THE KREIN–VON NEUMANN EXTENSION AND ITS CONNECTION TO AN ABSTRACT BUCKLING PROBLEM

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Dedicated to the memory of Erhard Schmidt (1876–1959).

Abstract. We prove the unitary equivalence of the inverse of the Krein–von Neumann extension (on the orthogonal complement of its kernel) of a densely defined, closed, strictly positive operator, $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ in a Hilbert space $\mathcal{H}$ to an abstract buckling problem operator.

In the concrete case where $S = \overline{-\Delta|_{C_0^\infty(\Omega)}}$ in $L^2(\Omega; d^n x)$ for $\Omega \subset \mathbb{R}^n$ an open, bounded (and sufficiently regular) domain, this recovers, as a particular case of a general result due to G. Grubb, that the eigenvalue problem for the Krein Laplacian S_K (i.e., the Krein–von Neumann extension of S),

$$S_K v = \lambda v, \quad \lambda \neq 0,$$

is in one-to-one correspondence with the problem of *the buckling of a clamped plate*,

$$(-\Delta)^2 u = \lambda(-\Delta) u \text{ in } \Omega, \quad \lambda \neq 0, \quad u \in H_0^2(\Omega),$$

where u and v are related via the pair of formulas

$$u = S_F^{-1}(-\Delta) v, \quad v = \lambda^{-1}(-\Delta) u,$$

with S_F the Friedrichs extension of S.

This establishes the Krein extension as a natural object in elasticity theory (in analogy to the Friedrichs extension, which found natural applications in quantum mechanics, elasticity, etc.).

1. Introduction

Suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in a separable complex Hilbert space $\mathcal{H}$ that satisfies

$$S \geq \varepsilon I_{\mathcal{H}} \text{ for some } \varepsilon > 0, \tag{1.1}$$

and denote by S_K and S_F the Krein–von Neumann and Friedrichs extensions of S, respectively (with $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$).

Then an abstract version of Proposition 1 in Grubb [20], describing an intimate connection between the nonzero eigenvalues of the Krein–von Neumann extension of an appropriate minimal elliptic differential operator of order $2m$, $m \in \mathbb{N}$, and

Date: July 12, 2009.

2000 *Mathematics Subject Classification.* Primary 35J25, 35J40, 47A05; Secondary 47A10, 47F05.

Key words and phrases. Krein–von Neumann extension, buckling problem.

Based upon work partially supported by the US National Science Foundation under Grant Nos. DMS-0400639 and FRG-0456306 and the Austrian Science Fund (FWF) under Grant No. Y330.

To appear in *Math. Nachrichten.*

nonzero eigenvalues of a suitable higher-order buckling problem (cf. Example 3.5), to be proved in Lemma 3.1, can be summarized as follows:

$$\text{There exists } 0 \neq v \in \operatorname{dom}(S_K) \text{ satisfying } S_K v = \lambda v, \quad \lambda \neq 0, \tag{1.2}$$

if and only if

$$\text{there exists a } 0 \neq u \in \operatorname{dom}(S^*S) \text{ such that } S^*Su = \lambda Su, \tag{1.3}$$

and the solutions v of (1.2) are in one-to-one correspondence with the solutions u of (1.3) given by the pair of formulas

$$u = (S_F)^{-1} S_K v, \quad v = \lambda^{-1} Su. \tag{1.4}$$

Next, we will go a step further and describe a unitary equivalence result going beyond the connection between the eigenvalue problems (1.2) and (1.3): Given S, we introduce the following sesquilinear forms in $\mathcal{H}$,

$$a(u,v) = (Su, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(a) = \operatorname{dom}(S), \tag{1.5}$$

$$b(u,v) = (u, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(b) = \operatorname{dom}(S). \tag{1.6}$$

Then S being densely defined and closed, implies that the sesquilinear form a is also densely defined and closed, and thus one can introduce the Hilbert space

$$\mathcal{W} = (\operatorname{dom}(S), (\cdot,\cdot)_{\mathcal{W}}) \tag{1.7}$$

with associated scalar product

$$(u,v)_{\mathcal{W}} = a(u,v) = (Su, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(S). \tag{1.8}$$

Suppressing for simplicity the continuous embedding operator of $\mathcal{W}$ into $\mathcal{H}$, we now introduce the following operator T in $\mathcal{W}$ by

$$(w_1, Tw_2)_{\mathcal{W}} = a(w_1, Tw_2) = b(w_1, w_2) = (w_1, Sw_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}. \tag{1.9}$$

One can prove that T is self-adjoint, nonnegative, and bounded and we will call T the *abstract buckling problem operator* associated with the Krein–von Neumann extension S_K of S.

Next, introducing the Hilbert space $\widehat{\mathcal{H}}$ by

$$\widehat{\mathcal{H}} = [\ker(S^*)]^{\perp} = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\mathcal{H} = \big[I_{\mathcal{H}} - P_{\ker(S_K)}\big]\mathcal{H} = [\ker(S_K)]^{\perp}, \tag{1.10}$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection onto the subspace $\mathcal{M} \subset \mathcal{H}$, we introduce the operator

$$\widehat{S} : \begin{cases} \mathcal{W} \to \widehat{\mathcal{H}}, \\ w \mapsto Sw, \end{cases} \tag{1.11}$$

and note that $\widehat{S} \in \mathcal{B}(\mathcal{W}, \widehat{\mathcal{H}})$ maps $\mathcal{W}$ unitarily onto $\widehat{\mathcal{H}}$.

Finally, defining the *reduced Krein–von Neumann operator* $\widehat{S}_K$ in $\widehat{\mathcal{H}}$ by

$$\widehat{S}_K := S_K|_{[\ker(S_K)]^{\perp}} \text{ in } \widehat{\mathcal{H}}, \tag{1.12}$$

we can state the principal unitary equivalence result to be proved in Theorem 3.4:

The inverse of the reduced Krein–von Neumann operator $\widehat{S}_K$ in $\widehat{\mathcal{H}}$ and the abstract buckling problem operator T in $\mathcal{W}$ are unitarily equivalent,

$$\big(\widehat{S}_K\big)^{-1} = \widehat{S}T(\widehat{S})^{-1}. \tag{1.13}$$

In addition,

$$\big(\widehat{S}_K\big)^{-1} = U_S\big[|S|^{-1}S|S|^{-1}\big](U_S)^{-1}. \tag{1.14}$$

Here we used the polar decomposition of S,

$$S = U_S|S|, \text{ with } |S| = (S^*S)^{1/2} \geq \varepsilon I_{\mathcal{H}}, \ \varepsilon > 0, \text{ and } U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big) \text{ unitary,} \tag{1.15}$$

and one observes that the operator $|S|^{-1}S|S|^{-1} \in \mathcal{B}(\mathcal{H})$ in (1.14) is self-adjoint in $\mathcal{H}$.

As discussed at the end of Section 3, one can readily rewrite the abstract linear pencil buckling eigenvalue problem (1.3), $S^*Su = \lambda Su$, $\lambda \neq 0$, in the form of the standard eigenvalue problem $|S|^{-1}S|S|^{-1}w = \lambda^{-1}w$, $\lambda \neq 0$, $w = |S|u$, and hence establish the connection between (1.2), (1.3) and (1.13), (1.14).

As mentioned in the abstract, the concrete case where S is given by $S = \overline{-\Delta|_{C_0^\infty(\Omega)}}$ in $L^2(\Omega; d^nx)$, then yields the spectral equivalence between the inverse of the reduced Krein–von Neumann extension $\widehat{S}_K$ of S and the problem of the buckling of a clamped plate. More generally, Grubb [20] actually treated the case where S is generated by an appropriate elliptic differential expression of order $2m$, $m \in \mathbb{N}$, and also introduced the higher-order analog of the buckling problem; we briefly summarize this in Example 3.5.

2. The Abstract Krein–von Neumann Extension

To get started, we briefly elaborate on the notational conventions used throughout this paper and especially throughout this section which collects abstract material on the Krein–von Neumann extension. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\operatorname{dom}(T)$, $\operatorname{ran}(T)$, and $\ker(T)$ denoting the domain, range, and kernel (i.e., null space) of T. The closure of a closable operator S is denoted by $\overline{S}$. The spectrum, essential spectrum, discrete spectrum, and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$, $\sigma_{\rm d}(\cdot)$, and $\rho(\cdot)$, respectively. The Banach spaces of bounded and compact linear operators in $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p(\mathcal{H})$, $p \in (0,\infty)$. Analogous notation $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$, $\mathcal{B}_\infty(\mathcal{H}_1,\mathcal{H}_2)$, etc., will be used for bounded, compact, etc., operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Whenever applicable, we retain the same type of notation in the context of Banach spaces. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$. $\mathcal{X}_1 \dotplus \mathcal{X}_2$ denotes the (not necessarily orthogonal) direct sum of the subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{X}$.

A linear operator $S : \operatorname{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$, is called *symmetric*, if

$$(u, Sv)_{\mathcal{H}} = (Su, v)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(S). \tag{2.1}$$

In this manuscript we will be particularly interested in this question within the class of densely defined (i.e., $\overline{\operatorname{dom}(S)} = \mathcal{H}$), non-negative operators (in fact, in most instances S will even turn out to be strictly positive) and we focus almost exclusively on self-adjoint extensions that are non-negative operators. In the latter scenario, there are two distinguished constructions which we review briefly next.

To set the stage, we recall that a linear operator $S : \operatorname{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called *non-negative* provided

$$(u, Su)_{\mathcal{H}} \geq 0, \quad u \in \operatorname{dom}(S). \tag{2.2}$$

(In particular, S is symmetric in this case.) S is called *strictly positive*, if for some $\varepsilon > 0$, $(u, Su)_{\mathcal{H}} \geq \varepsilon \|u\|_{\mathcal{H}}^2$, $u \in \mathrm{dom}(S)$. Next, we recall that $A \leq B$ for two self-adjoint operators in $\mathcal{H}$ if

$$\begin{aligned} &\mathrm{dom}\left(|A|^{1/2}\right) \supseteq \mathrm{dom}\left(|B|^{1/2}\right) \text{ and} \\ &\left(|A|^{1/2}u, U_A|A|^{1/2}u\right)_{\mathcal{H}} \leq \left(|B|^{1/2}u, U_B|B|^{1/2}u\right)_{\mathcal{H}}, \quad u \in \mathrm{dom}\left(|B|^{1/2}\right). \end{aligned} \tag{2.3}$$

Here U_C denotes the partial isometry in $\mathcal{H}$ in the polar decomposition $C = U_C|C|$, $|C| = (C^*C)^{1/2}$, of a densely defined closed operator C in $\mathcal{H}$. (If C is in addition self-adjoint, then $|C|$ and U_C commute.) We also recall that for $A \geq 0$ self-adjoint,

$$\ker(A) = \ker\left(A^{1/2}\right) \tag{2.4}$$

(with $D^{1/2}$ denoting the unique nonnegative square root of a nonnegative self-adjoint operator D in $\mathcal{H}$).

For simplicity we will always adhere to the conventions that S is a linear, unbounded, densely defined, nonnegative (i.e., $S \geq 0$) operator in $\mathcal{H}$, and that S has nonzero deficiency indices. Since S is bounded from below, the latter are necessarily equal. In particular,

$$\mathrm{def}(S) = \dim(\ker(S^* - zI_{\mathcal{H}})) \in \mathbb{N} \cup \{\infty\}, \quad z \in \mathbb{C}\backslash[0, \infty), \tag{2.5}$$

is well-known to be independent of z. Moreover, since S and its closure $\overline{S}$ have the same self-adjoint extensions in $\mathcal{H}$, we will without loss of generality assume that S is closed in the remainder of this paper.

The following is a fundamental result to be found in M. Krein's celebrated 1947 paper [27] (cf. also Theorems 2 and 5–7 in the English summary on page 492)[1]:

Theorem 2.1. *Assume that S is a densely defined, closed, nonnegative operator in $\mathcal{H}$. Then, among all non-negative self-adjoint extensions of S, there exist two distinguished ones, S_K and S_F, which are, respectively, the smallest and largest* (*in the sense of order between self-adjoint operators, cf.* (2.3)) *such extension. Furthermore, a non-negative self-adjoint operator $\widetilde{S}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies*

$$S_K \leq \widetilde{S} \leq S_F. \tag{2.6}$$

In particular, (2.6) *determines S_K and S_F uniquely.*

In addition, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, one has $S_F \geq \varepsilon I_{\mathcal{H}}$, and

$$\mathrm{dom}(S_F) = \mathrm{dom}(S) \dotplus (S_F)^{-1} \ker(S^*), \tag{2.7}$$

$$\mathrm{dom}(S_K) = \mathrm{dom}(S) \dotplus \ker(S^*), \tag{2.8}$$

$$\begin{aligned} \mathrm{dom}(S^*) &= \mathrm{dom}(S) \dotplus (S_F)^{-1} \ker(S^*) \dotplus \ker(S^*) \\ &= \mathrm{dom}(S_F) \dotplus \ker(S^*), \end{aligned} \tag{2.9}$$

in particular,

$$\ker(S_K) = \ker\left((S_K)^{1/2}\right) = \ker(S^*) = \mathrm{ran}(S)^{\perp}. \tag{2.10}$$

We also note that

$$S_F u = S^* u, \quad u \in \mathrm{dom}(S_F), \tag{2.11}$$

$$S_K v = S^* v, \quad v \in \mathrm{dom}(S_K). \tag{2.12}$$

[1] We are particularly indebted to Gerd Grubb for a clarification of the necessary and sufficient nature of the inequalities (2.6) (resp., (2.13)) for $\widetilde{S}$ to be a self-adjoint extension of S.

Here the operator inequalities in (2.6) are understood in the sense of (2.3) and they can equivalently be written as

$$(S_F + aI_{\mathcal{H}})^{-1} \leqslant \big(\widetilde{S} + aI_{\mathcal{H}}\big)^{-1} \leqslant (S_K + aI_{\mathcal{H}})^{-1} \text{ for some (and hence for all) } a > 0. \tag{2.13}$$

For classical references on the subject of self-adjoint extensions of semibounded operators (not necessarily restricted to the Krein–von Neumann extension) we refer to Birman [10], [11], Friedrichs [14], Freudenthal [13], Grubb [17], [18], Krein [28], Štraus [30], and Višik [31] (see also the monographs by Akhiezer and Glazman [1, Sect. 109], Faris [12, Part III], Fukushima, Oshima, and Takeda [15, Sect. 3.3], and the recent book by Grubb [21, Sect. 13.2]).

We will call the operator S_K the *Krein–von Neumann extension* of S. See [27] and also the discussion in [2] and [3]. It should be noted that the Krein–von Neumann extension was first considered by von Neumann [32] in 1929 in the case where S is strictly bounded from below, that is, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$. (His construction appears in the proof of Theorem 42 on pages 102–103.) However, von Neumann did not isolate the extremal property of this extension as described in (2.6) and (2.13). M. Krein [27], [28] was the first to systematically treat the general case $S \geq 0$ and to study all nonnegative self-adjoint extensions of S, illustrating the special role of the *Friedrichs extension* (i.e., the "hard" extension) S_F of S and the Krein–von Neumann (i.e., the "soft") extension S_K of S as extremal cases when considering all nonnegative extensions of S. For a recent exhaustive treatment of self-adjoint extensions of semibounded operators we refer to [4]–[7].

For convenience of the reader we also mention the following intrinsic description of the Friedrichs extension S_F of $S \geq 0$ (S densely defined and closed in $\mathcal{H}$) due to Freudenthal [13],

$$\begin{aligned} &S_F u := S^* u, \\ &u \in \mathrm{dom}(S_F) := \big\{v \in \mathrm{dom}(S^*) \,\big|\, \text{there exists } \{v_j\}_{j\in\mathbb{N}} \subset \mathrm{dom}(S), \\ &\quad \text{with } \lim_{j\to\infty} \|v_j - v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k))_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}, \end{aligned} \tag{2.14}$$

and an intrinsic description of the Krein–von Neumann extension S_K of $S \geq 0$ due to Ando and Nishio [3],

$$\begin{aligned} &S_K u := S^* u, \\ &u \in \mathrm{dom}(S_K) := \big\{v \in \mathrm{dom}(S^*) \,\big|\, \text{there exists } \{v_j\}_{j\in\mathbb{N}} \subset \mathrm{dom}(S), \\ &\quad \text{with } \lim_{j\to\infty} \|S v_j - S^* v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k)_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}. \end{aligned} \tag{2.15}$$

Throughout the rest of this paper we make the following assumptions:

Hypothesis 2.2. *Suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in $\mathcal{H}$ that satisfies*

$$S \geq \varepsilon I_{\mathcal{H}} \text{ for some } \varepsilon > 0. \tag{2.16}$$

We recall that the *reduced Krein–von Neumann operator* $\widehat{S}_K$ in the Hilbert space $\widehat{\mathcal{H}}$ (cf. (2.10)),

$$\widehat{\mathcal{H}} = [\ker(S^*)]^\perp = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\mathcal{H} = \big[I_{\mathcal{H}} - P_{\ker(S_K)}\big]\mathcal{H} = [\ker(S_K)]^\perp, \tag{2.17}$$

is given by

$$\widehat{S}_K : = S_K|_{[\ker(S_K)]^\perp} \tag{2.18}$$

$$\begin{aligned} &= S_K[I_{\mathcal{H}} - P_{\ker(S_K)}] \ \text{ in } \widehat{\mathcal{H}} \\ &= [I_{\mathcal{H}} - P_{\ker(S_K)}]S_K[I_{\mathcal{H}} - P_{\ker(S_K)}] \ \text{ in } \widehat{\mathcal{H}}, \end{aligned} \tag{2.19}$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection onto the subspace $\mathcal{M} \subset \mathcal{H}$, and we are alluding to the orthogonal direct sum decomposition of $\mathcal{H}$ into

$$\mathcal{H} = P_{\ker(S_K)}\mathcal{H} \oplus \widehat{\mathcal{H}} = \ker(S_K) \oplus [\ker(S_K)]^{\perp}. \tag{2.20}$$

We continue with the following elementary observation:

Lemma 2.3. *Assume Hypothesis* 2.2 *and let* $v \in \operatorname{dom}(S_K)$. *Then the decomposition,* $\operatorname{dom}(S_K) = \operatorname{dom}(S) \dot{+} \ker(S^*)$ (*cf.* (2.8)), *leads to the following decomposition of* v,

$$v = (S_F)^{-1}S_K v + w, \ \text{ where } (S_F)^{-1}S_K v \in \operatorname{dom}(S) \text{ and } w \in \ker(S^*). \tag{2.21}$$

As a consequence,

$$\big(\widehat{S}_K\big)^{-1} = [I_{\mathcal{H}} - P_{\ker(S_K)}](S_F)^{-1}[I_{\mathcal{H}} - P_{\ker(S_K)}]. \tag{2.22}$$

Proof. Let $v = u + w$, with $u \in \operatorname{dom}(S)$ and $w \in \ker(S^*)$. Then

$$\begin{aligned} v &= u + w = (S_F)^{-1}S_F u + w = (S_F)^{-1}Su + w \\ &= (S_F)^{-1}S_K u + w = (S_F)^{-1}S_K(u + w) + w \\ &= (S_F)^{-1}S_K v + w \end{aligned} \tag{2.23}$$

proves (2.21). Given $v \in \operatorname{dom}(S_K)$, one infers

$$S_K v = S_K(P_{\ker(S_K)} + P_{\widehat{\mathcal{H}}})v = S_K P_{\widehat{\mathcal{H}}} v, \tag{2.24}$$

since $S_K P_{\ker(S_K)} = 0$. In particular,

$$P_{\widehat{\mathcal{H}}} v \in \operatorname{dom}(S_K) \ \text{ whenever } \ v \in \operatorname{dom}(S_K). \tag{2.25}$$

Applying $P_{\widehat{\mathcal{H}}}$ to (2.21) then yields

$$\begin{aligned} P_{\widehat{\mathcal{H}}} v &= P_{\widehat{\mathcal{H}}}(S_F)^{-1}S_K[P_{\widehat{\mathcal{H}}} + P_{\ker(S_K)}]v = P_{\widehat{\mathcal{H}}}(S_F)^{-1}S_K P_{\widehat{\mathcal{H}}} v = P_{\widehat{\mathcal{H}}}(S_F)^{-1}\widehat{S}_K P_{\widehat{\mathcal{H}}} v \\ &= P_{\widehat{\mathcal{H}}}(S_F)^{-1}P_{\widehat{\mathcal{H}}}\widehat{S}_K P_{\widehat{\mathcal{H}}} v, \quad v \in \operatorname{dom}(S_K). \end{aligned} \tag{2.26}$$

Thus,

$$\big(\widehat{S}_K\big)^{-1}\big(\widehat{S}_K P_{\widehat{\mathcal{H}}} v\big) = P_{\widehat{\mathcal{H}}}(S_F)^{-1}P_{\widehat{\mathcal{H}}}\big(\widehat{S}_K P_{\widehat{\mathcal{H}}} v\big), \quad v \in \operatorname{dom}(S_K). \tag{2.27}$$

Since $\operatorname{ran}\big(\widehat{S}_K\big) = \widehat{\mathcal{H}}$, (2.27) proves (2.22). □

We note that equation (2.22) was proved by Krein in his seminal paper [27] (cf. the proof of Theorem 26 in [27]).

Next, we consider a self-adjoint operator

$$T : \operatorname{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}, \quad T = T^*, \tag{2.28}$$

which is bounded from below, that is, there exists $\alpha \in \mathbb{R}$ such that

$$T \geq \alpha I_{\mathcal{H}}. \tag{2.29}$$

We denote by $\{E_T(\lambda)\}_{\lambda \in \mathbb{R}}$ the family of strongly right-continuous spectral projections of T, and introduce, as usual, $E_T((a,b)) = E_T(b_-) - E_T(a)$, $E_T(b_-) = \text{s-lim}_{\varepsilon \downarrow 0} E_T(b - \varepsilon)$, $-\infty \leq a < b$. In addition, we set

$$\mu_{T,j} := \inf\big\{\lambda \in \mathbb{R} \,|\, \dim(\operatorname{ran}(E_T((-\infty,\lambda)))) \geq j\big\}, \quad j \in \mathbb{N}. \tag{2.30}$$

Then, for fixed $k \in \mathbb{N}$, either:
(i) $\mu_{T,k}$ is the kth eigenvalue of T counting multiplicity below the bottom of the essential spectrum, $\sigma_{\rm ess}(T)$, of T,
or
(ii) $\mu_{T,k}$ is the bottom of the essential spectrum of T,

$$\mu_{T,k} = \inf\{\lambda \in \mathbb{R} \,|\, \lambda \in \sigma_{\rm ess}(T)\}, \tag{2.31}$$

and in that case $\mu_{T,k+\ell} = \mu_{T,k}$, $\ell \in \mathbb{N}$, and there are at most $k-1$ eigenvalues (counting multiplicity) of T below $\mu_{T,k}$.

We now record the following basic result:

Theorem 2.4. *Assume Hypothesis* 2.2. *Then,*

$$\varepsilon \leq \mu_{S_F,j} \leq \mu_{\widehat{S}_K,j}, \quad j \in \mathbb{N}. \tag{2.32}$$

In particular, if the Friedrichs extension S_F of S has purely discrete spectrum, then, except possibly for $\lambda = 0$, the Krein–von Neumann extension S_K of S also has purely discrete spectrum in $(0,\infty)$, that is,

$$\sigma_{\rm ess}(S_F) = \emptyset \ \text{ implies } \ \sigma_{\rm ess}(S_K)\backslash\{0\} = \emptyset. \tag{2.33}$$

In addition, let $p \in (0,\infty) \cup \{\infty\}$, then

$$\begin{split} &(S_F - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_p(\mathcal{H}) \ \text{ for some } z_0 \in \mathbb{C}\backslash[\varepsilon,\infty) \\ &\text{implies } \ (S_K - z I_{\mathcal{H}})^{-1}[I_{\mathcal{H}} - P_{\ker(S_K)}] \in \mathcal{B}_p(\mathcal{H}) \ \text{ for all } z \in \mathbb{C}\backslash[\varepsilon,\infty). \end{split} \tag{2.34}$$

In fact, the $\ell^p(\mathbb{N})$-based trace ideals $\mathcal{B}_p(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ can be replaced by any two-sided symmetrically normed ideals of $\mathcal{B}(\mathcal{H})$.

Proof. Denote by $\mathcal{M}_j$ subspaces of $\mathcal{H}$ of dimension $j \in \mathbb{N}$, and similarly, $\widehat{\mathcal{M}}_j$ subspaces of $\widehat{\mathcal{H}}$ of dimension $j \in \mathbb{N}$. Then the inequalities (2.32) follow from $S_F \geq \varepsilon I_{\mathcal{H}}$, (2.22), and the minimax (better, maximin) theorem as follows: First we note that (cf., e.g., [22, Theorem 5.28], [23, Sect. 32])

$$\frac{1}{\mu_{S_F,j}} = \sup_{\mathcal{M}_j \subset \mathcal{H}} \min_{\substack{u \in \mathcal{M}_j \\ \|u\|_{\mathcal{H}}=1}} \big(u, (S_F)^{-1} u\big)_{\mathcal{H}}, \quad j \in \mathbb{N}. \tag{2.35}$$

As a consequence,

$$\frac{1}{\mu_{S_F,j}} \geq \min_{u \in \mathcal{M}_j \subset \mathcal{H}} \big(u, (S_F)^{-1} u\big)_{\mathcal{H}}, \quad j \in \mathbb{N}, \tag{2.36}$$

for any subspace $\mathcal{M}_j$ of $\mathcal{H}$ of dimension $j \in \mathbb{N}$. In particular,

$$\begin{split} \frac{1}{\mu_{S_F,j}} &\geq \min_{\substack{v \in \widehat{\mathcal{M}}_j \subset \widehat{\mathcal{H}} \\ \|v\|_{\widehat{\mathcal{H}}}=1}} \big(v, (S_F)^{-1} v\big)_{\widehat{\mathcal{H}}} \\ &= \min_{\substack{v \in \widehat{\mathcal{M}}_j \subset \widehat{\mathcal{H}} \\ \|v\|_{\widehat{\mathcal{H}}}=1}} \big(v, P_{\widehat{\mathcal{H}}} (S_F)^{-1} P_{\widehat{\mathcal{H}}} v\big)_{\widehat{\mathcal{H}}}, \quad j \in \mathbb{N}, \end{split} \tag{2.37}$$

for any subspace $\widehat{\mathcal{M}}_j$ of $\widehat{\mathcal{H}}$ of dimension $j \in \mathbb{N}$. Thus, one concludes

$$\frac{1}{\mu_{S_F,j}} \geq \sup_{\widehat{\mathcal{M}}_j \subset \widehat{\mathcal{H}}} \min_{\substack{v \in \widehat{\mathcal{M}}_j \\ \|v\|_{\widehat{\mathcal{H}}}=1}} \big(v, P_{\widehat{\mathcal{H}}} (S_F)^{-1} P_{\widehat{\mathcal{H}}} v\big)_{\widehat{\mathcal{H}}}$$

$$= \sup_{\widehat{\mathcal{M}}_j \subset \widehat{\mathcal{H}}} \min_{\substack{v \in \widehat{\mathcal{M}}_j \\ \|v\|_{\widehat{\mathcal{H}}}=1}} \big(v, \big(\widehat{S}_K\big)^{-1} v\big)_{\widehat{\mathcal{H}}}$$
$$= \frac{1}{\mu_{\widehat{S}_K, j}}, \quad j \in \mathbb{N}. \tag{2.38}$$

Next, let $\mathcal{J}(\mathcal{H})$ be a two-sided symmetrically normed ideal of $\mathcal{B}(\mathcal{H})$. Temporarily, we will identify operators of the type $P_{\widehat{\mathcal{H}}} T P_{\widehat{\mathcal{H}}}$ in $\widehat{\mathcal{H}}$ for $T \in \mathcal{B}(\mathcal{H})$, with 2×2 block operators of the type

$$\begin{pmatrix} 0 & 0 \\ 0 & P_{\widehat{\mathcal{H}}} T P_{\widehat{\mathcal{H}}}|_{\widehat{\mathcal{H}}} \end{pmatrix} \text{ in } \mathcal{H} = (\ker(S_K))^{\perp} \oplus \widehat{\mathcal{H}}. \tag{2.39}$$

By (2.22), and since $P_{\widehat{\mathcal{H}}}$ is bounded, one concludes that $(S_F)^{-1} \in \mathcal{J}(\mathcal{H})$ implies $\big(\widehat{S}_K\big)^{-1} = \text{n-lim}_{z\to 0}(S_K - zI_{\mathcal{H}})^{-1}[I_{\mathcal{H}} - P_{\ker(S_K)}] \in \mathcal{J}(\mathcal{H})$. The (first) resolvent equation applied to S_F, and subsequently, applied to S_K, then proves (2.34). $\square$

We note that (2.33) is a classical result of Krein [27], the more general fact (2.32) has not been mentioned explicitly in Krein's paper [27], although it immediately follows from the minimax principle and Krein's formula (2.22). On the other hand, in the special case $\operatorname{def}(S) < \infty$, Krein states an extension of (2.32) in his Remark 8.1 in the sense that he also considers self-adjoint extensions different from the Krein extension. Apparently, (2.32) has first been proven by Alonso and Simon [2] by a somewhat different method.

Concluding this section, we point out that a great variety of additional results for the Krein–von Neumann extension can be found in the very extensive list of references in [7] and [8].

3. The Krein–von Neumann Extension and its Unitary Equivalence to an Abstract Buckling Problem

In this section we prove our principal result, the unitary equivalence of the inverse of the Krein–von Neumann extension (on the orthogonal complement of its kernel) of a densely defined, closed, operator S satisfying $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, in a complex separable Hilbert space $\mathcal{H}$ to an abstract buckling problem operator.

We start by introducing an abstract version of Proposition 1 in Grubb's paper [20] devoted to Krein–von Neumann extensions of even order elliptic differential operators on bounded domains:

Lemma 3.1. *Assume Hypothesis* 2.2 *and let* $\lambda \neq 0$*. Then there exists* $0 \neq v \in \operatorname{dom}(S_K)$ *with*

$$S_K v = \lambda v \tag{3.1}$$

if and only if there exists $0 \neq u \in \operatorname{dom}(S^* S)$ *such that*

$$S^* S u = \lambda S u. \tag{3.2}$$

In particular, the solutions v *of* (3.1) *are in one-to-one correspondence with the solutions* u *of* (3.2) *given by the formulas*

$$u = (S_F)^{-1} S_K v, \tag{3.3}$$
$$v = \lambda^{-1} S u. \tag{3.4}$$

Of course, since $S_K \geq 0$*, any* $\lambda \neq 0$ *in* (3.1) *and* (3.2) *necessarily satisfies* $\lambda > 0$*.*

Proof. Let $S_K v = \lambda v$, $v \in \operatorname{dom}(S_K)$, $\lambda \neq 0$, and $v = u + w$, with $u \in \operatorname{dom}(S)$ and $w \in \ker(S^*)$. Then,

$$S_K v = \lambda v \Longleftrightarrow v = \lambda^{-1} S_K v = \lambda^{-1} S_K u = \lambda^{-1} S u. \tag{3.5}$$

Moreover, $u = 0$ implies $v = 0$ and clearly $v = 0$ implies $u = w = 0$, hence $v \neq 0$ if and only if $u \neq 0$. In addition, $u = (S_F)^{-1} S_K v$ by (2.21). Finally,

$$\begin{aligned} &\lambda w = Su - \lambda u \in \ker(S^*) \text{ implies} \\ &0 = \lambda S^* w = S^*(Su - \lambda u) = S^* S u - \lambda S^* u = S^* S u - \lambda S u. \end{aligned} \tag{3.6}$$

Conversely, suppose $u \in \operatorname{dom}(S^* S)$ and $S^* S u = \lambda S u$, $\lambda \neq 0$. Introducing $v = \lambda^{-1} S u$, then $v \in \operatorname{dom}(S^*)$ and

$$S^* v = \lambda^{-1} S^* S u = S u = \lambda v. \tag{3.7}$$

Noticing that

$$S^* S u = \lambda S u = \lambda S^* u \text{ implies } S^*(S - \lambda I_{\mathcal{H}}) u = 0, \tag{3.8}$$

and hence $(S - \lambda I_{\mathcal{H}}) u \in \ker(S^*)$, rewriting v as

$$v = u + \lambda^{-1}(S - \lambda I_{\mathcal{H}}) u \tag{3.9}$$

then proves that also $v \in \operatorname{dom}(S_K)$, using (2.8) again. □

Due to Example 3.5 and Remark 3.6 at the end of this section, we will call the linear pencil eigenvalue problem $S^* S u = \lambda S u$ in (3.2) the *abstract buckling problem* associated with the Krein–von Neumann extension S_K of S.

Next, we turn to a variational formulation of the correspondence between the inverse of the reduced Krein extension $\widehat{S}_K$ and the abstract buckling problem in terms of appropriate sesquilinear forms by following the treatment of Kozlov [24]–[26] in the context of elliptic partial differential operators. This will then lead to an even stronger connection between the Krein–von Neumann extension S_K of S and the associated abstract buckling eigenvalue problem (3.2), culminating in a unitary equivalence result in Theorem 3.4.

Given the operator S, we introduce the following sesquilinear forms in $\mathcal{H}$,

$$a(u, v) = (Su, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(a) = \operatorname{dom}(S), \tag{3.10}$$

$$b(u, v) = (u, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(b) = \operatorname{dom}(S). \tag{3.11}$$

Then S being densely defined and closed implies that the sesquilinear form a shares these properties and (2.16) implies its boundedness from below,

$$a(u, u) \geq \varepsilon^2 \|u\|_{\mathcal{H}}^2, \quad u \in \operatorname{dom}(S). \tag{3.12}$$

Thus, one can introduce the Hilbert space $\mathcal{W} = (\operatorname{dom}(S), (\cdot, \cdot)_{\mathcal{W}})$ with associated scalar product

$$(u, v)_{\mathcal{W}} = a(u, v) = (Su, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(S). \tag{3.13}$$

In addition, we denote by $\iota_{\mathcal{W}}$ the continuous embedding operator of $\mathcal{W}$ into $\mathcal{H}$,

$$\iota_{\mathcal{W}} : \mathcal{W} \hookrightarrow \mathcal{H}. \tag{3.14}$$

Hence we will use the notation

$$(w_1, w_2)_{\mathcal{W}} = a(\iota_{\mathcal{W}} w_1, \iota_{\mathcal{W}} w_2) = (S \iota_{\mathcal{W}} w_1, S \iota_{\mathcal{W}} w_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}, \tag{3.15}$$

in the following.

Given the sesquilinear forms a and b and the Hilbert space $\mathcal{W}$, we next define the operator T in $\mathcal{W}$ by

$$\begin{aligned}(w_1, Tw_2)_{\mathcal{W}} &= a(\iota_{\mathcal{W}}w_1, \iota_{\mathcal{W}}Tw_2) = (S\iota_{\mathcal{W}}w_1, S\iota_{\mathcal{W}}Tw_2)_{\mathcal{H}} \\ &= b(\iota_{\mathcal{W}}w_1, \iota_{\mathcal{W}}w_2) = (\iota_{\mathcal{W}}w_1, S\iota_{\mathcal{W}}w_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}.\end{aligned} \tag{3.16}$$

(In contrast to the informality of our introduction, we now explicitly write the embedding operator $\iota_{\mathcal{W}}$.) One verifies that T is well-defined and that

$$|(w_1, Tw_2)_{\mathcal{W}}| \leq \|\iota_{\mathcal{W}}w_1\|_{\mathcal{H}}\|S\iota_{\mathcal{W}}w_2\|_{\mathcal{H}} \leq \varepsilon^{-1}\|w_1\|_{\mathcal{W}}\|w_2\|_{\mathcal{W}}, \quad w_1, w_2 \in \mathcal{W}, \tag{3.17}$$

and hence that

$$0 \leq T = T^* \in \mathcal{B}(\mathcal{W}), \quad \|T\|_{\mathcal{B}(\mathcal{W})} \leq \varepsilon^{-1}. \tag{3.18}$$

For reasons to become clear at the end of this section, we will call T the *abstract buckling problem operator* associated with the Krein–von Neumann extension S_K of S.

Next, recalling the notation $\widehat{\mathcal{H}} = [\ker(S^*)]^{\perp} = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\mathcal{H}$ (cf. (2.17)), we introduce the operator

$$\widehat{S} : \begin{cases} \mathcal{W} \to \widehat{\mathcal{H}}, \\ w \mapsto S\iota_{\mathcal{W}}w, \end{cases} \tag{3.19}$$

and note that

$$\operatorname{ran}\big(\widehat{S}\big) = \operatorname{ran}(S) = \widehat{\mathcal{H}}, \tag{3.20}$$

since $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ and S is closed in $\mathcal{H}$ (see, e.g., [33, Theorem 5.32]). In fact, one has the following result:

Lemma 3.2. *Assume Hypothesis* 2.2. *Then* $\widehat{S} \in \mathcal{B}(\mathcal{W}, \widehat{\mathcal{H}})$ *maps* $\mathcal{W}$ *unitarily onto* $\widehat{\mathcal{H}}$.

Proof. Clearly $\widehat{S}$ is an isometry since

$$\big\|\widehat{S}w\big\|_{\widehat{\mathcal{H}}} = \big\|S\iota_{\mathcal{W}}w\big\|_{\mathcal{H}} = \|w\|_{\mathcal{W}}, \quad w \in \mathcal{W}. \tag{3.21}$$

Since $\operatorname{ran}\big(\widehat{S}\big) = \widehat{\mathcal{H}}$ by (3.20), $\widehat{S}$ is unitary. ☐

Next we recall the definition of the reduced Krein–von Neumann operator $\widehat{S}_K$ in $\widehat{\mathcal{H}}$ defined in (2.19), the fact that $\ker(S^*) = \ker(S_K)$ by (2.10), and state the following auxiliary result:

Lemma 3.3. *Assume Hypothesis* 2.2. *Then the map*

$$\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big] : \operatorname{dom}(S) \to \operatorname{dom}\big(\widehat{S}_K\big) \tag{3.22}$$

is a bijection. In addition, we note that

$$\begin{aligned}\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]S_K u &= S_K\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]u = \widehat{S}_K\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]u \\ &= \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]Su = Su \in \widehat{\mathcal{H}}, \quad u \in \operatorname{dom}(S).\end{aligned} \tag{3.23}$$

Proof. Let $u \in \operatorname{dom}(S)$, then $\ker(S^*) = \ker(S_K)$ implies that $\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]u \in \operatorname{dom}(S_K)$ and of course $\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]u \in \operatorname{dom}\big(\widehat{S}_K\big)$. To prove injectivity of the map (3.22) it suffices to assume $v \in \operatorname{dom}(S)$ and $\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]v = 0$. Then $\operatorname{dom}(S) \ni v = P_{\ker(S^*)}v \in \ker(S^*)$ yields $v = 0$ as $\operatorname{dom}(S) \cap \ker(S^*) = \{0\}$. To prove surjectivity of the map (3.22) we suppose $u \in \operatorname{dom}\big(\widehat{S}_K\big)$. The decomposition, $u = f + g$ with $f \in \operatorname{dom}(S)$ and $g \in \ker(S^*)$, then yields

$$u = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]u = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]f \in \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\operatorname{dom}(S) \tag{3.24}$$

and hence proves surjectivity of (3.22).

Equation (3.23) is clear from

$$S_K\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big] = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]S_K = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]S_K\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]. \quad (3.25)$$

□

Continuing, we briefly recall the polar decomposition of S,

$$S = U_S|S|, \quad (3.26)$$

with

$$|S| = (S^*S)^{1/2} \geq \varepsilon I_{\mathcal{H}},\ \varepsilon > 0, \quad U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big) \text{ is unitary.} \quad (3.27)$$

At this point we are in position to state our principal unitary equivalence result:

Theorem 3.4. *Assume Hypothesis* 2.2. *Then the inverse of the reduced Krein–von Neumann extension $\widehat{S}_K$ in $\widehat{\mathcal{H}} = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\mathcal{H}$ and the abstract buckling problem operator T in $\mathcal{W}$ are unitarily equivalent, in particular,*

$$\big(\widehat{S}_K\big)^{-1} = \widehat{S}T(\widehat{S})^{-1}. \quad (3.28)$$

Moreover, one has

$$\big(\widehat{S}_K\big)^{-1} = U_S\big[|S|^{-1}S|S|^{-1}\big](U_S)^{-1}, \quad (3.29)$$

where $U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big)$ is the unitary operator in the polar decomposition (3.26) *of S and the operator $|S|^{-1}S|S|^{-1} \in \mathcal{B}(\mathcal{H})$ is self-adjoint in $\mathcal{H}$.*

Proof. Let $w_1, w_2 \in \mathcal{W}$. Then,

$$\begin{aligned}
&\big(w_1, (\widehat{S})^{-1}\big(\widehat{S}_K\big)^{-1}\widehat{S}w_2\big)_{\mathcal{W}} = \big(\widehat{S}w_1, \big(\widehat{S}_K\big)^{-1}\widehat{S}w_2\big)_{\widehat{\mathcal{H}}} \\
&\quad = \big(\big(\widehat{S}_K\big)^{-1}\widehat{S}w_1, \widehat{S}w_2\big)_{\widehat{\mathcal{H}}} = \big(\big(\widehat{S}_K\big)^{-1}S\iota_{\mathcal{W}}w_1, \widehat{S}w_2\big)_{\widehat{\mathcal{H}}} \\
&\quad = \big(\big(\widehat{S}_K\big)^{-1}\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]S\iota_{\mathcal{W}}w_1, \widehat{S}w_2\big)_{\widehat{\mathcal{H}}} \quad \text{by (3.23)} \\
&\quad = \big(\big(\widehat{S}_K\big)^{-1}\widehat{S}_K\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\iota_{\mathcal{W}}w_1, \widehat{S}w_2\big)_{\widehat{\mathcal{H}}} \quad \text{again by (3.23)} \\
&\quad = \big(\big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\iota_{\mathcal{W}}w_1, \widehat{S}w_2\big)_{\widehat{\mathcal{H}}} \\
&\quad = \big(\iota_{\mathcal{W}}w_1, S\iota_{\mathcal{W}}w_2\big)_{\mathcal{H}} \\
&\quad = \big(w_1, Tw_2\big)_{\mathcal{W}} \quad \text{by definition of } T \text{ in (3.16)}, \qquad (3.30)
\end{aligned}$$

yields (3.28). In addition one verifies that

$$\begin{aligned}
\big(\widehat{S}w_1, \big(\widehat{S}_K\big)^{-1}\widehat{S}w_2\big)_{\widehat{\mathcal{H}}} &= \big(w_1, Tw_2\big)_{\mathcal{W}} \\
&= \big(\iota_{\mathcal{W}}w_1, S\iota_{\mathcal{W}}w_2\big)_{\mathcal{H}} \\
&= \big(|S|^{-1}|S|\iota_{\mathcal{W}}w_1, S|S|^{-1}|S|\iota_{\mathcal{W}}w_2\big)_{\mathcal{H}} \\
&= \big(|S|\iota_{\mathcal{W}}w_1, \big[|S|^{-1}S|S|^{-1}\big]|S|\iota_{\mathcal{W}}w_2\big)_{\mathcal{H}} \\
&= \big((U_S)^*S\iota_{\mathcal{W}}w_1, \big[|S|^{-1}S|S|^{-1}\big](U_S)^*S\iota_{\mathcal{W}}w_2\big)_{\mathcal{H}} \\
&= \big(S\iota_{\mathcal{W}}w_1, U_S\big[|S|^{-1}S|S|^{-1}\big](U_S)^*S\iota_{\mathcal{W}}w_2\big)_{\mathcal{H}} \\
&= \big(\widehat{S}w_1, U_S\big[|S|^{-1}S|S|^{-1}\big](U_S)^*\widehat{S}w_2\big)_{\widehat{\mathcal{H}}}, \qquad (3.31)
\end{aligned}$$

where we used $|S| = (U_S)^*S$. □

Equation (3.29) is of course motivated by rewriting the abstract linear pencil buckling eigenvalue problem (3.2), $S^*Su = \lambda Su$, $\lambda \neq 0$, in the form

$$\lambda^{-1}S^*Su = \lambda^{-1}(S^*S)^{1/2}\big[(S^*S)^{1/2}u\big] = S(S^*S)^{-1/2}\big[(S^*S)^{1/2}u\big] \tag{3.32}$$

and hence in the form of a standard eigenvalue problem

$$|S|^{-1}S|S|^{-1}w = \lambda^{-1}w, \quad \lambda \neq 0, \quad w = |S|u. \tag{3.33}$$

We conclude this section with a concrete example discussed explicitly in Grubb [20] (see also [17]–[19] for necessary background) and make the explicit connection with the buckling problem. It was this example which greatly motivated the abstract results in this note:

Example 3.5. ([20].) *Let $\mathcal{H} = L^2(\Omega; d^nx)$, with $\Omega \subset \mathbb{R}^n$, $n \geq 2$, open and bounded, with a smooth boundary $\partial\Omega$, and consider the minimal operator realization S of the differential expression $\mathscr{S}$ in $L^2(\Omega; d^nx)$, defined by*

$$\begin{aligned} &Su = \mathscr{S}u, \\ &u \in \mathrm{dom}(S) = H_0^{2m}(\Omega) = \big\{v \in H^{2m}(\Omega) \,\big|\, \gamma_k v = 0,\ 0 \leq k \leq 2m-1\big\}, \quad m \in \mathbb{N}, \end{aligned} \tag{3.34}$$

where

$$\mathscr{S} = \sum_{0\leq|\alpha|\leq 2m} a_\alpha(\cdot)D^\alpha, \tag{3.35}$$

$$D^\alpha = (-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}, \quad \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n, \tag{3.36}$$

$$a_\alpha(\cdot) \in C^\infty(\overline{\Omega}), \quad C^\infty(\overline{\Omega}) = \bigcap_{k\in\mathbb{N}_0} C^k(\overline{\Omega}), \tag{3.37}$$

and the coefficients a_α are chosen such that S is symmetric in $L^2(\mathbb{R}^n; d^nx)$, that is, the differential expression $\mathscr{S}$ is formally self-adjoint,

$$(\mathscr{S}u, v)_{L^2(\mathbb{R}^n;d^nx)} = (u, \mathscr{S}v)_{L^2(\mathbb{R}^n;d^nx)}, \quad u, v \in C_0^\infty(\Omega), \tag{3.38}$$

and $\mathscr{S}$ is strongly elliptic, that is, for some $c > 0$,

$$\mathrm{Re}\bigg(\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha\bigg) \geq c|\xi|^{2m}, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^n. \tag{3.39}$$

In addition we assume that $S \geq \varepsilon I_{L^2(\Omega;d^nx)}$ for some $\varepsilon > 0$. The trace operators γ_k are defined as follows: Consider

$$\mathring{\gamma}_k : \begin{cases} C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega) \\ u \mapsto (\partial_n^k u)|_{\partial\Omega}, \end{cases} \tag{3.40}$$

with ∂_n denoting the interior normal derivative. The map $\mathring{\gamma}$ then extends by continuity to a bounded operator

$$\gamma_k : H^s(\Omega) \to H^{s-k-(1/2)}(\partial\Omega), \quad s > k + (1/2), \tag{3.41}$$

in addition, the map

$$\gamma^{(r)} = (\gamma_0,\dots,\gamma_r) : H^s(\Omega) \to \prod_{k=0}^{r} H^{s-k-(1/2)}(\partial\Omega), \quad s > r + (1/2), \tag{3.42}$$

satisfies

$$\ker\big(\gamma^{(r)}\big) = H_0^s(\Omega), \quad \mathrm{ran}\big(\gamma^{(r)}\big) = \prod_{k=0}^{r} H^{s-k-(1/2)}(\partial\Omega). \tag{3.43}$$

Then S^*, the maximal operator realization of $\mathscr{S}$ in $L^2(\Omega; d^n x)$, is given by

$$S^* u = \mathscr{S} u, \quad u \in \operatorname{dom}(S^*) = \big\{ v \in L^2(\Omega; d^n x) \,\big|\, \mathscr{S} v \in L^2(\Omega; d^n x) \big\}, \tag{3.44}$$

and S_F is characterized by

$$S_F u = \mathscr{S} u, \quad u \in \operatorname{dom}(S_F) = \big\{ v \in H^{2m}(\Omega) \,\big|\, \gamma_k v = 0,\, 0 \le k \le m-1 \big\}. \tag{3.45}$$

The Krein–von Neumann extension S_K of S then has the domain

$$\operatorname{dom}(S_K) = H_0^{2m}(\Omega) \dotplus \ker(S^*), \quad \dim(\ker(S^*)) = \infty, \tag{3.46}$$

and elements $u \in \operatorname{dom}(S_K)$ satisfy the nonlocal boundary condition

$$\gamma_N u - P_{\gamma_D, \gamma_N} \gamma_D u = 0, \tag{3.47}$$

$$\gamma_D u = (\gamma_0 u, \dots, \gamma_{m-1} u), \quad \gamma_N u = (\gamma_m u, \dots, \gamma_{2m-1} u), \quad u \in \operatorname{dom}(S_K), \tag{3.48}$$

where

$$\begin{aligned} &P_{\gamma_D, \gamma_N} = \gamma_N \gamma_Z^{-1} : \prod_{k=0}^{m-1} H^{s-k-(1/2)}(\partial\Omega) \to \prod_{j=m}^{2m-1} H^{s-j-(1/2)}(\partial\Omega) \\ &\qquad \text{continuously for all } s \in \mathbb{R}, \end{aligned} \tag{3.49}$$

and γ_Z^{-1} denotes the inverse of the isomorphism γ_Z given by

$$\gamma_D : Z^s_{\mathscr{S}} \to \prod_{k=0}^{m-1} H^{s-k-(1/2)}(\partial\Omega), \tag{3.50}$$

$$Z^s_{\mathscr{S}} = \big\{ u \in H^s(\Omega) \,\big|\, \mathscr{S} u = 0 \text{ in } \Omega \text{ in the sense of distributions in } \mathcal{D}'(\Omega) \big\}, \quad s \in \mathbb{R}. \tag{3.51}$$

Moreover one has

$$\big(\widehat{S}\big)^{-1} = \iota_{\mathcal{W}} [I_{\mathcal{H}} - P_{\gamma_D, \gamma_N} \gamma_D] \big(\widehat{S}_K\big)^{-1}, \tag{3.52}$$

since $[I_{\mathcal{H}} - P_{\gamma_D, \gamma_N} \gamma_D] \operatorname{dom}(S_K) \subseteq \operatorname{dom}(S)$ and $S[I_{\mathcal{H}} - P_{\gamma_D, \gamma_N} \gamma_D] v = \lambda v$, $v \in \operatorname{dom}(S_K)$.

As discussed in detail in Grubb [20],

$$\sigma_{\rm ess}(S_K) = \{0\}, \quad \sigma(S_K) \cap (0, \infty) = \sigma_{\rm d}(S_K) \tag{3.53}$$

and the nonzero (and hence discrete) eigenvalues of S_K satisfy a Weyl-type asymptotics. The connection to a higher-order buckling eigenvalue problem established by Grubb then reads

$$\text{There exists } 0 \neq v \in S_K \text{ satisfying } \mathscr{S} v = \lambda v \text{ in } \Omega, \quad \lambda \neq 0 \tag{3.54}$$

if and only if

$$\text{there exists } 0 \neq u \in C^\infty(\overline{\Omega}) \text{ such that } \begin{cases} \mathscr{S}^2 u = \lambda \mathscr{S} u \text{ in } \Omega, \quad \lambda \neq 0, \\ \gamma_k u = 0,\ 0 \le k \le 2m-1, \end{cases} \tag{3.55}$$

where the solutions v of (3.54) *are in one-to-one correspondence with the solutions u of* (3.55) *via*

$$u = S_F^{-1} \mathscr{S} v, \quad v = \lambda^{-1} \mathscr{S} u. \tag{3.56}$$

Since S_F has purely discrete spectrum in Example 3.5, we note that Theorem 2.4 applies in this case.

Remark 3.6. In the particular case $m = 1$ and $\mathscr{S} = -\Delta$, the linear pencil eigenvalue problem (3.55) (i.e., the concrete analog of the abstract buckling eigenvalue problem $S^*Su = \lambda Su$, $\lambda \neq 0$, in (3.2)), then yields the *buckling of a clamped plate problem*,

$$(-\Delta)^2 u = \lambda(-\Delta)u \text{ in } \Omega, \quad \lambda \neq 0, \ u \in H_0^2(\Omega), \tag{3.57}$$

as distributions in $H^{-2}(\Omega)$. Here we used the fact that for any nonempty bounded open set $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, $(-\Delta)^m \in \mathcal{B}\big(H^k(\Omega), H^{k-2m}(\Omega)\big)$, $k \in \mathbb{Z}$, $m \in \mathbb{N}$. In addition, if Ω is a Lipschitz domain, then one has that $-\Delta \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism and similarly, $(-\Delta)^2 \colon H_0^2(\Omega) \to H^{-2}(\Omega)$ is an isomorphism. (For the natural norms on $H^k(\Omega)$, $k \in \mathbb{Z}$, see, e.g., [29, p. 73–75].) We refer, for instance, to [9, Sect. 4.3B] for a derivation of (3.57) from the fourth-order system of quasilinear von Kármán partial differential equations. To be precise, (3.57) should also be considered in the special case $n = 2$.

Remark 3.7. We emphasize that the smoothness hypotheses on $\partial\Omega$ can be relaxed in the special case of the second-order Schrödinger operator associated with the differential expression $-\Delta + V$, where $V \in L^\infty(\Omega; d^n x)$ is real-valued: Following the treatment of self-adjoint extensions of $S = \overline{(-\Delta + V)|_{C_0^\infty(\Omega)}}$ on quasi-convex domains Ω first introduced in [16], the case of the Krein–von Neumann extension S_K of S on such quasi-convex domains (which are close to minimally smooth) is treated in great detail in [8]. In particular, a Weyl-type asymptotics of the associated (nonzero) eigenvalues of S_K has been proven in [8]. In the higher-order smooth case described in Example 3.5, a Weyl-type asymptotics for the nonzero eigenvalues of S_K has been proven by Grubb [20] in 1983.

Acknowledgments. We are indebted to Gerd Grubb, Konstantin Makarov, and Eduard Tsekanovskii for a critical reading of our manuscript, and for providing us with numerous additional insights into this circle of ideas.

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