A criterion for the reality of the spectrum of $\mathcal{PT}$ symmetric Schrödinger operators with complex-valued periodic potentials

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Abstract

Consider in $L^2(\mathbb{R})$ the Schrödinger operator family $H(g) := -d_x^2 + V_g(x)$ depending on the real parameter g, where $V_g(x)$ is a complex-valued but PT symmetric periodic potential. An explicit condition on V is obtained which ensures that the spectrum of $H(g)$ is purely real and band shaped; furthermore, a further condition is obtained which ensures that the spectrum contains at least a pair of complex analytic arcs.

1 Introduction and statement of the results

There is currently an intense and ever increasing activity on an aspect of quantum theory known as PT-symmetric quantum mechanics (see e.g.[Be1], [BBM], [Be2], [BBJ], [CJT], [CJN], [LZ], [Sp], [ZCBR]). Mathematically speaking, in the simplest, one -dimensional case one deals with the stationary Schrödinger equation

$$H\psi := (-\frac{d^2}{dx^2} + V)\psi = E\psi \ . \tag{1.1}$$

where the potential $V(x)$ can be complex-valued but is invariant under the combined action of the linear parity operation P, $P\psi(x) = \psi(-x)$, and of the (anti)linear "time-reversal" symmetry, i.e. the complex-conjugation operation $T\psi(x) = \overline{\psi}(x)$; namely, $\overline{V}(-x) = V(x)$. The basic mathematical problem is to determine under what conditions, if any, on the complex PT symmetric potential V the spectrum of the corresponding Schrödinger operator is purely real.

Here we deal with this problem in the context of periodic potentials on $\mathbb{R}$, already considered in [Ah], [BDM],[Ce], [CR], [Jo], [Sh]. Without loss, the period is assumed to be 2π. If V is periodic and real valued it is well known (see e.g. [BS]) that, under mild regularity assumptions, the spectrum is absolutely continuous on $\mathbb{R}$ and band shaped.

It is then natural to ask whether or not there exist classes of PT symmetric, complex periodic potentials generating Schrödinger operators with real band spectrum. This question has been examined in [Ah], [BDM],[Ce], [CR], [Jo], by a combination of numerical and WKB techniques, in several particular examples. It was later proved in [Sh] that the

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above arguments cannot exclude the occurrence of complex spectra, and actually a condition has been isolated under which H admits complex spectrum consisting of a disjoint union of analytic arcs [Sh].

Our first result is the explicit determination a class of PT symmetric, complex periodic potentials admitting real band spectrum. Denote $\mathcal{T}(u)$ the non-negative quadratic form in $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$ defined by the kinetic energy:

$$\mathcal{T}(u) := \int_{\mathbb{R}} |u'|^2 \, dx, \qquad u \in H^1(\mathbb{R}) \tag{1.2}$$

Let q be a real-valued, tempered distribution. Assume:

(1) q is a 2π-periodic, P symmetric distribution belonging to $H^{-1}_{loc}(\mathbb{R})$;

(2) $W(x) : \mathbb{R} \to \mathbb{C}$ belongs to $L^\infty(\mathbb{R})$ and is PT-symmetric, $\overline{W(-x)} = -W(x)$, .

(3) q generates a real quadratic form $\mathcal{Q}(u)$ in $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$;

(4) $\mathcal{Q}(u)$ is relatively bounded with respect to $\mathcal{T}(u)$ with relative bound $b < 1$, i.e. there are $b < 1$ and $a > 0$ such that

$$\mathcal{Q}(u) \leq bT(u) + a\|u\|^2 \tag{1.3}$$

Under these assumptions the real quadratic form

$$\mathcal{H}_0(u) := \mathcal{T}(u) + \mathcal{Q}(u) \quad u \in H^1(\mathbb{R}) \tag{1.4}$$

is closed and bounded below in $L^2(\mathbb{R})$. We denote $H(0)$ the corresponding self-adjoint operator. This is the self-adjoint realization of the formal differential expression (note the abuse of notation)

$$H(0) = -\frac{d^2}{dx^2} + q(x)$$

Under these circumstances it is known (see e.g. [AGHKH]) that the spectrum of $H(0)$ is continuous and band shaped. For $n = 1, 2, \ldots$ we denote

$$B_{2n} := [\alpha_{2n}, \beta_{2n}], \quad B_{2n+1} := [\beta_{2n+1}, \alpha_{2n+1}]$$

the bands of $H(0)$, and $\Delta_n :=]\beta_{2n}, \beta_{2n+1}[, \ ]\alpha_{2n+1}, \alpha_{2(n+2)}[$ the gaps between the bands. Here:

$$0 \leq \alpha_0 \leq \beta_0 \leq \beta_1 \leq \alpha_1 \leq \alpha_2 \leq \beta_2 \leq \beta_3 \leq \alpha_3 \leq \alpha_4 \leq \ldots.$$

The maximal multiplication operator by W is continuous in L^2, and therefore so is the quadratic form $\langle u, Wu \rangle$. It follows that the quadratic form family

$$\mathcal{H}_g(u) := \mathcal{T}(u) + \mathcal{Q}(u) + g\langle u, Wu \rangle, \quad u \in H^1(\mathbb{R}) \tag{1.5}$$

is closed and sectorial in $L^2(\mathbb{R})$ for any $g \in \mathbb{C}$.

We denote $H(g)$ the uniquely associated $m-$ sectorial operator in $L^2(\mathbb{R})$. This is the realization of the formal differential operator family

$$H(g) = -\frac{d^2}{dx^2} + q(x) + gW(x)$$

By definition, $H(g)$ is a holomorphic family of operators of type B in the sense of Kato for $g \in \mathbb{C}$; by (1) it is also PT symmetric for $g \in \mathbb{R}$. Our first result deals with its spectral properties.

Theorem 1.1 *Let all gaps of $H(0)$ be open, namely: $\alpha_n < \beta_n < \alpha_{n+1}\ \forall n \in \mathbb{N}$, and let there exist $d > 0$ such that*

$$\frac{1}{2} \inf_{n \in \mathbb{N}} \Delta_n := d > 0 \tag{1.6}$$

Then, if

$$|g| < \frac{d^2}{2(1+d)\|W\|_\infty} := \overline{g} \tag{1.7}$$

there exist

$$0 \leq \alpha_0(g) < \beta_0(g) < \beta_1(g) < \alpha_1(g) < \alpha_2(g) < \beta_2(g) \dots$$

such that

$$\sigma(H(g)) = \left(\bigcup_{n \in \mathbb{N}} B_{2n}(g)\right) \bigcup \left(\bigcup_{n \in \mathbb{N}} B_{2n+1}(g)\right) \tag{1.8}$$

where, as above

$$B_{2n}(g) := [\alpha_{2n}(g), \beta_{2n}(g)], \quad B_{2n+1}(g) := [\beta_{2n+1}(g), \alpha_{2n+1}(g)].$$

Remark

The theorem states that for $|g|$ small enough the spectrum of the non self-adjoint operator $H(g)$ remains real and band-shaped. The proof is critically dependent on the validity of the lower bound (1.6). Therefore it cannot apply to smooth potentials $q(x)$, in which case the gaps vanish as $n \to \infty$. Actually we have the following

Example

A locally $H^{-1}(\mathbb{R})$ distribution $q(x)$ fulfilling the above conditions is

$$q(x) = \sum_{n \in \mathbb{Z}} \delta(x - 2\pi n)$$

the periodic δ function. Here we have:

$$\mathcal{Q}(u) = \sum_{n \in \mathbb{Z}} |u(2\pi n)|^2 = \int_{\mathbb{R}} q(x)|u(x)|^2\, dx, \quad u \in H^1(\mathbb{R})$$

This example is known as the Kronig-Penney model in the one-electron theory of solids. Let us verify that condition (1.3) is satisfied. As is known, this follows from the inequality (see e.g. [Ka], §VI.4.10):

$$|u(2\pi n)|^2 \le \epsilon \int_{2\pi n}^{2\pi(n+1)} |u'(y)|^2 \, dy + \delta \int_{2\pi n}^{2\pi(n+1)} |u(y)|^2$$

where ϵ can be chosen arbitrarity small for δ large enough. In fact, if $u \in H^1(\mathbb{R})$ this inequality yields:

$$\int_{\mathbb{R}} q(x)|u(x)|^2 \, dx = \sum_{n\in\mathbb{Z}} |u(2\pi n)|^2 \le \epsilon \int_{\mathbb{R}} |u'(y)|^2 \, dy + \delta \int_{R} |u(y)|^2$$

$$= \epsilon \mathcal{T}(u) + \delta \|u\|^2, \qquad u \in H^1(\mathbb{R})$$

which in turn entails the closedness of $\mathcal{T}(u) + \mathcal{Q}(u)$ defined on H^1 by the standard Kato criterion. The closedness and sectoriality of $\mathcal{H}_g(u)$ defined on $H^1(\mathbb{R})$ is an immediate consequence of the continuity of W as a maximal multiplication operator in L^2. For the verification of (1.6), see e.g.[AGHKH]. Hence any bounded PT-symmetric periodic perturbation of the Kronig-Penney potential has real spectrum for $g \in \mathbb{R}$, $|g| < \overline{g}$, where $\overline{g}$ is defined by (1.7). .

As a second result, we show that an elementary argument of perturbation theory allows us to sharpen the result of [Sh] about the existence of complex spectra for PT-symmetric periodic potentials.
Let indeed $W(x) \in L^\infty(\mathbb{R};\mathbb{C})$ be a 2π-periodic function. Then the continuity of W as a multiplication operator in $L^2(\mathbb{R})$ entails that the Schrödinger operator

$$K(g)u := -\frac{d^2u}{dx^2} + gWu, \quad u \in D(K) := H^2(\mathbb{R}), \quad g \in \mathbb{R} \tag{1.9}$$

is closed and has non-empty resolvent set. Consider the Fourier expansion of W:

$$W(x) = \sum_{n\in\mathbb{Z}} w_n e^{inx}, \quad w_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)e^{-inx} \, dx$$

which converges pointwise almost everywhere in $[-\pi, \pi]$. Then we have

Theorem 1.2 *Let $W(x)$ be PT-symmetric, namely $\overline{W(-x)} = W(x)$. Then*

1. $\overline{w}_n = w_n$, $\forall\, n \in \mathbb{Z}$;

2. *Furthermore, let there exist $k \in \mathbb{N}$, k odd, such that $w_k w_{-k} < 0$.*

Then there is $\delta > 0$ such that for $|g| < \delta$ the spectrum of $K(g)$ contains at least a pair of complex conjugate analytic arcs.

Remarks

1. This theorem sharpens the results of [Sh] in the sense that its assumptions are explicit because they involve only the given potential $W(x)$, while those of Theorem 3 and Corollary 4 of [Sh] involve some conditions on the Floquet discriminant of the equation $K(g)\psi = E\psi$. This requires some a priori information on the solutions of the equation itself.

2. Explicit examples of potentials fulfilling the above conditions are:

$$W(x) = i \sin^{2k+1} nx, \quad k = 0, 1 \ldots; \quad n \text{ odd}$$

   For $g = 1$ these potentials have been considered in [BDM], where is is claimed that the spectrum is purely real. A more careful examination by [Sh] shows that the appearance of complex spectra cannot be excluded.

2 Proof of the statements

Let us first state an elementary remark under the form of a lemma. Incidentally, this also proves Assertion 1 of Theorem 1.2.

Lemma 2.1 *Let $f(x) \in L^\infty(\mathbb{R};\mathbb{C})$ be 2π periodic, $f(x+2\pi) = f(x)$, $x \in \mathbb{R}$, and PT symmetric, $\overline{f}(-x) = f(x)$. Consider its Fourier coefficients*

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\, dx$$

Then $\overline{f}_n = f_n \ \forall n \in \mathbb{Z}$.

Proof

The assertion is an immediate consequence of the Carleson-Fefferman theorem, which states the pointwise convergence of the Fourier expansion

$$\overline{f}(-x) = \sum_{n\in\mathbb{Z}} \overline{f}_n e^{inx} = \sum_{n\in\mathbb{Z}} f_n e^{inx} = f(x)$$

almost everywhere in $[0, 2\pi]$. This proves the Lemma.

To prove Theorem 1.1, let us first recall that by the Floquet-Bloch theory (see e.g. [BS], [Ea]), $\lambda \in \sigma(H(g))$ if and only if the equation $H(g)\psi = \lambda\psi$ has a non-constant bounded solution. In turn, all bounded solutions have the (Bloch) form

$$\psi_p(x; \lambda, g) = e^{ipx}\phi_p(x; \lambda, g) \tag{2.1}$$

where $p \in]-1/2, 1/2] := B$ (the Brillouin zone) and ϕ_p is 2π-periodic. It is indeed immediately checked that $\psi_p(x;\lambda,g)$ solves $H(g)\psi = \lambda\psi$ if and only if $\phi_p(x;\lambda,g)$ is a solution of

$$H_p(g)\phi_p(x;\lambda,g) = \lambda\phi_p(x;\lambda,g).$$

Here $H_p(g)$ is the operator in $L^2(0,2\pi)$ given by

$$H_p(g)u = \left(-i\frac{d}{dx}+p\right)^2 u + qu + gWu, \quad u \in D(H_p(g)) \tag{2.2}$$

with periodic boundary conditions; its realization will be recalled below. More precisely, denote S^1 the one-dimensional torus, i.e. the interval $[-\pi,\pi]$ with the endpoints identified. By Assumptions (1) and (2) above the restriction of q to S^1, still denoted q by a standard abuse of notation, belongs to $H^{-1}(S^1)$ and generates a real quadratic form $\mathcal{Q}_p(u)$ in $L^2(S^1)$ with domain $H^1(S^1)$. By Assumption (3) $\mathcal{Q}_p(u)$ is relatively bounded, with relative bound zero, with respect to

$$\mathcal{T}_p(u) := \int_{-\pi}^{\pi} [-iu' + pu][i\overline{u}' + p\overline{u}]\,dx, \quad D(\mathcal{T}_p(u)) = H^1(S^1) \tag{2.3}$$

so that the real semibounded form $\mathcal{H}_p^0(u) := \mathcal{T}_p(u) + \mathcal{Q}_p(u)$ defined on $H^1(S^1)$ is closed. The corresponding self-adjoint operator in $L^2(S^1)$ is the self-adjoint realization of the formal differential expression (note again the abuse of notation)

$$H_p(0) = -\frac{d^2}{dx^2} - 2ip\frac{d}{dx} + p^2 + q$$

As above, the form $\mathcal{H}_p(g)(u) := \mathcal{H}_p^0(u) + \langle u, Wu\rangle$ defined on $H^1(S^1)$ is closed and sectorial in $L^2(S^1)$. Let $H_p(g)$ the associated m-sectorial operator in $L^2(S^1)$. On $u \in D(H_p(g))$ the action of the operator $H_p(g)$ is specified by (2.2); moreover, $H_p(g)$ has compact resolvent. Let

$$\sigma(H_p(g)) := \{\lambda_n(g;p) : \ n = 0,1,\dots\}$$

denote the spectrum of $H_p(g)$, with $p \in]-1/2, 1/2]$, $|g| < \overline{g}$. By the above remarks we have

$$\sigma(H(g)) = \bigcup_{p\in]-1/2,1/2]} \sigma(H_p(g)) = \bigcup_{p,n} \lambda_n(g;p)$$

To prove the reality of $\sigma(H(g))$, $|g| < \overline{g}$, it is therefore enough to prove the reality of all eigenvalues $\lambda_n(g;p) : \ n = 0,1,\dots; p \in]-1/2,1/2]; n = 0,1,\dots$.

To this end, let us further recall the construction of the bands for $g = 0$: it can be proved that under the present conditions all eigenvalues $\lambda_n(0;p)$ are simple $\forall\, p \in [-1/2, 1/2]$; the functions $\lambda_n(0;p)$ are continuous and even in $[-1/2,1/2]$ with respect to

p, so that one can restrict to $p \in [0, 1/2]$; the functions $\lambda_{2k}(0;p)$ are strictly increasing on $[0, 1/2]$ while the functions $\lambda_{2k+1}(0;p)$ are strictly decreasing, $k = 0, 1 \ldots$. Set:

$$\alpha_k = \lambda_k(0,0), \quad \beta_k = \lambda_k(0,1/2)$$

Then:

$$\alpha_0 < \beta_0 < \beta_1 < \alpha_1 < \alpha_2 < \beta_2 < \beta_3 \ldots$$

The intervals $[\alpha_{2n}, \beta_{2n}]$ and $[\beta_{2n+1}, \alpha_{2n+1}]$ coincide with the range of $\lambda_{2n}(0,p)$, $\lambda_{2n+1}(0,p)$, respectively, and represent the bands of $\sigma(H(0))$; the intervals

$$\Delta_n :=]\beta_{2n}, \beta_{2n+1}[, \quad]\alpha_{2n+1}, \alpha_{2(n+2)}[$$

the gaps between the bands.

The monotonicity of the functions $\lambda_n(0,p)$ and Assumption (1.6) entail

$$\inf_n \min_{p \in [0,1/2]} |\lambda_n(0,p) - \lambda_{n+1}(0,p)| \geq 2d \tag{2.4}$$

Let us now state the following preliminary result:

Proposition 2.2

1. *Let $g \in \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ is the disk $\{g \ : \ |g| < \overline{g}\}$. For any n, there is a function $\lambda_n(g,p) : \overline{\mathcal{D}} \times [0, 1/2] \to \mathbb{C}$, holomorphic in g and continuous in p, such that $\lambda_n(g,p)$ is a simple eigenvalue of $\mathcal{H}_p(g)$ for all $(g,p) \in \overline{\mathcal{D}} \times [0, 1/2]$.*

2. $$\sup_{g \in \overline{\mathcal{D}}, p \in [0,1/2]} |\lambda_n(g,p) - \lambda_n(0,p)| < \frac{d}{2}$$

3. *If $g \in \mathbb{R} \cap \overline{\mathcal{D}}$ all eigenvalues $\lambda_n(g,p)$ are real;*

4. *If $g \in \mathbb{R} \cap \overline{\mathcal{D}}$ then $\sigma(\mathcal{H}_p(g)) \equiv \{\lambda_n(g,p)\}_{n=0}^{\infty}$.*

Assuming the validity of this Proposition the proof of Theorem 1.1 is immediate.

Proof of Theorem 1.1

Since the functions $\lambda_n(g;p)$ are real and continuous for $g \in \mathbb{R} \cap \overline{\mathcal{D}}$, $p \in [0, 1/2]$, we can define:

$$\alpha_{2n}(g) := \min_{p \in [0,1/2]} \lambda_n(g;p); \quad \beta_{2n}(g) := \max_{p \in [0,1/2]} \lambda_{2n}(g;p) \tag{2.5}$$

$$\alpha_{2n+1}(g) := \max_{p \in [0,1/2]} \lambda_{2n+1}(g;p); \quad \beta_{2n+1}(g) := \min_{p \in [0,1/2]} \lambda_{2n+1}(g;p) \tag{2.6}$$

Then:

$$\sigma(H(g)) = \bigcup_{n=0}^{\infty} B_n(g)$$

where the bands $B_n(g)$ are defined, in analogy with the $g = 0$ case, by:

$$\begin{aligned} B_{2n}(g) &:= [\alpha_{2n}(g), \beta_{2n}(g)] \\ B_{2n+1}(g) &:= [\beta_{2n+1}(g), \alpha 2n+1(g)] \end{aligned}$$

By Assertion 2 of Proposition 2.2 we have, $\forall\, n = 0, 1, \ldots, \forall\, g \in \overline{D}$:

$$\begin{aligned} &\alpha_{2n}(g) - \frac{d}{2} = \lambda_{2n}(0,0) - \frac{d}{2} \le \lambda_{2n}(0,p) - \frac{d}{2} \\ &\le \lambda_{2n}(g,p) \le \lambda_{2n}(0,1/2) + \frac{d}{2} = \beta_{2n} + \frac{d}{2} \end{aligned}$$

whence

$$\alpha_{2n}(g) - \frac{d}{2} \le \lambda_{2n}(g,p) \le \beta_{2n} + \frac{d}{2}, \quad \forall\, n,\ \forall\, g \in \overline{\mathcal{D}}.$$

This yields:

$$B_{2n}(g) \subset \left[\alpha_{2n} - \frac{d}{2}, \beta_{2n} + \frac{d}{2}\right]$$

By an analogous argument:

$$B_{2n+1}(g) \subset \left[\beta_{2n+1} - \frac{d}{2}, \alpha_{2n+1} + \frac{d}{2}\right]$$

Therefore the bands are pairwise disjoint, because the gaps

$$\Delta_n(g) :=]\beta_{2n}(g), \alpha_{2n}(g)[, \quad]\alpha_{2n+1}(g), \beta_{2n+1}(g)[$$

are all open and their width is no smaller than d. In fact, by (1.6) we have:

$$|\alpha_n - \alpha_{n+1}| \ge 2d, \quad |\beta_n - \beta_{n+1}| \le 2d.$$

This concludes the proof of the Theorem

We now prove separately the assertions of Proposition 2.2

Proof of Proposition 2.2, Assertions 1 and 2

Since the maximal multiplication operator by W is continuous in $L^2(S^1)$ with norm $\|W\|_\infty$, the operator family $\mathcal{H}_p(g)$ is a type-A holomorphic family with respect to $g \in \mathbb{C}$, uniformly with respect to $p \in [0, 1/2]$. Hence we can direct apply regular perturbation theory (see e.g.[Ka]): the perturbation expansion near any eigenvalue $\lambda_n(0;p)$ of $\mathcal{H}(0,p)$ exists and is convergent in $\overline{\mathcal{D}}$ for $g \in \overline{\mathcal{D}}$ to a simple eigenvalue $\lambda_n(g;p)$ of $H(g;p)$:

$$\lambda_n(g;p) = \lambda_n(0,p) + \sum_{s=1}^{\infty} \lambda_n^s(0;p) g^s, \quad g \in \overline{\mathcal{D}} \tag{2.7}$$

The convergence radius $r_n(p)$ is no smaller than $\overline{g}$. Hence $\overline{g}$ represents a lower bound for $r_n(p)$ independent of n and p. Moreover $\lambda_n^s(0;p)$ is continuous for all $p \in [0,1/2]$, and hence the same is true for the sum $\lambda_n(g;p)$. This proves Assertion 1.

To prove Assertion 2, recall that the coefficients $\lambda_n^s(0;p)$ fulfill the majorization (see [Ka], §II.3)

$$|\lambda_n^s(0;p)| \leq \left(\frac{2\|W\|_\infty}{\inf_k \min_{[0,1/2]} |\lambda_k(0,p) - \lambda_{k\pm1}(0,p)|}\right)^s \leq (\|W\|_\infty/d)^s \tag{2.8}$$

Therefore, by (2.7):

$$|\lambda_n(g;p) - \lambda_n(0,p)| \quad \leq \quad \frac{|g|\,(\|W\|_\infty/d)}{1-(2|g|\|W\|_\infty/d)} = \frac{|g|\|W\|_\infty}{d-|g|\|W\|_\infty} < \frac{d}{2}$$

whence the stated majorization on account of (1.7).

Proof of Proposition 2.2, Assertion 3

As is known, and anyway very easy to verify, the PT symmetry entails that the eigenvalues of a PT-symmetric operator are either real or complex conjugate. By standard regular perturbation theory (see e.g. [Ka], §VII.2) any eigenvalue $\lambda_n(0;p)$ of $H_p(0)$ is stable with respect to $H_p(g)$; since $\lambda_n(0,p)$ is simple, for g suitably small there is one and only one eigenvalue $\lambda_n(g,p)$ of $H_p(g)$ near $\lambda_n(0,p)$, and $\lambda_n(g,p) \to \lambda_n(0,p)$ as $g \to 0$. This excludes the existence of the complex conjugate eigenvalue $\overline{\lambda}_n(g,p)$ distinct from $\lambda_n(g,p)$. Thus for $g \in \mathbb{R}$, $|g|$ suitably small, $\lambda_n(g,p)$ is real. This entails the reality of series expansion (2.7) for g small and hence $\forall\, g \in \overline{\mathcal{D}}$. This in turn implies the reality of $\lambda_n(g,p) \forall\, g \in \overline{\mathcal{D}}$.

Proof of Proposition 2.2, Assertion 4

We repeat here the argument introduced in [CGS],[CG] to prove the analogous result in different contexts. We describe all details to make the paper self contained. We have seen that for any $r \in \mathbb{N}$ the Rayleigh-Schrödinger perturbation expansion associated with the eigenvalue $\lambda_r(g;p)$ of $H_p(g)$ which converges to $\lambda_r(0;p)$ as $g \to 0$, has radius of convergence no smaller than $\overline{g}$. Hence, $\forall g \in \mathbb{R}$ such that $|g| < \overline{g}$, $H_p(g)$ admits a sequence of real eigenvalues $\lambda_r(g;p), r \in \mathbb{N}$. We want to prove that for $|g| < \overline{g}, g \in \mathbb{R}$, $H_p(g)$ has no other eigenvalues. Thus all its eigenvalues are real. To this end, for any $r \in \mathbb{N}$ let $\mathcal{Q}_r$ denote the circle centered at $\lambda_r(0;p)$ with radius d. Then if $g \in \mathbb{R}$, $|g| < \overline{g}$, and $\lambda(g)$ is an eigenvalue of $H_p(g)$:

$$\lambda(g) \in \bigcup_{r \in \mathbb{N}} \mathcal{Q}_r.$$

In fact, denoting

$$R_0(z) := (H_p(0) - z)^{-1}$$

for any $z \notin \bigcup_{r\in\mathbb{N}} \mathcal{Q}_r$ we have

$$\|gWR_0(z)\| \le |g|\|W\|_\infty\|R_0(z)\| < \overline{g}\|W\|_\infty[\mathrm{dist}(z,\sigma(H_0))]^{-1} \le \frac{\overline{g}\|W\|_\infty}{d} < 1. \tag{2.9}$$

The last inequality in (2.9) follows directly from the definition (1.7) of $\overline{g}$. Thus, $z \in \rho(H_p(g))$ and

$$R(g,z) := (H_p(g)-z)^{-1} = R_0(z)[1+gWR_0(z)]^{-1}\,.$$

Now let $g_0 \in \mathbb{R}$ be fixed with $|g| < \overline{g}$. Without loss of generality we assume that $g_0 > 0$. Let $\lambda(g_0)$ be a given eigenvalue of $H_p(g_0)$. Then $\lambda(g_0)$ must be contained in the interior (and not on the boundary) of $\mathcal{Q}_{n_0}$ for some $n_0 \in \mathbb{N}$. Moreover if m_0 is the multiplicity of $\lambda(g_0)$, for g close to g_0 there are m_0 eigenvalues (counting multiplicities) $\lambda^{(s)}(g), s = 1.\ldots,m_0$, of $H_p(g)$ which converge to $\lambda(g_0)$ as $g \to g_0$ and each function $\lambda^{(s)}(g)$ represents a branch of one or several holomorphic functions which have at most algebraic singularities at $g = g_0$ (see [Ka], Thm. VII.1.8). Let us now consider any one of such branches $\lambda^{(s)}(g)$ for $0 < g < g_0$, suppressing the index s from now on. First of all we notice that, by continuity, $\lambda(g)$ cannot lie outside $\mathcal{Q}_{n_0}$ for g close to g_0. Moreover, if we denote Γ_t the boundary of the circle centered at $\lambda_{n_0}(0;p)$ with radius t, $0 < t \le d$, we have, for $z \in \Gamma_t$ and $0 < g \le g_0$,

$$\|gWR_0(z)\| \le g\|W\|_\infty[\mathrm{dist}(z,\sigma(H_p(0)))]^{-1} \le g\|W\|_\infty/t\,. \tag{2.10}$$

Then $t > g\|W\|_\infty$ implies $z \notin \sigma(H_p(g))$, i.e. if $z \in \sigma(H_p(g)) \cap \Gamma_t$ then $t \le g\|W\|_\infty < g_0\|W\|_\infty < \overline{g}\|W\|_\infty < d$. Hence we observe that as $g \to g_0^-$, $\lambda(g)$ is contained in the circle centered at $\lambda_{n_0}(0;p)$ and radius $g\|W\|_\infty$. Suppose that the holomorphic function $\lambda(g)$ is defined on the interval $]g_1,g_0]$ with $g_1 > 0$. We will show that it can be continued up to $g = 0$, and in fact up to $g = -\overline{g}$. From what has been established so far the function $\lambda(g)$ is bounded as $g \to g_1^+$. Thus, by the well known properties on the stability of the eigenvalues of the analytic families of operators, $\lambda(g)$ must converge to an eigenvalue $\lambda(g_1)$ of $H_p(g_1)$ as $g \to g_1^+$ and $\lambda(g_1)$ is contained in the circle centered at $\lambda_{n_0}(0;p)$ and radius $g_1\|W\|_\infty$. Repeating the argument starting now from $\lambda(g_1)$, we can continue $\lambda(g)$ to a holomorphic function on an interval $]g_2,g_1]$, which has at most an algebraic singularity at $g = g_2$. We build in this way a sequence $g_1 > g_2 > \ldots > g_n > \ldots$ which can accumulate only at $g = -\overline{g}$. In particular the function $\lambda(g)$ is piecewise holomorphic on $]-\overline{g},\overline{g}[$. But while passing through $g = 0$, $\lambda(g)$ coincides with the eigenvalue $\lambda_r(g;p)$ generated by an unperturbed eigenvalue $\lambda_r(0;p)$ of $H_p(0)$ (namely $\lambda_{n_0}(0;p)$), which represents a real analytic function defined for $g \in]-\overline{g}, \overline{g}[$. Thus, $\lambda(g_0)$ arises from this function and is therefore real. This concludes the proof of Assertion 4.

Proof of Theorem 1.2

Consider the operator $K_p(g)$ acting in $L^2(S^1)$, defined on the domain $H^2(S^1)$. By the Floquet-Bloch theory recalled above, we have again

$$\sigma(K(g)) = \bigcup_{p\in]-1/2,1/2]} \sigma(K_p(g)).$$

It is then enough to prove that there is $\eta > 0$ such that $K_p(g)$ has complex eigenvalues for $p \in]1/2-\eta, 1/2]$. Since $K_p(g)$ is PT-symmetric, eigenvalues may occur only in complex-conjugate pairs. The eigenvalues of $K_p(0)$ are $\lambda_n(0,p) = (n+p)^2 : n \in \mathbb{Z}$. The eigenvalue $\lambda_0(0,p) = p^2$ is simple $\forall\, p \in [0,1/2]$; any other eigenvalues is simple for $p \neq 0$, $p \neq 1/2$ and has multiplicity 2 for $p = 0$ or $p = 1/2$ because $n^2 = (-n)^2$ and $(n+1/2)^2 = (-n-1+1/2)^2$, $n = 0,1,\ldots$. For $p = 1/2$ a set of orthonormal eigenfunctions corresponding to the double eigenvalue $(n+1/2)^2 = (-n-1+1/2)^2$, $n = 0,1,\ldots$ is given by $\{u_n, u_{-n-1}\}$, where:

$$u_n := \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z}$$

Remark that, for $0 < p < 1/2$, u_n and u_{-n-1} are the (normalized) eigenfunctions corresponding to the simple eigenvalues $(n+p)^2$, $(-n-1+p)^2$. Let now $n = k \in \mathbb{N}$, and consider the 2×2 matrix

$$T := \begin{pmatrix} \langle u_k, W u_k\rangle & \langle u_k, W u_{-k-1}\rangle \\ \langle u_{-k-1}, W u_k\rangle & \langle u_{-k-1}, W u_{-k-1}\rangle \end{pmatrix}$$

A trivial computation yields

$$T := \begin{pmatrix} 0 & w_{2k+1} \\ w_{-2k-1} & 0 \end{pmatrix}$$

with purely imaginary eigenvalues $\mu^{\pm} = \pm i\sqrt{-w_{2k+1}w_{-2k-1}}$. By standard degenerate perturbation theory, for g small enough $\mathcal{H}_{1/2}(g)$ admits the pair of complex conjugate eigenvalues

$$\lambda_k^{\pm}(g,1/2) = (k+1/2)^2 \pm ig\sqrt{-w_{2k+1}w_{-2k-1}} + O(g^2) \tag{2.11}$$

Under the above assumptions, for any fixed $g \in \mathbb{R}$ the operator family $p \mapsto K_p(g)$ is type-A holomorphic in the sense of Kato (see [Ka], §VII.2) $\forall\, p \in \mathbb{C}$ because its domain does not depend on p and the scalar products $\langle u, K_p(g)u\rangle : u \in H^1(S^1)$ are obviously holomorphic functions of p $\forall\, p \in \mathbb{C}$. This entails the continuity with respect to p of the eigenvalues $\lambda_n(g;p)$. Therefore, for $|g|$ suitably small, there is $\eta(g) > 0$ such that

$$\operatorname{Im} \lambda_k^{\pm}(g,p) \neq 0, \quad 1/2 - \eta \leq p \leq 1/2$$

It follows (see e.g.[BS], [Ea]) that the complex arcs $\mathcal{E}_k^{\pm} := \text{Range}(\lambda_k^{\pm}(g,p)) : p \in [1/2 - \eta(g), 1/2]$ belong to the spectrum of $K(g)$. This concludes the proof of the theorem.

References

[AGHKH] S.Albeverio, F.Gesztesy, R.Heght-Krohn, H.Holden *Solvable models in quantum mechanics* , Springer-Verlag 1988

[Ah] Z.Ahmed, *Energy band structure due to a complex, periodic, PT invariant potential*, Phys.Lett. A **286**, 231-235 (2001)

[Be1] C. M. Bender, S. Boettcher, and P. N. Meisinger, *PT-Symmetric Quantum Mechanics* Journal of Mathematical Physics **40**,2201-2229 (1999)

[Be2] C.M.Bender, *Making Sense of Non-Hermitian Hamiltoniians* (hep-th/0703096)

[BBM] C. M. Bender, M. V. Berry, and A. Mandilara *Generalized PT Symmetry and Real Spectra*. J.Phys. A: Math. Gen. **35**, L467-L471 (2002)

[BBJ] C. M. Bender, D. C. Brody, and H. F. Jones *Must a Hamiltonian be Hermitian?*, American Journal of Physics, **71**, 1039-1031 (2003)

[BDM] C. M. Bender, G.V.Dunne, and N.Meisinger, *Complex periodic potentials with real band spectra*, Phys.Lett. A **252**, 272-276 (1999)

[BS] F.Berezin and M.S.Shubin *The Schrödinger equation*, Kluwer 1991

[Ce] J.M.Cerverò, *PT-symmetry in one-dimensional periodic potentials*, Phys.Lett. A **317**, 26-31 (2003)

[CG] E.Caliceti and S.Graffi *On a class of non self-adjoint quantum non-linear oscillators with real spectrum*, J.Nonlinear Math.Phys. **12**, 138-145 (2005)

[CR] J.M.Cerverò and J.M.Rodriguez, *The band spectrum of periodic potentials with PT symmetry*, J.Phys.A (2003)

[CGS] *Spectra of $\mathcal{PT}$-symmetric operators and perturbation theory*, J.Physics A, Math&Gen, **38**, 185-193 (2005)

[CJN] F.Cannata, M.V.Ioffe, D.N.Nishniadinze, *Two-dimensional SUSY Pseudo-Hermiticity without Separation of Variables*. Phys. Lett. **A310**, 344-352 (2003)

[CJT] F.Cannata, G.Junker and J.Trost, *Schrödinger operators with complex potential but real spectrum*, Phys Lett **A246** 219-226 (1998)

[Ea] M.S.P.Eastham, *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, 1973

[Jo] H.F.Jones, *The energy spectrum of complex periodic potentials of Kronig-Penney type*, Phys.Lett. A **262**, 242-244 (1999)

[LZ] G. Levai and M. Znojil, *Systematic search for PT symmetric potentials with real energy spectra* J. Phys. A: Math. Gen. **33** (2000) 7165.

[Ka] T.Kato, *Perturbation Theory for Linear Operators*, 2nd Edition, Springer-Verlag, 1976

[Sh] K.C.Shin, *On the shape of the spectra of non self-adjoint periodic Schrödinger operators*, J.Phys.A (2004)

[Sp] J. Phys.A, Math&Gen, **39**,n,32 (2006) (Special Issue on $\mathcal{PT}$-Symmetric Quantum Mechanics)

[ZCBR] M.Znojil, F.Cannata, B.Bagchi, R.Roychoudhury, *Supersymmetry without Hermiticity within PT symmetric quantum mechanics*. Phys. Lett. **B483**, 284 (2000)