# LONG-TIME ASYMPTOTICS OF THE TODA LATTICE FOR DECAYING INITIAL DATA REVISITED

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Abstract. The purpose of this article is to give a streamlined and self-contained treatment of the long-time asymptotics of the Toda lattice for decaying initial data in the soliton and in the similarity region via the method of nonlinear steepest descent.

## 1. Introduction

In this paper we want to compute the long time asymptotics for the doubly infinite Toda lattice which reads in Flaschka's variables (see e.g. [22], [23], or [24])

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \tag{1.1}$$

$(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here the dot denotes differentiation with respect to time. We will consider solutions $(a,b)$ satisfying

$$\sum_n (1+|n|)^{l+2} (|a(n,t) - \frac{1}{2}| + |b(n,t)|) < \infty \tag{1.2}$$

for some $l \in \mathbb{N}$ for one (and hence for all, see [22]) $t \in \mathbb{R}$. It is well-known that the corresponding initial value problem has unique global solutions which can be computed via the inverse scattering transform [22].

The Toda lattice is one of the most prominent soliton equations. The classical result going back to Zabusky and Kruskal [27] states that a "short range" perturbation of the constant solution of a soliton equation eventually splits into a number of stable solitons and a decaying background radiation component. In particular, the solitons constitute the stable part of the solutions arising from arbitrary short range initial conditions. This is illustrated in Figure 1 which shows the numerically computed solution $a(n,t)$ corresponding to the initial condition $a(n,0) = \frac{1}{2}$, $b(n,0) = \delta_{0,n}$ at some large time $t = 150$. You can see the soliton region $|\frac{n}{t}| > 1$ with one single soliton on the very left and the similarity region $|\frac{n}{t}| < 1$ where there are some small oscillations which decay like $t^{-1/2}$. The goal is to mathematically justify this picture and to derive an asymptotic formula for the solution in the similarity region.

This was first done by Novokshenov and Habibullin [18] and was later made rigorous by Kamvissis [11] under the the additional assumption that no solitons are present and that the underlying Jacobi operator is resonant at the band edges of

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FIGURE 1. Numerically computed solution of the Toda lattice, with initial condition the constant solution perturbed at one point in the middle.

the spectrum (i.e., the reflection coefficient has absolute value strictly less than one, which is generically not the case). The case of solitons was recently investigated by us in [15].

We also want to mention that one could also replace the constant background solution by a periodic one. However, this case exhibits a much different behaviour as was pointed out by Kamvissis and one of us in [12] (see also [8], [9], [13], [14], and [16] for a rigorous mathematical treatment).

The main purpose of the present paper is to revisit this classical problem using the method of nonlinear steepest descent by Deift and Zhou [3] and give a complete and expository introduction. We are trying to present a streamlined and simplified approach with complete proofs. In particular, we have added two appendices which show how to solve the localized Riemann–Hilbert problem on a small cross via parabolic cylinder functions and how to rewrite Riemann–Hilbert problems as singular integral equations. Only some basic knowledge on Riemann–Hilbert problems, which can be found for example in the beautiful lecture notes by Deift [2], is required.

As one of our main simplifications in contradistinction to [11] we will work with the vector Riemann–Hilbert problem which arises naturally from the inverse scattering theory, thus avoiding the detour over the associated matrix Riemann–Hilbert problem. This also avoids the singularities appearing in the matrix Riemann–Hilbert problem in case the reflection coefficient is $-1$ at the band edges. In particular, this will remove the corresponding assumption $|R(z)| < 1$ imposed in [11].

To state the main results, we begin by recalling that the sequences $a(n,t)$, $b(n,t)$, $n \in \mathbb{Z}$, for fixed $t \in \mathbb{R}$, are uniquely determined by its scattering data, that is, by its right reflection coefficient $R_+(z,t)$, $|z| = 1$, and its eigenvalues $\lambda_j \in (-\infty,-1) \cup (1,\infty)$, $j = 1,\dots,N$, together with the corresponding right norming constants $\gamma_{+,j}(t) > 0$, $j = 1,\dots,N$. It is well-known that under the assumption (1.2) the reflection coefficients are $C^{l+1}(\mathbb{T})$. Rather than in the complex plane, we will work on the unit disc using the usual Joukowski transformation

$$\lambda = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad z = \lambda - \sqrt{\lambda^2 - 1}, \qquad \lambda \in \mathbb{C},\ |z| \le 1. \tag{1.3}$$

In these new coordinates the eigenvalues $\lambda_j \in (-\infty,-1) \cup (1,\infty)$ will be denoted by $\zeta_j \in (-1,0) \cup (0,1)$. The continuous spectrum $[-1,1]$ is mapped to the unit circle $\mathbb{T}$. Moreover, the phase of the associated Riemann–Hilbert problem is given by

$$\Phi(z) = z - z^{-1} + 2\frac{n}{t}\log(z) \tag{1.4}$$

and the stationary phase points, $\Phi'(z)=0$, are denoted by

$$z_0 = -\frac{n}{t} - \sqrt{(\frac{n}{t})^2 - 1}, \quad z_0^{-1} = -\frac{n}{t} + \sqrt{(\frac{n}{t})^2 - 1}, \qquad \lambda_0 = -\frac{n}{t}. \tag{1.5}$$

Here the branch of the square root is chosen such that $\mathrm{Im}(\sqrt{z}) \ge 0$. For $\frac{n}{t} < -1$ we have $z_0 \in (0,1)$, for $-1 \le \frac{n}{t} \le 1$ we have $z_0 \in \mathbb{T}$ (and hence $z_0^{-1} = \overline{z_0}$), and for $\frac{n}{t} > 1$ we have $z_0 \in (-1,0)$. For $|\frac{n}{t}| > 1$ we will also need the value $\zeta_0 \in (-1,0)\cup(0,1)$ defined via $\mathrm{Re}(\Phi(\zeta_0)) = 0$, that is,

$$\frac{n}{t} = -\frac{\zeta_0 - \zeta_0^{-1}}{2\log(|\zeta_0|)}. \tag{1.6}$$

We will set $\zeta_0 = -1$ if $|\frac{n}{t}| \le 1$ for notational convenience. A simple analysis shows that for $\frac{n}{t} < -1$ we have $0 < \zeta_0 < z_0 < 1$ and for $\frac{n}{t} > 1$ we have $-1 < z_0 < \zeta_0 < 0$.

Furthermore, recall that the transmission coefficient $T(z)$, $|z| \le 1$, is time independent and can be reconstructed using the Poisson–Jensen formula. In particular, we define the partial transmission coefficient with respect to $z_0$ by

$$T(z,z_0) = \tag{1.7}$$
$$\begin{cases} \prod_{\zeta_k \in (\zeta_0,0)} |\zeta_k| \frac{z-\zeta_k^{-1}}{z-\zeta_k}, & z_0 \in (-1,0), \\ \left(\prod_{\zeta_k \in (-1,0)} |\zeta_k| \frac{z-\zeta_k^{-1}}{z-\zeta_k}\right) \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\overline{z_0}}^{z_0} \log(|T(s)|) \frac{s+z}{s-z} \frac{ds}{s}\right), & |z_0| = 1, \\ \left(\prod_{\zeta_k \in (-1,0)\cup(\zeta_0,1)} |\zeta_k| \frac{z-\zeta_k^{-1}}{z-\zeta_k}\right) \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \log(|T(s)|) \frac{s+z}{s-z} \frac{ds}{s}\right), & z_0 \in (0,1). \end{cases}$$

Here the integral is to be taken along the arc $\Sigma(z_0) = \{z \in \mathbb{T} | \mathrm{Re}(z) < \mathrm{Re}(z_0)\}$ oriented counterclockwise. Observe that $T(z,z_0) = T(z)$ once $z_0 \in (0,1)$ and $(0,\zeta_0)$ contains no eigenvalues. Moreover, $T(z,z_0)$ can be computed in terms of the scattering data since $|T(z)|^2 = 1 - |R_+(z,t)|^2$.

Moreover, we set

$$\begin{aligned} T_0(z_0) &= T(0,z_0) \\ &= \begin{cases} \prod_{\zeta_k \in (\zeta_0,0)} |\zeta_k|^{-1}, & z_0 \in (-1,0), \\ \left(\prod_{\zeta_k \in (-1,0)} |\zeta_k|^{-1}\right) \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\overline{z_0}}^{z_0} \log(|T(s)|) \frac{ds}{s}\right), & |z_0| = 1, \\ \left(\prod_{\zeta_k \in (-1,0)\cup(\zeta_0,1)} |\zeta_k|^{-1}\right) \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \log(|T(s)|) \frac{ds}{s}\right), & z_0 \in (0,1), \end{cases} \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} T_1(z_0) &= \frac{\partial}{\partial z} \log T(z,z_0)\Big|_{z=0} \\ &= \begin{cases} \sum_{\zeta_k \in (\zeta_0,0)} (\zeta_k^{-1} - \zeta_k), & z_0 \in (-1,0), \\ \sum_{\zeta_k \in (-1,0)} (\zeta_k^{-1} - \zeta_k) + \frac{1}{\pi\mathrm{i}} \int_{\overline{z_0}}^{z_0} \log(|T(s)|) \frac{ds}{s^2}, & |z_0| = 1, \\ \sum_{\zeta_k \in (-1,0)\cup(\zeta_0,1)} (\zeta_k^{-1} - \zeta_k) + \frac{1}{\pi\mathrm{i}} \int_{\mathbb{T}} \log(|T(s)|) \frac{ds}{s^2}, & z_0 \in (0,1). \end{cases} \end{aligned} \tag{1.9}$$

In other words, $T(z,z_0) = T_0(z_0)(1 + T_1(z_0) z + O(z^2))$.

**Theorem 1.1** (Soliton region). *Assume (1.2) and abbreviate by* $c_k = -\frac{\zeta_k - \zeta_k^{-1}}{2\log(|\zeta_k|)}$ *the velocity of the k'th soliton determined by* $\mathrm{Re}(\Phi(\zeta_k)) = 0$. *Then the asymptotics in the soliton region,* $|n/t| \geq 1 + C/t \log(t)^2$ *for some* $C > 0$, *are as follows.*

*Let* $\varepsilon > 0$ *sufficiently small such that the intervals* $[c_k - \varepsilon, c_k + \varepsilon]$, $1 \leq k \leq N$, *are disjoint and lie inside* $(-1,0) \cup (0,1)$.

*If* $|\frac{n}{t} - c_k| < \varepsilon$ *for some k, one has*

$$\prod_{j=n}^{\infty} (2a(j,t)) = T_0(z_0) \left( \sqrt{\frac{1 - \zeta_k^2 + \gamma_k(n,t)}{1 - \zeta_k^2 + \gamma_k(n,t)\zeta_k^2}} + O(t^{-l}) \right),$$

$$\sum_{j=n+1}^{\infty} b(j,t) = \frac{1}{2} T_1(z_0) - \frac{\gamma_k(n,t)\zeta_k(\zeta_k^2 - 1)}{2((\gamma_k(n,t) - 1)\zeta_k^2 + 1)} + O(t^{-l}), \tag{1.10}$$

*for any* $l \geq 1$, *where*

$$\gamma_k(n,t) = \gamma_k T(\zeta_k, c_k)^{-2} \mathrm{e}^{t(\zeta_k - \zeta_k^{-1})} \zeta_k^{2n}. \tag{1.11}$$

*If* $|\frac{n}{t} - c_k| \geq \varepsilon$, *for all k, one has*

$$\prod_{j=n}^{\infty} (2a(j,t)) = T_0(z_0) \left(1 + O(t^{-l})\right),$$

$$\sum_{j=n+1}^{\infty} b(j,t) = \frac{1}{2} T_1(z_0) + O(t^{-l}), \tag{1.12}$$

*for any* $l \geq 1$.

In the remaining region, we will show

**Theorem 1.2** (Similarity region). *Assume (1.2) with* $l \geq 3$, *then, away from the soliton region,* $|n/t| \leq 1 - C$ *for some* $C > 0$. *Then the asymptotics are given by*

$$\begin{aligned}\prod_{j=n}^{\infty} (2a(j,t)) =& T_0(z_0) \Bigg( 1 + \left( \frac{\nu}{-2\sin(\theta_0) t} \right)^{1/2} \cos\left(t\Phi_0 + \nu \log(t) - \delta\right) \\ & + O(t^{-\alpha}) \Bigg),\end{aligned} \tag{1.13}$$

$$\begin{aligned}\sum_{j=n+1}^{\infty} b(j,t) =& \frac{1}{2} T_1(z_0) + \left( \frac{\nu}{-2\sin(\theta_0) t} \right)^{1/2} \cos\left(t\Phi_0 + \nu \log(t) - \delta + \theta_0\right) \\ & + O(t^{-\alpha}),\end{aligned} \tag{1.14}$$

*for any* $\alpha < 1$. *Here*

$$\begin{aligned} \theta_0 &= \arg(z_0), \\ \Phi_0 &= 2(\sin(\theta_0) - \theta_0 \cos(\theta_0)), \\ \delta &= \pi/4 - 3\nu \log|2\sin(\theta_0)| + 2\arg(\tilde{T}(z_0, z_0)) - \arg(R(z_0)) + \arg(\Gamma(\mathrm{i}\nu)), \\ \tilde{T}(z, z_0) &= \prod_{\zeta_k \in (-1,0)} |\zeta_k| \frac{z - \zeta_k^{-1}}{z - \zeta_k} \cdot \exp\left( \frac{1}{2\pi\mathrm{i}} \int_{\overline{z_0}}^{z_0} \log\left( \frac{|T(s)|}{|T(z_0)|} \right) \frac{s+z}{s-z} \frac{ds}{s} \right). \end{aligned} \tag{1.15}$$

We will not address the asymptotics in the missing region around $|n| \approx t$. In the case $|R(z)| < 1$ the solution can be given in terms of Painlevé II transcendents [11]. In the general case an outline using the $g$-function method was given in [6] (for the case of the Korteweg–de Vires equation). The method can also be used to obtain further terms in the asymptotic expansion [4].

Finally, note that one can obtain the asymptotics for $n \geq 0$ from the ones for $n \leq 0$ by virtue of a simple reflection. Similarly for $t \geq 0$ versus $t \leq 0$.

**Lemma 1.3.** *Suppose $a(n,t)$, $b(n,t)$ satisfy the Toda equation (1.1), then so do*

$$\tilde{a}(n,t) = a(-n-1,t), \quad \tilde{b}(n,t) = -b(-n,t)$$

*respectively*

$$\tilde{a}(n,t) = a(n,-t), \quad \tilde{b}(n,t) = -b(n,-t).$$

## 2. The Inverse scattering transform and the Riemann–Hilbert problem

In this section we want to derive the Riemann–Hilbert problem from scattering theory. The special case without eigenvalues was first given in Kamvissis [11]. How eigenvalues can be added was first shown in Deift, Kamvissis, Kriecherbauer, and Zhou [7]. We essentially follow [15] in this section.

For the necessary results from scattering theory respectively the inverse scattering transform for the Toda lattice we refer to [20], [21], [22].

Associated with $a(t), b(t)$ is a self-adjoint Jacobi operator

$$H(t) = a(t)S^{+} + a^{-}(t)S^{-} + b(t) \tag{2.1}$$

in $\ell^2(\mathbb{Z})$, where $S^{\pm}f(n) = f^{\pm}(n) = f(n\pm 1)$ are the usual shift operators and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. By our assumption (1.2) the spectrum of $H$ consists of an absolutely continuous part $[-1,1]$ plus a finite number of eigenvalues $\lambda_k \in \mathbb{R}\backslash[-1,1]$, $1 \leq k \leq N$. In addition, there exist two Jost functions $\psi_{\pm}(z,n,t)$ which solve the recurrence equation

$$H(t)\psi_{\pm}(z,n,t) = \frac{z+z^{-1}}{2}\psi_{\pm}(z,n,t), \qquad |z| \leq 1, \tag{2.2}$$

and asymptotically look like the free solutions

$$\lim_{n\to\pm\infty} z^{\mp n}\psi_{\pm}(z,n,t) = 1. \tag{2.3}$$

Both $\psi_{\pm}(z,n,t)$ are analytic for $0 < |z| < 1$ with smooth boundary values for $|z| = 1$. The asymptotics of the two projections of the Jost function are

$$\psi_{\pm}(z,n,t) = \frac{z^{\pm n}}{A_{\pm}(n,t)}\Big(1 + 2B_{\pm}(n,t)z + O(z^2)\Big), \tag{2.4}$$

as $z \to 0$, where

$$\begin{aligned} A_{+}(n,t) &= \prod_{j=n}^{\infty} 2a(j,t), \quad B_{+}(n,t) = -\sum_{j=n+1}^{\infty} b(j,t), \\ A_{-}(n,t) &= \prod_{j=-\infty}^{n-1} 2a(j,t), \quad B_{-}(n,t) = -\sum_{j=-\infty}^{n-1} b(j,t). \end{aligned} \tag{2.5}$$

One has the scattering relations

$$T(z)\psi_\mp(z,n,t) = \overline{\psi_\pm(z,n,t)} + R_\pm(z,t)\psi_\pm(z,n,t), \qquad |z|=1, \tag{2.6}$$

where $T(z)$, $R_\pm(z,t)$ are the transmission respectively reflection coefficients. The transmission and reflection coefficients have the following well-known properties:

**Lemma 2.1.** *The transmission coefficient $T(z)$ has a meromorphic extension to the interior of the unit circle with simple poles at the images of the eigenvalues $\zeta_j$. The residues of $T(z)$ are given by*

$$\operatorname{Res}_{\zeta_k} T(z) = -\zeta_k \frac{\gamma_{+,k}(t)}{\mu_k(t)} = -\zeta_k \gamma_{-,k}(t)\mu_k(t), \tag{2.7}$$

*where*

$$\gamma_{\pm,k}(t)^{-1} = \sum_{n\in\mathbb{Z}} |\psi_\pm(\zeta_k,n,t)|^2 \tag{2.8}$$

*and $\psi_-(\zeta_k,n,t) = \mu_k(t)\psi_+(\zeta_k,n,t)$.*

*Moreover,*

$$T(z)\overline{R_+(z,t)} + \overline{T(z)}R_-(z,t) = 0, \qquad |T(z)|^2 + |R_\pm(z,t)|^2 = 1. \tag{2.9}$$

In particular one reflection coefficient, say $R(z,t) = R_+(z,t)$, and one set of norming constants, say $\gamma_k(t) = \gamma_{+,k}(t)$, suffices. Moreover, the time dependence is given by:

**Lemma 2.2.** *The time evolutions of the quantities $R_+(z,t)$, $\gamma_{+,k}(t)$ are given by*

$$R(z,t) = R(z)\mathrm{e}^{t(z-z^{-1})} \tag{2.10}$$

$$\gamma_k(t) = \gamma_k \mathrm{e}^{t(\zeta_k-\zeta_k^{-1})}, \tag{2.11}$$

*where $R(z) = R(z,0)$ and $\gamma_k = \gamma_k(0)$.*

We will define a Riemann–Hilbert problem as follows:

$$m(z,n,t) = \left\{ \begin{array}{ll} \begin{pmatrix} T(z)\psi_-(z,n,t)z^n & \psi_+(z,n,t)z^{-n} \end{pmatrix}, & |z|<1, \\ \begin{pmatrix} \psi_+(z^{-1},n,t)z^n & T(z^{-1})\psi_-(z^{-1},n,t)z^{-n} \end{pmatrix}, & |z|>1. \end{array} \right. \tag{2.12}$$

We are interested in the jump condition of $m(z,n,t)$ on the unit circle $\mathbb{T}$ (oriented counterclockwise). To formulate our jump condition we use the following convention: When representing functions on $\mathbb{T}$, the lower subscript denotes the non-tangential limit from different sides,

$$m_\pm(z) = \lim_{\zeta\to z,\ |\zeta|^{\pm1}<1} m(\zeta), \qquad |z|=1. \tag{2.13}$$

Using the notation above implicitly assumes that these limits exist in the sense that $m(z)$ extends to a continuous function on the boundary.

**Theorem 2.3** (Vector Riemann–Hilbert problem)**.** *Let $\mathcal{S}_+(H(0)) = \{R(z),\ |z| = 1;\ (\zeta_k,\gamma_k),\ 1\le k\le N\}$ the left scattering data of the operator $H(0)$. Then $m(z) = m(z,n,t)$ defined in (2.12) is meromorphic away from the unit circle with simple poles at $\zeta_k$, $\zeta_k^{-1}$ and satisfies:*

(i) *The jump condition*

$$m_+(z) = m_-(z)v(z), \qquad v(z) = \begin{pmatrix} 1-|R(z)|^2 & -\overline{R(z)}\mathrm{e}^{-t\Phi(z)} \\ R(z)\mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}, \tag{2.14}$$

*for $z\in\mathbb{T}$,*

(ii) *the pole conditions*

$$\begin{aligned} \operatorname{Res}_{\zeta_k} m(z) &= \lim_{z\to\zeta_k} m(z) \begin{pmatrix} 0 & 0 \\ -\zeta_k \gamma_k \mathrm{e}^{t\Phi(\zeta_k)} & 0 \end{pmatrix}, \\ \operatorname{Res}_{\zeta_k^{-1}} m(z) &= \lim_{z\to\zeta_k^{-1}} m(z) \begin{pmatrix} 0 & \zeta_k^{-1}\gamma_k \mathrm{e}^{t\Phi(\zeta_k)} \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.15}$$

(iii) *the symmetry condition*

$$m(z^{-1}) = m(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.16}$$

(iv) *and the normalization*

$$m(0) = (m_1 \quad m_2), \quad m_1 \cdot m_2 = 1 \quad m_1 > 0. \tag{2.17}$$

*Here the phase is given by*

$$\Phi(z) = z - z^{-1} + 2\frac{n}{t} \log z. \tag{2.18}$$

*Proof.* The jump condition (2.14) is a simple calculation using the scattering relations (2.6) plus (2.9). The pole conditions follow since $T(z)$ is meromorphic in $|z| < 1$ with simple poles at $\zeta_k$ and residues given by (2.7). The symmetry condition holds by construction and the normalization (2.17) is immediate from the following lemma. □

Observe that the pole condition at $\zeta_k$ is sufficient since the one at $\zeta_k^{-1}$ follows by symmetry. Moreover, it can be shown that the solution of the above Riemann–Hilbert problem is unique [15]. However, we will not need this fact here and it will follow as a byproduct of our analysis at least for sufficiently large $t$.

Moreover, we have the following asymptotic behaviour near $z = 0$:

**Lemma 2.4.** *The function $m(z, n, t)$ defined in (2.12) satisfies*

$$m(z, n, t) = \Big( A(n, t)(1 - 2B(n - 1, t)z) \quad \tfrac{1}{A(n,t)}(1 + 2B(n, t)z) \Big) + O(z^2). \tag{2.19}$$

*Here $A(n, t) = A_+(n, t)$ and $B(n, t) = B_+(n, t)$ are defined in (2.5).*

*Proof.* This follows from (2.4) and $T(z) = A_+A_-(1-2(B_+-b+B_-)z+O(z^2))$. □

For our further analysis it will be convenient to rewrite the pole condition as a jump condition and hence turn our meromorphic Riemann–Hilbert problem into a holomorphic Riemann–Hilbert problem following [7]. Choose $\varepsilon$ so small that the discs $|z - \zeta_k| < \varepsilon$ are inside the unit circle and do not intersect. Then redefine $m$ in a neighborhood of $\zeta_k$ respectively $\zeta_k^{-1}$ according to

$$m(z) = \begin{cases} m(z) \begin{pmatrix} 1 & 0 \\ \frac{\zeta_k \gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 1 \end{pmatrix}, & |z - \zeta_k| < \varepsilon, \\ m(z) \begin{pmatrix} 1 & -\frac{z\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k^{-1}} \\ 0 & 1 \end{pmatrix}, & |z^{-1} - \zeta_k| < \varepsilon, \\ m(z), & \text{else}. \end{cases} \tag{2.20}$$

and for $|z| > 1$ by symmetry (2.16). Then a straightforward calculation using $\operatorname{Res}_\zeta m = \lim_{z\to\zeta}(z - \zeta)m(z)$ shows

**Lemma 2.5.** *Suppose $m(z)$ is redefined as in (2.20). Then $m(z)$ is holomorphic away from the unit circle and satisfies (2.14), (2.16), (2.17) and the pole condition is replaced by the jump condition*

$$m_+(z) = m_-(z) \begin{pmatrix} 1 & 0 \\ \frac{\zeta_k \gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 1 \end{pmatrix}, \quad |z-\zeta_k| = \varepsilon,$$
$$m_+(z) = m_-(z) \begin{pmatrix} 1 & -\frac{z\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k^{-1}} \\ 0 & 1 \end{pmatrix}, \quad |z^{-1}-\zeta_k| = \varepsilon, \tag{2.21}$$

*where the small circle around $\zeta_k$ is oriented counterclockwise.*

Finally, we note that the case of just one eigenvalue and zero reflection coefficient can be solved explicitly.

**Lemma 2.6** (One soliton solution)**.** *Suppose there is only one eigenvalue and a vanishing reflection coefficient, that is, $\mathcal{S}_+(H(t)) = \{R(z) \equiv 0,\ |z| = 1;\ (\zeta, \gamma)\}$. Then the solution of the Riemann–Hilbert problem (2.14)–(2.17) is given by*

$$m_0(z) = \begin{pmatrix} f(z) & f(1/z) \end{pmatrix} \tag{2.22}$$
$$f(z) = \frac{1}{\sqrt{1-\zeta^2+\gamma}\sqrt{1-\zeta^2+\zeta^2\gamma}} \left( \gamma\zeta^2 \frac{z-\zeta^{-1}}{z-\zeta} + 1 - \zeta^2 \right).$$

*In particular,*

$$A_+ = \sqrt{\frac{1-\zeta^2+\gamma}{1-\zeta^2+\gamma\zeta^2}}, \qquad B_+ = \frac{\gamma\zeta(\zeta^2-1)}{2((\gamma-1)\zeta^2+1)}. \tag{2.23}$$

## 3. Conjugation and deformation

This section demonstrates how to conjugate our Riemann–Hilbert problem and deform the jump contours, such that the jumps will be exponentially decreasing away from the stationary phase points. In order to do this we will assume that $R(z)$ has an analytic extension to a strip around the unit circle throughout this and the following section. This is for example the case if the decay in (1.2) is exponentially. We will eventually show how to remove this assumption in Section 5.

For easy reference we note the following result which can be checked by a straightforward calculation.

**Lemma 3.1** (Conjugation)**.** *Assume that $\widetilde{\Sigma} \subseteq \Sigma$. Let $D$ be a matrix of the form*

$$D(z) = \begin{pmatrix} d(z)^{-1} & 0 \\ 0 & d(z) \end{pmatrix}, \tag{3.1}$$

*where $d : \mathbb{C}\backslash\widetilde{\Sigma} \to \mathbb{C}$ is a sectionally analytic function. Set*

$$\tilde{m}(z) = m(z) D(z), \tag{3.2}$$

*then the jump matrix transforms according to*

$$\tilde{v}(z) = D_-(z)^{-1} v(z) D_+(z). \tag{3.3}$$

*If $d$ satisfies $d(z^{-1}) = d(z)^{-1}$ and $d(0) > 0$. Then the transformation $\tilde{m}(z) = m(z)D(z)$ respects our symmetry, that is, $\tilde{m}(z)$ satisfies (2.16) if and only if $m(z)$ does.*

In particular, we obtain

$$\tilde{v} = \begin{pmatrix} v_{11} & v_{12} d^2 \\ v_{21} d^{-2} & v_{22} \end{pmatrix}, \qquad z \in \Sigma \backslash \tilde{\Sigma}, \tag{3.4}$$

respectively

$$\tilde{v} = \begin{pmatrix} \frac{d_-}{d_+} v_{11} & v_{12} d_+ d_- \\ v_{21} d_+^{-1} d_-^{-1} & \frac{d_+}{d_-} v_{22} \end{pmatrix}, \qquad z \in \Sigma \cap \tilde{\Sigma}. \tag{3.5}$$

In order to remove the poles there are two cases to distinguish. If $\lambda_k > \frac{1}{2}(\zeta_0 + \zeta_0^{-1})$ the jump is exponentially decaying and there is nothing to do.

Otherwise we use conjugation to turn the jumps into exponentially decaying ones, again following Deift, Kamvissis, Kriecherbauer, and Zhou [7] (see also [15]). It turns out that we will have to handle the poles at $\zeta_k$ and $\zeta_k^{-1}$ in one step in order to preserve symmetry and in order to not add additional poles elsewhere.

**Lemma 3.2.** *Assume that the Riemann–Hilbert problem for $m$ has jump conditions near $\zeta$ and $\zeta^{-1}$ given by*

$$\begin{aligned} m_+(z) &= m_-(z) \begin{pmatrix} 1 & 0 \\ \frac{\gamma\zeta}{z-\zeta} & 1 \end{pmatrix}, \qquad |z-\zeta| = \varepsilon, \\ m_+(z) &= m_-(z) \begin{pmatrix} 1 & -\frac{\gamma z}{z-\zeta^{-1}} \\ 0 & 1 \end{pmatrix}, \quad |z^{-1}-\zeta| = \varepsilon. \end{aligned} \tag{3.6}$$

*Then this Riemann–Hilbert problem is equivalent to a Riemann–Hilbert problem for $\tilde{m}$ which has jump conditions near $\zeta$ and $\zeta^{-1}$ given by*

$$\begin{aligned} \tilde{m}_+(z) &= \tilde{m}_-(z) \begin{pmatrix} 1 & \frac{(\zeta z-1)^2}{\zeta(z-\zeta)\gamma} \\ 0 & 1 \end{pmatrix}, \qquad |z-\zeta| = \varepsilon, \\ \tilde{m}_+(z) &= \tilde{m}_-(z) \begin{pmatrix} 1 & 0 \\ -\frac{(z-\zeta)^2}{\zeta z(\zeta z-1)\gamma} & 1 \end{pmatrix}, \qquad |z^{-1}-\zeta| = \varepsilon, \end{aligned}$$

*and all remaining data conjugated (as in Lemma 3.1) by*

$$D(z) = \begin{pmatrix} \frac{z-\zeta}{\zeta z-1} & 0 \\ 0 & \frac{\zeta z-1}{z-\zeta} \end{pmatrix}. \tag{3.7}$$

*Proof.* To turn $\gamma$ into $\gamma^{-1}$, introduce $D$ by

$$D(z) = \begin{cases} \begin{pmatrix} 1 & \frac{1}{\gamma}\frac{z-\zeta}{\zeta} \\ -\gamma\frac{\zeta}{z-\zeta} & 0 \end{pmatrix} \begin{pmatrix} \frac{z-\zeta}{\zeta z-1} & 0 \\ 0 & \frac{\zeta z-1}{z-\zeta} \end{pmatrix}, & |z-\zeta| < \varepsilon, \\ \begin{pmatrix} 0 & \gamma\frac{z\zeta}{z\zeta-1} \\ -\frac{1}{\gamma}\frac{z\zeta-1}{z\zeta} & 1 \end{pmatrix} \begin{pmatrix} \frac{z-\zeta}{\zeta z-1} & 0 \\ 0 & \frac{\zeta z-1}{z-\zeta} \end{pmatrix}, & |z^{-1}-\zeta| < \varepsilon, \\ \begin{pmatrix} \frac{z-\zeta}{\zeta z-1} & 0 \\ 0 & \frac{\zeta z-1}{z-\zeta} \end{pmatrix}, & \text{else}, \end{cases}$$

and note that $D(z)$ is analytic away from the two circles. Now set $\tilde{m}(z) = m(z)D(z)$. □

The jumps along $\mathbb{T}$ are of oscillatory type and our aim is to apply a contour deformation which will move them into regions where the oscillatory terms will decay exponentially. Since the jump matrix $v$ contains both $\exp(t\Phi)$ and $\exp(-t\Phi)$

we need to separate them in order to be able to move them to different regions of the complex plane. For this we will need the following factorizations of the jump condition (2.14). First of all

$$v(z) = b_-(z)^{-1} b_+(z), \tag{3.8}$$

where

$$b_-(z) = \begin{pmatrix} 1 & \overline{R(z)} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, \qquad b_+(z) = \begin{pmatrix} 1 & 0 \\ R(z) \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}.$$

This will be the proper factorization for $z > z_0$. Here $z > z_0$ has to be understood as $\lambda(z) > \lambda_0$ (or equivalently $\mathrm{Re}(z) > \mathrm{Re}(z_0)$). Similarly, we have

$$v(z) = B_-(z)^{-1} \begin{pmatrix} 1 - |R(z)|^2 & 0 \\ 0 & \frac{1}{1-|R(z)|^2} \end{pmatrix} B_+(z), \tag{3.9}$$

where

$$B_-(z) = \begin{pmatrix} 1 & 0 \\ -\frac{R(z)\mathrm{e}^{t\Phi(z)}}{1-|R(z)|^2} & 1 \end{pmatrix}, \qquad B_+(z) = \begin{pmatrix} 1 & -\frac{\overline{R(z)}\mathrm{e}^{-t\Phi(z)}}{1-|R(z)|^2} \\ 0 & 1 \end{pmatrix}.$$

This will be the right factorization for $z < z_0$.

To get rid of the diagonal part we need to solve the corresponding scalar Riemann–Hilbert problem. Moreover, for $z_0 \in (-1, 0)$ we have $\mathrm{Re}(\Phi(z)) > 0$ for $z \in (\zeta_0, 0)$ and $\mathrm{Re}(\Phi(z)) < 0$ for $z \in (-1, \zeta_0) \cup (0, 1)$, for $z_0 \in \mathbb{T}$ we have $\mathrm{Re}(\Phi(z)) > 0$ for $z \in (-1, 0)$ and $\mathrm{Re}(\Phi(z)) < 0$ for $z \in (0, 1)$, and for $z_0 \in (0, 1)$ we have $\mathrm{Re}(\Phi(z)) > 0$ for $z \in (-1, 0) \cup (\zeta_0, 1)$ and $\mathrm{Re}(\Phi(z)) < 0$ for $z \in (0, \zeta_0)$ (compare Figure 2 and note that by $\mathrm{Re}(\Phi(z^{-1})) = -\mathrm{Re}(\Phi(z))$ the curves $\mathrm{Re}(\Phi(z)) = 0$ are symmetric with respect to $z \mapsto z^{-1}$).

Together with the Blaschke factors needed to conjugate the jumps near the eigenvalues, this is just the partial transmission coefficient $T(z, z_0)$ introduced in (1.7). In fact, it satisfies the following scalar meromorphic Riemann–Hilbert problem:

**Lemma 3.3.** *The partial transmission coefficient $T(z, z_0)$ is meromorphic in $\mathbb{C} \backslash \Sigma(z_0)$, where $\Sigma(z_0)$ is the arc given by $\Sigma(z_0) = \{z \in \mathbb{T} | \mathrm{Re}(z) < \mathrm{Re}(z_0)\}$, with simple poles at $\zeta_j$ and simple zeros at $\zeta_j^{-1}$ for all $j$ with $\frac{1}{2}(\zeta_j + \zeta_j^{-1}) < \lambda_0$, and satisfies the jump condition*

$$T_+(z, z_0) = T_-(z, z_0)(1 - |R(z)|^2), \qquad z \in \Sigma(z_0).$$

*Moreover,*

- (i) $T(z^{-1}, z_0) = T(z, z_0)^{-1}$, $z \in \mathbb{C} \backslash \Sigma(z_0)$, *and* $T(0, z_0) > 0$,
- (ii) $\overline{T(z, z_0)} = T(\overline{z}, z_0)$ *and in particular* $T(z, z_0)$ *is real-valued for* $z \in \mathbb{R}$,
- (iii) $T(z)/T(z, z_0) = O(1)$ *for* $z$ *in the interior of* $\Sigma(z_0)$.

*Proof.* That $\zeta_j$ are simple poles and $\zeta_j^{-1}$ are simple zeros is obvious from the Blaschke factors and that $T(z, z_0)$ has the given jump follows from Plemelj's formulas. (i)–(iii) are straightforward to check. □

Observe that for $\zeta_0 < \zeta_N$ if $\zeta_N \in (0, 1)$ respectively $\zeta_0 < 1$ else we have $T(z) = T(z, z_0)$.

Moreover, note that (i) and (ii) imply

$$|T(z, z_0)|^2 = T(\overline{z}, z_0) T(z, z_0) = T(z^{-1}, z_0) T(z, z_0) = 1, \qquad z \in \mathbb{T} \backslash \Sigma(z_0). \tag{3.10}$$

Now we are ready to perform our conjugation step. Introduce

$$D(z) = \begin{cases} \begin{pmatrix} 1 & \frac{z-\zeta_k}{\zeta_k\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}} \\ -\frac{\zeta_k\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 0 \end{pmatrix} D_0(z), & |z-\zeta_k|<\varepsilon,\ \lambda_k < \frac{1}{2}(\zeta_0+\zeta_0^{-1}), \\ \begin{pmatrix} 0 & \frac{z\zeta_k\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z\zeta_k-1} \\ -\frac{z\zeta_k-1}{z\zeta_k\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}} & 1 \end{pmatrix} D_0(z), & |z^{-1}-\zeta_k|<\varepsilon,\ \lambda_k < \frac{1}{2}(\zeta_0+\zeta_0^{-1}), \\ D_0(z), & \text{else}, \end{cases}$$

where

$$D_0(z) = \begin{pmatrix} T(z,z_0)^{-1} & 0 \\ 0 & T(z,z_0) \end{pmatrix}.$$

Note that we have

$$D(z^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we conjugate our problem using $D(z)$ and observe that, since $T(z,z_0)$ has the same behaviour as $T(z)$ for $z=\pm 1 > z_0$, the new vector

$$\tilde{m}(z) = m(z)D(z) \tag{3.11}$$

is again continuous near $z=\pm 1$ (even if $T(z)$ vanishes there).

Then using Lemma 3.1 and Lemma 3.2 the jump corresponding $\lambda_k < \frac{1}{2}(\zeta_0+\zeta_0^{-1})$ (if any) is given by

$$\begin{aligned} \tilde{v}(z) &= \begin{pmatrix} 1 & \frac{z-\zeta_k}{\zeta_k\gamma_k T(z,z_0)^{-2}\mathrm{e}^{t\Phi(\zeta_k)}} \\ 0 & 1 \end{pmatrix}, \qquad |z-\zeta_k|=\varepsilon, \\ \tilde{v}(z) &= \begin{pmatrix} 1 & 0 \\ -\frac{\zeta_k z-1}{\zeta_k z\gamma_k T(z,z_0)^{2}\mathrm{e}^{t\Phi(\zeta_k)}} & 1 \end{pmatrix}, \qquad |z^{-1}-\zeta_k|=\varepsilon, \end{aligned} \tag{3.12}$$

and corresponding $\lambda_k > \frac{1}{2}(\zeta_0+\zeta_0^{-1})$ (if any) by

$$\begin{aligned} \tilde{v}(z) &= \begin{pmatrix} 1 & 0 \\ \frac{\zeta_k\gamma_k T(z,z_0)^{-2}\mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 1 \end{pmatrix}, \qquad |z-\zeta_k|=\varepsilon, \\ \tilde{v}(z) &= \begin{pmatrix} 1 & -\frac{z\gamma_k T(z,z_0)^{2}\mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k^{-1}} \\ 0 & 1 \end{pmatrix}, \qquad |z^{-1}-\zeta_k|=\varepsilon. \end{aligned} \tag{3.13}$$

In particular, all jumps corresponding to poles, except for possibly one if $\zeta_k=\zeta_0$, are exponentially decreasing. In this case we will keep the pole condition which now reads

$$\begin{aligned} \operatorname{Res}_{\zeta_k} \tilde{m}(z) &= \lim_{z\to\zeta_k} \tilde{m}(z) \begin{pmatrix} 0 & 0 \\ -\zeta_k\gamma_k T(\zeta_k,z_0)^{-2}\mathrm{e}^{t\Phi(\zeta_k)} & 0 \end{pmatrix}, \\ \operatorname{Res}_{\zeta_k^{-1}} \tilde{m}(z) &= \lim_{z\to\zeta_k^{-1}} \tilde{m}(z) \begin{pmatrix} 0 & \zeta_k^{-1}\gamma_k T(\zeta_k,z_0)^{-2}\mathrm{e}^{t\Phi(\zeta_k)} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.14}$$

Furthermore, the jump along $\mathbb{T}$ is given by

$$\tilde{v}(z) = \begin{cases} \tilde{b}_-(z)^{-1}\tilde{b}_+(z), & \lambda(z)>\lambda_0, \\ \tilde{B}_-(z)^{-1}\tilde{B}_+(z), & \lambda(z)<\lambda_0, \end{cases} \tag{3.15}$$

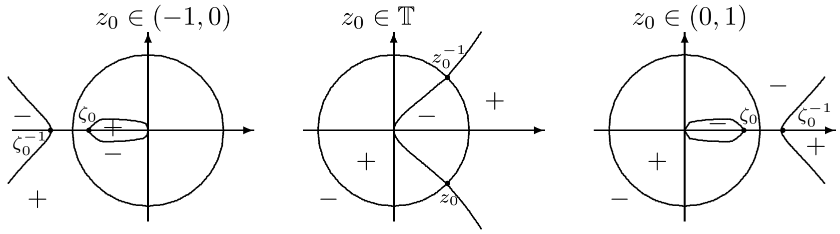


FIGURE 2. Sign of $\mathrm{Re}(\Phi(z))$ for different values of $z_0$

where

$$\tilde{b}_-(z) = \begin{pmatrix} 1 & \frac{R(z^{-1})\mathrm{e}^{-t\Phi(z)}}{T(z^{-1},z_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(z) = \begin{pmatrix} 1 & 0 \\ \frac{R(z)\mathrm{e}^{t\Phi(z)}}{T(z,z_0)^2} & 1 \end{pmatrix}, \tag{3.16}$$

and

$$\tilde{B}_-(z) = \begin{pmatrix} 1 & 0 \\ -\frac{T_-(z,z_0)^{-2}}{1-R(z)R(z^{-1})} R(z)\mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix},$$
$$\tilde{B}_+(z) = \begin{pmatrix} 1 & -\frac{T_+(z,z_0)^2}{1-R(z)R(z^{-1})} R(z^{-1})\mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}. \tag{3.17}$$

Here we have used $T_\pm(z^{-1},z_0) = T_\pm(\overline{z},z_0) = \overline{T_\pm(z,z_0)}$ and $R(z^{-1}) = R(\overline{z}) = \overline{R(z)}$ for $z \in \mathbb{T}$ to show that there exists an analytic continuation into a neighborhood of the unit circle. Moreover, using

$$T_\pm(z,z_0) = T_\mp(z^{-1},z_0)^{-1}, \qquad z \in \Sigma(z_0).$$

we can write

$$\frac{T_-(z,z_0)^{-2}}{1-R(z)R(z^{-1})} = \frac{\overline{T_-(z,z_0)}}{T_-(z,z_0)}, \quad \frac{T_+(z,z_0)^2}{1-R(z)R(z^{-1})} = \frac{T_+(z,z_0)}{\overline{T_+(z,z_0)}} \tag{3.18}$$

for $z \in \mathbb{T}$, which shows that the matrix entries are in fact bounded.

Now we deform the jump along $\mathbb{T}$ to move the oscillatory terms into regions where they are decaying. There are three cases to distinguish (see Figure 2):

**Case 1:** $z_0 \in (-1,0)$**.** In this case we will set $\Sigma_\pm = \{z \,|\, |z| = (1-\varepsilon)^{\pm 1}\}$ for some small $\varepsilon \in (0,1)$ such that $\Sigma_\pm$ lies in the region with $\pm\mathrm{Re}(\Phi(z)) < 0$ and such that we do not intersect any other contours. Then we can split our jump by redefining $\tilde{m}(z)$ according to

$$\hat{m}(z) = \begin{cases} \tilde{m}(z)\tilde{b}_+(z)^{-1}, & (1-\varepsilon) < |z| < 1, \\ \tilde{m}(z)\tilde{b}_-(z)^{-1}, & 1 < |z| < (1-\varepsilon)^{-1}, \\ \tilde{m}(z), & \text{else}. \end{cases} \tag{3.19}$$

It is straightforward to check that the jump along $\mathbb{T}$ disappears and the jump along $\Sigma_\pm$ is given by

$$\hat{v}(z) = \begin{cases} \tilde{b}_+(z), & z \in \Sigma_+, \\ \tilde{b}_-(z)^{-1}, & z \in \Sigma_-. \end{cases} \tag{3.20}$$

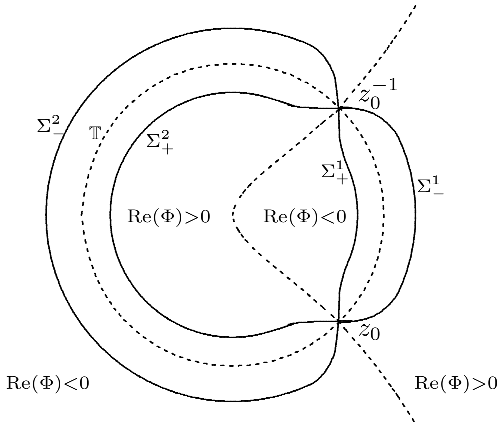


Figure 3. Deformed contour

All other jumps are unchanged. Note that the resulting Riemann–Hilbert problem still satisfies our symmetry condition (2.16) since we have

$$\tilde{b}_\pm(z^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{b}_\mp(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By construction the jump along $\Sigma_\pm$ is exponentially decreasing as $t \to \infty$.

**Case 2:** $z_0 \in \mathbb{T} \setminus \{\pm 1\}$**.** In this case we will set $\Sigma_\pm = \Sigma_\pm^1 \cup \Sigma_\pm^2$ as indicated in Figure 3. Again note that $\Sigma_\pm^1$ respectively $\Sigma_\mp^2$ lies in the region with $\pm\mathrm{Re}(\Phi(z)) < 0$. Then we can split our jump by redefining $\tilde{m}(z)$ according to

$$\hat{m}(z) = \begin{cases} \tilde{m}(z)\tilde{b}_+(z)^{-1}, & z \text{ between } \mathbb{T} \text{ and } \Sigma_+^1, \\ \tilde{m}(z)\tilde{b}_-(z)^{-1}, & z \text{ between } \mathbb{T} \text{ and } \Sigma_-^1, \\ \tilde{m}(z)\tilde{B}_+(z)^{-1}, & z \text{ between } \mathbb{T} \text{ and } \Sigma_+^2, \\ \tilde{m}(z)\tilde{B}_-(z)^{-1}, & z \text{ between } \mathbb{T} \text{ and } \Sigma_-^2, \\ \tilde{m}(z), & \text{else}. \end{cases} \tag{3.21}$$

One checks that the jump along $\mathbb{T}$ disappears and the jump along $\Sigma_\pm$ is given by

$$\hat{v}(z) = \begin{cases} \tilde{b}_+(z), & z \in \Sigma_+^1, \\ \tilde{b}_-(z)^{-1}, & z \in \Sigma_-^1, \\ \tilde{B}_+(z), & z \in \Sigma_+^2, \\ \tilde{B}_-(z)^{-1}, & z \in \Sigma_-^2. \end{cases} \tag{3.22}$$

All other jumps are unchanged. Again the resulting Riemann–Hilbert problem still satisfies our symmetry condition (2.16) and the jump along $\Sigma_\pm \setminus \{z_0, z_0^{-1}\}$ is exponentially decreasing as $t \to \infty$.

**Case 3:** $z_0 \in (0,1)$**.** In this case we will set $\Sigma_\pm = \{z \,|\, |z| = (1-\varepsilon)^{\pm 1}\}$ for some small $\varepsilon \in (0,1)$ such that $\Sigma_\pm$ lies in the region with $\mp\mathrm{Re}(\Phi(z)) < 0$ and such that we do not intersect any other contours. Then we can split our jump by redefining

$\tilde{m}(z)$ according to

$$\hat{m}(z) = \begin{cases} \tilde{m}(z)\tilde{B}_+(z)^{-1}, & (1-\varepsilon) < |z| < 1, \\ \tilde{m}(z)\tilde{B}_-(z)^{-1}, & 1 < |z| < (1-\varepsilon)^{-1}, \\ \tilde{m}(z), & \text{else}. \end{cases} \tag{3.23}$$

One checks that the jump along $\mathbb{T}$ disappears and the jump along $\Sigma_\pm$ is given by

$$\hat{v}(z) = \begin{cases} \tilde{B}_+(z), & z \in \Sigma_+, \\ \tilde{B}_-(z)^{-1}, & z \in \Sigma_-. \end{cases} \tag{3.24}$$

All other jumps are unchanged. Again the resulting Riemann–Hilbert problem still satisfies our symmetry condition (2.16) and the jump along $\Sigma_\pm$ is exponentially decreasing as $t \to \infty$.

In Case 1 and 3 we can immediately apply Theorem B.6 as follows: If $|\frac{n}{t} - c_k| > \varepsilon$ for all $k$ we can choose $\gamma_0 = 0$ and $w_0^t \equiv 0$. Since the error between $w^t$ and $w_0^t$ is exponentially small, this proves the second part of Theorem 1.1 in the analytic case. The changes necessary for the general case will be given in Section 5.

Otherwise, if $|\frac{n}{t} - c_k| < \varepsilon$ for some $k$, we choose $\gamma_0^t = \gamma_k(n,t)$ and $w_0^t \equiv 0$. Again we conclude that the error between $w^t$ and $w_0^t$ is exponentially small, proving the first part of Theorem 1.1. The changes necessary for the general case will also be given in Section 5.

In Case 2 the jump will not decay on the two small crosses containing the stationary phase points $z_0$ and $z_0^{-1}$. Hence we will need to continue the investigation of this problem in the next section.

## 4. Reduction to a Riemann–Hilbert problem on a small cross

In the previous section we have shown that for $z_0 \in \mathbb{T}\backslash\{\pm 1\}$ we can reduce everything to a Riemann–Hilbert problem for $\hat{m}(z)$ such that the jumps are of order $O(t^{-1})$ except in a small neighborhoods of the stationary phase points $z_0$ and $z_0^{-1}$. Denote by $\Sigma^C(z_0^{\pm 1})$ the parts of $\Sigma_+ \cup \Sigma_-$ inside a small neighborhood of $z_0^{\pm 1}$. In this section we will show that everything can reduced to solving the two problems in the two small crosses $\Sigma^C(z_0)$ respectively $\Sigma^C(z_0^{-1})$.

It will be slightly more convenient to use the alternate normalization

$$\check{m}(z) = \frac{1}{\tilde{A}}\hat{m}(z), \qquad A = T_0\tilde{A}, \tag{4.1}$$

such that

$$\check{m}(0) = \begin{pmatrix} 1 & \frac{1}{\tilde{A}^2} \end{pmatrix}. \tag{4.2}$$

Without loss of generality we can also assume that $\hat{\Sigma}$ consists of two straight lines in a sufficiently small neighborhood of $z_0$.

We will need the solution of the corresponding $2 \times 2$ matrix

$$\begin{aligned} M_+^C(z) &= M_-^C(z)\tilde{v}(z), \qquad z \in \Sigma^C, \\ M^C(\infty) &= \mathbb{I}, \end{aligned} \tag{4.3}$$

where the jump $\tilde{v}$ is the same as for $\tilde{m}(z)$ but restricted to a neighborhood of one of the two crosses $\Sigma^C = (\Sigma_+ \cup \Sigma_-) \cap \{z |\, |z - z_0| < \varepsilon/2\}$ for some small $\varepsilon > 0$.

As a first step we make a change of coordinates

$$\zeta = \frac{\sqrt{-2\sin(\theta_0)}}{z_0 \mathrm{i}}(z - z_0), \qquad z = z_0 + \frac{z_0 \mathrm{i}}{\sqrt{-2\sin(\theta_0)}}\zeta \tag{4.4}$$

such that the phase reads $\Phi(z) = \mathrm{i}\Phi_0 + \frac{\mathrm{i}}{2}\zeta^2 + O(\zeta^3)$. Here we have set

$$z_0 = \mathrm{e}^{\mathrm{i}\theta_0}, \qquad \theta_0 \in (-\pi, 0),$$

respectively $\cos(\theta_0) = -n/t$, which implies

$$\Phi_0 = 2(\sin(\theta_0) - \theta_0\cos(\theta_0)), \qquad \Phi''(z_0) = 2\mathrm{i}\mathrm{e}^{-2\mathrm{i}\theta_0}\sin(\theta_0).$$

The corresponding Riemann–Hilbert problem will be solved in Section A. To apply this result we need the behaviour of our jump matrices near $z_0$, that is, the behaviour of $T(z, z_0)$ near $z \to z_0$.

**Lemma 4.1.** *Let $z_0 \in \mathbb{T}$, then*

$$T(z, z_0) = \left(-\overline{z_0}\frac{z - z_0}{z - \overline{z_0}}\right)^{\mathrm{i}\nu} \tilde{T}(z, z_0) \tag{4.5}$$

*where $\nu = -\frac{1}{\pi}\log(|T(z_0)|)$ and the branch cut of the logarithm used to define $z^{\mathrm{i}\nu} = \mathrm{e}^{\mathrm{i}\nu\log(z)}$ is chosen along the negative real axis. Here*

$$\tilde{T}(z, z_0) = \prod_{\zeta_k \in (-1,0)} |\zeta_k| \frac{z - \zeta_k^{-1}}{z - \zeta_k} \cdot \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\overline{z_0}}^{z_0} \log\left(\frac{|T(s)|}{|T(z_0)|}\right) \frac{s+z}{s-z}\frac{ds}{s}\right),$$

*is Hölder continuous of any exponent less then 1 at $z = z_0$ and satisfies $\tilde{T}(z_0, z_0) \in \mathbb{T}$.*

*Proof.* This follows since

$$\exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\overline{z_0}}^{z_0} \log\left(|T(z_0)|\right) \frac{s+z}{s-z}\frac{ds}{s}\right) = \left(-\overline{z_0}\frac{z - z_0}{z - \overline{z_0}}\right)^{\mathrm{i}\nu}.$$

The property $\tilde{T}(z_0, z_0) \in \mathbb{T}$ follows after letting $z \to z_0$ in (3.10). □

Now if $z(\zeta)$ is defined as in (4.4) and $0 < \alpha < 1$, then there is an $L > 0$ such that

$$|T(z(\zeta), z_0) - \zeta^{\mathrm{i}\nu}\tilde{T}(z_0, z_0)\mathrm{e}^{-\frac{3}{2}\mathrm{i}\nu\log(-2\sin(\theta_0))}| \le L|\zeta|^\alpha,$$

where the branch cut of $\zeta^{\mathrm{i}\nu}$ is tangent to the negative real axis. Clearly we also have

$$|R(z(\zeta)) - R(z_0)| \le L|\zeta|^\alpha$$

and thus the assumptions of Theorem A.1 are satisfied with

$$r = R(z_0)\tilde{T}(z_0, z_0)^{-2}\mathrm{e}^{3\mathrm{i}\nu\log(-2\sin(\theta_0))}$$

and the solution of (4.3) is given by

$$M^C(z) = \mathbb{I} - \frac{z_0}{(-2\sin(\theta_0)t)^{1/2}}\frac{M_0}{z - z_0} + O\left(\frac{1}{t^\alpha}\right), \tag{4.6}$$

$$M_0 = \begin{pmatrix} 0 & -\overline{\beta} \\ \beta & 0 \end{pmatrix},$$

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(R(z_0)) + \arg(\Gamma(\mathrm{i}\nu)))}(-2\sin(\theta_0))^{-3\mathrm{i}\nu}\tilde{T}(z_0, z_0)^2\mathrm{e}^{-\mathrm{i}t\Phi_0}t^{-\mathrm{i}\nu}, \tag{4.7}$$

where $1/2 < \alpha < 1$, and $\cos(\theta_0) = -\lambda_0$.

Now we are ready to show

**Theorem 4.2.** *The solution $\check{m}(z)$ is given by*

$$\check{m}(z) = \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{(-2\sin(\theta_0)t)^{1/2}}(m_0(z) + \bar{m}_0(z)) + O\left(\frac{1}{t^\alpha}\right), \tag{4.8}$$

*where*

$$m_0(z) = \begin{pmatrix} \overline{\beta}\frac{z}{z-z_0} & -\beta\frac{z_0}{z-z_0} \end{pmatrix}, \qquad \bar{m}_0(z) = \overline{m_0(\overline{z})} = m_0(z^{-1})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.9}$$

*Proof.* Introduce $m(z)$ by

$$m(z) = \begin{cases} \check{m}(z)M^C(z)^{-1}, & |z - z_0| \le \varepsilon, \\ \check{m}(z)\tilde{M}^C(z)^{-1}, & |z^{-1} - z_0| \le \varepsilon, \\ \check{m}(z), & \text{else}, \end{cases}$$

where

$$\tilde{M}^C(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^C(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{I} - \frac{z}{(-2\sin(\theta_0)t)^{1/2}}\frac{\overline{M_0}}{z - z_0} + O\left(\frac{1}{t^\alpha}\right).$$

The Riemann–Hilbert problem for $m$ has jumps given by

$$v(z) = \begin{cases} M^C(z)^{-1}, & |z - z_0| = \varepsilon, \\ M^C(z)\hat{v}(z)M^C(z)^{-1}, & z \in \hat{\Sigma}, \frac{\varepsilon}{2} < |z - z_0| < \varepsilon, \\ \mathbb{I}, & z \in \Sigma, |z - z_0| < \frac{\varepsilon}{2}, \\ \tilde{M}^C(z)^{-1}, & |z^{-1} - z_0| = \varepsilon, \\ \tilde{M}^C(z)\hat{v}(z)\tilde{M}^C(z)^{-1}, & z \in \hat{\Sigma}, \frac{\varepsilon}{2} < |z^{-1} - z_0| < \varepsilon, \\ \mathbb{I}, & z \in \Sigma, |z^{-1} - z_0| < \frac{\varepsilon}{2}, \\ \hat{v}(z), & \text{else}. \end{cases}$$

The jumps are $\mathbb{I} + O(t^{-1/2})$ on the loops $|z - z_0| = \varepsilon$, $|z^{-1} - z_0| = \varepsilon$ and even $\mathbb{I} + O(t^{-\alpha})$ on the rest (in the $L^\infty$ norm, so also in the $L^\infty$ one). In particular, as in Lemma A.3 we infer

$$\|\mu - \begin{pmatrix} 1 & 1 \end{pmatrix}\|_2 = O(t^{-1/2}).$$

Thus we have with $\Omega_\infty$ as in (B.8)

$$\begin{aligned} m(z) &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mu(s)w(s)\Omega_\infty(s,z) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_{|s-z_0|=\varepsilon} \mu(s)(M^C(s)^{-1} - \mathbb{I})\Omega_\infty(s,z) \\ &\quad + \frac{1}{2\pi\mathrm{i}}\int_{|s^{-1}-z_0|=\varepsilon} \mu(s)(\tilde{M}^C(s)^{-1} - \mathbb{I})\Omega_\infty(s,z) + O(t^{-\alpha}) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{(-2\sin(\theta_0)t)^{1/2}}\begin{pmatrix} 1 & 1 \end{pmatrix} M_0 \frac{1}{2\pi\mathrm{i}}\int_{|s-z_0|=\varepsilon} \frac{z_0}{s - z_0}\Omega_\infty(s,z) \\ &\quad + \frac{1}{(-2\sin(\theta_0)t)^{1/2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \overline{M_0} \frac{1}{2\pi\mathrm{i}}\int_{|s^{-1}-z_0|=\varepsilon} \frac{s}{s - \overline{z_0}}\Omega_\infty(s,z) + O(t^{-\alpha}) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{(-2\sin(\theta_0)t)^{1/2}}(m_0(z) + \bar{m}_0(z)) + O\left(\frac{1}{t^\alpha}\right) \end{aligned}$$

and hence the claim is proven. □

Hence, using (2.19) and (4.1),

$$(\check{m}(z))_2 = \frac{1}{\tilde{A}^2}\left(1 + (T_1 + 2B)z + O(z^2)\right) \tag{4.10}$$

and comparing with

$$(\check{m}(z))_2 = \left(1 - \frac{2\mathrm{Re}(\beta)}{(-2\sin(\theta_0)t)^{1/2}}\right) - \left(\frac{2\mathrm{Re}(\overline{z_0}\beta)}{(-2\sin(\theta_0)t)^{1/2}}\right) z + O(z^2) + O\left(\frac{1}{t^\alpha}\right), \tag{4.11}$$

we obtain

$$\tilde{A}^2 = 1 + \frac{2\mathrm{Re}(\beta)}{(-2\sin(\theta_0)t)^{1/2}} + O\left(\frac{1}{t^\alpha}\right) \tag{4.12}$$

and

$$T_1 + 2B = -\frac{2\mathrm{Re}(\overline{z_0}\beta)}{(-2\sin(\theta_0)t)^{1/2}} + O\left(\frac{1}{t^\alpha}\right). \tag{4.13}$$

In summary we have

$$A = T_0\left(1 + \frac{\mathrm{Re}(\beta)}{(-2\sin(\theta_0)t)^{1/2}} + O\left(\frac{1}{t^\alpha}\right)\right), \tag{4.14}$$

$$B = -\frac{1}{2}T_1 - \frac{\mathrm{Re}(\overline{z_0}\beta)}{(-2\sin(\theta_0)t)^{1/2}} + O\left(\frac{1}{t^\alpha}\right), \tag{4.15}$$

which proves Theorem 1.2 in the analytic case.

**Remark 4.3.** *Note that, in contradistinction to Theorem B.6, Theorem 4.2 does not require uniform boundedness of the associated integral operators, but only some knowledge of the solution of the Riemann-Hilbert problem. However, it requires that the solution is of the form $\mathbb{I} + o(1)$ and hence cannot be used in the soliton region.*

## 5. Analytic Approximation

In this section we want to present the necessary changes in the case where the reflection coefficient does not have an analytic extension. The idea is to use an analytic approximation and split the reflection in an analytic part plus a small rest. The analytic part will be moved to the complex plane while the rest remains on the unit circle. This needs to be done in such a way that the rest is of $O(t^{-l})$ and the growth of the analytic part can be controlled by the decay of the phase.

In the soliton region a straightforward splitting based on the Fourier series

$$R(z) = \sum_{k=-\infty}^{\infty} \hat{R}(k) z^k. \tag{5.1}$$

will be sufficient. Recall that if $R \in C^l(\mathbb{T})$, then $\lim_{k\to\pm\infty} k^l \hat{R}(k) = 0$.

**Lemma 5.1.** *Suppose $R \in C^{l+1}(\mathbb{T})$ and let $0 < \varepsilon < 1$, $\beta > 0$ be given. Then we can split the reflection coefficient according to $R(z) = R_{a,t}(z) + R_{r,t}(z)$ such that $R_{a,t}(z)$ is analytic in $0 < |z| < 1$ and*

$$|R_{a,t}(z)\mathrm{e}^{-\beta t}| = O(t^{-l}), \quad 1-\varepsilon \le |z| \le 1, \qquad |R_{r,t}(z)| = O(t^{-l}), \quad |z| = 1. \tag{5.2}$$

*Proof.* We choose $R_{a,t}(z) = \sum_{k=-K(t)}^{\infty} \hat{R}(k) z^k$ with $K(t) = \lfloor \frac{\beta_0}{-\log(1-\varepsilon)} t \rfloor$ for some positive $\beta_0 < \beta$. Then, for $1-\varepsilon \le |z|$,

$$|R_{a,t}(z)\mathrm{e}^{-\beta t}| \le \sum_{k=-K(t)}^{\infty} |\hat{R}(k)| \mathrm{e}^{-\beta t}(1-\varepsilon)^k \le \|\hat{R}\|_1 \mathrm{e}^{-\beta t}(1-\varepsilon)^{-K(t)} \le \|\hat{R}\|_1 \mathrm{e}^{-(\beta-\beta_0)t}$$

Similarly, for $|z|=1$,

$$|R_{r,t}(z)| \le \sum_{k=-\infty}^{-K(t)-1} |\hat{R}(k)| \le const \sum_{k=-\infty}^{-K(t)+1} \frac{1}{k^{l+1}} \le \frac{const}{K(t)^l} \le \frac{const}{t^l}.$$

□

To apply this lemma in the soliton region $z_0 \in (-1,0)$ we choose

$$\beta = \min_{|z|=1-\varepsilon} \mathrm{Re}(\Phi(z)) > 0. \tag{5.3}$$

and split $R(z) = R_{a,t}(z) + R_{r,t}(z)$ according to Lemma 5.1 to obtain

$$\tilde{b}_\pm(z) = \tilde{b}_{a,t,\pm}(z)\tilde{b}_{r,t,\pm}(z) = \tilde{b}_{r,t,\pm}(z)\tilde{b}_{a,t,\pm}(z).$$

Here $\tilde{b}_{a,t,\pm}(z)$, $\tilde{b}_{r,t,\pm}(z)$ denote the matrices obtained from $\tilde{b}_\pm(z)$ as defined in (3.16) by replacing $R(z)$ with $R_{a,t}(z)$, $R_{r,t}(z)$, respectively. Now we can move the analytic parts into the complex plane as in Section 3 while leaving the rest on $\mathbb{T}$. Hence, rather then (3.20), the jump now reads

$$\hat{v}(z) = \begin{cases} \tilde{b}_{a,t,+}(z), & z \in \Sigma_+, \\ \tilde{b}_{a,t,-}(z)^{-1}, & z \in \Sigma_-, \\ \tilde{b}_{r,t,-}(z)^{-1}\tilde{b}_{r,t,+}(z), & z \in \mathbb{T}. \end{cases} \tag{5.4}$$

By construction we have $\hat{v}(z) = \mathbb{I} + O(t^{-l})$ on the whole contour and the rest follows as in Section 3.

In the other soliton region $z_0 \in (0,1)$ we proceed similarly, with the only difference that the jump matrices $\tilde{B}_\pm(z)$ have at first sight more complicated off diagonal entries. To remedy this we will rewrite them in terms of left rather then right scattering data. For this purpose let us use the notation $R_r(z) \equiv R_+(z)$ for the right and $R_l(z) \equiv R_-(z)$ for the left reflection coefficient. Moreover, let $T_r(z,z_0) \equiv T(z,z_0)$ be the right and $T_l(z,z_0) \equiv T(z)/T(z,z_0)$ be the left partial transmission coefficient.

With this notation we have

$$\tilde{v}(z) = \begin{cases} \tilde{b}_-(z)^{-1}\tilde{b}_+(z), & \lambda(z) > \lambda_0, \\ \tilde{B}_-(z)^{-1}\tilde{B}_+(z), & \lambda(z) < \lambda_0, \end{cases} \tag{5.5}$$

where

$$\tilde{b}_-(z) = \begin{pmatrix} 1 & \frac{R_r(z^{-1})\mathrm{e}^{-t\Phi(z)}}{T_r(z^{-1},z_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(z) = \begin{pmatrix} 1 & 0 \\ \frac{R_r(z)\mathrm{e}^{t\Phi(z)}}{T_r(z,z_0)^2} & 1 \end{pmatrix},$$

and

$$\tilde{B}_-(z) = \begin{pmatrix} 1 & 0 \\ -\frac{T_{r,-}(z,z_0)^{-2}}{|T(z)|^2} R_r(z) \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix},$$
$$\tilde{B}_+(z) = \begin{pmatrix} 1 & -\frac{T_{r,+}(z,z_0)^{2}}{|T(z)|^2} R_r(z^{-1}) \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}.$$

Using (2.9) together with (3.18) we can further write

$$\tilde{B}_-(z) = \begin{pmatrix} 1 & 0 \\ \frac{R_l(z^{-1}) \mathrm{e}^{-t\Phi(z)}}{T_l(z^{-1},z_0)^2} & 1 \end{pmatrix},$$
$$\tilde{B}_+(z) = \begin{pmatrix} 1 & \frac{R_l(z) \mathrm{e}^{t\Phi(z)}}{T_l(z,z_0)^2} \\ 0 & 1 \end{pmatrix}.$$

Now we can proceed as before with $\tilde{B}_\pm(z)$ as with $\tilde{b}_\pm(z)$ by splitting $R_l(z)$ rather than $R_r(z)$.

In the similarity region we need to take the small vicinities of the stationary phase points into account. Since the phase is quadratic near these points, we cannot use it to dominate the exponential growth of the analytic part away from the unit circle. Hence we will take the phase as a new variable and use the Fourier transform with respect to this new variable. Since this change of coordinates is singular near the stationary phase points, there is a price we have to pay, namely, requiring additional smoothness for $R(z)$. We begin with

**Lemma 5.2.** *Suppose $R(z) \in C^5(\mathbb{T})$. Then we can split $R(z)$ according to*

$$R(z) = R_0(z) + (z - z_0)(z - \overline{z_0}) H(z), \qquad z \in \Sigma(z_0), \tag{5.6}$$

*where $R_0(z)$ is a real polynomial in $z$ such that $H(z)$ vanishes at $z_0, \overline{z_0}$ of order three and has a Fourier series*

$$H(z) = \sum_{k=-\infty}^{\infty} \hat{H}_k \mathrm{e}^{k\omega_0 \Phi(z)}, \qquad \omega_0 = \frac{\pi}{\pi \cos(\theta_0) + \Phi_0} \tag{5.7}$$

*with $k\,\hat{H}_k$ summable. Here $\Phi_0 = \Phi(z_0)/\mathrm{i}$.*

*Proof.* By choosing a polynomial $R_0$ we can match the values of $R$ and its first four derivatives at $z_0, \overline{z_0}$. Hence $H(z) \in C^4(\mathbb{T})$ and vanishes together with its first three derivatives at $z_0, \overline{z_0}$.

When restricted to $\Sigma(z_0)$ the phase $\Phi(z)/\mathrm{i}$ gives a one to one coordinate transform $\Sigma(z_0) \to [\mathrm{i}\Phi_0, \mathrm{i}\Phi_0 + \mathrm{i}\omega_0]$ and we can hence express $H(z)$ in this new coordinate. The coordinate transform locally looks like a square root near $z_0$ and $\overline{z_0}$, however, due to our assumption that $H$ vanishes there, $H$ is still $C^2$ in this new coordinate and the Fourier transform with respect to this new coordinates exists and has the required properties. $\square$

Moreover, as in Lemma 5.1 we obtain:

**Lemma 5.3.** *Let $H(z)$ be as in the previous lemma. Then we can split $H(z)$ according to $H(z) = H_{a,t}(z) + H_{r,t}(z)$ such that $H_{a,t}(z)$ is analytic in the region* $\mathrm{Re}(\Phi(z)) < 0$ *and*

$$|H_{a,t}(z) \mathrm{e}^{\Phi(z)t/2}| = O(1), \mathrm{Re}(\Phi(z)) < 0, |z| \le 1, \quad |H_{r,t}(z)| = O(t^{-1}), |z| = 1. \tag{5.8}$$

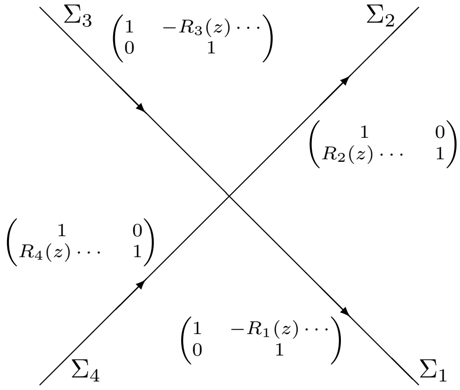


Figure 4. Contours of a cross

*Proof.* We choose $H_{a,t}(z) = \sum_{k=-K(t)}^{\infty} \hat{H}_k \mathrm{e}^{k\omega\Phi(z)}$ with $K(t) = \lfloor t/(2\omega) \rfloor$. The rest follows as in Lemma 5.1. $\square$

By construction $R_{a,t}(z) = R_0(z) + (z - z_0)(z - \overline{z_0}) H_{a,t}(z)$ will satisfy the required Lipschitz estimate in a vicinity of the stationary phase points (uniformly in $t$) and all jumps will be $\mathbb{I} + O(t^{-1})$. Hence we can proceed as in Section 4.

## Appendix A. The solution on a small cross

Introduce the cross $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_4$ (see Figure 4) by

$$\begin{aligned} &\Sigma_1 = \{u\mathrm{e}^{-\mathrm{i}\pi/4},\ u \in [0,\infty)\} && \Sigma_2 = \{u\mathrm{e}^{\mathrm{i}\pi/4},\ u \in [0,\infty)\} \\ &\Sigma_3 = \{u\mathrm{e}^{3\mathrm{i}\pi/4},\ u \in [0,\infty)\} && \Sigma_4 = \{u\mathrm{e}^{-3\mathrm{i}\pi/4},\ u \in [0,\infty)\}. \end{aligned} \tag{A.1}$$

Orient $\Sigma$ such that the real part of $z$ increases in the positive direction. Denote by $\mathbb{D} = \{z,\ |z| < 1\}$ the open unit disc. Throughout this section $z^{\mathrm{i}\nu}$ will denote the function $\mathrm{e}^{\mathrm{i}\nu\log(z)}$, where the branch cut of the logarithm is chosen along the negative real axis $(-\infty, 0)$.

Introduce the following jump matrices ($v_j$ for $z \in \Sigma_j$)

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 & -R_1(z) z^{2\mathrm{i}\nu} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, & v_2 &= \begin{pmatrix} 1 & 0 \\ R_2(z) z^{-2\mathrm{i}\nu} \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}, \\ v_3 &= \begin{pmatrix} 1 & -R_3(z) z^{2\mathrm{i}\nu} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, & v_4 &= \begin{pmatrix} 1 & 0 \\ R_4(z) z^{-2\mathrm{i}\nu} \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}. \end{aligned} \tag{A.2}$$

Now consider the RHP given by

$$\begin{aligned} m_+(z) &= m_-(z) v_j(z), && z \in \Sigma_j, \quad j = 1,2,3,4, \\ m(z) &\to \mathbb{I}, && z \to \infty. \end{aligned} \tag{A.3}$$

We have the next theorem, in which we follow the computations of Sections 3 and 4 in [3]. The method can be found in earlier literature, see for example [10]. One can also find arguments like this in Section 5 in [11] or (3.65) to (3.76) in [5].

We will allow some variation, in all parameters as indicated in the next result.

**Theorem A.1.** *There is some $\rho_0 > 0$ such that $v_j(z) = \mathbb{I}$ for $|z| > \rho_0$. Moreover, suppose that within $|z| \le \rho_0$ the following estimates hold:*

(i) *The phase satisfies* $\Phi(0) = \mathrm{i}\Phi_0 \in \mathrm{i}\mathbb{R}$, $\Phi'(0) = 0$, $\Phi''(0) = \mathrm{i}$ *and*

$$\pm\mathrm{Re}\big(\Phi(z) - \Phi(0)\big) \ge \frac{1}{4}|z|^2, \qquad \begin{cases} + & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\ - & \text{else}, \end{cases} \tag{A.4}$$

$$|\Phi(z) - \Phi(0) - \frac{\mathrm{i}z^2}{2}| \le C|z|^3. \tag{A.5}$$

(ii) *There is some* $r \in \mathbb{D}$ *and constants* $(\alpha, L) \in (0,1] \times (0,\infty)$ *such that* $R_j$, $j = 1, \dots, 4$, *satisfy Hölder conditions of the form*

$$\begin{aligned} &|R_1(z) - \overline{r}| \le L|z|^\alpha, && |R_2(z) - r| \le L|z|^\alpha, \\ &|R_3(z) - \frac{\overline{r}}{1-|r|^2}| \le L|z|^\alpha, && |R_4(z) - \frac{r}{1-|r|^2}| \le L|z|^\alpha. \end{aligned} \tag{A.6}$$

*Then the solution of the RHP (A.3) satisfies*

$$m(z) = \mathbb{I} + \frac{1}{z}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + O(t^{-\frac{1+\alpha}{2}}), \tag{A.7}$$

*for* $|z| > \rho_0$, *where*

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-\mathrm{i}t\Phi_0}t^{-\mathrm{i}\nu}, \qquad \nu = -\frac{1}{2\pi}\log(1-|r|^2). \tag{A.8}$$

*Furthermore, if* $R_j(z)$ *and* $\Phi(z)$ *depend on some parameter, the error term is uniform with respect to this parameter as long as* $r$ *remains within a compact subset of* $\mathbb{D}$ *and the constants in the above estimates can be chosen independent of the the parameters.*

Note that the actual value of $\rho_0$ is of no importance. In fact, if we choose $0 < \rho_1 < \rho_0$, then the solution $\tilde{m}$ of the problem with jump $\tilde{v}$, where $\tilde{v}$ is equal to $v$ for $|z| < \rho_1$ and $\mathbb{I}$ otherwise, differs from $m$ only by an exponentially small error.

This already indicates, that we should be able to replace $R_j(z)$ by their respective values at $z = 0$. To see this we start by rewriting our RHP as a singular integral equation. We will use the theory developed in Appendix B for the case of $2 \times 2$ matrix valued functions with $m_0(z) = \mathbb{I}$ and the usual Cauchy kernel (since we won't require symmetry in this section)

$$\Omega(s,z) = \mathbb{I}\frac{ds}{s-z}.$$

Moreover, since our contour is unbounded, we will assume $w \in L^1(\Sigma) \cap L^2(\Sigma)$. All results from Appendix B still hold in this case with some straightforward modifications if one observes that $\mu - \mathbb{I} \in L^2$. Indeed, as in Theorem B.3, in the special case $b_+(z) = v_j(z)$ and $b_-(z) = \mathbb{I}$ for $z \in \Sigma_j$, we obtain

$$m(z) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mu(s)w(s)\frac{ds}{s-z}, \tag{A.9}$$

where $\mu - \mathbb{I}$ is the solution of the singular integral equation

$$(\mathbb{I} - C_w)(\mu - \mathbb{I}) = C_w\mathbb{I}, \tag{A.10}$$

that is,

$$\mu = \mathbb{I} + (\mathbb{I} - C_w)^{-1}C_w\mathbb{I}, \qquad C_w f = \mathcal{C}_-(wf). \tag{A.11}$$

Here $\mathcal{C}$ denotes the usual Cauchy operator and we set $w(z) = w_+(z)$ (since $w_-(z) = 0$).

As our first step we will get rid of some constants and rescale the entire problem by setting

$$\hat{m}(z) = D(t)^{-1} m(zt^{-1/2}) D(t), \tag{A.12}$$

where

$$D(t) = \begin{pmatrix} d(t)^{-1} & 0 \\ 0 & d(t) \end{pmatrix}, \qquad d(t) = \mathrm{e}^{\mathrm{i}t\Phi_0/2} t^{\mathrm{i}\nu/2}, \quad d(t)^{-1} = \overline{d(t)}. \tag{A.13}$$

Then one easily checks that $\hat{m}(z)$ solves the RHP

$$\begin{aligned} \hat{m}_+(z) &= \hat{m}_-(z)\hat{v}_j(z), && z \in \Sigma_j, \quad j = 1,2,3,4, \\ \hat{m}(z) &\to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma, \end{aligned} \tag{A.14}$$

where $\hat{v}_j(z) = D(t)^{-1} v_j(zt^{-1/2}) D(t)$, $j = 1, \dots, 4$, explicitly

$$\begin{aligned} \hat{v}_1(z) &= \begin{pmatrix} 1 & -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2}) - \Phi(0))} \\ 0 & 1 \end{pmatrix}, \\ \hat{v}_2(z) &= \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2\mathrm{i}\nu} \mathrm{e}^{t(\Phi(zt^{-1/2}) - \Phi(0))} & 1 \end{pmatrix}, \\ \hat{v}_3(z) &= \begin{pmatrix} 1 & -R_3(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2}) - \Phi(0))} \\ 0 & 1 \end{pmatrix}, \\ \hat{v}_4(z) &= \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2\mathrm{i}\nu} \mathrm{e}^{t(\Phi(zt^{-1/2}) - \Phi(0))} & 1 \end{pmatrix}. \end{aligned} \tag{A.15}$$

Our next aim is to show that the solution $\hat{m}(z)$ of the rescaled problem is close to the solution $\hat{m}^c(z)$ of the RHP

$$\begin{aligned} \hat{m}^c_+(z) &= \hat{m}^c_-(z)\hat{v}^c_j(z), && z \in \Sigma_j, \quad j = 1,2,3,4, \\ \hat{m}^c(z) &\to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma, \end{aligned} \tag{A.16}$$

associated with the following jump matrices

$$\begin{aligned} \hat{v}^c_1(z) &= \begin{pmatrix} 1 & -\overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \\ 0 & 1 \end{pmatrix}, & \hat{v}^c_2(z) &= \begin{pmatrix} 1 & 0 \\ r z^{-2\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/2} & 1 \end{pmatrix}, \\ \hat{v}^c_3(z) &= \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \\ 0 & 1 \end{pmatrix}, & \hat{v}^c_4(z) &= \begin{pmatrix} 1 & 0 \\ \frac{r}{1-|r|^2} z^{-2\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/2} & 1 \end{pmatrix}. \end{aligned} \tag{A.17}$$

The difference between these jump matrices can be estimated as follows.

**Lemma A.2.** *The matrices $\hat{w}^c$ and $\hat{w}$ are close in the sense that*

$$\hat{w}_j(z) = \hat{w}^c_j(z) + O(t^{-\alpha/2} \mathrm{e}^{-|z|^2/8}), \quad z \in \Sigma_j, \quad j = 1, \dots 4. \tag{A.18}$$

*Furthermore, the error term is uniform with respect to parameters as stated in Theorem A.1.*

*Proof.* We only give the proof $z \in \Sigma_1$, the other cases being similar. There is only one nonzero matrix entry in $\hat{w}_j(z) - \hat{w}^c_j(z)$ given by

$$W = \begin{cases} -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2}) - \Phi(0))} + \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2}, & |z| \le \rho_0 t^{1/2}, \\ \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} & |z| > \rho_0 t^{1/2}. \end{cases}$$

A straightforward estimate for $|z| \le \rho_0 t^{1/2}$ shows

$$\begin{aligned}|W| &= \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2})\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} - \overline{r}| \mathrm{e}^{-|z|^2/2}\\ &\le \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{\mathrm{Re}(-t\hat\Phi(zt^{-1/2}))-|z|^2/2} + \mathrm{e}^{\nu\pi/4}|\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} - 1|\mathrm{e}^{-|z|^2/2}\\ &\le \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/4} t|\hat\Phi(zt^{-1/2})|\mathrm{e}^{-|z|^2/4},\end{aligned}$$

where $\hat\Phi(z) = \Phi(z) - \Phi(0) - \frac{\mathrm{i}}{2}z^2 = \frac{\Phi'''(0)}{6}z^3 + \dots$. Here we have used $\frac{\mathrm{i}}{2}z^2 = \frac{1}{2}|z|^2$ for $z \in \Sigma_1$ and $\mathrm{Re}(-t\hat\Phi(zt^{-1/2})) \le |z|^2/4$ by (A.4). Furthermore, by (A.5) and (A.6),

$$|W| \le \mathrm{e}^{\nu\pi/4} L t^{-\alpha/2}|z|^\alpha \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/4} C t^{-1/2}|z|^3 \mathrm{e}^{-|z|^2/4},$$

for $|z| \le \rho_0 t^{1/2}$. For $|z| > \rho_0 t^{1/2}$ we have

$$|W| \le \mathrm{e}^{\nu\pi/4}\mathrm{e}^{-|z|^2/2} \le \mathrm{e}^{\nu\pi/4}\mathrm{e}^{-\rho_0^2 t/4}\mathrm{e}^{-|z|^2/4}$$

which finishes the proof. □

The next lemma allows us, to replace $\hat m(z)$ by $\hat m^c(z)$.

**Lemma A.3.** *Consider the RHP*

$$\begin{aligned} m_+(z) &= m_-(z) v(z), && z \in \Sigma,\\ m(z) &\to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma. \end{aligned} \tag{A.19}$$

*Assume that $w \in L^2(\Sigma) \cap L^\infty(\Sigma)$. Then*

$$\|\mu - \mathbb{I}\|_2 \le \frac{c\|w\|_2}{1 - c\|w\|_\infty} \tag{A.20}$$

*provided $c\|w\|_\infty < 1$, where $c$ is the norm of the Cauchy operator on $L^2(\Sigma)$.*

*Proof.* This follows since $\tilde\mu = \mu - \mathbb{I} \in L^2(\Sigma)$ satisfies $(\mathbb{I} - C_w)\tilde\mu = C_w\mathbb{I}$. □

**Lemma A.4.** *The solution $\hat m(z)$ has a convergent asymptotic expansion*

$$\hat m(z) = \mathbb{I} + \frac{1}{z}\hat M(t) + O(\frac{1}{z^2}) \tag{A.21}$$

*for $|z| > \rho_0 t^{1/2}$ with the error term uniformly in $t$. Moreover,*

$$\hat M(t) = \hat M^c + O(t^{-\alpha/2}). \tag{A.22}$$

*Proof.* Consider $\hat m^d(z) = \hat m(z)\hat m^c(z)^{-1}$, whose jump matrix is given by

$$\hat v^d(z) = \hat m^c_-(z)\hat v(z)\hat v^c(z)^{-1}\hat m^c_-(z)^{-1} = \mathbb{I} + \hat m^c_-(z)\big(\hat w(z) - \hat w^c(z)\big)\hat m^c_-(z)^{-1}.$$

By Lemma A.2, we have that $\hat w - \hat w^c$ is decaying of order $t^{-\alpha/2}$ in the norms of $L^1$ and $L^\infty$ and thus the same is true for $\hat w^d = \hat v^d - \mathbb{I}$. Hence by the previous lemma

$$\|\hat\mu^d - \mathbb{I}\|_2 = O(t^{-\alpha/2}).$$

Furthermore, by $\hat\mu^d = \hat m^d_- = \hat m_-(\hat m^c_-)^{-1} = \hat\mu(\hat\mu^c)^{-1}$ we infer

$$\|\hat\mu - \hat\mu^c\|_2 = O(t^{-\alpha/2})$$

since $\hat\mu^c$ is bounded. Now

$$\hat m(z) = \mathbb{I} - \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma \hat\mu(s)\hat w(s)ds + \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma s\hat\mu(s)\hat w(s)\frac{ds}{s-z}$$

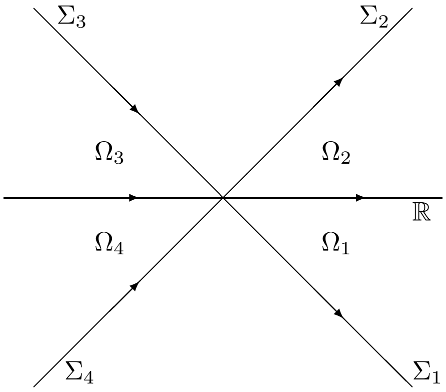


FIGURE 5. Deforming back the cross

shows (recall that $\hat{w}$ is supported inside $|z| \le \rho_0 t^{1/2}$)

$$\hat{m}(z) = \mathbb{I} + \frac{1}{z}\hat{M}(t) + O(\frac{\|\hat{\mu}(s)\|_2 \|s\hat{w}(s)\|_2}{z^2}),$$

where

$$\hat{M}(t) = -\frac{1}{2\pi\mathrm{i}} \int_\Sigma \hat{\mu}(s)\hat{w}(s) ds.$$

Now the rest follows from

$$\hat{M}(t) = \hat{M}^c - \frac{1}{2\pi\mathrm{i}} \int_\Sigma (\hat{\mu}(s)\hat{w}(s) - \hat{\mu}^c(s)\hat{w}^c(s)) ds$$

using $\|\hat{\mu}\hat{w} - \hat{\mu}^c\hat{w}^c\|_1 \le \|\hat{w} - \hat{w}^c\|_1 + \|\hat{\mu} - \mathbb{I}\|_2 \|\hat{w} - \hat{w}^c\|_2 + \|\hat{\mu} - \hat{\mu}^c\|_2 \|\hat{w}^c\|_2$. □

Finally, it remains to solve (A.16) and to show:

**Theorem A.5.** *The solution of the RHP (A.16) is of the form*

$$\hat{m}^c(z) = \mathbb{I} + \frac{1}{z}\hat{M}^c + O(\frac{1}{z^2}), \tag{A.23}$$

*where*

$$\hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix}, \quad \beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}. \tag{A.24}$$

*The error term is uniform with respect to $r$ in compact subsets of $\mathbb{D}$. Moreover, the solution is bounded (again uniformly with respect to $r$).*

Given this result, Theorem A.1 follows from Lemma A.4

$$\begin{aligned} m(z) &= D(t)\hat{m}(zt^{1/2})D(t)^{-1} = \mathbb{I} + \frac{1}{t^{1/2}z} D(t)\hat{M}(t)D(t)^{-1} + O(z^{-2}t^{-1}) \\ &= \mathbb{I} + \frac{1}{t^{1/2}z} D(t)\hat{M}^c D(t)^{-1} + O(t^{-(1+\alpha)/2}) \end{aligned} \tag{A.25}$$

for $|z| > \rho_0$, since $D(t)$ is bounded.

The proof of this result will be given in the remainder of this section. In order to solve (A.16) we begin with a deformation which moves the jump to $\mathbb{R}$ as follows. Denote the region enclosed by $\mathbb{R}$ and $\Sigma_j$ as $\Omega_j$ (cf. Figure 5) and define

$$\tilde{m}^c(z) = \hat{m}^c(z) \begin{cases} D_0(z) D_j, & z \in \Omega_j,\ j = 1, \dots, 4, \\ D_0(z), & \text{else}, \end{cases} \tag{A.26}$$

where

$$D_0(z) = \begin{pmatrix} z^{\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & z^{-\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix},$$

and

$$D_1 = \begin{pmatrix} 1 & \overline{r} \\ 0 & 1 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \quad D_3 = \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} \\ 0 & 1 \end{pmatrix} \quad D_4 = \begin{pmatrix} 1 & 0 \\ -\frac{r}{1-|r|^2} & 1 \end{pmatrix}.$$

**Lemma A.6.** *The function $\tilde{m}^c(z)$ defined in (A.26) satisfies the RHP*

$$\begin{aligned} \tilde{m}^c_+(z) &= \tilde{m}^c_-(z) \begin{pmatrix} 1 - |r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}, && z \in \mathbb{R} \\ \tilde{m}^c(z) &= (\mathbb{I} + \frac{1}{z} \hat{M}^c + \dots) D_0(z), && z \to \infty,\ \frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}. \end{aligned} \tag{A.27}$$

*Proof.* First, one checks that $\tilde{m}^c_+(z) = \tilde{m}^c_-(z) D_0(z)^{-1} \hat{v}^c_1(z) D_0(z) D_1 = \tilde{m}^c_-(z)$, $z \in \Sigma_1$ and similarly for $z \in \Sigma_2, \Sigma_3, \Sigma_4$. To compute the jump along $\mathbb{R}$ observe that, by our choice of branch cut for $z^{\mathrm{i}\nu}$, $D_0(z)$ has a jump along the negative real axis given by

$$D_{0,\pm}(z) = \begin{pmatrix} \mathrm{e}^{(\log|z| \pm \mathrm{i}\pi)\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & \mathrm{e}^{-(\log|z| \pm \mathrm{i}\pi)\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix}, \qquad z < 0.$$

Hence the jump along $\mathbb{R}$ is given by

$$D_1^{-1} D_2, \quad z > 0 \quad \text{and} \quad D_4^{-1} D_{0,-}^{-1}(z) D_{0,+}(z) D_3, \quad z < 0,$$

and (A.27) follows after recalling $\mathrm{e}^{-2\pi\nu} = 1 - |r|^2$. □

Now, we can follow (4.17) to (4.51) in [3] to construct an approximate solution.

The idea is as follows, since the jump matrix for (A.27), the derivative $\frac{d}{dz}\tilde{m}^c(z)$ has the same jump and hence is given by $n(z)\tilde{m}^c(z)$, where the entire matrix $n(z)$ can be determined from the behaviour $z \to \infty$. Since this will just serve as a motivation for our ansatz, we will not worry about justifying any steps.

For $z$ in the sector $\frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}$ (enclosed by $\Sigma_2$ and $\Sigma_3$) we have $\tilde{m}^c(z) = \hat{m}^c(z) D_0(z)$ and hence

$$\begin{aligned} &\left( \frac{d}{dz} \tilde{m}^c(z) + \frac{\mathrm{i}z}{2} \sigma_3 \tilde{m}^c(z) \right) \tilde{m}^c(z)^{-1} \\ &\quad = \left( \mathrm{i}(\frac{\nu}{z} - \frac{z}{2}) \hat{m}^c(z) \sigma_3 + \frac{d}{dz} \hat{m}^c(z) + \mathrm{i}\frac{z}{2} \sigma_3 \hat{m}^c(z) \right) \hat{m}^c(z)^{-1} \\ &\quad = \frac{\mathrm{i}}{2} [\sigma_3, \hat{M}^c] + O(\frac{1}{z}), \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Since the left hand side has no jump, it is entire and hence by Liouville's theorem a constant given by the right hand side. In other words,

$$\frac{d}{dz} \tilde{m}^c(z) + \frac{\mathrm{i}z}{2} \sigma_3 \tilde{m}^c(z) = \beta \tilde{m}^c(z), \quad \beta = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix} = \frac{\mathrm{i}}{2} [\sigma_3, \hat{M}^c]. \tag{A.28}$$

This differential equation can be solved in terms of parabolic cylinder function which then gives the solution of (A.27).

**Lemma A.7.** *The RHP (A.27) has a unique solution, and the term $\hat{M}^c$ is given by*

$$\hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta_{12} \\ \beta_{21} & 0 \end{pmatrix}, \qquad \beta_{12} = \overline{\beta_{21}} = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(r)+\arg(\Gamma(\mathrm{i}\nu)))}. \tag{A.29}$$

*Proof.* Uniqueness follows by the standard Liouville argument since the determinant of the jump matrix is equal to 1. To find the solution we use the ansatz

$$\tilde{m}^c(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix},$$

where the functions $\psi_{jk}(z)$ satisfy

$$\psi_{11}''(z) = -\left(\frac{\mathrm{i}}{2} + \frac{1}{4}z^2 - \beta_{12}\beta_{21}\right)\psi_{11}(z), \quad \psi_{12}(z) = \frac{1}{\beta_{21}}\left(\frac{d}{dz} - \frac{\mathrm{i}z}{2}\right)\psi_{22}(z),$$
$$\psi_{21}(z) = \frac{1}{\beta_{12}}\left(\frac{d}{dz} + \frac{\mathrm{i}z}{2}\right)\psi_{11}(z), \qquad \psi_{22}''(z) = \left(\frac{\mathrm{i}}{2} - \frac{1}{4}z^2 + \beta_{12}\beta_{21}\right)\psi_{22}(z).$$

That is, $\psi_{11}(\mathrm{e}^{3\pi\mathrm{i}/4}\zeta)$ satisfies the parabolic cylinder equation

$$D''(\zeta) + \left(a + \frac{1}{2} - \frac{1}{4}\zeta^2\right)D(\zeta) = 0$$

with $a = \mathrm{i}\beta_{12}\beta_{21}$ and $\psi_{22}(\mathrm{e}^{\mathrm{i}\pi/4}\zeta)$ satisfies the parabolic cylinder equation with $a = -\mathrm{i}\beta_{12}\beta_{21}$.

Let $D_a$ be the entire parabolic cylinder function of §16.5 in [25] and set

$$\psi_{11}(z) = \begin{cases} \mathrm{e}^{-3\pi\nu/4}D_{\mathrm{i}\nu}(-\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) > 0, \\ \mathrm{e}^{\pi\nu/4}D_{\mathrm{i}\nu}(\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) < 0, \end{cases}$$
$$\psi_{22}(z) = \begin{cases} \mathrm{e}^{\pi\nu/4}D_{-\mathrm{i}\nu}(-\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) > 0, \\ \mathrm{e}^{-3\pi\nu/4}D_{-\mathrm{i}\nu}(\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) < 0. \end{cases}$$

Using the asymptotic behavior

$$D_a(z) = z^a\mathrm{e}^{-z^2/4}\big(1 - \frac{a(a-1)}{2z^2} + O(z^{-4})\big), \quad z\to\infty, \quad |\arg(z)| \le 3\pi/4,$$

shows that the choice $\beta_{12}\beta_{21} = \nu$ ensures the correct asymptotics

$$\psi_{11}(z) = z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(1 + O(z^{-2})), \qquad \psi_{12}(z) = -\mathrm{i}\beta_{12}z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(z^{-1} + O(z^{-3})),$$
$$\psi_{21}(z) = \mathrm{i}\beta_{21}z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(z^{-1} + O(z^{-3})), \quad \psi_{22}(z) = z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1 + O(z^{-2})),$$

as $z\to\infty$ inside the half plane $\mathrm{Im}(z) \ge 0$. In particular,

$$\tilde{m}^c(z) = \big(\mathbb{I} + \frac{1}{z}\hat{M}^c + O(z^{-2})\big)D_0(z) \quad \text{with} \quad \hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.$$

It remains to check that we have the correct jump. Since by construction both limits $\tilde{m}_+^c(z)$ and $\tilde{m}_-^c(z)$ satisfy the same differential equation (A.28), there is a constant matrix $v$ such that $\tilde{m}_+^c(z) = \tilde{m}_-^c(z)v$. Moreover, since the coefficient matrix of the linear differential equation (A.28) has trace 0, the determinant of $\tilde{m}_\pm^c(z)$ is constant and hence $\det(\tilde{m}_\pm^c(z)) = 1$ by our asymptotics. Moreover, evaluating

$$v = \tilde{m}_-^c(0)^{-1}\tilde{m}_+^c(0) = \begin{pmatrix} \mathrm{e}^{-2\pi\nu} & -\frac{\sqrt{2\pi}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\sqrt{\nu}\Gamma(\mathrm{i}\nu)}\gamma^{-1} \\ \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\sqrt{\nu}\Gamma(-\mathrm{i}\nu)}\gamma & 1 \end{pmatrix}$$

where $\gamma = \frac{\sqrt{\nu}}{\beta_{12}} = \frac{\beta_{21}}{\sqrt{\nu}}$. Here we have used

$$D_a(0) = \frac{2^{a/2}\sqrt{\pi}}{\Gamma((1-a)/2)}, \qquad D_a'(0) = -\frac{2^{(1+a)/2}\sqrt{\pi}}{\Gamma(-a/2)}$$

plus the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ for the Gamma function. Hence, if we choose

$$\gamma = \frac{\sqrt{\nu}\Gamma(-\mathrm{i}\nu)}{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}} r,$$

we have

$$v = \begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}$$

since $|\gamma|^2 = 1$. To see this use $|\Gamma(-\mathrm{i}\nu)|^2 = \frac{\Gamma(1-\mathrm{i}\nu)\Gamma(\mathrm{i}\nu)}{-\mathrm{i}\nu} = \frac{\pi}{\nu\sinh(\pi\nu)}$ which follows from Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ for the Gamma function.

In particular,

$$\beta_{12} = \overline{\beta_{21}} = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(r)+\arg(\Gamma(\mathrm{i}\nu)))}$$

which finishes the proof. □

**Remark A.8.** *An inspection of the proof shows that $\hat{m}^c$ is given by the solution of a differential equation depending analytically on $\nu$. Hence, $\hat{m}^c$ depends analytically on $\nu = -\frac{1}{2\pi}\log(1-|r|^2)$. This implies local Lipschitz dependence on $r$ as long as $r \in \mathbb{D}$.*

## Appendix B. Singular integral equations

In this section we show how to transform a meromorphic vector Riemann–Hilbert problem with simple poles at $\zeta$, $\zeta^{-1}$,

$$\begin{aligned}
& m_+(z) = m_-(z)v(z), \qquad z\in\Sigma,\\
& \operatorname{Res}_\zeta m(z) = \lim_{z\to\zeta} m(z)\begin{pmatrix} 0 & 0 \\ -\zeta\gamma & 0\end{pmatrix}, \quad \operatorname{Res}_{\zeta^{-1}} m(z) = \lim_{z\to\zeta^{-1}} m(z)\begin{pmatrix} 0 & \zeta^{-1}\gamma \\ 0 & 0\end{pmatrix},\\
& m(z^{-1}) = m(z)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\\
& m(0) = \begin{pmatrix} 1 & m_2\end{pmatrix},
\end{aligned} \tag{B.1}$$

into a singular integral equation. Since we require the symmetry condition (2.16) for our Riemann–Hilbert problems we need to adapt the usual Cauchy kernel to preserve this symmetry. Moreover, we keep the single soliton as an inhomogeneous term which will play the role of the leading asymptotics in our applications.

**Hypothesis B.1.** *Let $\Sigma$ consist of a finite number of smooth oriented finite curves in $\mathbb{C}$ which intersect at most finitely many times with all intersections being transversal. Assume that the contour $\Sigma$ does not contain $0$ and is invariant under $z\mapsto z^{-1}$. Moreover, suppose the jump matrix $v$ can be factorized according to $v = b_-^{-1}b_+ = (\mathbb{I}-w_-)^{-1}(\mathbb{I}+w_+)$, where $w_\pm$ are continuous and satisfy*

$$w_\pm(z^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} w_\mp(z)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad z\in\Sigma. \tag{B.2}$$

The classical Cauchy-transform of a function $f : \Sigma \to \mathbb{C}$ which is square integrable is the analytic function $Cf : \mathbb{C}\backslash\Sigma \to \mathbb{C}$ given by

$$Cf(z) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{f(s)}{s-z} ds, \qquad z \in \mathbb{C}\backslash\Sigma. \tag{B.3}$$

Denote the non-tangential boundary values from both sides (taken possibly in the $L^2$-sense — see e.g. [2, eq. (7.2)]) by $C_+ f$ respectively $C_- f$. Then it is well-known that $C_+$ and $C_-$ are bounded operators $L^2(\Sigma) \to L^2(\Sigma)$, which satisfy $C_+ - C_- = \mathbb{I}$ and $C_+ C_- = 0$ (see e.g. [1]). Moreover, one has the Plemelj–Sokhotsky formula ([17])

$$C_\pm = \frac{1}{2}(\mathrm{i}H \pm \mathbb{I}),$$

where

$$Hf(t) = \frac{1}{\pi} ⨍_\Sigma \frac{f(s)}{t-s} ds, \qquad t \in \Sigma, \tag{B.4}$$

is the Hilbert transform and $⨍$ denotes the principal value integral.

In order to respect the symmetry condition we will restrict our attention to the set $L^2_s(\Sigma)$ of square integrable functions $f : \Sigma \to \mathbb{C}^2$ such that

$$f(z^{-1}) = f(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.5}$$

Clearly this will only be possible if we require our jump data to be symmetric as well (i.e., Hypothesis B.1 holds).

Next we introduce the Cauchy operator

$$(Cf)(z) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma f(s) \Omega_\zeta(s,z) \tag{B.6}$$

acting on vector-valued functions $f : \Sigma \to \mathbb{C}^2$. Here the Cauchy kernel is given by

$$\Omega_\zeta(s,z) = \begin{pmatrix} \frac{z-\zeta^{-1}}{s-\zeta^{-1}} \frac{1}{s-z} & 0 \\ 0 & \frac{z-\zeta}{s-\zeta} \frac{1}{s-z} \end{pmatrix} ds = \begin{pmatrix} \frac{1}{s-z} - \frac{1}{s-\zeta^{-1}} & 0 \\ 0 & \frac{1}{s-z} - \frac{1}{s-\zeta} \end{pmatrix} ds, \tag{B.7}$$

for some fixed $\zeta \notin \Sigma$. In the case $\zeta = \infty$ we set

$$\Omega_\infty(s,z) = \begin{pmatrix} \frac{1}{s-z} - \frac{1}{s} & 0 \\ 0 & \frac{1}{s-z} \end{pmatrix} ds. \tag{B.8}$$

and one easily checks the symmetry property:

$$\Omega_\zeta(1/s, 1/z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Omega_\zeta(s,z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.9}$$

The properties of $C$ are summarized in the next lemma.

**Lemma B.2.** *Assume Hypothesis B.1. The Cauchy operator $C$ has the properties, that the boundary values $C_\pm$ are bounded operators $L^2_s(\Sigma) \to L^2_s(\Sigma)$ which satisfy*

$$C_+ - C_- = \mathbb{I} \tag{B.10}$$

*and*

$$(Cf)(\zeta^{-1}) = (0 \quad *), \qquad (Cf)(\zeta) = (* \quad 0). \tag{B.11}$$

*Furthermore, $C$ restricts to $L^2_s(\Sigma)$, that is*

$$(Cf)(z^{-1}) = (Cf)(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in \mathbb{C}\backslash\Sigma \tag{B.12}$$

*for $f \in L^2_s(\Sigma)$ and if $w_\pm$ satisfy (H.B.1) we also have*

$$C_\pm(f w_\mp)(1/z) = C_\mp(f w_\pm)(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in \Sigma. \tag{B.13}$$

*Proof.* Everything follows from (B.9) and the fact that $C$ inherits all properties from the classical Cauchy operator. ☐

We have thus obtained a Cauchy transform with the required properties. Following Section 7 and 8 of [1], we can solve our Riemann–Hilbert problem using this Cauchy operator.

Introduce the operator $C_w : L^2_s(\Sigma) \to L^2_s(\Sigma)$ by

$$C_w f = C_+(f w_-) + C_-(f w_+), \quad f \in L^2_s(\Sigma) \tag{B.14}$$

and recall from Lemma 2.6 that the solution corresponding to $v \equiv \mathbb{I}$ is given by

$$m_0(z) = \begin{pmatrix} f(z) & f(\frac{1}{z}) \end{pmatrix}, \quad f(z) = \frac{1}{1-\zeta^2+\gamma}\left(\gamma\zeta^2 \frac{z-\zeta^{-1}}{z-\zeta} + 1 - \zeta^2\right)$$

Observe that for $\gamma = 0$ we have $f(z) = 1$ and for $\gamma = \infty$ we have $f(z) = \zeta^2 \frac{z-\zeta^{-1}}{z-\zeta}$. In particular, $m_0(z)$ is uniformly bounded away from $\zeta$ for all $\gamma \in [0, \infty]$.

Then we have the next result.

**Theorem B.3.** *Assume Hypothesis B.1.*

*Suppose $m$ solves the Riemann–Hilbert problem (B.1). Then*

$$m(z) = (1 - c_0) m_0(z) + \frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s))\Omega_\zeta(s, z), \tag{B.15}$$

*where*

$$\mu = m_+ b_+^{-1} = m_- b_-^{-1} \quad \text{and} \quad c_0 = \left(\frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s))\Omega_\zeta(s, 0)\right)_1.$$

*Here $(m)_j$ denotes the $j$'th component of a vector. Furthermore, $\mu$ solves*

$$(\mathbb{I} - C_w)\mu = (1 - c_0) m_0(z). \tag{B.16}$$

*Conversely, suppose $\tilde{\mu}$ solves*

$$(\mathbb{I} - C_w)\tilde{\mu} = m_0(z), \tag{B.17}$$

*and*

$$\tilde{c}_0 = \left(\frac{1}{2\pi\mathrm{i}} \int_\Sigma \tilde{\mu}(s)(w_+(s) + w_-(s))\Omega_\zeta(s, 0)\right)_1 \neq 1,$$

*then $m$ defined via (B.15), with $(1 - c_0) = (1 - \tilde{c}_0)^{-1}$ and $\mu = (1 - \tilde{c}_0)^{-1}\tilde{\mu}$, solves the Riemann–Hilbert problem (B.1) and $\mu = m_\pm b_\pm^{-1}$.*

*Proof.* First of all note hat by (B.13) $(\mathbb{I} - C_w)$ satisfies the symmetry condition and hence so do $(\mathbb{I} - C_w)^{-1} m_0$ and $m$.

So if $m$ solves (B.1) and we set $\mu = m_\pm b_\pm^{-1}$, then $m$ satisfies an additive jump given by

$$m_+ - m_- = \mu(w_+ + w_-).$$

Hence, if we denote the left hand side of (B.15) by $\tilde{m}$, both functions satisfy the same additive jump. So $m - \tilde{m}$ has no jump and must thus solve (B.1) with $v \equiv \mathbb{I}$. By uniqueness (cf. [15]) $m - \tilde{m} = \alpha m_0$ for some $\alpha \in \mathbb{C}$ and by looking at the first component at $z = 0$ we see $\alpha = 0$, that is $m = \tilde{m}$.

Moreover, if $m$ is given by (B.15), then (B.10) implies

$$\begin{aligned} m_\pm &= (1-c_0)m_0 + C_\pm(\mu w_-) + C_\pm(\mu w_+) \\ &= (1-c_0)m_0 + C_w(\mu) \pm \mu w_\pm \\ &= (1-c_0)m_0 - (\mathbb{I} - C_w)\mu + \mu b_\pm. \end{aligned} \tag{B.18}$$

From this we conclude that $\mu = m_\pm b_\pm^{-1}$ solves (B.16).

Conversely, if $\tilde{\mu}$ solves (B.17) then (B.18) implies $m_\pm = \mu b_\pm$ which shows that $m$ defined via (B.15) solves the Riemann–Hilbert problem (B.1). □

Note that in the special case $\gamma = 0$ we have $m_0(z) = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and we can choose $\zeta$ as we please, say $\zeta = \infty$ such that $c_0 = \tilde{c}_0 = 0$ in the above theorem.

Hence we have a formula for the solution of our Riemann–Hilbert problem $m(z)$ in terms of $(\mathbb{I} - C_w)^{-1} m_0$ and this clearly raises the question of bounded invertibility of $\mathbb{I} - C_w$. This follows from Fredholm theory (cf. e.g. [26]):

**Lemma B.4.** *Assume Hypothesis B.1. The operator* $\mathbb{I} - C_w$ *is Fredholm of index zero,*

$$\mathrm{ind}(\mathbb{I} - C_w) = 0. \tag{B.19}$$

*Proof.* Since one can easily check

$$(\mathbb{I} - C_w)(\mathbb{I} - C_{-w}) = (\mathbb{I} - C_{-w})(\mathbb{I} - C_w) = \mathbb{I} - T_w, \tag{B.20}$$

where

$$T_w = T_{++} + T_{+-} + T_{-+} + T_{--}, \qquad T_{\sigma_1\sigma_2}(f) = C_{\sigma_1}[C_{\sigma_2}(f w_{-\sigma_2}) w_{-\sigma_1}],$$

it suffices to check that the operators $T_{\sigma_1\sigma_2}$ are compact ([19, Thm. 1.4.3]). By Mergelyan's theorem we can approximate $w_\pm$ by rational functions and, since the norm limit of compact operators is compact, we can assume without loss that $w_\pm$ have an analytic extension to a neighborhood of $\Sigma$.

Indeed, suppose $f_n \in L^2(\Sigma)$ converges weakly to zero. Without loss we can assume $f_n$ to be continuous. We will show that $\|T_w f_n\|_{L^2} \to 0$.

Using the analyticity of $w$ in a neighborhood of $\Sigma$ and the definition of $C_\pm$, we can slightly deform the contour $\Sigma$ to some contour $\Sigma_\pm$ close to $\Sigma$, on the left, and have, by Cauchy's theorem,

$$T_{++} f_n(z) = \frac{1}{2\pi\mathrm{i}} \int_{\Sigma_+} (C(f_n w_-)(s) w_-(s)) \Omega_\zeta(s,z).$$

Now $(C(f_n w_-)w_-)(z) \to 0$ as $n \to \infty$. Also

$$|(C(f_n w_-)w_-)(z)| < const\, \|f_n\|_{L^2} \|w_-\|_{L^\infty} < const$$

and thus, by the dominated convergence theorem, $\|T_{++} f_n\|_{L^2} \to 0$ as desired.

Moreover, considering $\mathbb{I} - \varepsilon C_w = \mathbb{I} - C_{\varepsilon w}$ for $0 \le \varepsilon \le 1$ we obtain $\mathrm{ind}(\mathbb{I} - C_w) = \mathrm{ind}(\mathbb{I}) = 0$ from homotopy invariance of the index. □

By the Fredholm alternative, it follows that to show the bounded invertibility of $\mathbb{I} - C_w$ we only need to show that $\ker(\mathbb{I} - C_w) = 0$. The latter being equivalent to unique solvability of the corresponding vanishing Riemann–Hilbert problem.

**Corollary B.5.** *Assume Hypothesis B.1. A unique solution of the Riemann–Hilbert problem (B.1) exists if and only if the corresponding vanishing Riemann–Hilbert problem, where the normalization condition is replaced by* $m(0) = \begin{pmatrix} 0 & m_2 \end{pmatrix}$, *has at most one solution.*

We are interested in comparing two Riemann–Hilbert problems associated with respective jumps $w_0$ and $w$ with $\|w - w_0\|_\infty$ small, where

$$\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)}. \tag{B.21}$$

For such a situation we have the following result:

**Theorem B.6.** *Assume that for some data $w_0^t$ the operator*

$$\mathbb{I} - C_{w_0^t} : L^2_s(\Sigma) \to L^2_s(\Sigma) \tag{B.22}$$

*has a bounded inverse, where the bound is independent of t, and let $\zeta = \zeta_0$, $\gamma^t = \gamma_0^t$.*

*Furthermore, assume $w^t$ satisfies*

$$\|w^t - w_0^t\|_\infty \le \alpha(t) \tag{B.23}$$

*for some function $\alpha(t) \to 0$ as $t \to \infty$. Then $(\mathbb{I} - C_{w^t})^{-1} : L^2_s(\Sigma) \to L^2_s(\Sigma)$ also exists for sufficiently large t and the associated solutions of the Riemann–Hilbert problems (B.1) only differ by $O(\alpha(t))$.*

*Proof.* By boundedness of the Cauchy transform, one has

$$\|(C_{w^t} - C_{w_0^t})\| \le const \|w\|_\infty.$$

Thus, by the second resolvent identity, we infer that $(\mathbb{I} - C_{w^t})^{-1}$ exists for large $t$ and

$$\|(\mathbb{I} - C_{w^t})^{-1} - (\mathbb{I} - C_{w_0^t})^{-1}\| = O(\alpha(t)).$$

From which the claim follows since this implies $\|\mu^t - \mu_0^t\|_{L^2} = O(\alpha(t))$ and thus $m^t(z) - m_0^t(z) = O(\alpha(t))$ uniformly in $z$ away from $\Sigma$. □

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