$.*

Proof. Consider the translations $\mathcal{J}_q(x,y) = (J_q(x), y)$, with $J_q(x) = x + q$ and $q \in \mathbb{R}^d$, defined on $\mathbb{C}^d \times \mathrm{SL}(2,\mathbb{C})$. For vector fields $X \in \mathcal{A}_\varrho(\mathcal{V}_0)$, with $X(x,y) = (\omega, f(x)y)$, we denote by $\mathcal{J}_q^* X$ the pullback under $\mathcal{J}_q$, as we did earlier. For maps $F \in \mathcal{F}_\varrho(\mathcal{V}_0)$, define $J_q^\star$ by $J_q^\star(F) = F \circ J_q$. By examining the construction of the map $\mathcal{U}_X$ and the scaling map $\mathcal{T}$, one can verify that

$$\mathcal{J}_q \circ \mathcal{U}_{\mathcal{J}_q^* X} = \mathcal{U}_X \circ \mathcal{J}_q\,, \qquad \mathcal{J}_{Tq} \circ \mathcal{T} = \mathcal{T} \circ \mathcal{J}_q\,. \tag{5.11}$$

Thus, one further has

$$\mathcal{R}_{\mathcal{J}_q^* X} \circ \mathcal{J}_q^* = \mathcal{J}_{T^{-1}q}^* \circ \mathcal{R}_X\,, \qquad M_{\mathcal{J}_q^* X} = J_q^\star \circ M_X \circ (J_{T^{-1}q}^\star)^{-1}. \tag{5.12}$$

We have also used that the translations $\mathcal{J}_q^*$ are isometries on the spaces $\mathcal{A}_\varrho(\mathcal{V}_0)$, and that the domain of $\mathcal{R}$ and the manifold $\mathcal{W}$ are invariant under these translations.

For convenience, we extend the map $X \mapsto \Psi_X$ to an open neighborhood of K, by setting $\Psi_X = \Psi_{X'}$, where $X' = (\mathbb{I} + W)(X - \mathbb{E}X)$. With this extension, the map $X \mapsto \Psi_X$ is analytic and bounded on a sufficiently small open ball $B \subset \mathcal{A}_\varrho(\mathcal{V}_0)$.

By examining the construction of Ψ_X and using the invariance property of $\mathcal{W}$ under $\mathcal{J}_q^*$, we can verify that, for any $q \in \mathbb{R}^d$, we have $\Psi_{\mathcal{J}_q^* X} = J_q^\star \Psi_X$, and therefore

$$\Psi_X(q) = \Psi_{\mathcal{J}_q^* X}(0)\,. \tag{5.13}$$

The idea now is to use the analyticity of map $X \mapsto \Psi_X$, to extend the right hand side of equation (5.13) to the complex domain $\|\mathrm{Im}\, q\| < \delta$. Let $0 < \varrho' < \varrho - \delta$. Let B' be an open ball around K in $\mathcal{A}_{\varrho'}(\mathcal{V}_0)$ such that the function $X \mapsto \Psi_X$ is well-defined for every $X \in B'$. Now, let B be an open ball in $\mathcal{A}_\varrho(\mathcal{V}_0)$ such that $\mathcal{J}_q^* B \subset B'$, for all $q \in \mathbb{C}^d$ of norm smaller than or equal to $\varrho' - \varrho$. Regarded as a function of (X, q), the right hand side of (5.13), is analytic and bounded on the product of B with the strip $\|\mathrm{Im}\, q\| < \varrho - \varrho'$. This show that Ψ_X belongs to $\mathcal{F}_\delta(\mathcal{V}_0)$ for all $X \in B$. **QED**

Theorem 5.4 *If κ' and κ are chosen sufficiently large then there exists an open neighborhood B of K in $\mathcal{A}_\varrho(\mathcal{V}_0)$ and for every $X \in \mathcal{W} \cap B$, of the form $X(x,y) = (\omega, f(x)y)$, there exists an analytic map $\psi_X : D_\delta \times \mathrm{SL}(2,\mathbb{C}) \to D_\delta \times \mathrm{SL}(2,\mathbb{C})$ of the form $\psi_X(x,y) = (x, \Psi_X(x)^{-1} y)$ with $\Psi_X \in \mathcal{F}'_\delta(\mathrm{SL}(2,\mathbb{C}), \mathcal{V}_0)$, and $0 < \delta < \varrho$, such that $\psi_X^* X = K$. If f is real-matrix valued when restricted to $\mathbb{R}^d$, so is Ψ_X.*

Proof. Let $\mathrm{G} = \mathrm{SL}(2, \mathbb{C})$. By Proposition 5.2 and Proposition 5.3, there exists an open neighborhood B of K in $\mathcal{A}_\varrho(\mathcal{V}_0)$ and, for every $X \in \mathcal{W} \cap B$, the map $\Psi_X \in \mathcal{F}_\delta(\mathrm{G}, \mathcal{V}_0)$, defined in Proposition 5.2, such that the map $X \mapsto \Psi_X$ is analytic from $B \cap \mathcal{W}$ to $\mathcal{F}_\delta(\mathrm{G}, \mathcal{V}_0)$. Since for every $x \in D_\delta$, $\Psi_X(x) \in \mathrm{G}$ is invertible, the map $\psi_X : D_\delta \times \mathrm{G} \to D_\delta \times \mathrm{G}$ is well defined by $\psi_X(x, y) = (x, \Psi_X(x)^{-1} y)$, and also invertible. Notice also that the conclusions of Proposition 5.1 and Proposition 5.2 are valid if in the definitions of the norms (5.4) and (5.5), $\|f_v\|$ is replaced by $(1+|\omega \cdot v|)\|f_v\|$, since $\|U_{n-1} - \mathrm{I}\|_\varrho$ can be replaced by $\|U_{n-1} - \mathrm{I}\|'_\varrho$ in (5.8). Therefore, $\Psi_X \in \mathcal{F}'_\delta(\mathrm{G}, \mathcal{V}_0)$.

It remains to show that $\psi_X^*(X) = K$, or equivalently (5.1). The maps Φ^t_X and $\Phi^t_{\mathcal{R}(X)}$, associated to vector fields X and $\mathcal{R}(X)$, respectively, are related via

$$\Phi^t_{\mathcal{R}(X)} = [U(\cdot + \eta^{-1}\omega t)^{-1} \Phi_X^{\eta^{-1} t} U] \circ T. \tag{5.14}$$

Using this identity we can obtain the following relation, for all integer $m \geq 0$,

$$(\Phi^t_m)^{-1} \Psi_m(\cdot + \omega t) e^{tu} = U_m [(\Phi^{\eta_{m+1} t}_{m+1})^{-1} \Psi_{m+1}(\cdot + \eta_{m+1}\omega t) e^{tu}] \circ T^{-1}_{m+1}. \tag{5.15}$$

Iterating this identity we find

$$\begin{aligned}(\Phi^t_0)^{-1} \Psi_0(\cdot + \omega t) e^{-tu} = {} & [U_0 U_1 \circ T_1^{-1} \cdots U_m \circ T_m^{-1} \circ \cdots \circ T_1^{-1}] \\ & \cdot [(\Phi^{\lambda_{m+1} t}_{m+1})^{-1} \Psi_{m+1}(\cdot + \lambda_{m+1}\omega t) e^{tu}] \circ T^{-1}_{m+1} \circ \cdots \circ T_1^{-1}.\end{aligned} \tag{5.16}$$

We know that the expression in the first square bracket converges to Ψ_0 in $\mathcal{F}_0(\mathrm{G}, \mathcal{V}_0)$, as $m \to \infty$, and that $\|U_0 U_1 \cdots U_m - \mathrm{I}\|_0 < 1$. Here, we have used that $\|F \circ T\|_0 = \|F\|_0$. We have previously obtained that $\Psi_m \in B_m$, and thus

$$\|\Psi_m - \mathrm{I}\|_0 < \chi_m^{1/3}. \tag{5.17}$$

Moreover, by writing the integral equation that $\varphi^t(x) = e^{-ut}\Phi^t(x) - \mathrm{I}$ satisfies,

$$\varphi^t(x) = \int_0^t e^{-us} [f(x + \omega s) - u] e^{us} [\varphi^s(x) + \mathrm{I}] \, ds\,, \tag{5.18}$$

and applying the contraction mapping principle, we can obtain the following estimate

$$\|e^{-ut}\Phi^t(x) - \mathrm{I}\|_0 \leq 2 e^{2\|u\||t|} \|f - u\|_0, \tag{5.19}$$

for t in a small open interval of $\mathbb{R}$ containing zero.

Similarly, starting from the equation (5.2) one can obtain, for a sufficiently small open interval of values t around zero, an estimate $\|\Phi^t(x) - \mathrm{I}\|_0 \leq 2|t|\|f\|_0$. Finally, using $u_{m+1} = \lambda^{-1}_{m+1} u$, we obtain the following estimate

$$\|(\Phi^{\lambda_{m+1} t}_{m+1})^{-1} \Psi_{m+1}(\cdot + \lambda_{m+1}\omega t) e^{tu} - \mathrm{I}\|_0 \leq c_1 \|f_{m+1} - u_{m+1}\|_0 + c_2 \|\Psi_{m+1} - \mathrm{I}\|_0 < c \chi^{1/3}_{m+1}. \tag{5.20}$$

Here $c, c_1, c_2 > 0$ are universal constants that depend only on $\|u\|$, an upper bound on $|t|$ and an upper bound on $\|f\|_0$.

Using (5.20), we find that the right hand side of (5.16), and thus the left one, converges to $\Psi_0(x)$ in $\mathcal{F}_0(\mathrm{G}, \mathcal{V}_0)$. Since the convergence in $\mathcal{F}_0(G, \mathcal{V}_0)$ also applies pointwise convergence (see Proposition 3.2), we have

$$\Phi_0^t(x)^{-1}\Psi_0(x+\omega t)e^{-tu} = \Psi_0(x), \tag{5.21}$$

for t in a small open interval containing zero, and all $x \in \mathbb{R}^d$. The identity can now be extended for all $t \in \mathbb{R}$, using the cocycle property (1.4). **QED**

Proof of Theorem 1.4. This theorem follows from Theorem 1.5. To see this, consider a family of vector fields X_v with $X_v(x, y) = X(x, y) + (0, vy)$, with $v \in \mathfrak{sl}(2, \mathbb{R})$. X_v belongs to $\mathcal{W}$ if and only if

$$\mathbb{E}(X_v) - W((\mathbb{I} - \mathbb{E})X_v) = 0\,, \tag{5.22}$$

or equivalently, for every $(x, y) \in \mathbb{T}^d \times \mathrm{SL}(2, \mathbb{R})$,

$$\mathbb{E}(X)(x, y) + (0, vy) - W((\mathbb{I} - \mathbb{E})X)(x, y) = 0\,. \tag{5.23}$$

Clearly, this equation has a unique solution v, for X close to K (such that $(\mathbb{I} - \mathbb{E})X$ is in the domain of W). The first assertion now follows from the fact that the vector fields on $\mathcal{W}$ are analytically reducible to K. The analyticity of the map $X \mapsto v$ follows from the analyticity of W (Theorem 1.5). **QED**

6 A normal form theorem for skew-product flows

Let G be an arbitrary Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$. Let $\mathbb{I}^-$ be a fixed projection operator on $\mathcal{F}_\varrho$, and let $K(x, y) = (\omega, u(x)y)$ be a fixed vector field in $\mathcal{A}_\varrho$, with $u \in \mathcal{F}_\varrho(\mathfrak{g})$, that satisfies $\mathbb{I}^- u = 0$. Recall the definition of the operator

$$\hat{f}g(x) = D_\omega g(x) - \mathrm{ad}_{f(x)}g(x), \tag{6.1}$$

where $\mathrm{ad}_{f(x)}g(x) = f(x)g(x) - g(x)f(x)$, for an arbitrary $g \in \mathcal{F}_\varrho$.

We will assume that the operator $\hat{u}$ maps $\mathbb{I}^-\mathcal{F}'_\varrho(\mathfrak{g})$ into $\mathbb{I}^-\mathcal{F}_\varrho(\mathfrak{g})$, and has an inverse which is bounded in norm by a positive constant $\zeta^{-1} > 1$, i.e. for every $g \in \mathbb{I}^-\mathcal{F}_\varrho(\mathfrak{g})$,

$$\left\|\hat{u}^{-1}g\right\|'_\varrho \le \zeta^{-1}\|g\|_\varrho\,. \tag{6.2}$$

In the case when $\mathrm{G} = \mathrm{SL}(2, \mathbb{C})$, $K(x, y) = (\omega, uy)$, with $u \in \mathfrak{g}$, and $\mathbb{I}^-$ is the projection operator onto the non-resonant subspace of $\mathcal{F}_\varrho(\mathfrak{g})$, this assumption is verified by Proposition 3.3.

Given a vector field $X : D_\varrho \times \mathrm{G} \to D_\varrho \times \mathrm{G}$, of the form $X(x,y) = (\omega, f(x)y)$, near K, with $f \in \mathcal{F}_\varrho(\mathfrak{g})$, the goal of this section is to construct a change of variables $\mathcal{U}_X(x,y) = (x, U_f(x)^{-1}y)$, with $U_f : D_\varrho \mapsto \mathrm{G}$, such that the pullback $\mathcal{U}_X^* X(x,y) = (\omega, U_f^\star f(x)y)$ of X under $\mathcal{U}_X$, belongs to $\mathcal{A}_\varrho$ and satisfies

$$\mathbb{I}^- U_f^\star f = 0\,. \tag{6.3}$$

We will construct a solution to (6.3) in the following way. First, we will construct an approximation to $\mathcal{U}_X$ given by the time-one map ϕ_Z^1 of the flow of a vector field Z of the form $Z(x,y) = (0, g(x)y)$ with $g \in \mathbb{I}^- \mathcal{F}_\varrho(\mathfrak{g})$ that satisfies the equation

$$\mathbb{I}^- (f - \hat{f} g) = 0\,. \tag{6.4}$$

Here and in the following the flow of a vector field Z is denoted by ϕ_Z and satisfies the equation $\frac{d}{dt}\phi_Z^t = Z \circ \phi_Z^t$, with the initial condition $\phi_Z^0 = \mathrm{I}$. Then, we will find the pullback of X under ϕ_Z^1, and iterate this procedure. Notice that $\phi_Z^t(x,y) = (x, \Phi_g^{-t}(x)y)$, where $\Phi_g^t(x) = e^{g(x)t}$ is a matrix exponential, defined for an arbitrary $A \in \mathbb{C}^{n\times n}$ by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{6.5}$$

We begin by proving some estimates for pullbacks under a time t map of the flow of a vector field $Z(x,y) = (0, g(x)y)$, with $g \in \mathcal{F}_\varrho(\mathfrak{g})$.

Proposition 6.1 *Let $t \in \mathbb{R}$ be fixed. Let $g \in \mathcal{F}_\varrho(\mathfrak{g})$ and $f \in \mathcal{F}_\varrho(\mathfrak{g})$ satisfy $\|tg\|_\varrho \le \varepsilon$ and $\|tD_\omega g\|_\varrho \le s\varepsilon$, with $\varepsilon \le 1/6$, and $s > 0$. Then $(\Phi_g^t)^\star f$ belongs to $\mathcal{F}_\varrho(\mathfrak{g})$, and satisfies*

$$\begin{aligned} \left\|(\Phi_g^t)^\star f - f\right\|_\varrho &\le 2(2\|f\|_\varrho + s)\,\varepsilon\,, \\ \left\|(\Phi_g^t)^\star f - f + t\hat{f} g\right\|_\varrho &\le (2\|f\|_\varrho + s)\,\varepsilon^2\,. \end{aligned} \tag{6.6}$$

Proof. Writing Φ_g^t explicitly and evaluating

$$(\Phi_g^t)^\star f = \Phi_g^{-t}(f - D_\omega)\Phi_g^t\,, \tag{6.7}$$

at point $x \in D_\varrho$, we find

$$(\Phi_g^t)^\star f(x) = e^{-g(x)t}(f(x) - D_\omega)e^{g(x)t}. \tag{6.8}$$

Differentiating with respect to t, one can verify the identities

$$e^{-g(x)t} f(x) e^{g(x)t} = e^{\mathrm{ad}_{-g(x)t}} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}_{-g(x)t})^n f(x), \tag{6.9}$$

and

$$e^{-g(x)t}D_\omega e^{g(x)t} = \sum_{n=1}^{\infty} \frac{1}{n!}(\mathrm{ad}_{-g(x)t})^{n-1} D_\omega g(x)t. \tag{6.10}$$

These two equalities guarantee that $(\Phi_g^t)^\star f \in \mathcal{F}_\varrho(\mathfrak{g})$, since $D_\omega g \in \mathcal{F}_\varrho(\mathfrak{g})$. Moreover,

$$[(\Phi_g^t)^\star f - f](x) = \sum_{n=1}^{\infty} \frac{1}{n!}(\mathrm{ad}_{-g(x)t})^{n-1}(\mathrm{ad}_{-g(x)t} f(x) - D_\omega g(x)t). \tag{6.11}$$

Using Proposition 3.2, the last identity implies

$$\|(\Phi_g^t)^\star f - f\|_\varrho \le \frac{e^{2\|tg\|_\varrho} - 1}{2\|g\|_\varrho}(2\|f\|_\varrho \|g\|_\varrho + \|D_\omega g\|_\varrho). \tag{6.12}$$

Similarly, from the same identity, we find that

$$\|(\Phi_g^t)^\star f - f + t\hat{f}g\|_\varrho \le \frac{e^{2\|tg\|_\varrho} - 2\|tg\|_\varrho - 1}{2\|g\|_\varrho}(2\|f\|_\varrho \|g\|_\varrho + \|D_\omega g\|_\varrho). \tag{6.13}$$

The first and second bound in (6.6) follow from the last two inequalities. **QED**

We start with our first step which is to solve the equation (6.4).

Proposition 6.2 *If $f \in \mathcal{F}_\varrho(\mathfrak{g})$, where $\varrho > 0$, satisfies*

$$\|f - u\|_\varrho \le \tfrac{1}{4}\zeta, \qquad \|\mathbb{I}^- f\|_\varrho \le \tfrac{1}{12}\zeta, \tag{6.14}$$

then the equation (6.4) *has a unique solution $g \in \mathbb{I}^- \mathcal{F}'_\varrho(\mathfrak{g})$, that satisfies $\|g\|'_\varrho \le \frac{2}{\zeta}\|\mathbb{I}^- f\|_\varrho$. Moreover, $(\Phi_g^1)^\star f$ belongs to $\mathcal{F}_\varrho(\mathfrak{g})$ and satisfies*

$$\begin{aligned} \left\|(\Phi_g^1)^\star f - f\right\|_\varrho &\le \frac{4}{\zeta}(2\|f\|_\varrho + 1)\|\mathbb{I}^- f\|_\varrho, \\ \left\|(\Phi_g^1)^\star f - f + \hat{f}g\right\|_\varrho &\le \frac{4}{\zeta^2}(2\|f\|_\varrho + 1)\|\mathbb{I}^- f\|_\varrho^2. \end{aligned} \tag{6.15}$$

Proof. Let $\tilde{f} = f - u$. We can rewrite the equation (6.4) in the form

$$\mathbb{I}^-(\mathbb{I} - \mathrm{ad}_{\tilde{f}}\hat{u}^{-1})\hat{u}g = \mathbb{I}^- f. \tag{6.16}$$

For every $g \in \mathcal{F}_\varrho(\mathfrak{g})$, using the first condition in (6.14), we have that

$$\|\mathrm{ad}_{\tilde{f}} g\|_\varrho \le 2\|g\|_\varrho \|\tilde{f}\|_\varrho \le \frac{\zeta}{2}\|g\|_\varrho. \tag{6.17}$$

Also, for every $g \in \mathbb{I}^{-}\mathcal{F}_{\varrho}(\mathfrak{g})$,

$$\|\mathbb{I}^{-}\mathrm{ad}_{\tilde{f}}\hat{u}^{-1}g\|_{\varrho} \leq \frac{\zeta}{2}\|\hat{u}^{-1}g\|_{\varrho} \leq \frac{1}{2}\|g\|_{\varrho}. \tag{6.18}$$

Thus, the operator $\mathbb{I}^{-}\mathrm{ad}_{\tilde{f}}\hat{u}^{-1}$ is bounded on $\mathbb{I}^{-}\mathcal{F}_{\varrho}(\mathfrak{g})$, with the operator norm smaller than or equal to $1/2$. The equation (6.16) can then be solved by inverting the operator $(\mathbb{I} - \mathrm{ad}_{\tilde{f}}\hat{u}^{-1})\mathbb{I}^{-}$ by means of Neumann series. The solution $g \in \mathbb{I}^{-}\mathcal{F}'_{\varrho}(\mathfrak{g})$ is unique and satisfies the desired bound. The second bound in (6.14), now allows us to apply Proposition 6.1 and the bounds (6.15) follow directly from it, by setting $\varepsilon = (2/\zeta)\|\mathbb{I}^{-}f\|_{\varrho}$, $s = 1$ and $t = 1$. **QED**

Our next step is to iterate the map $f \mapsto (\Phi^1_g)^{\star} f$ described in Proposition 6.2. We start with $f = f_0$ and set

$$f_{n+1} = (\Phi^1_{g_n})^{\star} f_n\,, \qquad \mathbb{I}^{-}(f_n - \hat{f}_n g_n) = 0\,, \tag{6.19}$$

for all non-negative integer n. The expectation is that the maps U_n defined by

$$U_n = \Phi^1_{g_0}\Phi^1_{g_1}\ldots\Phi^1_{g_{n-1}}, \tag{6.20}$$

for $n \in \mathbb{N}$ and $U_0 = \mathrm{I}$, converge to a solution U_f of the equation (6.3), as n tends to infinity. This leads to the main result of this section. To prove it, the following estimate will be needed.

Proposition 6.3 *If $\varrho, t > 0$ and $g \in \mathcal{F}'_{\varrho}$ is given, then $\Phi^t_g - \mathrm{I} \in \mathcal{F}'_{\varrho}$ and*

$$\|\Phi^t_g - \mathrm{I}\|'_{\varrho} \leq e^{\|tg\|'_{\varrho}} - 1\,. \tag{6.21}$$

Proof. The claim follows directly from $\Phi^t_g(x) = e^{g(x)t}$, using the Banach algebra property of $\mathcal{F}'_{\varrho}$. **QED**

Let $\varrho > 0$, and let us choose $R \geq \|u\|_{\varrho} + \zeta/4 + 1/2$ and $\varepsilon > 0$, subject to the constraints

$$\varepsilon \leq 2^{-4}\zeta\,, \qquad \varepsilon \leq 2^{-6}\zeta^2 R^{-1}\,. \tag{6.22}$$

In the following, we state and prove a normal form theorem that is the main result of this section.

Theorem 6.4 *If $f \in \mathcal{F}_{\rho}(\mathfrak{g})$ such that*

$$\|f - u\|_{\varrho} \leq 2^{-3}\zeta\,, \qquad \|\mathbb{I}^{-}f\|_{\varrho} \leq \varepsilon\,, \tag{6.23}$$

with $\varepsilon > 0$ satisfying conditions (6.22), then there exists an analytic change of coordinates $U_f : D_{\varrho} \to \mathrm{G}$, such that $U_f^{\star} f$ belongs to $\mathcal{F}_{\varrho}(\mathfrak{g})$ and satisfies equation (6.3). The map

$f \mapsto U_f$ is continuous in the region defined by inequalities (6.23)*, analytic in the interior of this region, and satisfies the bounds*

$$\begin{aligned} \|U_f - \mathrm{I}\|'_\varrho &\leq \frac{3}{\zeta}\|\mathbb{I}^- f\|_\varrho\,, \\ \|U_f^\star f - f\|_\varrho &\leq \frac{64R}{7\zeta}\|\mathbb{I}^- f\|_\varrho\,, \\ \left\|U_f^\star f - f + \hat{f}g\right\|_\varrho &\leq \Bigl(\frac{64R}{7\zeta}+1\Bigr)\frac{8R}{\zeta^2}\|\mathbb{I}^- f\|_\varrho^2\,. \end{aligned} \tag{6.24}$$

Here, $g \in \mathbb{I}^-\mathcal{F}'_\varrho(\mathfrak{g})$ is defined by (6.4) *and satisfies the bound $\|g\|'_\varrho \leq \frac{2}{\zeta}\|\mathbb{I}^- f\|_\varrho$. If f is real matrix valued when restricted to $\mathbb{R}^d$, so is U_f.*

Proof. Our first goal is to prove that (6.19) defines a sequence of maps $f_m \in \mathcal{F}_\varrho$, satisfying

$$\|f_m - f_{m-1}\|_\varrho \leq 2^{-m-3}\zeta\,, \qquad \|\mathbb{I}^- f_m\|_\varrho \leq 8^{-m}\varepsilon\,. \tag{6.25}$$

We proceed by induction. If we define $f_{-1} = u$ and $f_0 = f$, then these bounds hold for $m = 0$ by (6.23). Assume now that (6.25) holds for all integer m with $0 \leq m \leq n$. Then, by summing up the bounds on $f_m - f_{m-1}$ for $0 \leq m \leq n$, we obtain the first of the inequalities

$$\|f_n - u\|_\varrho \leq \frac{1}{4}\zeta\,, \qquad \|\mathbb{I}^- f_n\|_\varrho \leq 4^{-n-2}\zeta\,. \tag{6.26}$$

The second inequality follows from (6.25), by substituting the first bound in (6.22) on ε. Thus, Proposition 6.2 guarantees a unique solution to (6.19), and yields the bounds

$$\|f_{n+1} - f_n\|_\varrho \leq \frac{1}{\zeta}8^{-n+1}R\varepsilon\,, \qquad \|\mathbb{I}^- f_{n+1}\|_\varrho \leq \frac{1}{\zeta^2}8^{-2n+1}R\varepsilon^2\,. \tag{6.27}$$

Here, we have also used that $2\|f_n\|_\varrho + 1 \leq 2R$, which follows from the first inequality in (6.26). By using the second condition in (6.22), we now obtain (6.25) for $m = n+1$ from the bounds (6.27).

Next, consider the functions $\varphi_j = \Phi^1_{g_j} - \mathrm{I}$. By Proposition 6.3 and Proposition 6.2,

$$\|\varphi_j\|'_\varrho \leq e^{\|g_j\|'_\varrho} - 1 \leq e^{\frac{2}{\zeta}\|\mathbb{I}^- f_j\|_\varrho} - 1\,. \tag{6.28}$$

Using (6.25) we have the bounds

$$\|\varphi_j\|'_\varrho \leq e^{\frac{2\varepsilon}{\zeta}8^{-j}} - 1\,. \tag{6.29}$$

Define the maps $U_{m,n} = \Phi^1_{g_m}\Phi^1_{g_{m+1}}\ldots\Phi^1_{g_{n-1}}$ in $\mathcal{F}'_\varrho(\mathrm{G})$. Here, and in what follows, it is assumed that $0 \leq m < n$. Setting $U_{j,j} = \mathrm{I}$, and using the bounds (6.29) we obtain

$$\|U_n - U_m\|'_\varrho = \left\|\sum_{j=m}^{n-1}\varphi_j U_{j+1,n}\right\|'_\varrho \leq e^{\frac{16\varepsilon}{7\zeta}8^{-m}} - 1\,. \tag{6.30}$$

This shows that the sequence $n \mapsto U_n$ converges uniformly in $\mathcal{F}'_\varrho(\mathrm{G})$ to a limit $U_f \in \mathcal{F}'_\varrho(\mathrm{G})$. The analyticity of the maps U_n and the uniform convergence of the sequence $n \mapsto U_n$ implies the analyticity of the map $f \mapsto U_f$. Clearly, f_n converge to $U_f^\star(f)$ in $\mathcal{F}_\varrho(\mathfrak{g})$. The first inequality in (6.24) follows from the estimate (6.30), if we set $m = 0$ and $\varepsilon = \|\mathbb{I}^- f\|_\varrho$, and use the first bound in (6.22). The second inequality in (6.24) is now obtained by summation over n of the first bound in (6.27).

Since $U_f^\star f = U_{f_1}^\star f_1$ with $f_1 = (\Phi_g^1)^\star f$, we have

$$U_f^\star f - f + \hat{f} g = \left(U_{f_1}^\star f_1 - f_1\right) + \left((\Phi_g^1)^\star f - f + \hat{f} g\right). \tag{6.31}$$

The norm of the term in the first parentheses of (6.31) can be estimated in the same way as the term $\|U_f^\star f - f\|_\varrho$, yielding the bound

$$\left\|U_{f_1}^\star f_1 - f_1\right\|_\varrho \leq \frac{64R}{7\zeta} \|\mathbb{I}^- f_1\|_\varrho . \tag{6.32}$$

Since $\|\mathbb{I}^- f_1\|_\varrho \leq \|(\Phi_g^1)^\star f - f + \hat{f} g\|_\varrho$, the third bound in (6.24) now follows from the second inequality in (6.15). **QED**

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