Localization for a Class of Linear Systems[1]

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Abstract

We consider a class of continuous-time stochastic growth models on d-dimensional lattice with non-negative real numbers as possible values per site. The class contains examples such as binary contact path process and potlatch process. We show the equivalence between the slow population growth and localization property that the time integral of the replica overlap diverges. We also prove, under reasonable assumptions, a localization property in a stronger form that the spatial distribution of the population does not decay uniformly in space.

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1 Introduction

We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \; ; \; x \in \mathbb{N}\}$. For $x = (x_1, .., x_d) \in \mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\eta = (\eta_x)_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\eta| = \sum_{x\in\mathbb{Z}^d} |\eta_x|$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For events $A, B \subset \Omega$, $A \subset B$ a.s. means that $P(A\backslash B) = 0$.

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Similarly, $A = B$ a.s. mean that $P(A\backslash B) = P(B\backslash A) = 0$. By a *constant*, we always means a *non-random* constant.

We consider a class of continuous-time stochastic growth models on d-dimensional lattice $\mathbb{Z}^d$ with non-negative real numbers as possible values per site, so that the configuration at time t can be written as $\eta_t = (\eta_{t,x})_{x\in\mathbb{Z}^d}$, $\eta_{t,x} \geq 0$. We interpret the coordinate $\eta_{t,x}$ as the "population" at time-space (t,x), though it need *not* be an integer. The class of growth models considered here is a reasonably ample subclass of the one considered in [Lig85, Chapter IX] as "linear systems". For example, it contains examples such as binary contact path process and potlatch process. The basic feature of the class is that the configurations are updated by applying the random linear transformation of the following form, when the Poisson clock rings at time-space (t,z):

$$\eta_{t,x} = \begin{cases} K_0\eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K_{x-z}\eta_{t-,z} & \text{if } x \neq z, \end{cases} \tag{1.1}$$

where $K = (K_x)_{x\in\mathbb{Z}^d}$ is a random vector with non-negative entries, and independent copies of K are used for each update (See section 1.1 for more detail). These models are known to exhibit, roughly speaking, the following phase transition [Lig85, Chapter IX, sections 3–5]:

i) If the dimension is high $d \geq 3$, *and* if the vector K is not too random, then, with positive probability, the growth of the population is as fast as its expected value as time the t tends to infinity, as such the *regular growth phase*.

ii) If the dimension is low $d = 1, 2$, *or* if the vector K is random enough, then, almost surely, the growth of the population strictly slower than its expected value as the time t tends to infinity, as such the *slow growth phase*.

We denote the spatial distribution of the population by:

$$\rho_{t,x} = \frac{\eta_{t,x}}{|\eta_t|}\mathbf{1}_{\{|\eta_t|>0\}}, \quad t > 0, x \in \mathbb{Z}^d. \tag{1.2}$$

In our previous paper [NY09], we investigated the case (i) above and showed under some technical assumptions that the spatial distribution (1.2) obeys the central limit theorem. We also proved the delocalization property which says that the spatial distribution (1.2) decays uniformly in space like $t^{-d/2}$ as time t tends to infinity.

In the present paper, we turn to the case (ii) above. We first prove the equivalence between the slow growth and a certain localization property in terms of the divergence of integrated replica overlap (Theorem 1.3.1 below). We also show that, under reasonable assumptions, the localization occurs in stronger form that the spatial distribution (1.2) does not decay uniformly in space as time t tends to infinity (Theorem 1.3.2 below). These, together with our previous work [NY09], verifies the delocalization/localization transition in correspondence with regular/slow growth transition for the class of model considered here.

It should be mentioned that the delocalization/localization transition in the same spirit has been discussed recently in various context, e.g., [CH02, CH06, CSY03, CY05, HY09, Sh09, Yo08a, Yo08b]. In particular, the last paper [Yo08b] by the second author of the present article can be considered as the discrete-time counterpart of the present paper. Still, we believe it worth while verifying the delocalization/localization transition for the continuous-time growth models discussed here, in view of its classical importance of the model.

1.1 The model

We introduce a random vector $K = (K_x)_{x\in\mathbb{Z}^d}$ which is bounded and of finite range in the sense that

$$0 \le K_x \le b_K \mathbf{1}_{\{|x|\le r_K\}} \text{ a.s. for some constants } b_K, r_K \in [0,\infty). \tag{1.3}$$

Let $\tau^{z,i}$, $(z \in \mathbb{Z}^d,\ i \in \mathbb{N}^*)$ be i.i.d. mean-one exponential random variables and $T^{z,i} = \tau^{z,1} + ... + \tau^{z,i}$. Let also $K^{z,i} = (K^{z,i}_x)_{x\in\mathbb{Z}^d}$ $(z \in \mathbb{Z}^d,\ i \in \mathbb{N}^*)$ be i.i.d. random vectors with the same distributions as K, independent of $\{\tau^{z,i}\}_{z\in\mathbb{Z}^d, i\in\mathbb{N}^*}$. Unless otherwise stated, we suppose for simplicity that the process $(\eta_t)_{t\ge0}$ starts from a single particle at the origin:

$$\eta_0 = (\eta_{0,x})_{x\in\mathbb{Z}^d}, \quad \eta_{0,x} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{1.4}$$

At time $t = T^{z,i}$, η_{t-} is replaced by η_t, where

$$\eta_{t,x} = \begin{cases} K^{z,i}_0 \eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K^{z,i}_{x-z}\eta_{t-,z} & \text{if } x \neq z. \end{cases} \tag{1.5}$$

A formal construction of the process $(\eta_t)_{t\ge0}$ can be given as a special case of [Lig85, page 427, Theorem 1.14] via Hille-Yosida theory. In section 1.4, we will also give an alternative construction of the process in terms of a stochastic differential equation.

To exclude uninteresting cases from the viewpoint of this article, we also assume that

$$\text{the set } \{x \in \mathbb{Z}^d \ ;\ E[K_x] \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d, \tag{1.6}$$

$$P(|K| = 1) < 1. \tag{1.7}$$

The first assumption (1.6) makes the model "truly d-dimensional". The reason for the second assumption (1.7) is to exclude the case $|\eta_t| \equiv 1$ a.s.

Here are some typical examples which fall into the above set-up:

• **The binary contact path process (BCPP):** The binary contact path process (BCPP), originally introduced by D. Griffeath [Gri83] is a special case the model, where

$$K = \begin{cases} (\delta_{x,0} + \delta_{x,e})_{x\in\mathbb{Z}^d} & \text{with probability } \frac{\lambda}{2d\lambda+1}, \text{ for each } 2d \text{ neighbor } e \text{ of } 0 \\ 0 & \text{with probability } \frac{1}{2d\lambda+1}. \end{cases} \tag{1.8}$$

The process is interpreted as the spread of an infection, with $\eta_{t,x}$ infected individuals at time t at the site x. The first line of (1.8) says that, with probability $\frac{\lambda}{2d\lambda+1}$ for each $|e| = 1$, all the infected individuals at site $x - e$ are duplicated and added to those on the site x. On the other hand, the second line of (1.8) says that, all the infected individuals at a site become healthy with probability $\frac{1}{2d\lambda+1}$. A motivation to study the BCPP comes from the fact that the projected process

$$(\eta_{t,x} \wedge 1)_{x\in\mathbb{Z}^d}, \quad t \ge 0$$

is the basic contact process [Gri83].

• **The potlatch process:** The potlatch process discussed in e.g. [HL81] and [Lig85, Chapter IX] is also a special case of the above set-up, in which

$$K_x = W k_x, \quad x \in \mathbb{Z}^d. \tag{1.9}$$

Here, $k = (k_x)_{x\in\mathbb{Z}^d} \in [0,\infty)^{\mathbb{Z}^d}$ is a non-random vector and W is a non-negative, bounded, mean-one random variable such that $P(W = 1) < 1$ (so that the notation k here is consistent with the definition (1.10) below). The potlatch process was first introduced in [Spi81] for the case $W \equiv 1$ and discussed further in [LS81]. It was in [HL81] where case with $W \not\equiv 1$ was introduced and discussed. Note that we do not restrict ourselves to the case $|k| = 1$ unlike in [HL81] and [Lig85, Chapter IX].

1.2 The regular and slow growth phases

We now recall the following facts and notion from [Lig85, page 433, Theorems 2.2 and 2.3], although our terminologies are somewhat different from the ones in [Lig85]. Let $\mathcal{F}_t$ be the σ-field generated by η_s, $s \le t$.

Lemma 1.2.1 *We set*

$$\begin{aligned} k &= (k_x)_{x \in \mathbb{Z}^d} = (E[K_x])_{x \in \mathbb{Z}^d} && (1.10)\\ \overline{\eta}_t &= (e^{-(|k|-1)t} \eta_{t,x})_{x \in \mathbb{Z}^d}. && (1.11) \end{aligned}$$

Then,

a) $(|\overline{\eta}_t|, \mathcal{F}_t)_{t \ge 0}$ *is a martingale, and therefore, the following limit exists a.s.*

$$|\overline{\eta}_\infty| = \lim_{t \to \infty} |\overline{\eta}_t|. \tag{1.12}$$

b) *Either*

$$E[|\overline{\eta}_\infty|] = 1 \ \ or \ \ 0. \tag{1.13}$$

Moreover, $E[|\overline{\eta}_\infty|] = 1$ *if and only if the limit (1.12) is convergent in* $\mathbb{L}^1(P)$.

We will refer to the former case of (1.13) as *regular growth phase* and the latter as *slow growth phase*.

The regular growth means that, at least with positive probability, the growth of the "total number" $|\eta_t|$ of the particles is of the same order as its expectation $e^{(|k|-1)t}|\eta_0|$. On the other hand, the slow growth means that, almost surely, the growth of $|\eta_t|$ is slower than its expectation.

Since we are mainly interested in the slow growth phase in this paper, we now present sufficient conditions for the slow growth.

Proposition 1.2.2 **a)** *For* $d = 1, 2$, $|\overline{\eta}_\infty| = 0$ *a.s. In particular for* $d = 1$, *there exists a constant* $c > 0$ *such that*

$$|\overline{\eta}_t| = O(e^{-ct}), \quad as\ t \to \infty,\ a.s. \tag{1.14}$$

b) *For any* $d \ge 1$, *suppose that*

$$\sum_{x \in \mathbb{Z}^d} E\left[K_x \ln K_x\right] > |k| - 1 \tag{1.15}$$

Then, again, there exists a constant $c > 0$ *such that (1.14) holds.*

Proof: Except for (1.14), these sufficient conditions are presented in [Lig85, Chapter IX, sections 4–5]. The exponential decay (1.14) follows from similar arguments as in discrete-time models discussed in [Yo08a, Theorems 3.1.1 and 3.2.1]. □

Remarks: 1) For BCPP, (1.15) is equivalent to $\lambda < (2d)^{-1}$, in which case it is known that $|\eta_t| \equiv 0$ for large enough t's a.s. [Lig85, Example 4.3.(c) on page 33, together with Theorem

1.10 (a) on page 267]. Thus, Proposition 1.2.2(b) applies only in a trivial manner for BCPP. In fact, we do not know if there is a value λ for which BCPP with $d \geq 3$ is in slow growth phase, without getting extinct a.s. For potlatch process,

$$(1.15) \iff E[W \ln W] > \frac{|k| - 1 - \sum_x k_x \ln k_x}{|k|}.$$

Thus, (1.15) and hence (1.14) is true if W is "random enough".
2) A sufficient condition for the regular growth phase will be given by (1.25) below.

1.3 Results

Recall that we have defined the spatial distribution of the population by (1.2). Interesting objects related to the density would be

$$\rho_t^* = \max_{x \in \mathbb{Z}^d} \rho_{t,x}, \text{ and } \mathcal{R}_t = \sum_{x \in \mathbb{Z}^d} \rho_{t,x}^2. \tag{1.16}$$

ρ_t^* is the density at the most populated site, while $\mathcal{R}_t$ is the probability that a given pair of particles at time t are at the same site. We call $\mathcal{R}_t$ the *replica overlap*, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of ρ_t^* or $\mathcal{R}_t$ indicate that the most of the particles are concentrated on small number of "favorite sites" (*localization*), whereas small values of them imply that the particles are spread out over a large number of sites (*delocalization*).

We first show that the regular and slow growth are characterized, respectively by convergence (delocalization) and divergence (localization) of the integrated replica overlap: $\int_0^\infty \mathcal{R}_s ds$.

Theorem 1.3.1 a) *Suppose that* $P(|\overline{\eta}_\infty| > 0) > 0$. *Then,*

$$\int_0^\infty \mathcal{R}_s ds < \infty \ \ a.s.$$

b) *Suppose on the contrary that* $P(|\overline{\eta}_\infty| = 0) = 1$. *Then,*

$$\{ \text{ survival } \} = \left\{ \int_0^\infty \mathcal{R}_s ds = \infty \right\}, \quad a.s. \tag{1.17}$$

where $\{\text{survival}\} = \{|\eta_t| \neq 0 \text{ for all } t \geq 0\}$. *Moreover, there exists a constant* $c > 0$ *such that*

$$|\overline{\eta}_t| \leq \exp\left(-c \int_0^t \mathcal{R}_s ds\right) \text{ for all large enough } t\text{'s, a.s.} \tag{1.18}$$

Results of this type are fundamental in analyzing a certain class of spatial random growth models, such as directed polymers in random environment [CH02, CH06, CSY03, CY05], linear stochastic evolutions [Yo08b], branching random walks and Brownian motions in random environment [HY09, Sh09]. Until quite recently, however, this type of results were available only when no extinction at finite time is allowed, i.e., $|\eta_t| > 0$ for all $t \geq 0$, e.g., [CH02, CH06, CSY03, CY05, HY09, Sh09]. In fact, the proof there relies on the analysis of the supermartingale $\ln |\overline{\eta}_t|$, which is not even defined if extinction at finite time is possible. To overcome this problem, we will adapt a more general approach introduced in [Yo08b].

Next, we present a result (Theorem 1.3.2 below) which says that, under reasonable assumptions, we can strengthen the localization property

$$\int_0^\infty \mathcal{R}_s ds = \infty$$

in (1.17) to:

$$\int_0^\infty \mathbf{1}\{\mathcal{R}_s \geq c\} ds = \infty,$$

where $c > 0$ is a constant. To state the theorem, we define

$$\beta_{x,y} = E[(K - \delta_0)_x (K - \delta_0)_y], \quad x, y \in \mathbb{Z}^d. \tag{1.19}$$

We also introduce:

$$G(x) = \int_0^\infty P_S^0(S_t = x) dt, \tag{1.20}$$

where $((S_t)_{t \geq 0}, P_S^x)$ is the continuous-time random walk on $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$, with the generator

$$L_S f(x) = \tfrac{1}{2} \sum_{y \in \mathbb{Z}^d} (k_{x-y} + k_{y-x}) (f(y) - f(x)), \quad \text{cf. (1.10)}. \tag{1.21}$$

Theorem 1.3.2 *Referring to (1.19)–(1.20), suppose either of*

a) $d = 1, 2$.

b) $d \geq 3$, $P(|\overline{\eta}_\infty| = 0) = 1$ *and*

$$\sum_{x,y \in \mathbb{Z}^d} G(x - y) \beta_{x,y} > 2. \tag{1.22}$$

Then there exist a constant $c \in (0, 1]$ *such that*

$$\{ \textit{survival} \} = \left\{ \int_0^\infty \mathbf{1}\{\mathcal{R}_s \geq c\} ds = \infty \right\} \quad \textit{a.s.} \tag{1.23}$$

Our proof of Theorem 1.3.2 is based on the idea of P. Carmona and Y. Hu in [CH02, CH06], where they prove similar results for directed polymers in random environment. Although the arguments in [CH02, CH06] are rather complicated and uses special structure of the model, it was possible to extract the main idea from [CH02, CH06] in a way applicable to our setting. Also, we could considerably reduce the technical complexity in the argument as compared with [CH02, CH06].

Remarks: **1)** We prove (1.23) by way of the following stronger estimate:

$$\{ \text{survival} \} \subset \left\{ \varliminf_{t \nearrow \infty} \frac{\int_0^t \mathcal{R}_s^{3/2} ds}{\int_0^t \mathcal{R}_s ds} \geq c_1 \right\} \quad \text{a.s.} \tag{1.24}$$

for some constant $c_1 > 0$. The inequality $r^{3/2} \leq \mathbf{1}\{r \geq c\} + \sqrt{c} r$ for $r, c \in [0, 1]$ can be used to conclude (1.23) from (1.24).
2) We note that $P(|\overline{\eta}_\infty| > 0) > 0$ if

$$d \geq 3 \ \text{ and } \sum_{x,y \in \mathbb{Z}^d} G(x - y) \beta_{x,y} < 2. \tag{1.25}$$

This, together with Theorem 1.3.1(a), shows that the condition (1.22) is necessary, up to the equality, for (1.23) to be true whenever survival occurs with positive probability. We see that (1.25) implies $P(|\overline{\eta}_\infty| > 0) > 0$ via the same line of argument as in [Lig85, page 464, Theorem 6.16], where the special case of the potlatch process is discussed. We consider the *dual process* $\zeta_t \in [0,\infty)^{\mathbb{Z}^d}$, $t \geq 0$ which evolves in the same way as $(\eta_t)_{t\geq 0}$ except that (1.1) is replaced by its transpose:

$$\zeta_{t,x} = \begin{cases} \sum_{y\in\mathbb{Z}^d} K_{y-x}\zeta_{t-,y} & \text{if } x = z, \\ \zeta_{t-,x} & \text{if } x \neq z. \end{cases} \tag{1.26}$$

By [Lig85, page 445, Theorem 3.12], a sufficient condition for $P(|\overline{\eta}_\infty| > 0) > 0$ is that there exists a function $h : \mathbb{Z}^d \to (0,\infty)$ such that $\lim_{|x|\to\infty} h(x) = 1$ and that

$$\sum_y q(x,y)h(y) = 0, \quad x \in \mathbb{Z}^d. \tag{1.27}$$

Here, $q(x,y)$ is the matrix given by [Lig85, page 445, (3.8)–(3.9)] for the dual process. In our setting, it is computed as:

$$q(x,y) = k_{x-y} + k_{y-x} - 2|k|\delta_{x,y} + \delta_{0,x}\sum_z \beta_{z,z+y},$$

so that (1.27) becomes:

$$(L_S h)(x) + \tfrac{1}{2}\delta_{0,x}\sum_{y,z} h(y-z)\beta_{y,z} = 0, \quad x \in \mathbb{Z}^d, \quad \text{cf. (1.21).}$$

Under the assumption (1.25), a choice of such function h is given by $h = 1 + cG$, where

$$c = \frac{E[(|K|-1)^2]}{1 - \frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} G(x-y)\beta_{x,y}}.$$

3) Let π_d be the return probability for the simple random walk on $\mathbb{Z}^d$. Also, let $\langle\,\cdot,\,\cdot\,\rangle$ and $*$ be the inner product of $\ell^2(\mathbb{Z}^d)$ and the discrete convolution respectively. We then have that

$$(1.22) \quad \Longleftrightarrow \quad \begin{cases} \lambda < \frac{1}{2d(1-2\pi_d)} & \text{for BCPP,} \\ E[W^2] > \frac{(2|k|-1)G(0)}{\langle\, G*k, k\,\rangle} & \text{for the potlatch process.} \end{cases} \tag{1.28}$$

For BCPP, (1.28) can be seen from that (cf. [NY09, page 965])

$$\beta_{x,y} = \frac{\mathbf{1}\{x=0\} + \lambda\mathbf{1}\{|x|=1\}}{2d\lambda+1}\delta_{x,y}, \quad \text{and} \quad G(0) = \frac{2d\lambda+1}{2d\lambda}\frac{1}{1-\pi_d}.$$

To see (1.28) for the potlatch process, we note that $\frac{1}{2}(k+\check{k})*G = |k|G - \delta_0$, with $\check{k}_x = k_{-x}$ and that

$$\beta_{x,y} = E[W^2]k_x k_y - k_x\delta_{y,0} - k_y\delta_{x,0} + \delta_{x,0}\delta_{y,0}.$$

Thus,

$$\begin{aligned} \sum_{x,y\in\mathbb{Z}^d} G(x-y)\beta_{x,y} &= E[W^2]\langle\, G*k, k\,\rangle - \langle\, G, k+\check{k}\,\rangle + G(0) \\ &= E[W^2]\langle\, G*k, k\,\rangle + 2 - (2|k|-1)G(0), \end{aligned}$$

from which (1.28) for the potlatch process follows.

1.4 SDE description of the process

We now give an alternative description of the process in terms of a stochastic differential equation (SDE). We introduce random measures on $[0,\infty) \times [0,\infty)^{\mathbb{Z}^d}$ by

$$N^z(dsd\xi) = \sum_{i\geq 1} \mathbf{1}\{(T^{z,i}, K^{z,i}) \in dsd\xi\}, \quad N^z_t(dsd\xi) = \mathbf{1}_{\{s\leq t\}} N^z(dsd\xi). \tag{1.29}$$

Then, N^z, $z \in \mathbb{Z}^d$ are independent Poisson random measures on $[0,\infty) \times [0,\infty)^{\mathbb{Z}^d}$ with the intensity

$$ds \times P(K \in \cdot).$$

The precise definition of the process $(\eta_t)_{t\geq 0}$ is then given by the following stochastic differential equation:

$$\eta_{t,x} = \eta_{0,x} + \sum_{z\in\mathbb{Z}^d} \int N^z_t(dsd\xi)\,(\xi_{x-z} - \delta_{x,z})\,\eta_{s-,z}. \tag{1.30}$$

By (1.3), it is standard to see that (1.30) defines a unique process $\eta_t = (\eta_{t,x})$, $(t \geq 0)$ and that (η_t) is Markovian.

2 Proofs

It is convenient to introduce the following notation:

$$\nu = P(K \in \cdot) \in \mathcal{P}([0,\infty)^{\mathbb{Z}^d}), \text{ the law of } K. \tag{2.1}$$

$$\widetilde{N}^z(dsd\xi) = N^z(dsd\xi) - ds\nu(d\xi), \quad \widetilde{N}^z_t(dsd\xi) = \mathbf{1}_{s\leq t}\widetilde{N}^z(dsd\xi). \tag{2.2}$$

2.1 Proof of Theorem 1.3.1

The proof of Theorem 1.3.1 is based on the following

Lemma 2.1.1

$$\{\, |\overline{\eta}_\infty| = 0, \ \ survival \,\} = \left\{ \int_0^\infty \mathcal{R}_s ds = \infty \right\}, \quad a.s. \tag{2.3}$$

Moreover, there exists a constant $c > 0$ *such that (1.18) holds a.s. on the event* $\{ \int_0^\infty \mathcal{R}_s ds = \infty \}$.

Proof: We see from (1.30) that

$$\begin{aligned} |\overline{\eta}_t| &= |\overline{\eta}_0| + \sum_z \int \widetilde{N}^z_t(dsd\xi)|\overline{\eta}_{s-}|(|\xi|-1)\rho_{s-,z} \quad \text{(cf. (2.2))} \\ &= |\overline{\eta}_0| + \int_0^t |\overline{\eta}_{s-}| dM_s \end{aligned}$$

where

$$M_t = \sum_z \int \widetilde{N}^z_t(dsd\xi)(|\xi|-1)\rho_{s-,z}.$$

Then, by the Doléans-Dale exponential formula (e.g., [HWY92, page 248, 9.39]),

$$|\overline{\eta}_t| = \exp(M_t)\,D_t,$$

where

$$D_t = \prod_{s\leq t}(1+\Delta M_s)\exp(-\Delta M_s), \quad \text{with } \Delta M_t = M_t - M_{t-}.$$

Note also the predictable quadratic variation of $M.$ is given by

1) $\quad \langle M \rangle_t = E[(|K|-1)^2] \int_0^t \mathcal{R}_s ds.$

Since $-1 \leq \Delta M_t \leq b_K - 1 < \infty$, we have that (See e.g.[HWY92, page 222, 8.32])

2) $\quad \{ \langle M \rangle_\infty < \infty \} \subset \{[M]_\infty < \infty, \ M_t \text{ converges as } t \nearrow \infty\} \quad \text{a.s.}$

3) $\quad \{ \langle M \rangle_\infty = \infty \} \subset \left\{ \lim_{t\to\infty} \frac{\langle M \rangle_t}{[M]_t} = 1, \ \lim_{t\to\infty} \frac{M_t}{\langle M \rangle_t} = 0 \right\} \quad \text{a.s.}$

where

$$[M]_t = \sum_{s \leq t} (\Delta M_s)^2$$

We start with the "$\supset$" part of (2.3): Note that $(1+u)e^{-u} \leq e^{-c_1 u^2}$ for $-1 \leq u \leq b_K - 1$, where $c_1 > 0$ is a constant. We suppose that $\int_0^\infty \mathcal{R}_s ds = \infty$, or equivalently that, $\langle M \rangle_\infty = \infty$. Then, for large t,

$$\exp(M_t) D_t \leq \exp(M_t - c_1 [M]_t) \stackrel{(3)}{\leq} \exp\left(-\frac{c_1}{2}\langle M \rangle_t\right) \stackrel{(1)}{\leq} \exp\left(-c_2 \int_0^t \mathcal{R}_s ds\right)$$

This shows that $\int_0^\infty \mathcal{R}_s ds = \infty$ implies $|\eta_\infty| = 0$, together with the bound (1.18).
We now turn to the "$\subset$" part of (2.3): We need to prove that

4) $\{ \int_0^\infty \mathcal{R}_s ds < \infty \ \ \text{survival}\} \stackrel{\text{a.s.}}{\subset} \{ |\overline{\eta}_\infty| > 0\}.$

We have

5) $\{ \int_0^\infty \mathcal{R}_s ds < \infty\} \stackrel{(1)\text{–}(2)}{\subset} \{M_t \text{ converges as } t \nearrow \infty\} \quad \text{a.s.}$

On the other hand,

$$\sum_{s \leq t} |(1 + \Delta M_s)\exp(-\Delta M_s) - 1| \leq \frac{e}{2}[M]_t,$$

since $|(1+u)e^{-u} - 1| \leq eu^2/2$ for $u \geq -1$. Thus,

6) $\{ \int_0^\infty \mathcal{R}_s ds < \infty, \ \ \text{survival}\} \subset \{D_t \text{ converges to a positive limit as } t \nearrow \infty\} \quad \text{a.s.}$

We now obtain (4) by (5)–(6). □

Proof of Theorem 1.3.1: a): If $P(|\overline{\eta}_\infty| > 0) > 0$, then,

$$\{\text{survival}\} = \{|\overline{\eta}_\infty| > 0\} \quad \text{a.s.}$$

This can be seen easily by the argument in [Gri83, page 701, proof of Proposition]. We see from this and (2.3) that $\int_0^\infty \mathcal{R}_s ds < \infty$ a.s. on the event of survival, while $\int_0^\infty \mathcal{R}_s ds < \infty$ is obvious outside the event of survival.
b): This follows from Lemma 2.1.1 □

2.2 Proof of Theorem 1.3.2

Let p be a transition function of a symmetric discrete-time random walk defined by

$$p(x) = \begin{cases} \dfrac{k_x + k_{-x}}{2(|k| - k_0)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

and p_n be the n-step transition function. We set

$$g_n(x) = \delta_{x,0} + \sum_{k=1}^{n} p_k(x).$$

Lemma 2.2.1 *Under the assumptions of Theorem 1.3.2, there exists n such that*

$$\sum_{x,y} g_n(x-y)\beta_{x,y} > 2(|k| - k_0). \tag{2.4}$$

Proof: Since the discrete-time random walk with the transition probability p is the jump chain of the continuous-time random walk $((S_t)_{t\geq 0}, P_S^x)$ with the generator (1.21), we have that

1) $$\lim_{n\to\infty} g_n(x) = (|k| - k_0)G(x) \quad \text{for all } x \in \mathbb{Z}^d.$$

For $d \geq 3$, $G(x) < \infty$ for any $x \in \mathbb{Z}^d$ and $\beta_{x,y} \neq 0$ only when $|x|, |y| \leq r_K$, we see from (1) that

$$\lim_{n\to\infty} \sum_{x,y} g_n(x-y)\beta_{x,y} = (|k| - k_0) \sum_{x,y} G(x-y)\beta_{x,y}.$$

Thus, (2.4) holds for all large enough n's.
To show (2.4) for $d = 1, 2$, we will prove that

$$\lim_{n\to\infty} \sum_{x,y} g_{2n-1}(x-y)\beta_{x,y} = \infty.$$

For $f \in \ell^1(\mathbb{Z}^d)$, we denote its Fourier transform by

$$\widehat{f}(\theta) = \sum_{x\in\mathbb{Z}^d} f(x)\exp(\mathbf{i}x\cdot\theta), \quad \theta \in I \stackrel{\text{def}}{=} [-\pi, \pi]^d.$$

We then have that

$$g_{2n-1}(x) = \frac{1}{(2\pi)^d} \int_I \frac{1 - \widehat{p}(\theta)^{2n}}{1 - \widehat{p}(\theta)} \exp(\mathbf{i}x\cdot\theta)d\theta$$

and hence that

$$\begin{aligned} \sum_{x,y} g_{2n-1}(x-y)\beta_{x,y} &= \frac{1}{(2\pi)^d} \int_I \frac{1 - \widehat{p}(\theta)^{2n}}{1 - \widehat{p}(\theta)} \sum_{x,y} \exp(\mathbf{i}(x-y)\cdot\theta) E[(K - \delta_0)_x (K - \delta_0)_y] d\theta \\ &= \frac{1}{(2\pi)^d} \int_I \frac{1 - \widehat{p}(\theta)^{2n}}{1 - \widehat{p}(\theta)} E[|\widehat{K}(\theta) - 1|^2] d\theta. \end{aligned}$$

Since $p(\cdot)$ is even, we see that $\widehat{p}(\theta) \in [-1, 1]$ for all $\theta \in I$. Also, by (1.6), there exist constants $c_i > 0$ $(i = 1, 2, 3)$ such that

$$0 \leq 1 - c_1|\theta|^2 \leq \widehat{p}(\theta) \leq 1 - c_2|\theta|^2 \quad \text{for } |\theta| \leq c_3.$$

These imply that

$$\varliminf_{n\to\infty}\sum_{x,y} g_{2n-1}(x-y)\beta_{x,y} \ge \frac{1}{(2\pi)^d c_1}\int_{|\theta|\le c_2}\frac{E[|\widehat{K}(\theta)-1|^2]}{|\theta|^2}d\theta.$$

The integral on the right-hand-side diverges if $d \ge 2$, since

$$E[|\widehat{K}(0)-1|^2] = E[(|K|-1)^2] \neq 0.$$

□

We take an n in Lemma 2.2.1 and fix it. We then set

$$g = g_n \ \text{ and } \ \mathcal{S}_t = \langle\, g * \rho_t, \rho_t \,\rangle, \tag{2.5}$$

where the bracket $\langle\, \cdot, \ \cdot \,\rangle$ and $*$ stand for the inner product of $\ell^2(\mathbb{Z}^d)$ and the discrete convolution respectively. In what follows, we will often use the Hausdorff-Young inequality:

$$|(f*h)^2|^{1/2} \le |f||h^2|^{1/2} \quad f \in \ell^1(\mathbb{Z}^d),\ h \in \ell^2(\mathbb{Z}^d). \tag{2.6}$$

For example, we have that

$$0 \le \mathcal{S}_t \overset{\text{Schwarz}}{\le} |(g*\rho_t)^2|^{1/2}|(\rho_t)^2|^{1/2} \overset{(2.6)}{\le} |g||(\rho_t)^2| = |g|\mathcal{R}_t < \infty. \tag{2.7}$$

The proof of Theorem 1.3.2 is based on the following

Lemma 2.2.2 *Let*

$$\mathcal{S}_t = \mathcal{S}_0 + \mathcal{M}_t + \mathcal{A}_t$$

be the Doob decomposition, where $\mathcal{M}.$ and $\mathcal{A}.$ are a martingale and a predictable process, respectively. Then,

a) *There is constants $c_1, c_2 \in (0,\infty)$ such that*

$$\mathcal{A}_t \ge \int_0^t \left(c_1\mathcal{R}_s - c_2\mathcal{R}_s^{3/2}\right) ds \tag{2.8}$$

b)

$$\left\{\int_0^\infty \mathcal{R}_s ds = \infty\right\} \subset \left\{\lim_{t\to\infty}\frac{\mathcal{M}_t}{\int_0^t \mathcal{R}_s ds} = 0\right\} \quad a.s. \tag{2.9}$$

Proof of Theorem 1.3.2: By Theorem 1.3.1 and the remark after Theorem 1.3.2, it is enough to prove that

1) $$\varliminf_{t\nearrow\infty}\frac{\int_0^t \mathcal{R}_s^{3/2}ds}{\int_0^t \mathcal{R}_s ds} \ge c \quad \text{a.s. on } D \overset{\text{def}}{=} \left\{\int_0^\infty \mathcal{R}_t dt = \infty\right\}$$

for a positive constant c. It follows from (2.7) and (2.9) that

$$\lim_{t\to\infty}\frac{\mathcal{A}_t}{\int_0^t \mathcal{R}_s ds} = 0 \quad \text{a.s. on } D$$

and hence from (2.8) that

$$\varliminf_{t\to\infty}\frac{\int_0^t \mathcal{R}_s^{3/2}ds}{\int_0^t \mathcal{R}_s ds} \ge \frac{c_1}{c_2} \quad \text{a.s. on } D.$$

This proves (1) and hence Theorem 1.3.2. □

2.3 Proof of Lemma 2.2.2

Proof of part (a): To make the expressions below easier to read, we introduce the following shorthand notation:

$$\begin{aligned} J_{t,x,z}(\xi) &= \rho_{t,x} + (\xi - \delta_0)_{x-z}\rho_{t,z}, \\ \overline{J}_{t,x,z}(\xi) &= \frac{\eta_{t,x} + (\xi-\delta_0)_{x-z}\eta_{t,z}}{|\eta_t| + (|\xi|-1)\eta_{t,z}} = \frac{J_{t,x,z}(\xi)}{1+(|\xi|-1)\rho_{t,z}}. \end{aligned}$$

We then rewrite $\mathcal{S}_t$ as:

$$\begin{aligned} \mathcal{S}_t &= \mathcal{S}_0 + \sum_z \int N_t^z(dud\xi) \sum_{x,y} g(x-y)\left(\overline{J}_{u-,x,z}(\xi)\overline{J}_{u-,y,z}(\xi) - \rho_{u-,x}\rho_{u-,y}\right) \\ &= \mathcal{S}_0 + \mathcal{M}_t + \mathcal{A}_t \end{aligned}$$

where $\mathcal{A}_t = \int_0^t A_s ds$ has been defined by

$$A_s = \sum_{x,y,z} g(x-y) \int \nu(d\xi)\left(\overline{J}_{s,x,z}(\xi)\overline{J}_{s,y,z}(\xi) - \rho_{s,x}\rho_{s,y}\right)$$

To bound A_s from below, we note that $(1+x)^{-2} \geq 1-2x$ for $x \geq -1$. Then,

$$\begin{aligned} &\overline{J}_{s,x,z}(\xi)\overline{J}_{s,y,z}(\xi) - \rho_{s,x}\rho_{s,y} \\ &\geq J_{t,x,z}(\xi)J_{t,y,z}(\xi) - 2(|\xi|-1)\rho_{s,z}J_{t,x,z}(\xi)J_{t,y,z}(\xi) - \rho_{s,x}\rho_{s,y} \\ &= U_{s,x,y,z}(\xi) - 2V_{s,x,y,z}(\xi) - 2W_{s,x,y,z}(\xi), \end{aligned} \tag{2.10}$$

where

$$U_{s,x,y,z}(\xi) = J_{s,x,z}(\xi)J_{s,y,z}(\xi) - \rho_{s,x}\rho_{s,y} \tag{2.11}$$

$$V_{s,x,y,z}(\xi) = (|\xi|-1)U_{s,x,y,z}(\xi)\rho_{s,z} \tag{2.12}$$

$$W_{s,x,y,z}(\xi) = (|\xi|-1)\rho_{s,x}\rho_{s,y}\rho_{s,z}. \tag{2.13}$$

We will see that

$$\sum_{x,y,z} g(x-y)\int V_{s,x,y,z}(\xi)\nu(d\xi) \leq c\mathcal{R}_s^{3/2}. \tag{2.14}$$

Here and in what follows, c denotes a multiplicative constant, which does not depends on time variable s and space variables $x, y, ...$ To prove (2.14), we can bound the factor $|\xi|-1$ by a constant. We write

$$U_{s,x,y,z}(\xi) = (\xi-\delta_0)_{y-z}\rho_{s,x}\rho_{s,z} + (\xi-\delta_0)_{x-z}\rho_{s,y}\rho_{s,z} + (\xi-\delta_0)_{x-z}(\xi-\delta_0)_{y-z}\rho_{s,z}^2 \tag{2.15}$$

We look at the contribution from the second term on the right-hand-side of (2.15) to the left-hand-side of (2.14).

$$\begin{aligned} \sum_{x,y,z} g(x-y)(\xi-\delta_0)_{x-z}\rho_{s,z}^2\rho_{s,y} &= \langle\, g*\rho_s, (\xi-\delta_0)*\rho_s^2\,\rangle \\ &\leq |(g*\rho_s)^2|^{1/2}|((\xi-\delta_0)*\rho_s^2)^2|^{1/2} \\ &\leq |g|\mathcal{R}_s^{1/2}|(\xi-\delta_0)^2|^{1/2}|\rho_s^2| \leq c\mathcal{R}_s^{3/2} \end{aligned}$$

Contributions from the other two terms on the right-hand-side of (2.15) can be bounded similarly. Hence we get (2.14).

On the other hand,

$$\begin{aligned}
&\sum_{x,y,z} g(x-y)\int U_{s,x,y,z}d\nu \\
&= \sum_{x,y,z} g(x-y)\Big((k-\delta_0)_{y-z}\rho_{s,x}\rho_{s,z} + (k-\delta_0)_{x-z}\rho_{s,y}\rho_{s,z} + \beta_{x-z,y-z}\rho_{s,z}^2\Big) \\
&= \langle\, g*(k-\delta_0)*\rho_s, \rho_s\,\rangle + \langle\, g*(\check{k}-\delta_0)*\rho_s, \rho_s\,\rangle + \sum_{x,y} g(x-y)\beta_{x,y}\mathcal{R}_s, \qquad (2.16)
\end{aligned}$$

where $\check{k}_x = k_{-x}$. Also,

$$\sum_{x,y,z} g(x-y)\int W_{s,x,y,z}d\nu = (|k|-1)\langle\, g*\rho_s, \rho_s\,\rangle. \qquad (2.17)$$

Note that

$$(k-\delta_0) + (\check{k}-\delta_0) - 2(|k|-1)\delta_0 = 2(|k|-k_0)(p-\delta_0),$$

and that

$$g*(p-\delta_0) = p_{n+1} - \delta_0 \geq -\delta_0.$$

Thus,

$$\begin{aligned}
&\langle\, g*(k-\delta_0)*\rho_s, \rho_s\,\rangle + \langle\, g*(\check{k}-\delta_0)*\rho_s, \rho_s\,\rangle - 2(|k|-1)\langle\, g*\rho_s, \rho_s\,\rangle \\
&= 2(|k|-k_0)\langle\, g*(p-\delta_0)*\rho_s, \rho_s\,\rangle \geq 2(|k|-k_0)\mathcal{R}_s.
\end{aligned}$$

By this, (2.16) and (2.17), we get

$$\sum_{x,y,z} g(x-y)\int (U_{s,x,y,z} - 2W_{s,x,y,z})\,d\nu \geq \left(\sum_{x,y} g(x-y)\beta_{x,y} - 2(|k|-k_0)\right)\mathcal{R}_s. \qquad (2.18)$$

By (2.10), (2.14), (2.18) and Lemma 2.2.1, we obtain (2.8) . □

Proof of part (b): The predictable quadratic variation of the martingale $\mathcal{M}.$ can be given by:

1) $$\langle\, \mathcal{M}\,\rangle_t = \sum_z \int_0^t ds \int F_{s,z}(\xi)^2\nu(d\xi)$$

where

$$F_{s,z}(\xi) = \sum_{x,y} g(x-y)(\bar{J}_{s,x,z}(\xi)\bar{J}_{s,y,z}(\xi) - \rho_{s,x}\rho_{s,y})$$

Recall that

$$\begin{aligned}
\{\langle\, \mathcal{M}\,\rangle_\infty < \infty\} &\subset \{\mathcal{M}_t \text{ converges as } t\to\infty\} \ \text{ a.s.} \\
\{\langle\, \mathcal{M}\,\rangle_\infty = \infty\} &\subset \left\{\lim_{t\to\infty}\frac{\mathcal{M}_t}{\langle\, \mathcal{M}\,\rangle_t} = 0\right\} \ \text{ a.s.}
\end{aligned}$$

Thus, to prove (2.9), it is enough to show that there is a constant $c\in(0,\infty)$ such that

2) $$\langle\, \mathcal{M}\,\rangle_t \leq c\int_0^t \mathcal{R}_s ds.$$

We will do so via two different bounds for $|F_{s,z}(\xi)|$:

3) $\qquad |F_{s,z}(\xi)| \le 2|g| \quad \text{for all } s,z,\xi,$

4) $\qquad |F_{s,z}(\xi)| \le c\rho_{s,z} \quad \text{if } \rho_{s,z} \le 1/2.$

To get (3), we note that $0 \le \bar{J}_{s,x,z}(\xi) \le 1$ and $\sum_x \bar{J}_{s,x,z} = 1$ for each z. Thus,

$$\begin{aligned}
|F_{s,z}(\xi)| &\le \langle\, g * \bar{J}_{s,\cdot,z}, \bar{J}_{s,\cdot,z}\,\rangle + \langle\, g * \rho_s, \rho_s\,\rangle \\
&\le |(g * \bar{J}_{s,\cdot,z})^2|^{1/2}|\bar{J}^2_{s,\cdot,z}|^{1/2} + |(g*\rho_s)^2|^{1/2}|\rho_s^2|^{1/2} \\
&\le |g||\bar{J}^2_{s,\cdot,z}| + |g|\mathcal{R}_s \le 2|g|.
\end{aligned}$$

To get (4), we assume $\rho_{s,z} \le 1/2$. Then, $1 + (|\xi| - 1)\rho_{s,z} \ge 1/2$ and thus, recalling (2.11) and (2.13),

$$\begin{aligned}
|F_{s,z}(\xi)| &\le \sum_{x,y} g(x-y)\frac{|U_{s,x,y,z}(\xi) - W_{s,x,y,z}(\xi)|}{1+(|\xi|-1)\rho_{s,z}} \\
&\le 2\sum_{x,y} g(x-y)(|U_{s,x,y,z}(\xi)| + |W_{s,x,y,z}(\xi)|),
\end{aligned}$$

By (2.13) and (2.15), it is clear that the last summation is bounded by $c\rho_{s,z}$ for some c.
(3)–(4) can be used to obtain (2) as follows. For each s, there is at most one z such that $\rho_{s,z} > 1/2$, and $\mathcal{R}_s > 1/4$ if there is such z. Thus,

$$\sum_z \mathbf{1}\{\rho_{s,z} > 1/2\} < 4\mathcal{R}_s.$$

By this and (3)–(4), we have

$$\sum_z F_{s,z}(\xi)^2 \le 4|g|^2 \sum_z \mathbf{1}\{\rho_{s,z} > 1/2\} + c^2 \sum_z \mathbf{1}\{\rho_{s,z} \le 1/2\}\rho^2_{s,z} \le (16|g|^2 + c^2)\mathcal{R}_s.$$

Plugging this into (1), we are done. □

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References

[CH02] Carmona, P., Hu Y.: On the partition function of a directed polymer in a Gaussian random environment, Probab.Theory Related Fields **124** (2002), no. 3, 431–457.

[CH06] Carmona,P., Hu, Y.: Strong disorder implies strong localization for directed polymers in a random environment ALEA Lat. Am. J. Probab. Math. Stat. **2** (2006), 217–229.

[CSY03] Comets, F., Shiga, T., Yoshida, N. Directed polymers in random environment: path localization and strong disorder, Bernoulli, **9** (2003), No. 4, 705–723.

[CY05] Comets, F., Yoshida, N.: Brownian directed polymers in random environment, Comm. Math. Phys. **254** (2005), no. 2, 257–287.

[Gri83] Griffeath, D.: The binary contact path process, Ann. Probab. **11** (1983) no. 3, 692–705.

[HWY92] Sheng-wu He, Jia-gang Wang, Jia-an Yan: "Semimartingale theory and stochastic calculus", Science Press, Beijing New York; CRC Press Inc., Boca Raton Ann Arbor London Tokyo, (1992).

[HL81] Holley, R., Liggett, T. M. : Generalized potlatch and smoothing processes, Z. Wahrsch. Verw. Gebiete **55** (1981), no. 2, 165–195.

[HY09] Hu, Y., Yoshida, N. : Localization for branching random walks in random environment, Stochastic Process. Appl. **119** (2009), no. 5, 1632–1651.

[Lig85] Liggett, T. M. : “Interacting Particle Systems”, Springer Verlag, Berlin-Heidelberg-Tokyo (1985).

[LS81] Liggett, T. M., Spitzer, F. : Ergodic theorems for coupled random walks and other systems with locally interacting components. Z. Wahrsch. Verw. Gebiete **56** (1981), no. 4, 443–468.

[NY09] Nagahata, Y., Yoshida, N.: Central limit theorem for a class of linear systems, Electron. J. Probab. **14** (2009), 960–977.

[Sh09] Shiozawa, Y.: Localization for branching Brownian motions in random environment, preprint (2009)

[Spi81] Spitzer, F. : Infinite systems with locally interacting components. Ann. Probab. **9**, (1981), no. 3, 349–364.

[Yo08a] Yoshida, N.: Phase transitions for the growth rate of linear stochastic evolutions, J. Stat. Phys. **133** (2008), no.6, 1033–1058.

[Yo08b] Yoshida, N.: Localization for linear stochastic evolutions, preprint, arXiv:0810.4218, (2008).