LOCAL SPECTRAL PROPERTIES OF REFLECTIONLESS JACOBI, CMV, AND SCHRÖDINGER OPERATORS

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Dedicated with great pleasure to Ludwig Streit on the occasion of his 70th birthday.

Abstract. We prove that Jacobi, CMV, and Schrödinger operators, which are reflectionless on a homogeneous set $\mathcal{E}$ (in the sense of Carleson), under the assumption of a Blaschke-type condition on their discrete spectra accumulating at $\mathcal{E}$, have purely absolutely continuous spectrum on $\mathcal{E}$.

1. Introduction

In this paper we consider self-adjoint Jacobi and Schrödinger operators H on $\mathbb{Z}$ and $\mathbb{R}$, respectively, and unitary CMV operators U on $\mathbb{Z}$, which are reflectionless on a homogeneous set $\mathcal{E}$ contained in the essential spectrum. We prove that under the assumption of a Blaschke-type condition on their discrete spectra accumulating at $\mathcal{E}$, the operators H, respectively, U, have purely absolutely continuous spectrum on $\mathcal{E}$.

We note that homogeneous sets were originally discussed by Carleson [6]; we also refer to [32], [53], and [76] in this context. Morever, by results of Kotani [37]–[39] (also recorded in detail in [51, Theorem 12.5]), it is known that CMV, Jacobi, and Schrödinger operators, reflectionless on a set $\mathcal{E}$ of positive Lebesgue measure, have absolutely continuous spectrum on the essential closure of $\mathcal{E}$, denoted by $\overline{\mathcal{E}}^e$ (with uniform multiplicity two on $\mathcal{E}$). This result has recently been revisited in [20]. The focal point of this paper is to show that under suitable additional conditions on $\mathcal{E}$, such as $\mathcal{E}$ homogeneous, and a Blaschke-type condition on the discrete spectrum accumulating at $\mathcal{E}$, the spectrum is actually purely absolutely continuous on $\mathcal{E}$.

To put this result in some perspective, we briefly single out Schrödinger operators and illustrate the notion of being reflectionless: *Reflectionless* (self-adjoint) Schrödinger operators H in $L^2(\mathbb{R}; dx)$ can be characterized, for instance, by the fact that for all $x \in \mathbb{R}$ and for a.e. $\lambda \in \sigma_{\mathrm{ess}}(H)$, the diagonal Green's function of H has purely imaginary normal boundary values,

$$G(\lambda + i0, x, x) \in i\mathbb{R}. \tag{1.1}$$

Here $\sigma_{\mathrm{ess}}(H)$ denotes the essential spectrum of H (we assume $\sigma_{\mathrm{ess}}(H) \neq \emptyset$) and

$$G(z, x, x') = (H - zI)^{-1}(x, x'), \quad z \in \mathbb{C}\backslash\sigma(H), \tag{1.2}$$

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denotes the integral kernel of the resolvent of H. This global notion of reflectionless Schrödinger operators can of course be localized and extends to subsets of $\sigma_{\rm ess}(H)$ of positive Lebesgue measure. In the actual body of our paper we will use an alternative definition of the notion of reflectionless Schrödinger operators convenientl formulated directly in terms of half-line Weyl–Titchmarsh functions; we refer to Definitions 2.2, 3.2, and 4.2 for more details. For various discussions of classes of reflectionless differential and difference operators, we refer, for instance, to Craig [10], De Concini and Johnson [11], Deift and Simon [12], Gesztesy, Krishna, and Teschl [19], Gesztesy and Yuditskii [22], Johnson [29], Kotani [37], [38], Kotani and Krishna [40], Peherstorfer and Yuditskii [54], Remling [59], [60], Sims [70], and Sodin and Yuditskii [71]–[73]. In particular, we draw attention to the recent papers by Remling [59], [60], that illustrate in great depth the ramifications of the existence of absolutely continuous spectra in one-dimensional problems.

The trivial case $H_0 = -d^2/dx^2$, and the N-soliton potentials V_N, $N \in \mathbb{N}$, that is, exponentially decreasing solutions in $C^\infty(\mathbb{R})$ of some (and hence infinitely many) equations of the stationary Korteweg–de Vries (KdV) hierarchy, yield well-known examples of reflectionless Schrödinger operators $H_N = -d^2/dx^2 + V_N$. Similarly, all periodic Schrödinger operators are reflectionless. Indeed, if V_a is periodic with some period $a > 0$, that is, $V_a(x + a) = V_a(x)$ for a.e. $x \in \mathbb{R}$, then standard Floquet theoretic considerations show that the spectrum of $H_a = -d^2/dx^2 + V_a$ is a countable union of compact intervals (which may degenerate into a union of finitely-many compact intervals and a half-line) and the diagonal Green's function of H_a is purely imaginary for every point in the open interior of $\sigma(H_a)$. More generally, certain classes of quasi-periodic and almost periodic potentials also give rise to reflectionless Schrödinger operators with homogeneous spectra. The prime example of such quasi-periodic potentials is represented by the class of real-valued bounded algebro-geometric KdV potentials corresponding to an underlying (compact) hyperelliptic Riemann surface (see, e.g., [5, Ch. 3], [14], [18, Ch. 1], [30], [45, Chs. 8, 10], [47, Ch. 4], [50, Ch. II] and the literature cited therein). These examples yield reflectionless operators in a global sense, that is, they are reflectionless on the whole spectrum. On the other hand, as discussed, recently by Remling [59], the notion of being reflectionless also makes sense locally on subsets of the spectrum. More general classes of almost periodic Schrödinger operators, reflectionless on sets where the Lyapunov exponent vanishes, were studied by Avron and Simon [4], Carmona and Lacroix [7, Ch. VII], Chulaevskii [9], Craig [10], Deift and Simon [12], Egorova [15], Johnson [29], Johnson and Moser [31], Kotani [37]–[39], Kotani and Krishna [40], Levitan [41]–[44], [45, Chs. 9, 11], Levitan and Savin [46], Moser [48], Pastur and Figotin [51, Chs. V, VII], Pastur and Tkachenko [52], and Sodin and Yuditskii [71]–[73].

Analogous considerations apply to Jacobi operators (see, e.g., [7], [75] and the literature cited therein) and CMV operators (see [62]–[67] and the extensive list of references provided therein and [24] for the notion of reflectionless CMV operators).

In Section 2 we consider the case of Jacobi operators; CMV operators are studied in Section 3 followed by Schrödinger operators in Section 4. Herglotz and Weyl–Titchmarsh functions in connection with Jacobi and Schrödinger Operators are discussed in Appendix A; Caratheodory and Weyl–Titchmarsh functions for CMV operators are summarized in Appendix B.

2. Reflectionless Jacobi Operators

In this section we investigate spectral properties of self-adjoint Jacobi operators reflectionless on compact homogeneous subsets of the real line.

We start with some general considerations of self-adjoint Jacobi operators. Let $a = \{a(n)\}_{n\in\mathbb{Z}}$ and $b = \{b(n)\}_{n\in\mathbb{Z}}$ be two sequences (Jacobi parameters) satisfying

$$a, b \in \ell^\infty(\mathbb{Z}), \quad a(n) > 0,\ b(n) \in \mathbb{R},\ n \in \mathbb{Z}, \tag{2.1}$$

and denote by L the second-order difference expression defined by

$$L = aS^+ + a^- S^- + b, \tag{2.2}$$

where we use the notation for $f = \{f(n)\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$,

$$(S^\pm f)(n) = f(n\pm 1) = f^\pm(n),\ n \in \mathbb{Z}, \quad S^{++} = (S^+)^+,\ S^{--} = (S^-)^-, \text{ etc.} \tag{2.3}$$

Moreover, we introduce the associated bounded self-adjoint Jacobi operator H in $\ell^2(\mathbb{Z})$ by

$$(Hf)(n) = (Lf)(n), \quad n \in \mathbb{Z}, \quad f = \{f(n)\}_{n\in\mathbb{Z}} \in \operatorname{dom}(H) = \ell^2(\mathbb{Z}). \tag{2.4}$$

Next, let $g(z,\cdot)$ denote the diagonal Green's function of H, that is,

$$g(z,n) = G(z,n,n), \quad G(z,n,n') = (H - zI)^{-1}(n,n'),\ z \in \mathbb{C}\backslash\sigma(H),\ n,n' \in \mathbb{Z}. \tag{2.5}$$

Since for each $n \in \mathbb{Z}$, $g(\cdot,n)$ is a Herglotz function (i.e., it maps the open complex upper half-plane analytically to itself),

$$\xi(\lambda,n) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im}[\ln(g(\lambda+i\varepsilon,n))] \text{ for a.e. } \lambda \in \mathbb{R} \tag{2.6}$$

is well-defined for each $n \in \mathbb{Z}$. In particular, for all $n \in \mathbb{Z}$,

$$0 \le \xi(\lambda,n) \le 1 \text{ for a.e. } \lambda \in \mathbb{R}. \tag{2.7}$$

In the following we will frequently use the convenient abbreviation

$$h(\lambda_0 + i0) = \lim_{\varepsilon\downarrow 0} h(\lambda_0 + i\varepsilon), \quad \lambda_0 \in \mathbb{R}, \tag{2.8}$$

whenever the limit in (2.8) is well-defined and hence (2.6) can then be written as $\xi(\lambda,n) = (1/\pi)\operatorname{Arg}(g(\lambda+i0,n))$. Moreover, in this section we will use the convention that whenever the phrase a.e. is used without further qualification, it always refers to Lebesgue measure on $\mathbb{R}$.

Associated with H in $\ell^2(\mathbb{Z})$, we also introduce the two half-lattice Jacobi operators $H_{\pm,n_0}$ in $\ell^2([n_0,\pm\infty)\cap\mathbb{Z})$ by

$$H_{\pm,n_0} = P_{\pm,n_0} H P_{\pm,n_0}|_{\ell^2([n_0,\pm\infty)\cap\mathbb{Z})}, \tag{2.9}$$

where $P_{\pm,n_0}$ are the orthogonal projections onto the subspaces $\ell^2([n_0,\pm\infty)\cap\mathbb{Z})$. By inspection, $H_{\pm,n_0}$ satisfy Dirchlet boundary conditions at $n_0 \mp 1$, that is,

$$\begin{aligned} &(H_{\pm,n_0} f)(n) = (Lf)(n), \quad n \gtrless n_0, \\ &f \in \operatorname{dom}(H_{\pm,n_0}) = \ell^2([n_0,\pm\infty)\cap\mathbb{Z}), \quad f(n_0 \mp 1) = 0. \end{aligned} \tag{2.10}$$

The half-lattice Weyl–Titchmarsh m-functions associated with $H_{\pm,n_0}$ are denoted by $m_\pm(\cdot,n_0)$ and $M_\pm(\cdot,n_0)$,

$$m_\pm(z,n_0) = (\delta_{n_0}, (H_{\pm,n_0} - zI)^{-1}\delta_{n_0})_{\ell^2([n_0,\pm\infty)\cap\mathbb{Z})}, \quad z \in \mathbb{C}\backslash\sigma(H_{\pm,n_0}), \tag{2.11}$$

$$M_+(z,n_0) = -m_+(z,n_0)^{-1} - z + b(n_0), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.12}$$

$$M_-(z,n_0) = m_-(z,n_0)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.13}$$

where $\delta_k = \{\delta_{k,n}\}_{n\in\mathbb{Z}}$, $k \in \mathbb{Z}$. An equivalent definition of $M_\pm(\cdot,n_0)$ is

$$M_\pm(z,n_0) = -a(n_0)\frac{\psi_\pm(z,n_0+1)}{\psi_\pm(z,n_0)}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.14}$$

where $\psi_\pm(z,\cdot)$ are the Weyl–Titchmarsh solutions of $(L-z)\psi_\pm(z,\cdot) = 0$ with $\psi_\pm(z,\cdot) \in \ell^2([n_0,\pm\infty)\cap\mathbb{Z})$. Then it follows that the diagonal Green's function $g(\cdot,n_0)$ is related to the m-functions $M_\pm(\cdot,n_0)$ via

$$g(z,n_0) = [M_-(z,n_0) - M_+(z,n_0)]^{-1}. \tag{2.15}$$

For subsequent purpose we note the universal asymptotic z-behavior of $g(z,n_0)$, valid for all $n_0 \in \mathbb{Z}$,

$$g(z,n_0) \underset{|z|\to\infty}{=} -\frac{1}{z}[1+o(1)]. \tag{2.16}$$

Definition 2.1. *Let $\mathcal{E} \subset \mathbb{R}$ be a compact set which we may write as*

$$\mathcal{E} = [E_0,E_\infty]\Big\backslash\bigcup_{j\in J}(E_{2j-1},E_{2j}), \quad J \subseteq \mathbb{N}, \tag{2.17}$$

for some $E_0, E_\infty \in \mathbb{R}$ and $E_0 \leq E_{2j-1} < E_{2j} \leq E_\infty$, where $(E_{2j-1},E_{2j}) \cap (E_{2j'-1},E_{2j'}) = \emptyset$, $j,j' \in J$, $j \neq j'$. Then $\mathcal{E}$ is called homogeneous if

$$\begin{aligned} &\text{there exists an } \varepsilon > 0 \text{ such that for all } \lambda \in \mathcal{E} \\ &\text{and all } 0 < \delta < \mathrm{diam}(\mathcal{E}), \quad |\mathcal{E}\cap(\lambda-\delta,\lambda+\delta)| \geq \varepsilon\delta. \end{aligned} \tag{2.18}$$

Here $\mathrm{diam}(\mathcal{M})$ denotes the diameter of the set $\mathcal{M} \subset \mathbb{R}$.

Next, following [19], we introduce a special class of reflectionless Jacobi operators (cf. also [59] and [75, Lemma 8.1]).

Definition 2.2. *Let $\Lambda \subset \mathbb{R}$ be of positive Lebesgue measure. Then we call H reflectionless on Λ if for some $n_0 \in \mathbb{Z}$*

$$M_+(\lambda+i0,n_0) = \overline{M_-(\lambda+i0,n_0)} \ \text{ for a.e. } \lambda \in \Lambda. \tag{2.19}$$

Equivalently (cf. [21]), H is called reflectionless on Λ if for all $n \in \mathbb{Z}$,

$$\xi(\lambda,n) = 1/2 \ \text{ for a.e. } \lambda \in \Lambda. \tag{2.20}$$

In the following hypothesis we describe a special class $\mathcal{R}(\mathcal{E})$ of reflectionless Jacobi operators associated with a homogeneous set $\mathcal{E}$, that will be our main object of investigation in this section.

Hypothesis 2.3. *Let $\mathcal{E} \subset \mathbb{R}$ be a compact homogeneous set. Then $H \in \mathcal{R}(\mathcal{E})$ if*

(i) H is reflectionless on $\mathcal{E}$. (In particular, this implies $\mathcal{E} \subseteq \sigma_{\mathrm{ess}}(H)$.)

(ii) Either $\mathcal{E} = \sigma_{\mathrm{ess}}(H)$ or the set $\sigma_{\mathrm{ess}}(H)\backslash\mathcal{E}$ is closed. (In particular, this implies that there is an open set $\mathcal{O} \subseteq \mathbb{R}$ such that $\mathcal{E} \subset \mathcal{O}$ and $\overline{\mathcal{O}} \cap (\sigma_{\mathrm{ess}}(H)\backslash\mathcal{E}) = \emptyset$.)

(iii) The discrete eigenvalues of H that accumulate to $\mathcal{E}$ satisfy a Blaschke-type condition, that is,

$$\sum_{\lambda\in\sigma(H)\cap(\mathcal{O}\backslash\mathcal{E})} G_{\mathcal{E}}(\lambda,\infty) < \infty, \tag{2.21}$$

where $\mathcal{O}$ is the set defined in (ii) (hence $\sigma(H)\cap(\mathcal{O}\backslash\mathcal{E}) \subseteq \sigma_{\mathrm{disc}}(H)$ is a discrete countable set) and $G_{\mathcal{E}}(\cdot,\infty)$ is the potential theoretic Green's function for the *domain* $(\mathbb{C}\cup\{\infty\})\backslash\mathcal{E}$ *with logarithmic singularity at infinity* (*cf., e.g.,* [58, Sect. 5.2]), $G_{\mathcal{E}}(z,\infty) \underset{|z|\to\infty}{=} \log|z| - \log(\operatorname{cap}(\mathcal{E})) + o(1)$.

Here $\operatorname{cap}(\mathcal{E})$ denotes the (logarithmic) capacity of $\mathcal{E}$ (see, e.g., [68, App. A]).

One particularly interesting situation in which the above hypothesis is satisfied occurs when $\sigma(H) = \sigma_{\rm ess}(H)$ is a homogeneous set and H is reflectionless on $\sigma(H)$. This case has been studied in great detail by Sodin and Yuditskii [74].

Next, we present the main result of this section. For a Jacobi operator H in the class $\mathcal{R}(\mathcal{E})$, we will show the absence of the singular spectrum on the set $\mathcal{E}$. The proof of this fact relies on certain techniques developed in harmonic analysis and potential theory associated with domains $(\mathbb{C}\cup\{\infty\})\backslash\mathcal{E}$ studied by Peherstorfer, Sodin, and Yuditskii in [53] and [74]. For completeness, we provide the necessary result in Theorem A.5.

Theorem 2.4. *Assume Hypothesis* 2.3*, that is,* $H \in \mathcal{R}(\mathcal{E})$*. Then, the spectrum of* H *is purely absolutely continuous on* $\mathcal{E}$*,*

$$\sigma_{\rm ac}(H) \supseteq \mathcal{E}, \quad \sigma_{\rm sc}(H)\cap\mathcal{E} = \sigma_{\rm pp}(H)\cap\mathcal{E} = \emptyset. \tag{2.22}$$

Moreover, $\sigma(H)$ *has uniform multiplicity equal to two on* $\mathcal{E}$*.*

Proof. Fix $n\in\mathbb{Z}$. By the asymptotic behavior of the diagonal Green's function $g(\cdot,n)$ in (2.16) one concludes that $g(z,n)$ is a Herglotz function of the type (cf. (A.3) and (A.13)–(A.15))

$$g(z,n) = \int_{\sigma(H)} \frac{d\Omega_{0,0}(\lambda,n)}{\lambda - z}, \quad z\in\mathbb{C}\backslash\sigma(H). \tag{2.23}$$

Next, we introduce two Herglotz functions $r_j(z,n)$, $j=1,2$, by

$$r_1(z,n) = \int_{\mathcal{O}} \frac{d\Omega_{0,0}(\lambda,n)}{\lambda - z} = \sum_{\lambda\in\mathcal{O}\backslash\mathcal{E}} \frac{\Omega_{0,0}(\{\lambda\},n)}{\lambda - z} + \int_{\mathcal{E}} \frac{d\Omega_{0,0}(\lambda,n)}{\lambda - z}, \quad z\in\mathbb{C}\backslash\mathcal{O}, \tag{2.24}$$

$$r_2(z,n) = \int_{\sigma(H)\backslash\mathcal{O}} \frac{d\Omega_{0,0}(\lambda,n)}{\lambda - z}, \quad z\in(\mathbb{C}\backslash\sigma(H))\cup\mathcal{O}, \tag{2.25}$$

where $\mathcal{O}$ is the set defined in Hypothesis 2.3 (ii). Then it is easy to see that

$$g(z,n) = r_1(z,n) + r_2(z,n), \quad z\in\mathbb{C}\backslash\sigma(H). \tag{2.26}$$

Since $H\in\mathcal{R}(\mathcal{E})$, one has $\xi(\lambda+i0,n) = 1/2$ and hence $\operatorname{Re}[g(\lambda+i0,n)] = 0$ for a.e. $\lambda\in\mathcal{E}$. This yields

$$\operatorname{Re}[r_1(\lambda+i0,n)] = -\operatorname{Re}[r_2(\lambda+i0,n)] \text{ for a.e. } \lambda\in\mathcal{E}. \tag{2.27}$$

Observing that the function $r_2(\cdot,n)$ is analytic on $(\mathbb{C}\backslash\sigma(H))\cup\mathcal{O}$ and $\mathcal{E}\subset\mathcal{O}$, one concludes that $r_2(\cdot,n)$ is bounded on $\mathcal{E}$, and hence,

$$\operatorname{Re}[r_1(\cdot+i0,n)] = -\operatorname{Re}[r_2(\cdot+i0,n)] \in L^1\big(\mathcal{E};dx\big). \tag{2.28}$$

Moreover, it follows from Theorem A.4 that the set of mass points of $d\Omega_{0,0}$ is a subset of the set of discrete eigenvalues of H, hence (2.21), (2.24), and (2.28) imply that the function $r_1(\cdot,n)$ satisfies the assumptions of Theorem A.5. Thus, the restriction $d\Omega_{0,0}|_{\mathcal{E}}$ of the measure $d\Omega_{0,0}$ to the set $\mathcal{E}$ is purely absolutely continuous,

$$d\Omega_{0,0}(\cdot,n)\big|_{\mathcal{E}} = d\Omega_{0,0,\rm ac}(\cdot,n)\big|_{\mathcal{E}} = \frac{1}{\pi}\operatorname{Im}[r_1(\cdot+i0,n)]d\lambda\big|_{\mathcal{E}}, \quad n\in\mathbb{Z}. \tag{2.29}$$

Finally, utilizing the formulas

$$d\Omega^{\rm tr}(\cdot,n)=d\Omega_{0,0}(\cdot,n)+d\Omega_{1,1}(\cdot,n) \text{ and } d\Omega_{1,1}(\cdot,n)=d\Omega_{0,0}(\cdot,n+1) \tag{2.30}$$

one concludes from (2.29)

$$d\Omega^{\rm tr}(\cdot,n)\big|_{\mathcal{E}}=d\Omega^{\rm tr}_{\rm ac}(\cdot,n)\big|_{\mathcal{E}}, \tag{2.31}$$

for the restriction of the trace measure $d\Omega^{\rm tr}(\cdot,n)$ associated with H. By (A.19) and Theorem A.4 (i) this completes the proof of (2.22).

Finally, equations (2.15) and (2.19) imply

$$-1/g(\lambda+i0,n)=\pm 2i\,{\rm Im}[M_\pm(\lambda+i0,n)] \text{ for a.e. } \lambda\in\mathcal{E}. \tag{2.32}$$

Thus, combining (2.19), (2.32), and (A.24) then yields that the absolutely continuous spectrum of H has uniform spectral multiplicity two on $\mathcal{E}$ since

$$\text{for a.e. } \lambda\in\mathcal{E},\; 0<\pm{\rm Im}[M_\pm(\lambda+i0,n)]<\infty. \tag{2.33}$$

□

For reflectionless measures with singular components we refer to the recent preprint [49] (see also [22]).

3. Reflectionless CMV Operators

In this section we investigate spectral properties of unitary CMV operators reflectionless on compact homogeneous subsets of the unit circle.

We start with some general considerations of unitary CMV operators. Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be a complex-valued sequence of Verblunsky coefficients satisfying

$$\alpha_n\in\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\},\quad n\in\mathbb{Z}, \tag{3.1}$$

and denote by $\{\rho_n\}_{n\in\mathbb{Z}}$ an auxiliary real-valued sequence defined by

$$\rho_n=\big[1-|\alpha_n|^2\big]^{1/2},\quad n\in\mathbb{Z}. \tag{3.2}$$

Then we introduce the associated unitary CMV operator U in $\ell^2(\mathbb{Z})$ by its matrix representation in the standard basis of $\ell^2(\mathbb{Z})$,

$$U=\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 0 & -\alpha_0\rho_{-1} & -\overline{\alpha_{-1}}\alpha_0 & -\alpha_1\rho_0 & \rho_0\rho_1 & & & \\ & & \rho_{-1}\rho_0 & \overline{\alpha_{-1}}\rho_0 & -\overline{\alpha_0}\alpha_1 & \overline{\alpha_0}\rho_1 & 0 & & \\ & & & 0 & -\alpha_2\rho_1 & -\overline{\alpha_1}\alpha_2 & -\alpha_3\rho_2 & \rho_2\rho_3 & \\ & & & & \rho_1\rho_2 & \overline{\alpha_1}\rho_2 & -\overline{\alpha_2}\alpha_3 & \overline{\alpha_2}\rho_3 & 0 \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.3}$$

Here terms of the form $-\overline{\alpha_n}\alpha_{n+1}$ represent the diagonal (n,n)-entries, $n\in\mathbb{Z}$, in the infinite matrix (3.3). Equivalently, one can define U by (cf. (2.3))

$$\begin{aligned} U&=\rho^-\rho\,\delta_{\rm even}\,S^{--}+(\overline{\alpha^-}\rho\,\delta_{\rm even}-\alpha^+\rho\,\delta_{\rm odd})S^- -\overline{\alpha}\alpha^+\\ &\quad+(\overline{\alpha}\rho^+\,\delta_{\rm even}-\alpha^{++}\rho^+\,\delta_{\rm odd})S^+ +\rho^+\rho^{++}\,\delta_{\rm odd}\,S^{++}, \end{aligned} \tag{3.4}$$

where $\delta_{\rm even}$ and $\delta_{\rm odd}$ denote the characteristic functions of the even and odd integers,

$$\delta_{\rm even}=\chi_{2\mathbb{Z}},\quad \delta_{\rm odd}=1-\delta_{\rm even}=\chi_{2\mathbb{Z}+1}. \tag{3.5}$$

Moreover, let $M_{1,1}(z,n)$ denote the diagonal element of the Cayley transform of U, that is,

$$M_{1,1}(z,n) = ((U+zI)(U-zI)^{-1})(n,n) = \oint_{\partial\mathbb{D}} d\Omega_{1,1}(\zeta,n)\,\frac{\zeta+z}{\zeta-z}, \quad z\in\mathbb{C}\backslash\sigma(U),\ n\in\mathbb{Z}, \tag{3.6}$$

where $d\Omega_{1,1}(\cdot,n)$, $n\in\mathbb{Z}$, are scalar-valued probability measures on $\partial\mathbb{D}$ (cf. [23, Section 3] for more details). Since $M_{1,1}(\cdot,n)$ is a Caratheodory function (i.e., it maps the open unit disk analytically to the complex right half-plane),

$$\Xi_{1,1}(\zeta,n) = \frac{1}{\pi}\lim_{r\uparrow 1}\operatorname{Im}[\ln(M_{1,1}(r\zeta,n))] \text{ for a.e. } \zeta\in\partial\mathbb{D} \tag{3.7}$$

is well-defined for each $n\in\mathbb{Z}$. In particular, for all $n\in\mathbb{Z}$,

$$-1/2 \le \Xi_{1,1}(\zeta,n) \le 1/2 \text{ for a.e. } \zeta\in\partial\mathbb{D} \tag{3.8}$$

(cf. [24, Section 2] for more details).

In the following we will frequently use the convenient abbreviation

$$h(\zeta) = \lim_{r\uparrow 1} h(r\zeta), \quad \zeta\in\partial\mathbb{D}, \tag{3.9}$$

whenever the limit in (3.9) is well-defined and hence (3.7) can then be written as $\Xi_{1,1}(\zeta,n) = (1/\pi)\operatorname{Arg}(M_{1,1}(\zeta,n))$. Moreover, in this section we will use the convention that whenever the phrase a.e. is used without further qualification, it always refers to Lebesgue measure on $\partial\mathbb{D}$.

Associated with U in $\ell^2(\mathbb{Z})$, we also introduce the two half-lattice CMV operators $U_{\pm,n_0}$ in $\ell^2([n_0,\pm\infty)\cap\mathbb{Z})$ by setting $\alpha_{n_0}=1$ which splits the operator U into a direct sum of two half-lattice operators U_{-,n_0-1} and U_{+,n_0}, that is,

$$U = U_{-,n_0-1}\oplus U_{+,n_0} \text{ in } \ell^2((-\infty,n_0-1]\cap\mathbb{Z})\oplus\ell^2([n_0,\infty)\cap\mathbb{Z}). \tag{3.10}$$

The half-lattice Weyl–Titchmarsh m-functions associated with $U_{\pm,n_0}$ are denoted by $m_\pm(\cdot,n_0)$ and $M_\pm(\cdot,n_0)$,

$$m_\pm(z,n_0) = ((U_{\pm,n_0}+zI)(U_{\pm,n_0}-zI)^{-1})(n_0,n_0), \quad z\in\mathbb{C}\backslash\sigma(U_{\pm,n_0}), \tag{3.11}$$

$$M_+(z,n_0) = m_+(z,n_0), \quad z\in\mathbb{C}\backslash\partial\mathbb{D}, \tag{3.12}$$

$$M_-(z,n_0) = \frac{\operatorname{Re}(1+\alpha_{n_0}) + i\operatorname{Im}(1-\alpha_{n_0})m_-(z,n_0-1)}{i\operatorname{Im}(1+\alpha_{n_0}) + \operatorname{Re}(1-\alpha_{n_0})m_-(z,n_0-1)}, \quad z\in\mathbb{C}\backslash\partial\mathbb{D}. \tag{3.13}$$

Then it follows that $m_\pm(\cdot,n_0)$ and $\pm M_\pm(\cdot,n_0)$ are Caratheodory functions (cf. [23, Section 2]). Moreover, the function $M_{1,1}(\cdot,n_0)$ is related to the m-functions $M_\pm(\cdot,n_0)$ by (cf. [23, Lemma 3.2])

$$M_{1,1}(z,n_0) = \frac{1-M_+(z,n_0)M_-(z,n_0)}{M_+(z,n_0)-M_-(z,n_0)}. \tag{3.14}$$

Definition 3.1. *Let $\mathcal{E}\subseteq\partial\mathbb{D}$ be a compact set which we may write as*

$$\mathcal{E} = \partial\mathbb{D}\backslash\bigcup_{j\in J}\operatorname{Arc}(e^{i\theta_{2j-1}},e^{i\theta_{2j}}), \quad J\subseteq\mathbb{N}, \tag{3.15}$$

where $\operatorname{Arc}(e^{i\theta_{2j-1}},e^{i\theta_{2j}}) = \{e^{i\theta}\in\partial\mathbb{D} \mid \theta_{2j-1}<\theta<\theta_{2j}\}$, $\theta_{2j-1}\in[0,2\pi)$, $\theta_{2j-1}<\theta_{2j}\le\theta_{2j-1}+2\pi$, $\operatorname{Arc}(e^{i\theta_{2j-1}},e^{i\theta_{2j}})\cap\operatorname{Arc}(e^{i\theta_{2j'-1}},e^{i\theta_{2j'}})=\emptyset$, $j,j'\in J$, $j\ne j'$.

Then $\mathcal{E}$ is called homogeneous if

$$\begin{aligned} &\text{there exists an } \varepsilon > 0 \text{ such that for all } e^{i\theta} \in \mathcal{E} \\ &\text{and all } \delta > 0, \quad |\mathcal{E} \cap \mathrm{Arc}(e^{i(\theta-\delta)}, e^{i(\theta+\delta)})| \geq \varepsilon\delta. \end{aligned} \tag{3.16}$$

Next, we introduce a special class of reflectionless CMV operators (cf. [24] for a similar definition).

Definition 3.2. *Let $\Lambda \subseteq \partial\mathbb{D}$ be of positive Lebesgue measure. Then we call U reflectionless on Λ if for some (equivalently, for all) $n_0 \in \mathbb{Z}$*

$$M_+(\zeta, n_0) = -\overline{M_-(\zeta, n_0)} \text{ for a.e. } \zeta \in \Lambda. \tag{3.17}$$

We note that if U is reflectionless on Λ, then by (3.7), (3.14), and (3.17), one has for all $n \in \mathbb{Z}$,

$$\Xi_{1,1}(\zeta, n) = 0 \text{ for a.e. } \zeta \in \Lambda. \tag{3.18}$$

In the following hypothesis we introduce a special class $\mathcal{R}(\mathcal{E})$ of reflectionless CMV operators associated with a homogeneous set $\mathcal{E}$, that will be the main object of investigation in this section.

Hypothesis 3.3. *Let $\mathcal{E} \subseteq \partial\mathbb{D}$ be a compact homogeneous set. Then $U \in \mathcal{R}(\mathcal{E})$ if*

(i) U is reflectionless on $\mathcal{E}$. (In particular, this implies $\mathcal{E} \subseteq \sigma_{\mathrm{ess}}(U)$.)

(ii) Either $\mathcal{E} = \sigma_{\mathrm{ess}}(U)$ or the set $\sigma_{\mathrm{ess}}(U)\backslash\mathcal{E}$ is closed. (In particular, this implies that there is an open set $\mathcal{O} \subseteq \partial\mathbb{D}$ such that $\mathcal{E} \subseteq \mathcal{O}$ and $\overline{\mathcal{O}} \cap (\sigma_{\mathrm{ess}}(U)\backslash\mathcal{E}) = \emptyset$.)

(iii) If $\sigma_{\mathrm{ess}}(U) \neq \partial\mathbb{D}$ then the discrete eigenvalues of U that accumulate to $\mathcal{E}$ satisfy a Blaschke-type condition, that is,

$$\sum_{\zeta \in \sigma(U) \cap (\mathcal{O}\backslash\mathcal{E})} G_{\mathcal{E}}(\zeta, \zeta_0) < \infty, \tag{3.19}$$

where $\mathcal{O}$ is the set defined in (ii) (hence $\sigma(U) \cap (\mathcal{O}\backslash\mathcal{E}) \subseteq \sigma_{\mathrm{disc}}(U)$ is a discrete countable set), $\zeta_0 \in \partial\mathbb{D}\backslash\sigma(U)$ is some fixed point, and $G_{\mathcal{E}}(\cdot, \zeta_0)$ is the potential theoretic Green's function for the domain $(\mathbb{C} \cup \{\infty\})\backslash\mathcal{E}$ with logarithmic singularity at ζ_0 (cf., e.g., [58, Sect. 4.4]), $G_{\mathcal{E}}(z, \zeta_0) \underset{z \to \zeta_0}{=} \log|z - \zeta_0|^{-1} + O(1)$.

One particularly interesting situation in which the above hypothesis is satisfied occurs when $\sigma(U) = \sigma_{\mathrm{ess}}(U)$ is a homogeneous set and U is reflectionless on $\sigma(U)$. This case has first been studied by Peherstorfer and Yudiskii [54].

Next, we turn to the principal result of this section. For a CMV operator U in the class $\mathcal{R}(\mathcal{E})$, we will show that U has purely absolutely continuous spectrum on $\mathcal{E}$, that is, we intend to prove that

$$\sigma_{\mathrm{ac}}(U) \supseteq \mathcal{E}, \quad \sigma_{\mathrm{sc}}(U) \cap \mathcal{E} = \sigma_{\mathrm{pp}}(U) \cap \mathcal{E} = \emptyset. \tag{3.20}$$

We start with an elementary lemma which permits one to apply Theorem A.5 to Caratheodory functions.

Lemma 3.4. *Let f be a Caratheodory function with representation*

$$\begin{aligned} &f(w) = ic + \oint_{\partial\mathbb{D}} d\omega(\zeta)\, \frac{\zeta + w}{\zeta - w}, \quad w \in \mathbb{D}, \\ &c = \mathrm{Im}(f(0)), \quad \oint_{\partial\mathbb{D}} d\omega(\zeta) = \mathrm{Re}(f(0)) < \infty, \;\; \mathrm{supp}\,(d\omega) \neq \partial\mathbb{D}, \end{aligned} \tag{3.21}$$

where $d\omega$ denotes a nonnegative measure on $\partial\mathbb{D}$. Consider the change of variables

$$w \mapsto z = -i\frac{w+w_0}{w-w_0}, \quad w = w_0\frac{z-i}{z+i}, \quad z \in \mathbb{C}\cup\{\infty\}, \tag{3.22}$$

for some fixed $w_0 \in \partial\mathbb{D}\backslash\operatorname{supp}(d\omega)$.

Then, the function $r(z) = if(w(z))$ is a Herglotz function with the representation

$$r(z) = if(w(z)) = d + \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+, \tag{3.23}$$

$$d = if(w_0), \ \ d\mu(\lambda) = (1+\lambda^2)\, d\omega(w(\lambda)), \quad \lambda \in \mathbb{R}, \tag{3.24}$$

$$\operatorname{supp}(d\mu) \subseteq \big[-1 - 2\operatorname{dist}(w_0, \operatorname{supp}(d\omega))^{-1}, 1 + 2\operatorname{dist}(w_0, \operatorname{supp}(d\omega))^{-1}\big]. \tag{3.25}$$

In particular, $d\mu$ is purely absolutely continuous on Λ if and only if $d\omega$ is purely absolutely continuous on $w(\Lambda)$, and

$$\text{if } \ d\omega(e^{i\theta})\big|_{w(\Lambda)} = \omega'(e^{i\theta})d\theta\big|_{w(\Lambda)}, \ \text{ then } \ d\mu(\lambda)\big|_{\Lambda} = 2\omega'(w(\lambda))\, d\lambda\big|_{\Lambda}. \tag{3.26}$$

Proof. This is a straightforward computation. We note that

$$\oint_{\partial\mathbb{D}} d\omega(\zeta) < \infty \ \text{ is equivalent to } \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2} < \infty. \tag{3.27}$$

□

The principal result of this section then reads as follows.

Theorem 3.5. *Assume Hypothesis* 3.3*, that is, $U \in \mathcal{R}(\mathcal{E})$. Then, the spectrum of U is purely absolutely continuous on $\mathcal{E}$,*

$$\sigma_{\rm ac}(U) \supseteq \mathcal{E}, \quad \sigma_{\rm sc}(U)\cap\mathcal{E} = \sigma_{\rm pp}(U)\cap\mathcal{E} = \emptyset. \tag{3.28}$$

Moreover, $\sigma(U)$ has uniform multiplicity equal to two on $\mathcal{E}$.

Proof. We consider two cases. First, suppose that $\sigma_{\rm ess}(U) \neq \partial\mathbb{D}$. Then using Lemma 3.4, we introduce the Herglotz function $r(\cdot,n)$, $n\in\mathbb{Z}$, by

$$r(z,n) = iM_{1,1}(w(z),n) = iM_{1,1}(\zeta_0,n) + \int_{\mathbb{R}} \frac{d\mu(\lambda,n)}{\lambda - z}, \quad z\in\mathbb{C}_+, \tag{3.29}$$

where ζ_0 is defined in Hypothesis 3.3 (iii) and $w(z) = \zeta_0\frac{z-i}{z+i}$. Abbreviating by $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{O}}$ the preimages of the sets $\mathcal{E}$ and $\mathcal{O}$ under the bijective map w,

$$\widetilde{\mathcal{E}} = w^{-1}(\mathcal{E}), \quad \widetilde{\mathcal{O}} = w^{-1}(\mathcal{O}), \tag{3.30}$$

we also introduce functions $r_j(z,n)$, $j=1,2$, by

$$r_1(z,n) = \int_{\widetilde{\mathcal{O}}} \frac{d\mu(\lambda,n)}{\lambda-z} = \sum_{\lambda\in\widetilde{\mathcal{O}}\backslash\widetilde{\mathcal{E}}} \frac{\mu(\{\lambda\},n)}{\lambda-z} + \int_{\widetilde{\mathcal{E}}} \frac{d\mu(\lambda,n)}{\lambda-z}, \quad z\in\mathbb{C}\backslash\widetilde{\mathcal{O}}, \tag{3.31}$$

$$r_2(z,n) = \int_{\mathbb{R}\backslash\widetilde{\mathcal{O}}} \frac{d\mu(\lambda,n)}{\lambda-z}, \quad z\in(\mathbb{C}\backslash\mathbb{R})\cup\widetilde{\mathcal{O}}. \tag{3.32}$$

Then

$$r(z,n) = r_1(z,n) + r_2(z,n), \quad z\in\mathbb{C}\backslash\mathbb{R}, \ n\in\mathbb{Z}. \tag{3.33}$$

Since $U \in \mathcal{R}(\mathcal{E})$, one has $\Xi_{1,1}(\zeta,n) = 0$ and hence $\mathrm{Im}[M_{1,1}(\zeta,n)] = 0$ for all $n \in \mathbb{Z}$ and a.e. $\zeta \in \mathcal{E}$. This yields for each $n \in \mathbb{Z}$, $\mathrm{Re}[r(\lambda+i0,n)] = 0$ for a.e. $\lambda \in \widetilde{\mathcal{E}}$, and hence,

$$\mathrm{Re}[r_1(\lambda+i0,n)] = -\mathrm{Re}[r_2(\lambda+i0,n)] \ \text{ for a.e. } \lambda \in \widetilde{\mathcal{E}}. \tag{3.34}$$

Observing that the function $r_2(\cdot,n)$ is analytic on $(\mathbb{C}\backslash\mathbb{R}) \cup \widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{O}}$, one concludes that $r_2(\cdot,n)$ is bounded on $\widetilde{\mathcal{E}}$, and hence,

$$\mathrm{Re}[r_1(\cdot+i0,n)] = -\mathrm{Re}[r_2(\cdot+i0,n)] \in L^1\big(\widetilde{\mathcal{E}};dx\big), \quad n \in \mathbb{Z}. \tag{3.35}$$

Moreover, it follows from [16, Proposition 5.1] that (3.19) is equivalent to

$$\sum_{\lambda\in\widetilde{\mathcal{O}}\backslash\widetilde{\mathcal{E}}} G_{\widetilde{\mathcal{E}}}(\lambda,\infty) < \infty, \tag{3.36}$$

and from [23, Corollary 3.5] that the set of discrete mass points of $d\mu(\cdot,n)$ is a subset of $\widetilde{\mathcal{O}}\backslash\widetilde{\mathcal{E}}$, hence (3.31), (3.35), and (3.36) imply that the function $r_1(\cdot,n)$ satisfies the assumptions of Theorem A.5 for each $n \in \mathbb{Z}$. Thus, the restriction $d\mu(\cdot,n)|_{\widetilde{\mathcal{E}}}$ of the measure $d\mu(\cdot,n)$ to the set $\widetilde{\mathcal{E}}$ is purely absolutely continuous,

$$d\mu(\cdot,n)\big|_{\widetilde{\mathcal{E}}} = d\mu_{\mathrm{ac}}(\cdot,n)\big|_{\widetilde{\mathcal{E}}} = \frac{1}{\pi}\mathrm{Im}[r_1(\cdot+i0,n)]d\lambda\big|_{\widetilde{\mathcal{E}}}, \quad n \in \mathbb{Z}, \tag{3.37}$$

and hence, it follows from Lemma 3.4 that

$$d\Omega_{1,1}(\cdot,n)\big|_{\mathcal{E}} = d\Omega_{1,1,\mathrm{ac}}(\cdot,n)\big|_{\mathcal{E}}, \quad n \in \mathbb{Z}. \tag{3.38}$$

By Theorem B.4, and in particular (B.30), this proves (3.28) in the case $\sigma_{\mathrm{ess}}(U) \neq \partial\mathbb{D}$.

Next, suppose $\sigma_{\mathrm{ess}}(U) = \partial\mathbb{D}$. In this case it follows from a special case of the Borg-type theorem proven in [24, Theorem 5.1] (cf. also [64, Sect. 11.14] for a more restrictive version of this theorem) that the Verblunsky coefficients $\alpha_n = 0$ for all $n \in \mathbb{Z}$. Hence, (3.4) implies that U is unitarily equivalent to a direct sum of two shift operators in $\ell^2(\mathbb{Z})$ (U shifts odd entries to the left and even entries to the right). Thus U has purely absolutely continuous spectrum on $\partial\mathbb{D}$, which proves (3.28) in the case $\sigma_{\mathrm{ess}}(U) = \partial\mathbb{D}$.

Finally, equations (3.14) and (3.17) imply

$$\frac{1}{M_{1,1}(\zeta,n)} = \frac{\pm 2\,\mathrm{Re}[M_\pm(\zeta,n)]}{1+|M_\pm(\zeta,n)|^2} \ \text{ for a.e. } \zeta \in \mathcal{E}. \tag{3.39}$$

Combining (3.17), (3.39), and (B.32) then yields that the absolutely continuous spectrum of U has uniform spectral multiplicity two on $\mathcal{E}$ since

$$\text{for a.e. } \zeta \in \mathcal{E}, \ 0 < \pm\mathrm{Re}[M_\pm(\zeta,n)] < \infty. \tag{3.40}$$

□

4. Reflectionless Schrödinger Operators

In this section we discuss spectral properties of self-adjoint Schrödinger operators reflectionless on a homogeneous subsets of the real line bounded from below.

We start with some general considerations of one-dimensional Schrödinger operators. Let

$$V \in L^\infty(\mathbb{R};dx), \quad V \ \text{ real-valued}, \tag{4.1}$$

and consider the differential expression

$$L = -d^2/dx^2 + V(x), \quad x \in \mathbb{R}. \tag{4.2}$$

We denote by H the corresponding self-adjoint realization of L in $L^2(\mathbb{R}; dx)$ given by

$$Hf = Lf, \quad f \in \mathrm{dom}(H) = H^2(\mathbb{R}), \tag{4.3}$$

with $H^2(\mathbb{R})$ the usual Sobolev space. Let $g(z,\cdot)$ denote the diagonal Green's function of H, that is,

$$g(z,x) = G(z,x,x), \quad G(z,x,x') = (H-zI)^{-1}(x,x'), \quad z \in \mathbb{C}\backslash\sigma(H),\ x,x' \in \mathbb{R}. \tag{4.4}$$

Since for each $x \in \mathbb{R}$, $g(\cdot,x)$ is a Herglotz function,

$$\xi(\lambda,x) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}[\ln(g(\lambda+i\varepsilon,x))] \text{ for a.e. } \lambda\in\mathbb{R} \tag{4.5}$$

is well-defined for each $x \in \mathbb{R}$. In particular, for all $x \in \mathbb{R}$,

$$0 \leq \xi(\lambda,x) \leq 1 \text{ for a.e. } \lambda \in \mathbb{R}. \tag{4.6}$$

In the following we will frequently use the convenient abbreviation

$$h(\lambda_0+i0) = \lim_{\varepsilon\downarrow 0} h(\lambda_0+i\varepsilon), \quad \lambda_0 \in \mathbb{R}, \tag{4.7}$$

whenever the limit in (4.7) is well-defined and hence (4.5) can then be written as $\xi(\lambda,x) = (1/\pi)\mathrm{Arg}(g(\lambda+i0,x))$. Moreover, in this section we will use the convention that whenever the phrase a.e. is used without further qualification, it always refers to Lebesgue measure on $\mathbb{R}$.

Associated with H in $L^2(\mathbb{R}; dx)$ we also introduce the two half-line Schrödinger operators $H_{\pm,x_0}$ in $L^2([x_0,\pm\infty); dx)$ with Dirchlet boundary conditions at the finite endpoint $x_0 \in \mathbb{R}$,

$$\begin{aligned} &H_{\pm,x_0} f = Lf, \\ &f \in \mathrm{dom}(H_{\pm,x_0}) = \big\{g \in L^2([x_0,\pm\infty); dx) \,|\, g, g' \in AC([x_0, x_0 \pm R]) \text{ for all } R > 0; \\ &\qquad \lim_{\varepsilon\downarrow 0} g(x_0 \pm \varepsilon) = 0;\ Lg \in L^2([x_0,\pm\infty); dx)\big\}. \end{aligned} \tag{4.8}$$

Denoting by $\psi_\pm(z,\cdot)$ the Weyl–Titchmarsh solutions of $(L-z)\psi(z,\,\cdot\,) = 0$, satisfying

$$\psi_\pm(z,\cdot) \in L^2([x_0,\pm\infty); dx), \tag{4.9}$$

the half-line Weyl–Titchmarsh functions associated with $H_{\pm,x_0}$ are given by

$$m_\pm(z,x_0) = \frac{\psi'_\pm(z,x_0)}{\psi_\pm(z,x_0)}, \quad z \in \mathbb{C}\backslash\sigma(H_{\pm,x_0}). \tag{4.10}$$

Then the diagonal Green's function of H satisfies

$$g(z,x_0) = [m_-(z,x_0) - m_+(z,x_0)]^{-1}. \tag{4.11}$$

For subsequent purpose we also introduce two Herglotz functions,

$$M_{0,0}(z,x_0) = \frac{1}{m_-(z,x_0) - m_+(z,x_0)} = \int_{\mathbb{R}} \frac{d\Omega_{0,0}(\lambda,x_0)\,d\lambda}{\lambda - z}, \tag{4.12}$$

$$M_{1,1}(z,x_0) = \frac{m_-(z,x_0)m_+(z,x_0)}{m_-(z,x_0) - m_+(z,x_0)} = \int_{\mathbb{R}} \frac{d\Omega_{1,1}(\lambda,x_0)\,d\lambda}{\lambda - z}. \tag{4.13}$$

Definition 4.1. *Let $\mathcal{E} \subset \mathbb{R}$ be a closed set bounded from below which we may write as*

$$\mathcal{E} = [E_0, \infty) \Big\backslash \bigcup_{j \in J} (a_j, b_j), \quad J \subseteq \mathbb{N}, \tag{4.14}$$

for some $E_0 \in \mathbb{R}$ and $a_j < b_j$, where $(a_j, b_j) \cap (a_{j'}, b_{j'}) = \emptyset$, $j, j' \in J$, $j \neq j'$. Then $\mathcal{E}$ is called homogeneous if

$$\begin{aligned} &\text{there exists an } \varepsilon > 0 \text{ such that for all } \lambda \in \mathcal{E} \\ &\text{and all } \delta > 0, \quad |\mathcal{E} \cap (\lambda - \delta, \lambda + \delta)| \geq \varepsilon \delta. \end{aligned} \tag{4.15}$$

Next, we introduce a special class of reflectionless Schrödinger operators.

Definition 4.2. *Let $\Lambda \subset \mathbb{R}$ be of positive Lebesgue measure. Then we call H reflectionless on Λ if for some $x_0 \in \mathbb{R}$*

$$m_+(\lambda + i0, x_0) = \overline{m_-(\lambda + i0, x_0)} \text{ for a.e. } \lambda \in \Lambda. \tag{4.16}$$

Equivalently (*cf.* [21], [22]), *H is called reflectionless on Λ if for each $x \in \mathbb{R}$,*

$$\xi(\lambda, x) = 1/2 \text{ for a.e. } \lambda \in \Lambda. \tag{4.17}$$

In the following hypothesis we describe a special class $\mathcal{R}(\mathcal{E})$ of reflectionless Schrödinger operators associated with a homogeneous set $\mathcal{E}$, that will be our main object of investigation in this section.

Hypothesis 4.3. *Let $\mathcal{E} \subset \mathbb{R}$ be a homogeneous set. Then $H \in \mathcal{R}(\mathcal{E})$ if*

(*i*) *H is reflectionless on $\mathcal{E}$.* (*In particular, this implies $\mathcal{E} \subseteq \sigma_{\rm ess}(H)$.*)
(*ii*) *Either $\mathcal{E} = \sigma_{\rm ess}(H)$ or the set $\sigma_{\rm ess}(H) \backslash \mathcal{E}$ is closed.* (*In particular, this implies that there is an open set $\mathcal{O} \subseteq \mathbb{R}$ such that $\mathcal{E} \subset \mathcal{O}$ and $\overline{\mathcal{O}} \cap (\sigma_{\rm ess}(H) \backslash \mathcal{E}) = \emptyset$.*)
(*iii*) *The discrete eigenvalues of H that accumulate to $\mathcal{E}$ satisfy a Blaschke-type condition, that is,*

$$\sum_{\lambda \in \sigma(H) \cap (\mathcal{O} \backslash \mathcal{E})} G_{\mathcal{E}}(\lambda, \lambda_0) < \infty, \tag{4.18}$$

where $\mathcal{O}$ is the set defined in (*ii*) (*hence $\sigma(H) \cap (\mathcal{O} \backslash \mathcal{E}) \subseteq \sigma_{\rm disc}(H)$ is a discrete countable set*), *$\lambda_0 \in \mathbb{R} \backslash \sigma(H)$ is some fixed point, and $G_{\mathcal{E}}(\cdot, \infty)$ is the potential theoretic Green's function for the domain $(\mathbb{C} \cup \{\infty\}) \backslash \mathcal{E}$ with logarithmic singularity at λ_0* (*cf., e.g.,* [58, Sect. 4.4]), *$G_{\mathcal{E}}(z, \lambda_0) \underset{z \to \lambda_0}{=} \log |z - \lambda_0|^{-1} + O(1)$.*

One particularly interesting situation in which the above hypothesis is satisfied occurs when $\sigma(H) = \sigma_{\rm ess}(H)$ is a homogeneous set and H is reflectionless on $\sigma(H)$. This case has been studied in great detail by Sodin and Yuditskii [71]–[73], and more recently, in [22].

Next, we turn to the principal result of this section. For a Schrödinger operator H in the class $\mathcal{R}(\mathcal{E})$, we will show that H has purely absolutely continuous spectrum on $\mathcal{E}$, that is, we intend to prove that

$$\sigma_{\rm ac}(H) \supseteq \mathcal{E}, \quad \sigma_{\rm sc}(H) \cap \mathcal{E} = \sigma_{\rm pp}(H) \cap \mathcal{E} = \emptyset. \tag{4.19}$$

We start with an elementary lemma which will permit us to reduce the discussion of unbounded homogeneous sets $\mathcal{E}$ (typical for Schrödinger operators) to the case of compact homogeneous sets $\widetilde{\mathcal{E}}$ (typical for Jacobi operators).

Lemma 4.4. *Let m be a Herglotz function with representation*

$$\begin{aligned} &m(z)=c+\int_{\mathbb{R}} d\omega(\lambda)\bigg(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\bigg), \quad z\in\mathbb{C}_+, \\ &c=\operatorname{Re}[m(i)], \quad \int_{\mathbb{R}}\frac{d\omega(\lambda)}{1+\lambda^2}<\infty, \quad \mathbb{R}\backslash\operatorname{supp}(d\omega)\neq\emptyset. \end{aligned} \tag{4.20}$$

Consider the change of variables

$$\begin{aligned} &z\mapsto\zeta=(\lambda_0-z)^{-1}, \;\; z=\lambda_0-\zeta^{-1}, \quad z\in\mathbb{C}\cup\{\infty\}, \\ &\text{for some fixed } \lambda_0\in\mathbb{R}\backslash\operatorname{supp}(d\omega). \end{aligned} \tag{4.21}$$

Then, the function $r(\zeta)=m(z(\zeta))$ is a Herglotz function with the representation

$$r(\zeta)=m(z(\zeta))=d+\int_{\mathbb{R}}\frac{d\mu(\eta)}{\eta-\zeta}, \quad \zeta\in\mathbb{C}_+, \tag{4.22}$$

$$d\mu(\eta)=x^2 d\omega\big(\lambda_0-\eta^{-1}\big)\big|_{\operatorname{supp}(d\mu)}, \tag{4.23}$$

$$\operatorname{supp}(d\mu)\subseteq\big[-\operatorname{dist}(\lambda_0,\operatorname{supp}(d\omega))^{-1},\operatorname{dist}(\lambda_0,\operatorname{supp}(d\omega))^{-1}\big], \tag{4.24}$$

$$d=c+\int_{\operatorname{supp}(d\mu)} d\mu(\eta)\,\frac{\lambda_0-\big(1+\lambda_0^2\big)\eta}{1-2\lambda_0\eta+\big(1+\lambda_0^2\big)\eta^2}. \tag{4.25}$$

In particular, $d\mu$ is purely absolutely continuous on Λ if and only if $d\omega$ is purely absolutely continuous on $z(\Lambda)$, and

$$\text{if } d\omega(\lambda)|_{z(\Lambda)}=\omega'(\lambda)d\lambda\big|_{z(\Lambda)}, \text{ then } d\mu(\eta)|_{\Lambda}=\omega'\big(\lambda_0-\eta^{-1}\big)d\eta\big|_{\Lambda}. \tag{4.26}$$

Proof. This is a straightforward computation. We note that

$$\int_{\mathbb{R}}\frac{d\omega(\lambda)}{1+\lambda^2}<\infty \text{ is equivalent to } \int_{\mathbb{R}} d\mu(\eta)<\infty. \tag{4.27}$$

□

The principal result of this section then reads as follows.

Theorem 4.5. *Assume Hypothesis 4.3, that is, $H\in\mathcal{R}(\mathcal{E})$. Then, the spectrum of H is purely absolutely continuous on $\mathcal{E}$,*

$$\sigma_{\rm ac}(H)\supseteq\mathcal{E}, \quad \sigma_{\rm sc}(H)\cap\mathcal{E}=\sigma_{\rm pp}(H)\cap\mathcal{E}=\emptyset. \tag{4.28}$$

Moreover, $\sigma(H)$ has uniform multiplicity equal to two on $\mathcal{E}$.

Proof. Without loss of generality we may assume that either $\mathcal{E}$ is a compact set or $\mathcal{E}$ contains an infinite interval $[a,\infty)$ for some $a\in\mathbb{R}$. Indeed, if $\mathcal{E}$ does not contain an infinite interval then there is an increasing subsequence of gaps (a_{j_k},b_{j_k}) with $b_{j_k}<a_{j_{k+1}}$ and $a_{j_k}\to\infty$ as $k\to\infty$ which splits the set $\mathcal{E}$ into a countable disjoint union of compact homogeneous sets $\mathcal{E}_0=\mathcal{E}\cap[E_0,a_{j_1}]$, $\mathcal{E}_k=\mathcal{E}\cap[b_{j_k},a_{j_{k+1}}]$, $k\in\mathbb{N}$. Moreover, it follows from the proof of [56, Theorem 2.7] that the ratio $G_{\mathcal{E}_k}(z,\lambda_0)/G_{\mathcal{E}}(z,\lambda_0)$ of the Green's functions associated with $\mathcal{E}_k$ and $\mathcal{E}$ is bounded in some sufficiently small neighborhood of $\mathcal{E}_k$, hence one easily verifies that $H\in\mathcal{R}(\mathcal{E}_k)$ for all $k\geq 0$.

Next, fix $x_0\in\mathbb{R}$. Then using Lemma 4.4, we introduce the Herglotz function $r(\cdot,x_0)$ by

$$r(\zeta,x_0)=M_{0,0}(z(\zeta),x_0)=d(x_0)+\int_{\mathbb{R}}\frac{d\mu(\lambda,x_0)}{\lambda-z}, \quad z\in\mathbb{C}_+, \tag{4.29}$$

where $M_{0,0}$ is defined in (4.12) and $z(\zeta) = (\lambda_0 - \zeta)^{-1}$ with $\lambda_0 \in \mathbb{R}\backslash\sigma(H)$ introduced in Hypothesis 4.3 (*iii*). Abbreviating by $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{O}}$ the preimages of the sets $\mathcal{E}$ and $\mathcal{O}$ under the bijective map z,

$$\widetilde{\mathcal{E}} = z^{-1}(\mathcal{E}), \quad \widetilde{\mathcal{O}} = z^{-1}(\mathcal{O}), \tag{4.30}$$

and adding the point zero to $\widetilde{\mathcal{E}}$, and a sufficiently small neighborhood of zero to $\widetilde{\mathcal{O}}$ in the case of an unbounded set $\mathcal{E}$, we note that $\widetilde{\mathcal{E}}$ is compact and homogeneous, $\widetilde{\mathcal{O}}$ is open, $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{O}}$, and $\overline{\widetilde{\mathcal{O}}} \cap z^{-1}(\sigma_{\rm ess}(H)\backslash\mathcal{E}) = \emptyset$. Then the functions $r_j(z, x_0)$, $j = 1, 2$, defined by

$$r_1(z, x_0) = \int_{\widetilde{\mathcal{O}}} \frac{d\mu(\lambda, x_0)}{\lambda - z} = \sum_{\lambda \in \widetilde{\mathcal{O}}\backslash\widetilde{\mathcal{E}}} \frac{\mu(\{\lambda\}, x_0)}{\lambda - z} + \int_{\widetilde{\mathcal{E}}} \frac{d\mu(\lambda, x_0)}{\lambda - z}, \quad z \in \mathbb{C}\backslash\widetilde{\mathcal{O}}, \tag{4.31}$$

$$r_2(z, x_0) = \int_{\mathbb{R}\backslash\widetilde{\mathcal{O}}} \frac{d\mu(\lambda, x_0)}{\lambda - z}, \quad z \in (\mathbb{C}\backslash\mathbb{R}) \cup \widetilde{\mathcal{O}}, \tag{4.32}$$

satisfy

$$r(z, x_0) = r_1(z, x_0) + r_2(z, x_0), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{4.33}$$

Since $H \in \mathcal{R}(\mathcal{E})$, one has $\xi(\lambda, x_0) = 1/2$ and hence $\mathrm{Re}[M_{0,0}(\lambda, x_0)] = 0$ for a.e. $\lambda \in \mathcal{E}$. This yields $\mathrm{Re}[r(\lambda + i0, x_0)] = 0$ for a.e. $\lambda \in \widetilde{\mathcal{E}}$, and hence,

$$\mathrm{Re}[r_1(\lambda + i0, x_0)] = -\mathrm{Re}[r_2(\lambda + i0, x_0)] \text{ for a.e. } \lambda \in \widetilde{\mathcal{E}}. \tag{4.34}$$

Observing that the function $r_2(\cdot, x_0)$ is analytic on $(\mathbb{C}\backslash\mathbb{R}) \cup \widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{O}}$, one concludes that $r_2(\cdot, x_0)$ is bounded on $\widetilde{\mathcal{E}}$, and hence,

$$\mathrm{Re}[r_1(\cdot + i0, x_0)] = -\mathrm{Re}[r_2(\cdot + i0, x_0)] \in L^1\big(\widetilde{\mathcal{E}}; dx\big). \tag{4.35}$$

Moreover, it follows from [16, Proposition 5.1] that (4.18) is equivalent to

$$\sum_{\lambda \in \widetilde{\mathcal{O}}\backslash\widetilde{\mathcal{E}}} G_{\widetilde{\mathcal{E}}}(\lambda, \infty) < \infty. \tag{4.36}$$

In addition, as a consequence of Theorem A.4, the set of mass points of $d\Omega_{0,0}$ is a subset of the set of discrete eigenvalues of H, hence the set of discrete mass points of $d\mu(\cdot, x_0)$ is a subset of $\widetilde{\mathcal{O}}\backslash\widetilde{\mathcal{E}}$. Then it follows from (4.31), (4.35), and (4.36) that the function $r_1(\cdot, x_0)$ satisfies the assumptions of Theorem A.5. Thus, the restriction $d\mu(\cdot, x_0)|_{\widetilde{\mathcal{E}}}$ of the measure $d\mu(\cdot, x_0)$ to the set $\widetilde{\mathcal{E}}$ is purely absolutely continuous,

$$d\mu(\cdot, x_0)\big|_{\widetilde{\mathcal{E}}} = d\mu_{\rm ac}(\cdot, x_0)\big|_{\widetilde{\mathcal{E}}} = \frac{1}{\pi}\mathrm{Im}[r_1(\cdot + i0, x_0)]d\lambda\big|_{\widetilde{\mathcal{E}}}, \tag{4.37}$$

and hence, it follows from Lemma 3.4 that

$$d\Omega_{0,0}(\cdot, x_0)\big|_{\mathcal{E}} = d\Omega_{0,0,\rm ac}(\cdot, x_0)\big|_{\mathcal{E}}. \tag{4.38}$$

Next, performing a similar analysis for the function $M_{1,1}$ defined in (4.13), one obtains

$$d\Omega_{1,1}(\cdot, x_0)\big|_{\mathcal{E}} = d\Omega_{1,1,\rm ac}(\cdot, x_0)\big|_{\mathcal{E}}, \tag{4.39}$$

and hence,

$$d\Omega^{\rm tr}(\cdot, x_0)\big|_{\mathcal{E}} = d\Omega^{\rm tr}_{\rm ac}(\cdot, x_0)\big|_{\mathcal{E}}, \tag{4.40}$$

for the restriction of the trace measure $d\Omega^{\rm tr}(\cdot, x_0) = d\Omega_{0,0}(\cdot, x_0) + d\Omega_{1,1}(\cdot, x_0)$ associated with H. By (A.19) and Theorem A.4 (*i*) this completes the proof of (4.28).

Finally, equations (4.11) and (4.16) imply

$$-1/g(\lambda + i0, x_0) = \pm 2i\,\mathrm{Im}[m_\pm(\lambda + i0, x_0)] \text{ for a.e. } \lambda \in \mathcal{E}. \tag{4.41}$$

Thus, combining (4.16), (4.41), and (A.24) then yields that the absolutely continuous spectrum of H has uniform spectral multiplicity two on $\mathcal{E}$ since

$$\text{for a.e. } \lambda \in \mathcal{E},\ 0 < \pm\mathrm{Im}[m_\pm(\lambda + i0, x_0)] < \infty. \tag{4.42}$$

□

Appendix A. Herglotz and Weyl–Titchmarsh Functions for Jacobi and Schrödinger Operators in a Nutshell

The material in this appendix is known, but since we use it repeatedly at various places in this paper, we thought it worthwhile to collect it in an appendix.

Definition A.1. *Let* $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \gtrless 0\}$. $m : \mathbb{C}_+ \to \mathbb{C}$ *is called a Herglotz function* (*or Nevanlinna or Pick function*) *if* m *is analytic on* $\mathbb{C}_+$ *and* $m(\mathbb{C}_+) \subseteq \mathbb{C}_+$.

One then extends m to $\mathbb{C}_-$ by reflection, that is, one defines

$$m(z) = \overline{m(\overline{z})}, \quad z \in \mathbb{C}_-. \tag{A.1}$$

Of course, generally, (A.1) does not represent an analytic continuation of $m\big|_{\mathbb{C}_+}$ into $\mathbb{C}_-$.

Fundamental results on Herglotz functions and their representations on Borel transforms, in part, are due to Fatou, Herglotz, Luzin, Nevanlinna, Plessner, Privalov, de la Vallée Poussin, Riesz, and others. Here we just summarize a few of these results:

Theorem A.2. ([2, Sect. 69], [3], [13, Chs. II, IV], [35], [36, Ch. 6], [57, Chs. II, IV], [61, Ch. 5]). *Let* m *be a Herglotz function. Then,*
(*i*) *There exists a nonnegative measure* $d\omega$ *on* $\mathbb{R}$ *satisfying*

$$\int_{\mathbb{R}} \frac{d\omega(\lambda)}{1 + \lambda^2} < \infty \tag{A.2}$$

such that the Nevanlinna, respectively, Riesz-Herglotz representation

$$\begin{aligned} &m(z) = c + dz + \int_{\mathbb{R}} d\omega(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\bigg), \quad z \in \mathbb{C}_+, \\ &c = \mathrm{Re}[m(i)], \quad d = \lim_{\eta\uparrow\infty} m(i\eta)/(i\eta) \geq 0 \end{aligned} \tag{A.3}$$

holds. Conversely, any function m *of the type* (A.3) *is a Herglotz function.*
(*ii*) *The absolutely continuous* (*ac*) *part* $d\omega_{ac}$ *of* $d\omega$ *with respect to Lebesgue measure* $d\lambda$ *on* $\mathbb{R}$ *is given by*

$$d\omega_{ac}(\lambda) = \pi^{-1}\mathrm{Im}[m(\lambda + i0)]\, d\lambda. \tag{A.4}$$

Next, we denote by

$$d\mu = d\mu_{\mathrm{ac}} + d\mu_{\mathrm{sc}} + d\mu_{\mathrm{pp}} \tag{A.5}$$

the decomposition of a measure $d\mu$ into its absolutely continuous (ac), singularly continuous (sc), and pure point (pp) parts with respect to Lebesgue measure on $\mathbb{R}$.

Theorem A.3. ([25], [28]). *Let m be a Herglotz function with representation* (A.3) *and denote by Λ the set*

$$\Lambda = \{\lambda \in \mathbb{R} \,|\, \mathrm{Im}[m(\lambda + i0)] \text{ exists (finitely or infinitely)}\}. \tag{A.6}$$

Then, S, $S_{\rm ac}$, $S_{\rm s}$, $S_{\rm sc}$, $S_{\rm pp}$ are essential supports of $d\omega$, $d\omega_{\rm ac}$, $d\omega_{\rm s}$, $d\omega_{\rm sc}$, $d\omega_{\rm pp}$, respectively, where

$$S = \{\lambda \in \Lambda \,|\, 0 < \mathrm{Im}[m(\lambda + i0)] \leq \infty\}, \tag{A.7}$$

$$S_{\rm ac} = \{\lambda \in \Lambda \,|\, 0 < \mathrm{Im}[m(\lambda + i0)] < \infty\}, \tag{A.8}$$

$$S_{\rm s} = \{\lambda \in \Lambda \,|\, \mathrm{Im}[m(\lambda + i0)] = \infty\}, \tag{A.9}$$

$$S_{\rm sc} = \Big\{\lambda \in \Lambda \,|\, \mathrm{Im}[m(\lambda + i0)] = \infty, \, \lim_{\varepsilon \downarrow 0} (-i\varepsilon) m(\lambda + i\varepsilon) = 0\Big\}, \tag{A.10}$$

$$S_{\rm pp} = \Big\{\lambda \in \Lambda \,|\, \mathrm{Im}[m(\lambda + i0)] = \infty, \, \lim_{\varepsilon \downarrow 0} (-i\varepsilon) m(\lambda + i\varepsilon) = \omega(\{\lambda\}) > 0\Big\}. \tag{A.11}$$

Next, consider Herglotz functions $\pm m_\pm$ of the type (A.3),

$$\begin{aligned} &\pm m_\pm(z) = c_\pm + d_\pm z + \int_{\mathbb{R}} d\omega_\pm(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\bigg), \quad z \in \mathbb{C}_+, \\ &c_\pm \in \mathbb{R}, \quad d_\pm \geq 0, \end{aligned} \tag{A.12}$$

and introduce the 2×2 matrix-valued Herglotz function M

$$M(z) = \big(M_{j,k}(z)\big)_{j,k=0,1}, \quad z \in \mathbb{C}_+, \tag{A.13}$$

$$M(z) = \frac{1}{m_-(z) - m_+(z)} \begin{pmatrix} 1 & \frac{1}{2}[m_-(z) + m_+(z)] \\ \frac{1}{2}[m_-(z) + m_+(z)] & m_-(z) m_+(z) \end{pmatrix} \tag{A.14}$$

$$= C + Dz + \int_{\mathbb{R}} d\Omega(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\bigg), \quad z \in \mathbb{C}_+, \tag{A.15}$$

$$C = C^*, \quad D \geq 0$$

with $C = (C_{j,k})_{j,k=0,1}$ and $D = (D_{j,k})_{j,k=0,1}$ 2×2 matrices and $d\Omega = (d\Omega_{j,k})_{j,k=0,1}$ a 2×2 matrix-valued nonnegative measure satisfying

$$\int_{\mathbb{R}} \frac{d|\Omega_{j,k}(\lambda)|}{1 + \lambda^2} < \infty, \quad j,k = 0,1. \tag{A.16}$$

Moreover, we introduce the trace Herglotz function $M^{\rm tr}$

$$M^{\rm tr}(z) = M_{0,0}(z) + M_{1,1}(z) = \frac{1 + m_-(z) m_+(z)}{m_-(z) - m_+(z)} \tag{A.17}$$

$$= c + dz + \int_{\mathbb{R}} d\Omega^{\rm tr}(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\bigg), \quad z \in \mathbb{C}_+, \tag{A.18}$$

$$c \in \mathbb{R}, \ d \geq 0, \quad d\Omega^{\rm tr} = d\Omega_{0,0} + d\Omega_{1,1}.$$

Then,

$$d\Omega \ll d\Omega^{\rm tr} \ll d\Omega \tag{A.19}$$

(where $d\mu \ll d\nu$ denotes that $d\mu$ is absolutely continuous with respect to $d\nu$).

The next result holds both for the Jacobi and Schrödinger cases. In the Jacobi case we identify

$$m_\pm(z) \text{ and } M_\pm(z, n_0), \quad z \in \mathbb{C}_+, \tag{A.20}$$

where $M_\pm(z,n_0)$ denote the half-lattice Weyl–Titchmarsh m-functions defined in (2.12)–(2.14) and in the Schrödinger case

$$m_\pm(z) \text{ and } m_\pm(z,x_0), \quad z \in \mathbb{C}_+, \tag{A.21}$$

where $m_\pm(z,x_0)$ are the half-line Weyl–Titchmarsh m-functions defined in (4.10). One then has the following basic result.

Theorem A.4. ([27], [33], [34], [66], [75]).
(*i*) *The operator* H (*in the Jacobi case* H *is defined in* (2.4) *and in the Schrödinger case in* (4.3)) *is unitarily equivalent to the operator of multiplication by* $I_2\mathrm{id}$ (*where* I_2 *is the* 2×2 *identity matrix and* $\mathrm{id}(\lambda)=\lambda$, $\lambda\in\mathbb{R}$) *on* $L^2(\mathbb{R};d\Omega(\cdot))$, *and hence,*

$$\sigma(H) = \mathrm{supp}\,(d\Omega) = \mathrm{supp}\,(d\Omega^{\mathrm{tr}}), \tag{A.22}$$

where $d\Omega$ *and* $d\Omega^{\mathrm{tr}}$ *are introduced in* (A.15) *and* (A.18), *respectively.*
(*ii*) *The spectral multiplicity of* H *is two if and only if*

$$|\mathcal{M}_2| > 0, \tag{A.23}$$

where

$$\mathcal{M}_2 = \{\lambda\in\Lambda_+ \,|\, m_+(\lambda+i0)\in\mathbb{C}\backslash\mathbb{R}\} \cap \{\lambda\in\Lambda_- \,|\, m_-(\lambda+i0)\in\mathbb{C}\backslash\mathbb{R}\}. \tag{A.24}$$

If $|\mathcal{M}_2|=0$, *the spectrum of* H *is simple. Moreover,* $\mathcal{M}_2$ *is a maximal set on which* H *has uniform multiplicity two.*
(*iii*) *A maximal set* $\mathcal{M}_1$ *on which* H *has uniform multiplicity one is given by*

$$\begin{aligned}\mathcal{M}_1 = \{&\lambda\in\Lambda_+\cap\Lambda_- \,|\, m_+(\lambda+i0)=m_-(\lambda+i0)\in\mathbb{R}\}\\ &\cup\{\lambda\in\Lambda_+\cap\Lambda_- \,|\, |m_+(\lambda+i0)|=|m_-(\lambda+i0)|=\infty\}\\ &\cup\{\lambda\in\Lambda_+\cap\Lambda_- \,|\, m_+(\lambda+i0)\in\mathbb{R}, m_-(\lambda+i0)\in\mathbb{C}\backslash\mathbb{R}\}\\ &\cup\{\lambda\in\Lambda_+\cap\Lambda_- \,|\, m_-(\lambda+i0)\in\mathbb{R}, m_+(\lambda+i0)\in\mathbb{C}\backslash\mathbb{R}\}.\end{aligned} \tag{A.25}$$

In particular, $\sigma_{\mathrm{s}}(H)=\sigma_{\mathrm{sc}}(H)\cup\sigma_{\mathrm{pp}}(H)$ *is always simple.*

Finally, we give a proof of a basic result due to Peherstorfer and Yuditskii [53, Lemma 2.4].

Theorem A.5. ([53, Lemma 2.4]). *Let* $\mathcal{E}\subset\mathbb{R}$ *be a compact homogenous set and* $m(z)$ *a Herglotz function with the representation*

$$\begin{aligned}&m(z) = a + \sum_{j\in J}\frac{\mu(\{\lambda_j\})}{\lambda_j - z} + \int_{\mathcal{E}}\frac{d\mu(\lambda)}{\lambda - z}, \quad z\in\mathbb{C}_+,\\ &a\in\mathbb{R},\;\; J\subseteq\mathbb{N},\;\; d\mu \text{ a finite measure},\;\; \mathrm{supp}\,(d\mu)\subseteq\mathcal{E}\cup\{\lambda_j\}_{j\in J}.\end{aligned} \tag{A.26}$$

Denote by $m(\lambda+i0)=\lim_{\varepsilon\downarrow 0}m(\lambda+i\varepsilon)$ *the a.e. normal boundary values of* m *and assume that*

$$\mathrm{Re}[m(\cdot+i0)]\in L^1\big(\mathcal{E};d\lambda\big), \tag{A.27}$$

$$\sum_{j\in J} G_{\mathcal{E}}(\lambda_j,\infty) < \infty, \tag{A.28}$$

where $G_{\mathcal{E}}(\cdot,\infty)$ *is the potential theoretic Green's function for the domain* $(\mathbb{C}\cup\{\infty\})\backslash\mathcal{E}$ *with logarithmic singularity at infinity* (*cf., e.g.,* [58, Sect. 5.2]),

$$G_{\mathcal{E}}(z,\infty) \underset{|z|\to\infty}{=} \log|z| - \log(\mathrm{cap}(\mathcal{E})) + o(1). \tag{A.29}$$

Then, $d\mu$ is purely absolutely continuous and hence

$$d\mu\big|_{\mathcal{E}} = d\mu_{\rm ac}\big|_{\mathcal{E}} = \frac{1}{\pi}\mathrm{Im}[m(\cdot + i0)]\, d\lambda\big|_{\mathcal{E}}. \tag{A.30}$$

Proof. First, we briefly introduce some important notation (cf., [8], [53], [69, Ch. 8], [74] for a comprehensive discussion). Let Γ be the Fuchsian group of linear-fractional transformations of the unit disc $\mathbb{D}$, uniformizing the domain $\Omega = (\mathbb{C} \cup \{\infty\})\backslash\mathcal{E}$, and denote by Γ^* the group of unimodular characters associated with the Fuchsian group Γ (i.e., each $\alpha \in \Gamma^*$ is a homomorphism from Γ to $\partial\mathbb{D}$). By

$$\mathbb{F} = \{z \in \mathbb{D} \mid |\gamma'(z)| < 1 \text{ for all } \gamma \in \Gamma\backslash\{\mathrm{id}\}\} \tag{A.31}$$

we denote the Ford orthocircular fundamental domain of Γ (see [17], [74]), and as usual, keeping the same notation, we add to $\mathbb{F}$ half of its boundary circles (say, the boundary circles lying in $\mathbb{C}_+ \cap \mathbb{D}$). By $\mathbf{x}(z)$ we denote the uniformizing map (also known as the universal covering map for Ω), that is, $\mathbf{x}(z)$ is the unique map satisfying the following properties:

(*i*) $\mathbf{x}(z)$ maps $\mathbb{D}$ meromorphically onto Ω.
(*ii*) $\mathbf{x}(z)$ is Γ-automorphic, that is, $\mathbf{x} \circ \gamma = \mathbf{x}$, $\gamma \in \Gamma$.
(*iii*) $\mathbf{x}(z)$ is locally a bijection ($\mathbf{x}$ maps $\mathbb{F}$ bijectively to Ω).
(*iv*) $\mathbf{x}(0) = \infty$ and $\lim_{z\to 0} z\mathbf{x}(z) > 0$.

We also introduce the notion of Blaschke products associated with the group Γ,

$$B(z,w) = \prod_{\gamma\in\Gamma} \frac{\gamma(w) - z}{1 - \overline{\gamma(w)}z}\frac{|\gamma(w)|}{\gamma(w)}, \quad z \in \mathbb{D}, \tag{A.32}$$

where we set $\frac{|\gamma(w)|}{\gamma(w)} \equiv -1$ if $\gamma(w) = 0$. Then we define

$$B(z) = B(z,0), \quad B_\infty(z) = \prod_{j\in J} B(z, z_j), \quad z \in \mathbb{D}, \tag{A.33}$$

(condition (A.28) guarantees convergence of the above product), where $\{z_j\}_{j\in J} \subset \mathbb{F}$ are the points satisfying $\mathbf{x}(z_j) = \lambda_j$, $j \in J$. It follows that the functions B and B_∞ are character-automorphic (i.e., there are characters $\alpha, \beta \in \Gamma^*$ such that $B \circ \gamma = \alpha(\gamma)B$ and $B_\infty \circ \gamma = \beta(\gamma)B_\infty$ for all $\gamma \in \Gamma$) and the following formulas hold

$$G_{\mathcal{E}}(\mathbf{x}(z), \infty) = -\log\big(|B(z)|\big), \tag{A.34}$$

$$\lim_{z\to 0} B(z)\mathbf{x}(z) = \mathrm{cap}(\mathcal{E}), \quad \lim_{z\to 0}\frac{B(z)^2\mathbf{x}'(z)}{B'(z)} = -\mathrm{cap}(\mathcal{E}), \tag{A.35}$$

where $\mathrm{cap}(\mathcal{E})$ denotes logarithmic capacity of the set $\mathcal{E}$ (cf. (A.29)). Moreover, we define

$$\phi(z) = \frac{zB'(z)}{B(z)}, \quad z \in \mathbb{D}, \tag{A.36}$$

then one verifies

$$\phi(\zeta) = \sum_{\gamma\in\Gamma} |\gamma'(\zeta)| \ \text{ for a.e. } \ \zeta \in \partial\mathbb{D}, \tag{A.37}$$

and

$$\phi(\gamma(\zeta))|\gamma'(\zeta)| = \phi(\zeta) \ \text{ for all } \ \gamma \in \Gamma, \ \zeta \in \partial\mathbb{D}. \tag{A.38}$$

After these preliminaries we commence with the proof of Theorem A.5. One observes that it suffices to prove

$$\mu(\mathbb{R}) = \sum_{j\in J} \mu(\{\lambda_j\}) + \frac{1}{\pi}\int_{\mathcal{E}} \mathrm{Im}[m(\lambda+i0)]\,d\lambda. \tag{A.39}$$

Let $r(z)$ denote the Herglotz function

$$r(z) = m(z) - a = \sum_{j\in J} \frac{\mu(\{\lambda_j\})}{\lambda_j - z} + \int_{\mathcal{E}} \frac{d\mu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+. \tag{A.40}$$

Then one verifies the following asymptotic formula,

$$r(z) \underset{z\to\infty}{=} -\frac{\mu(\mathbb{R})}{z} + O(z^{-2}). \tag{A.41}$$

Using the symmetry $r(\overline{z}) = \overline{r(z)}$ and (A.40) one also derives

$$r(\lambda+i0) - r(\lambda-i0) = 2i\mathrm{Im}[r(\lambda+i0)] = 2i\mathrm{Im}[m(\lambda+i0)] \text{ for a.e. } \lambda\in\mathbb{R}. \tag{A.42}$$

Moreover, it follows from condition (A.27) that $r(\cdot \pm i0) \in L^1\big(\mathcal{E}; d\lambda\big)$.

Next, one computes,

$$\begin{aligned}\frac{1}{\pi}\int_{\mathcal{E}} \mathrm{Im}[m(\lambda+i0)]\,d\lambda &= \frac{1}{2\pi i}\int_{\mathcal{E}}[r(\lambda+i0) - r(\lambda-i0)]\,d\lambda = \frac{-1}{2\pi i}\oint_{\partial\Omega} r(\lambda+i0)\,d\lambda\\ &= \frac{1}{2\pi i}\int_{\partial\mathbb{F}\cap\partial\mathbb{D}} r(\mathbf{x}(\zeta))\mathbf{x}'(\zeta)\,d\zeta = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} r(\mathbf{x}(\zeta))\frac{\mathbf{x}'(\zeta)}{\phi(\zeta)}\,d\zeta.\end{aligned} \tag{A.43}$$

To evaluate the last integral one utilizes the Direct Cauchy Theorem (cf. [53, Lemma 1.1], [74, Theorem H]),

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} r(\mathbf{x}(\zeta))\frac{\mathbf{x}'(\zeta)}{\phi(\zeta)}\,d\zeta &= \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{\frac{r(\mathbf{x}(\zeta))B_\infty(\zeta)}{B(\zeta)}\frac{B(\zeta)^2\mathbf{x}'(\zeta)}{\phi(\zeta)}}{B(\zeta)B_\infty(\zeta)}\,d\zeta\\ &= \frac{r(\mathbf{x}(0))B(0)^2\mathbf{x}'(0)}{B(0)B'(0)} + \sum_{j\in J}\frac{r(\mathbf{x}(z_j))B_\infty(z_j)\mathbf{x}'(z_j)}{B'_\infty(z_j)}\\ &= \mu(\mathbb{R}) - \sum_{j\in J}\mu(\{\lambda_j\}).\end{aligned} \tag{A.44}$$

Thus, combining (A.43) and (A.44) yields (A.39). □

Appendix B. Caratheodory and Weyl–Titchmarsh Functions for CMV Operators in a Nutshell

In this appendix we provide some basic facts on Caratheodory functions and prove the analog of Theorem A.4 for CMV operators.

Definition B.1. *Let $\mathbb{D}$ and $\partial\mathbb{D}$ denote the open unit disk and the counterclockwise oriented unit circle in the complex plane $\mathbb{C}$,*

$$\mathbb{D} = \{z\in\mathbb{C} \mid |z|<1\}, \quad \partial\mathbb{D} = \{\zeta\in\mathbb{C} \mid |\zeta|=1\}, \tag{B.1}$$

and $\mathbb{C}_\ell$ and $\mathbb{C}_r$ the open left and right complex half-planes, respectively,

$$\mathbb{C}_\ell = \{z\in\mathbb{C} \mid \mathrm{Re}(z)<0\}, \quad \mathbb{C}_r = \{z\in\mathbb{C} \mid \mathrm{Re}(z)>0\}. \tag{B.2}$$

A function $f : \mathbb{D} \to \mathbb{C}$ is called Caratheodory if f is analytic on $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{C}_r$. One then extends f to $\mathbb{C}\backslash\overline{\mathbb{D}}$ by reflection, that is, one defines

$$f(z) = -\overline{f(1/\overline{z})}, \quad z \in \mathbb{C}\backslash\overline{\mathbb{D}}. \tag{B.3}$$

Of course, generally, (B.3) *does not represent the analytic continuation of $f|_{\mathbb{D}}$ into $\mathbb{C}\backslash\overline{\mathbb{D}}$.*

The fundamental result on Caratheodory functions reads as follows.

Theorem B.2. ([1, Sect. 3.1], [2, Sect. 69], [64, Sect. 1.3]). *Let f be a Caratheodory function. Then,*
(i) $f(z)$ has finite normal limits $f(\zeta) = \lim_{r\uparrow 1} f(r\zeta)$ for a.e. $\zeta \in \partial\mathbb{D}$.
(ii) Suppose $f(r\zeta)$ has a zero normal limit on a subset of $\partial\mathbb{D}$ having positive Lebesgue measure. Then $f \equiv 0$.
(iii) There exists a nonnegative finite measure $d\omega$ on $\partial\mathbb{D}$ such that the Herglotz representation

$$\begin{aligned} &f(z) = ic + \oint_{\partial\mathbb{D}} d\omega(\zeta)\, \frac{\zeta + z}{\zeta - z}, \quad z \in \mathbb{D}, \\ &c = \operatorname{Im}(f(0)), \quad \oint_{\partial\mathbb{D}} d\omega(\zeta) = \operatorname{Re}(f(0)) < \infty, \end{aligned} \tag{B.4}$$

holds. Conversely, any function f of the type (B.4) *is a Caratheodory function.*
(iv) The absolutely continuous (ac) part $d\omega_{ac}$ of $d\omega$ with respect to the normalized Lebesgue measure $d\omega_0$ on $\partial\mathbb{D}$ is given by

$$d\omega_{ac}(\zeta) = \pi^{-1}\operatorname{Re}[f(\zeta)]\, d\omega_0(\zeta). \tag{B.5}$$

Next, we denote by

$$d\omega = d\omega_{\rm ac} + d\omega_{\rm sc} + d\omega_{\rm pp} \tag{B.6}$$

the decomposition of $d\omega$ into its absolutely continuous (ac), singularly continuous (sc), and pure point (pp) parts with respect to Lebesgue measure on $\partial\mathbb{D}$.

Theorem B.3. ([64, Sects. 1.3, 1.4]). *Let f be a Caratheodory function with representation* (B.4) *and denote by Λ the set*

$$\Lambda = \{\zeta \in \partial\mathbb{D} \mid \operatorname{Re}[f(\zeta)] \text{ exists (finitely or infinitely)}\}. \tag{B.7}$$

Then, S, $S_{\rm ac}$, $S_{\rm s}$, $S_{\rm sc}$, $S_{\rm pp}$ are essential supports of $d\omega$, $d\omega_{\rm ac}$, $d\omega_{\rm s}$, $d\omega_{\rm sc}$, $d\omega_{\rm pp}$, respectively, where

$$S = \{\zeta \in \Lambda \mid 0 < \operatorname{Re}[f(\zeta)] \le \infty\}, \tag{B.8}$$
$$S_{\rm ac} = \{\zeta \in \Lambda \mid 0 < \operatorname{Re}[f(\zeta)] < \infty\}, \tag{B.9}$$
$$S_{\rm s} = \{\zeta \in \Lambda \mid \operatorname{Re}[f(\zeta)] = \infty\}, \tag{B.10}$$
$$S_{\rm sc} = \Big\{\zeta \in \Lambda \mid \operatorname{Re}[f(\zeta)] = \infty,\, \lim_{r\uparrow 1}(1-r)f(r\zeta) = 0\Big\}, \tag{B.11}$$
$$S_{\rm pp} = \Big\{\zeta \in \Lambda \mid \operatorname{Re}[f(\zeta)] = \infty,\, \lim_{r\uparrow 1}\left(\frac{1-r}{2}\right) f(r\zeta) = \omega(\{\zeta\}) > 0\Big\}. \tag{B.12}$$

Next, consider Caratheodory functions $\pm m_\pm$ of the type (B.4),

$$\begin{aligned} &\pm m_\pm(z) = ic_\pm + \oint_{\partial\mathbb{D}} d\omega_\pm(\zeta)\, \frac{\zeta + z}{\zeta - z}, \quad z \in \mathbb{D}, \\ &c_\pm \in \mathbb{R}, \end{aligned} \tag{B.13}$$

and introduce the 2×2 matrix-valued Caratheodory function $\widetilde{M}$ by

$$\widetilde{M}(z) = \big(\widetilde{M}_{j,k}(z)\big)_{j,k=0,1}, \quad z\in\mathbb{D}, \tag{B.14}$$

$$\widetilde{M}(z) = \frac{1}{m_+(z)-m_-(z)}\begin{pmatrix} 1 & \frac{1}{2}[m_+(z)+m_-(z)] \\ -\frac{1}{2}[m_+(z)+m_-(z)] & -m_+(z)m_-(z)\end{pmatrix}, \tag{B.15}$$

$$= i\widetilde{C} + \oint_{\partial\mathbb{D}} d\widetilde{\Omega}(\zeta)\frac{\zeta+z}{\zeta-z}, \quad z\in\mathbb{D}, \tag{B.16}$$

$$\widetilde{C} = \widetilde{C}^* = \mathrm{Im}[\widetilde{M}(0)],$$

where $d\widetilde{\Omega} = (d\widetilde{\Omega}_{j,k})_{j,k=0,1}$ a 2×2 matrix-valued nonnegative measure satisfying

$$\oint_{\partial\mathbb{D}} d\,|\widetilde{\Omega}_{j,k}(\zeta)| < \infty, \quad j,k=0,1. \tag{B.17}$$

Moreover, we introduce the trace Caratheodory function $\widetilde{M}^{\mathrm{tr}}$

$$\widetilde{M}^{\mathrm{tr}}(z) = \widetilde{M}_{0,0}(z) + \widetilde{M}_{1,1}(z) = \frac{1-m_+(z)m_-(z)}{m_+(z)-m_-(z)} \tag{B.18}$$

$$= i\widetilde{c} + \oint_{\partial\mathbb{D}} d\widetilde{\Omega}^{\mathrm{tr}}(\zeta)\frac{\zeta+z}{\zeta-z}, \quad z\in\mathbb{D}, \tag{B.19}$$

$$\widetilde{c}\in\mathbb{R}, \quad d\widetilde{\Omega}^{\mathrm{tr}} = d\widetilde{\Omega}_{0,0} + d\widetilde{\Omega}_{1,1}.$$

Then,

$$d\widetilde{\Omega} \ll d\widetilde{\Omega}^{\mathrm{tr}} \ll d\widetilde{\Omega} \tag{B.20}$$

(where $d\mu \ll d\nu$ denotes that $d\mu$ is absolutely continuous with respect to $d\nu$). This implies that there is a self-adjoint integrable 2×2 matrix $\widetilde{R}(\zeta)$ such

$$d\widetilde{\Omega}(\zeta) = \widetilde{R}(\zeta)d\widetilde{\Omega}^{\mathrm{tr}}(\zeta) \tag{B.21}$$

by the Radon–Nikodym theorem. Moreover, the matrix $\widetilde{R}(\zeta)$ is nonnegative and given by

$$\widetilde{R}_{j,k}(\zeta) = \lim_{r\uparrow 1}\frac{\mathrm{Re}[\widetilde{M}_{j,k}(r\zeta)]}{\mathrm{Re}[\widetilde{M}_{0,0}(r\zeta)+\widetilde{M}_{1,1}(r\zeta)]} \text{ for a.e. } \zeta\in\partial\mathbb{D},\ j,k=0,1. \tag{B.22}$$

Next, we identify

$$m_\pm(z) \text{ and } M_\pm(z,n_0), \quad n_0\in\mathbb{Z},\ z\in\mathbb{D}, \tag{B.23}$$

where $M_\pm(z,n_0)$ denote the half-lattice Weyl–Titchmarsh m-functions defined in (3.12)–(3.13) and introduce another 2×2 matrix-valued Caratheodory function

$$\begin{aligned} M(z,n_0) &= \begin{pmatrix} M_{0,0}(z,n_0) & M_{0,1}(z,n_0) \\ M_{1,0}(z,n_0) & M_{1,1}(z,n_0)\end{pmatrix} \\ &= \begin{pmatrix} (\delta_{n_0-1},(U+zI)(U-zI)^{-1}\delta_{n_0-1}) & (\delta_{n_0-1},(U+zI)(U-zI)^{-1}\delta_{n_0}) \\ (\delta_{n_0},(U+zI)(U-zI)^{-1}\delta_{n_0-1}) & (\delta_{n_0},(U+zI)(U-zI)^{-1}\delta_{n_0})\end{pmatrix} \\ &= \oint_{\partial\mathbb{D}} d\Omega(\zeta,n_0)\,\frac{\zeta+z}{\zeta-z}, \quad z\in\mathbb{D}, \end{aligned} \tag{B.24}$$

where U denotes a CMV operator of the form (3.4) and $d\Omega = (d\Omega_{j,k})_{j,k=0,1}$ a 2×2 matrix-valued nonnegative measure satisfying

$$\oint_{\partial\mathbb{D}} d\,|\Omega_{j,k}(\zeta)| < \infty, \quad j,k=0,1. \tag{B.25}$$

Then the two Caratheodory matrices $M(z,n_0)$ and $\widetilde{M}(z)$ are related by (cf. [23, Equation (3.62)])

$$\begin{aligned}
&\widetilde{M}(z) + \begin{pmatrix} \frac{i}{2}\mathrm{Im}(\alpha_{n_0}) & \frac{1}{2}\mathrm{Re}(\alpha_{n_0}) \\ -\frac{1}{2}\mathrm{Re}(\alpha_{n_0}) & -\frac{i}{2}\mathrm{Im}(\alpha_{n_0}) \end{pmatrix} \\
&\quad = \begin{cases} \frac{1}{4}\begin{pmatrix} \rho_{n_0} & \rho_{n_0} \\ -b_{n_0} & a_{n_0} \end{pmatrix}^* M(z,n_0) \begin{pmatrix} \rho_{n_0} & \rho_{n_0} \\ -b_{n_0} & a_{n_0} \end{pmatrix}, & n_0 \text{ odd}, \\ \frac{1}{4}\begin{pmatrix} -\rho_{n_0} & \rho_{n_0} \\ \overline{b_{n_0}} & \overline{a_{n_0}} \end{pmatrix}^* M(z,n_0) \begin{pmatrix} -\rho_{n_0} & \rho_{n_0} \\ \overline{b_{n_0}} & \overline{a_{n_0}} \end{pmatrix}, & n_0 \text{ even}, \end{cases} \quad z \in \mathbb{D}. \qquad \text{(B.26)}
\end{aligned}$$

Moreover, it follows from [23, Lemma 3.2] that

$$M_{1,1}(z,n_0) = \frac{1 - M_+(z,n_0)M_-(z,n_0)}{M_+(z,n_0) - M_-(z,n_0)}, \quad z \in \mathbb{D}, \qquad \text{(B.27)}$$

hence by (B.18) and (B.23)

$$M_{1,1}(z,n_0) = \widetilde{M}^{\mathrm{tr}}(z), \quad z \in \mathbb{D}. \qquad \text{(B.28)}$$

One then has the following basic result (see also [66]).

Theorem B.4.
(i) The CMV operator U on $\ell^2(\mathbb{Z})$ defined in (3.4) *is unitarily equivalent to the operator of multiplication by $I_2\mathrm{id}$ (where I_2 is the 2×2 identity matrix and $\mathrm{id}(\zeta) = \zeta$, $\zeta \in \partial\mathbb{D}$) on $L^2(\partial\mathbb{D}; d\widetilde{\Omega}(\cdot))$, and hence,*

$$\sigma(U) = \mathrm{supp}\,(d\widetilde{\Omega}) = \mathrm{supp}\,(d\widetilde{\Omega}^{\mathrm{tr}}), \qquad \text{(B.29)}$$

where $d\widetilde{\Omega}$ and $d\widetilde{\Omega}^{\mathrm{tr}}$ are introduced in (B.16) *and* (B.19)*, respectively.*
(i') The operator U is also unitarily equivalent to the operator of multiplication by $I_2\mathrm{id}$ on $L^2(\partial\mathbb{D}; d\Omega(\cdot))$, and hence by (i), (B.20)*, and* (B.28)*,*

$$d\Omega \ll d\Omega_{1,1} \ll d\Omega \;\text{ and }\; \sigma(U) = \mathrm{supp}\,(d\Omega) = \mathrm{supp}\,(d\Omega_{1,1}), \qquad \text{(B.30)}$$

where $d\Omega$ and $d\Omega_{1,1}$ are introduced in (B.24).
(ii) The spectral multiplicity of U is two if and only if

$$|\mathcal{M}_2| > 0, \qquad \text{(B.31)}$$

where

$$\mathcal{M}_2 = \{\zeta \in \Lambda_+ \,|\, m_+(\zeta) \in \mathbb{C}\backslash i\mathbb{R}\} \cap \{\zeta \in \Lambda_- \,|\, m_-(\zeta) \in \mathbb{C}\backslash i\mathbb{R}\}. \qquad \text{(B.32)}$$

If $|\mathcal{M}_2| = 0$, the spectrum of U is simple. Moreover, $\mathcal{M}_2$ is a maximal set on which U has uniform multiplicity two.
(iii) A maximal set $\mathcal{M}_1$ on which U has uniform multiplicity one is given by

$$\begin{aligned}
\mathcal{M}_1 = {} & \{\zeta \in \Lambda_+ \cap \Lambda_- \,|\, m_+(\zeta) = m_-(\zeta) \in i\mathbb{R}\} \\
& \cup \{\zeta \in \Lambda_+ \cap \Lambda_- \,|\, |m_+(\zeta)| = |m_-(\zeta)| = \infty\} \\
& \cup \{\zeta \in \Lambda_+ \cap \Lambda_- \,|\, m_+(\zeta) \in i\mathbb{R}, m_-(\zeta) \in \mathbb{C}\backslash i\mathbb{R}\} \\
& \cup \{\zeta \in \Lambda_+ \cap \Lambda_- \,|\, m_-(\zeta) \in i\mathbb{R}, m_+(\zeta) \in \mathbb{C}\backslash i\mathbb{R}\}. \qquad \text{(B.33)}
\end{aligned}$$

In particular, $\sigma_{\mathrm{s}}(U) = \sigma_{\mathrm{sc}}(U) \cup \sigma_{\mathrm{pp}}(U)$ is always simple.

Proof. We refer to Lemma 3.6 and Corollary 3.5 in [23] for a proof of (i) and (i'), respectively. To prove (ii) and (iii) observe that by (i) and (B.21)

$$\mathcal{N}_k = \{\zeta \in \sigma(U) \mid \operatorname{rank}[\widetilde{R}(\zeta)] = k\}, \quad k = 1, 2, \tag{B.34}$$

denote the maximal sets where the spectrum of U has multiplicity one and two, respectively. Using (B.15) and (B.22) one verifies that $\mathcal{N}_k = \mathcal{M}_k$, $k = 1, 2$. □

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References

[1] N. I. Akhiezer, *The Classical Moment Problem*, Oliver & Boyd., Edinburgh, 1965.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Operators in Hilbert Space*, Vol. I, Pitman, Boston, 1981.

[3] N. Aronszajn and W. F. Donoghue, *On exponential representations of analytic functions in the upper half-plane with positive imaginary part*, J. Analyse Math. **5**, 321–388 (1956–57).

[4] J. Avron and B. Simon, *Almost periodic Schrödinger operators I. Limit periodic potentials*, Commun. Math. Phys. **82**, 101–120 (1981).

[5] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[6] L. Carleson, *On H^∞ in multiply connected domains*, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, Vol. II, W. Beckner, A. P. Calderón, R. Fefferman, and P. W. Jones (eds.), Wadsworth, CA, 1983, pp. 349–372.

[7] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Basel, 1990.

[8] J. S. Christiansen, B. Simon, and M. Zinchenko, *Finite gap Jacobi matrices, I. The isospectral torus*, in preparation.

[9] V. A. Chulaevskii, *Inverse spectral problem for limit-periodic Schrödinger operators*, Funct. Anal. Appl. **18**, 230–233 (1984).

[10] W. Craig, *The trace formula for Schrödinger operators on the line*, Commun. Math. Phys. **126**, 379–407 (1989).

[11] C. De Concini and R. A. Johnson, *The algebraic-geometric AKNS potentials*, Ergod. Th. Dyn. Syst. **7**, 1–24 (1987).

[12] P. Deift and B. Simon, *Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension*, Commun. Math. Phys. **90**, 389–411 (1983).

[13] W. F. Donoghue, *Monotone Matrix Functions and Analytic Continuation*, Springer, Berlin, 1974.

[14] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties*, Russian Math. Surv. **31:1**, 59–146 (1976).

[15] I. E. Egorova, *On a class of almost periodic solutions of the KdV equation with a nowhere dense spectrum*, Russian Acad. Sci. Dokl. Math. **45**, 290–293 (1990).

[16] S. Fisher, *Function Theory on Planar Domains*, John Wiley & Sons, New York, 1983.

[17] L. R. Ford, *Automorphic Functions*, 2nd ed., Chelsea, New York, 1951.

[18] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I: $(1+1)$-Dimensional Continuous Models*, Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge Univ. Press, 2003.

[19] F. Gesztesy, M. Krishna, and G. Teschl, *On isospectral sets of Jacobi operators,* Commun. Math. Phys. **181**, 631–645 (1996).

[20] F. Gesztesy, K. A. Makarov, and M. Zinchenko, *Local AC spectrum for reflectionless Jacobi, CMV, and Schrödinger operators*, preprint, 2008.

[21] F. Gesztesy and B. Simon, *The ξ function*, Acta Math. **176**, 49–71 (1996).

[22] F. Gesztesy and P. Yuditskii, *Spectral properties of a class of reflectionless Schrödinger operators*, J. Funct. Anal. **241**, 486–527 (2006).

[23] F. Gesztesy and M. Zinchenko, *Weyl–Titchmarsh theory for CMV operators associated with orthogonal polynomials on the unit circle*, J. Approx. Th. **139**, 172–213 (2006).

[24] F. Gesztesy and M. Zinchenko, *A Borg-type theorem associated with orthogonal polynomials on the unit circle*, J. London Math. Soc. **74**, 757–777 (2006).
[25] D. J. Gilbert, *Subordinacy and Spectral Analysis of Schrödinger Operators*, Ph.D. Thesis, University of Hull, 1984.
[26] D. J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh **112A**, 213-229 (1989).
[27] D. J. Gilbert, *On subordinacy and spectral multiplicity for a class of singular differential operators*, Proc. Roy. Soc. Edinburgh **A 128**, 549–584 (1998).
[28] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128**, 30-56 (1987).
[29] R. A. Johnson, *The recurrent Hill's equation*, J. Diff. Eqs. **46**, 165–193 (1982).
[30] R. A. Johnson, *On the Sato–Segal–Wilson solutions of the K–dV equation*, Pac. J. Math. **132**, 343–3355 (1988).
[31] R. Johnson and J. Moser, *The rotation number for almost periodic potentials*, Commun. Math. Phys. **84**, 403–438 (1982).
[32] P. W. Jones and D. E. Marshall, *Critical points of Green's function, harmonic measure, and the corona problem*, Ark. Mat. **23**, 281–314 (1985).
[33] I. S. Kac, *On the multiplicity of the spectrum of a second-order differential operator*, Sov. Math. Dokl. **3**, 1035–1039 (1962).
[34] I. S. Kac, *Spectral multiplicity of a second order differential operator and expansion in eigenfunctions*, Izv. Akad. Nauk SSSR **27**, 1081–11112 (1963). Erratum, Izv. Akad. Nauk SSSR **28**, 951–952 (1964). (Russian.)
[35] I. S. Kac and M. G. Krein, *R-functions–analytic functions mapping the upper halfplane into itself*, Amer. Math. Soc. Transl. (2) **103**, 1-18 (1974).
[36] P. Koosis, *Introduction to H_p Spaces*, 2nd ed., Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1998.
[37] S. Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, in "*Stochastic Analysis*", K. Itǒ (ed.), North-Holland, Amsterdam, 1984, pp. 225–247.
[38] S. Kotani, *One-dimensional random Schrödinger operators and Herglotz functions*, in "*Probabilistic Methods in Mathematical Physics*", K. Itǒ and N. Ikeda (eds.), Academic Press, New York, 1987, pp. 219–250.
[39] S. Kotani, *Link between periodic potentials and random potentials in one-dimensional Schrödinger operators*, in *Differential Equations and Mathematical Physics*, I. W. Knowles (ed.), Springer, Berlin, 1987, pp. 256–269.
[40] S. Kotani and M. Krishna, *Almost periodicity of some random potentials*, J. Funct. Anal. **78**, 390–405 (1988).
[41] B. M. Levitan, *Almost periodicity of infinite-zone potentials*, Math. USSR Izvestija **18**, 249–273 (1982).
[42] B. M. Levitan, *Approximation of infinite-zone potentials by finite-zone potentials*, Math USSR Izvestija **20**, 55–87 (1983).
[43] B. M. Levitan, *An inverse problem for the Sturm-Liouville operator in the case of finite-zone and infinite-zone potentials*, Trans. Moscow Math. Soc. **45:1**, 1–34 (1984).
[44] B. M. Levitan, *On the closure of the set of finite-zone potentials*, Math. USSR Sbornik **51**, 67–89 (1985).
[45] B. M. Levitan, *Inverse Sturm–Liouville Problems*, VNU Science Press, Utrecht, 1987.
[46] B. M. Levitan and A. V. Savin, *Examples of Schrödinger operators with almost periodic potentials and nowhere dense absolutely continuous spectrum*, Sov. Math. Dokl. **29**, 541–544 (1984).
[47] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.
[48] J. Moser, *An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum*, Comment. Math. Helvetici **56**, 198–224 (1981).
[49] F. Nazarov, A. Volberg, and P. Yuditskii, *Reflectionless measures with singular components*, preprint, arXiv:0711.0948.
[50] S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York, 1984.
[51] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer, Berlin, 1992.

[52] L. A. Pastur and V. A. Tkachenko, *Spectral theory of a class of one-dimensional Schrödinger operators with limit-periodic potentials*, Trans. Moscow Math. Soc. 1989, 115–166.

[53] F. Peherstorfer and P. Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, J. Analyse Math. **89**, 113–154 (2003).

[54] F. Peherstorfer and P. Yuditskii, *Almost periodic Verblunsky coefficients and reproducing kernels on Riemann surfaces*, J. Approx. Th. **139**, 91–106 (2006).

[55] F. Peherstorfer and P. Yuditskii, *Finite difference operators with a finite-band spectrum*, arXiv:math.SP/0611816.

[56] I. Popovici and A. Volberg, *Boundary Harnack principle for Denjoy domains*, Complex Var. Theory Appl. **37**, 471–490 (1998).

[57] I. I. Priwalow, *Randeigenschaften analytischer Funktionen*, 2nd ed., VEB Verlag, Berlin, 1956.

[58] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[59] C. Remling, *The absolutely continuous spectrum of Jacobi matrices*, arXiv:0706.1101.

[60] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators*, arXiv:0710.4128.

[61] M. Rosenblum and J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*, Birkhäuser, Basel, 1994.

[62] B. Simon, *Analogs of the m-function in the theory of orthogonal polynomials on the unit circle*, J. Comp. Appl. Math. **171**, 411-424 (2004).

[63] B. Simon, *Orthogonal polynomials on the unit circle: New results*, Intl. Math. Res. Notices, **53**, 2837–2880 (2004).

[64] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Part 2: Spectral Theory*, AMS Colloquium Publication Series, Vol. 54, Providence, R.I., 2005.

[65] B. Simon, *OPUC on one foot*, Bull. Amer. Math. Soc. **42**, 431–460 (2005).

[66] B. Simon, *On a theorem of Kac and Gilbert*, J. Funct. Anal. **223**, 109–115 (2005).

[67] B. Simon, *CMV matrices: Five years later*, J. Comp. Appl. Math. **208**, 120–154 (2007).

[68] B. Simon, *Equilibrium measures and capacities in spectral theory*, preprint, arXiv:0711.2700.

[69] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, in preparation; to be published by Princeton Univ. Press.

[70] R. Sims, *Reflectionless Sturm–Liouville equations*, J. Comp. Appl. Math. **208**, 207–225 (2007).

[71] M. Sodin and P. Yuditskii, *Almost periodic Sturm-Liouville operators with Cantor homogeneous spectrum and pseudoextendible Weyl functions*, Russ. Acad. Sci. Dokl. Math. **50**, 512–515 (1995).

[72] M. Sodin and P. Yuditskii, *Almost periodic Sturm-Liouville operators with Cantor homogeneous spectrum*, Comment. Math. Helvetici **70**, 639–658 (1995).

[73] M. Sodin and P. Yuditskii, *Almost-periodic Sturm-Liouville operators with homogeneous spectrum*, in *Algebraic and Geometric Methods in Mathematical Physics*, A. Boutel de Monvel and A. Marchenko (eds.), Kluwer, 1996, pp. 455–462.

[74] M. Sodin and P. Yuditskii, Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions, J. Geom. Anal. **7**, 387–435 (1997).

[75] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 2000.

[76] M. Zinsmeister, *Espaces de Hardy et domaines de Denjoy*, Ark. Mat. **27**, 363–378 (1989).

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