SPECTRAL ANALYSIS OF A CLASS OF HERMITIAN JACOBI MATRICES IN A CRITICAL (DOUBLE ROOT) HYPERBOLIC CASE: PERIODICALLY MODULATED MATRIX ENTRIES.

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Abstract. We consider a class of Jacobi matrices with periodically modulated diagonal in a critical hyperbolic ("double root") situation. For the model with "non-smooth" matrix entries we obtain the asymptotics of generalized eigenvectors and analyze the spectrum of the corresponding Jacobi matrix. In addition, we reformulate a very helpful Theorem from [9] in its full generality in order to serve the needs of our method.

1. Introduction

In the present paper, we consider a class of Jacobi matrices with periodically modulated growing diagonal giving an example of the "double root" problem. This means a critical situation for asymptotics of the generalized eigenvectors related to the matrix. Such a situation particularly arises in the spectral phase transition phenomenon. If the matrix depends on some parameters, the decomposition of its spectrum into different types (absolutely continuous, singular continuous, pure point, discrete) may be independent of these parameters. But if the structure of this decomposition changes under a variation of the parameters, a spectral phase transition occurs. If this change happens by a jump, such a phenomenon is called a spectral phase transition of the first type, whereas if this change is smooth with the change of the parameters when they move across some hyper-surface in a space, it is called a spectral phase transition of the second type. The transition in types of spectrum is tightly related to the change of form of asymptotics of generalized eigenvectors due to subordinacy theory (Gilbert-Pearson [6]), which was generalized to the case of Jacobi matrices in [12].

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Last decade the spectral analysis of Jacobi matrices attracted the attention of many specialists in operator theory and mathematical physics.

We consider the Jacobi matrix J with diagonal entries b_n:

$$b_n = \begin{cases} bn^\alpha \text{ for odd values of } n \\ 0 \text{ for even values of } n \end{cases}$$

(where α and b are real parameters $\alpha \in (\frac{2}{3}; 1)$ and $b \neq 0$) and off-diagonal entries (weights)

$$a_n = n^\alpha.$$

The model demonstrates the situation of the spectral phase transition of the first order corresponding to the "moment of transition" exactly. We are interested in asymptotics of generalized eigenvectors, i.e. solutions of the spectral recurrence relation

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n,\ n \geq 2. \tag{1}$$

This model has been studied in [5] and in particular the following result was obtained: for $\lambda < 0$ there are two solutions u_n^+ and u_n^- of (1) with the following asymptotics as $n \to \infty$:

$$\begin{aligned} &u_{2n}^\pm \sim (-1)^n n^{-\frac{\alpha}{4}} \exp\left(\pm\sqrt{\tfrac{b\lambda}{2^\alpha}}\tfrac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right) \\ &u_{2n+1}^\pm \sim \pm\sqrt{\tfrac{\lambda}{2^\alpha b}}\left(1-\tfrac{\alpha}{2}\right)(-1)^n n^{-\frac{3\alpha}{4}} \exp\left(\pm\sqrt{\tfrac{b\lambda}{2^\alpha}}\tfrac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right) \end{aligned}$$

The problem of determining the asymptotics for $\lambda > 0$ was stated, and in the present paper, we show that the answer has the same form. However this interesting question is not the main concern of this paper. The principal difficulty in our analysis is that the situation is "critical hyperbolic" unlike the "critical elliptic" situation in [5], which roughly means that exponents in the answer grow and decay if $\lambda > 0$ and oscillate if $\lambda < 0$.

Let us explain the problem of the critical situation in more detail. Whenever one deals with the three-term recurrence relation (1), it is often useful to write it in the vector form introducing the sequence $\overrightarrow{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$ and the transfer-matrix $B_n := \begin{pmatrix} 0 & 1 \\ -\frac{a_{n-1}}{a_n} & \frac{\lambda - b_n}{a_n} \end{pmatrix}$. So (1) is equivalent to the discrete linear system in $\mathbb{C}^2$:

$$\overrightarrow{u}_{n+1} = B_n \overrightarrow{u}_n,\ n \geq 2. \tag{2}$$

The solution for such a system is obtained by taking the chronological product of transfer matrices, $\left(\prod_{k=2}^{n} B_k\right)\overrightarrow{u}_2 = B_n \cdot \ldots \cdot B_2 \overrightarrow{u}_2$. The analysis becomes much easier if the matrix of the system is in some sense smooth-in-n (say, has a limit and asymptotic expansion in inverse powers of n as $n \to \infty$). In our case the transfer-matrix is not smooth in this sense because the coefficients of the spectral equation "jump" all the time. This is the first (simple) problem that we face and it may be solved by taking the product of two consecutive transfer-matrices, introducing the new linear system with the sequence of the coefficient matrices

$$M_n := B_{2n} B_{2n-1} \tag{3}$$

which turns out to be smooth-in-n. It is easy to see that the sequence M_n has a limit $M := \lim_{n\to\infty} M_n$ with $\det M = 1$. The are tree possibilities for the eigenvalues of M: they can be

- unimodular and complex conjugate (the elliptic situation),
- real and different (the hyperbolic situation; hence one of them is greater than 1 and another less than 1 in absolute value), or
- coincide and equal 1 or -1 (the critical situation=the double root case).

In the hyperbolic situation solutions are supposed to grow or decay, in the elliptic situation they are supposed to oscillate having similar behavior of their norms $\|\overrightarrow{u}_n\|$. The numerous variants of analogues to Levinson Theorem (also known as Bernzaid-Lutz Theorem [4]) for differential linear systems [3] can be applied in these two cases, cf. [9], [13].

In the critical situation one also can distinguish the "critical elliptic" and the "critical hyperbolic" cases. This separation depends on the lower orders behavior of M_n as $n \to \infty$, namely on the asymptotic sign of the discriminant ($\mathrm{discr} M := (\mathrm{tr} M)^2 - 4 \det M$) of the matrix M_n. In the critical case, the matrix M is similar to the Jordan block, which powers are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

and the worst thing here is not that the powers of the matrix grow with n (so they do also in the hyperbolic situation), but the fact that the large entry is off-diagonal. This mixes upper and lower components of the solution, which makes the system unstable and sensitive to small perturbations. The problem of the "critical hyperbolic" situation was considered for smooth matrix elements in [10]. In the present paper, considering the model of a Jacobi matrix with "oscillating" diagonal we intend to simplify greatly the scheme of successful transformations of the matrix system which was used there by making it more general and transparent (Section 3). This approach differs from that of [5] for "critical elliptic" situation, because in our case we have to deal with growing exponents (see Step 3 on page 10). Moreover, we consider in the Appendix a Theorem from [9] which is necessary for the final step of our method. The problem is that the original formulation of this Theorem states asymptotics of only one (principal) solution. Although it is surely the most difficult part of the problem considered, we want to state the result in its full generality in order to use it.

The method of the present paper works for a much wider class of Jacobi matrices. However our goal is to present a simple formulation of the method and show by example how it works.

A similar problem (the critical hyperbolic situation) was considered in [7] for a "smooth" model where the author used a completely different method related to Kelley's paper [11].

2. Preliminaries

As usual, the operator J in the Hilbert space $l^2(\mathbb{N})$ is first defined (as $\mathcal{J}$) on the linear set of vectors which have only a finite number of non-zero components, $l_{fin}(\mathbb{N})$, by the rule

$$\begin{aligned}(\mathcal{J}u)_1 &= a_1u_1 + b_1u_2\\ (\mathcal{J}u)_n &= b_{n-1}u_{n-1} + a_nu_n + b_nu_{n+1}, \text{ for } n \geq 2.\end{aligned}$$

Then its closure $J = \overline{\mathcal{J}}$ is a self-adjoint operator provided the Carleman condition $\sum\limits_{n=0}^{\infty} \frac{1}{a_n} = +\infty$ [2] is satisfied. In the standard basis $\{e_n\}_{n=1}^{\infty}$ (where e_n is the vector with all the components zeros except the n-th) the operator J admits the following matrix representation:

$$\begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We consider the smooth system

$$\begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} = M_n \begin{pmatrix} u_{2n-2} \\ u_{2n-1} \end{pmatrix} \tag{4}$$

(see (9), (3)). One can obtain the solution of the system directly via the matrix M_n since $\begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} = \left[\prod\limits_{k=2}^{n} M_k\right] \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$. Let us calculate the matrix $M_n = B_{2n}B_{2n-1}$:

$$M_n = \begin{pmatrix} 0 & 1 \\ -\left(1-\frac{1}{2n}\right)^\alpha & \frac{\lambda}{(2n)^\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\left(1-\frac{1}{2n-1}\right)^\alpha & \frac{\lambda - b(2n-1)^\alpha}{(2n-1)^\alpha} \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} + \frac{\lambda}{(2n)^\alpha} \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} + \frac{\alpha}{2n} I + O\left(\frac{1}{n^{2\alpha}}\right).$$

We see that it has smooth-in-n asymptotics as $n \to \infty$.

The eigenvalues of the limit matrix $M := \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ coincide, so the situation is critical (the double root case). By an easy calculation, one can see that discriminants of matrices M_n equal $\operatorname{discr} M_n = \frac{4b\lambda}{(2n)^\alpha} + O\left(\frac{1}{n}\right)$, hence indeed $\lambda < 0$ corresponds to the elliptic situation and $\lambda > 0$ corresponds to the hyperbolic situation (both critical).

Our method is based upon a sequence of transformations which are determined by an anzats. Similar consideration for the anzats constructing may be also found in [10].

Remark 1. *In what follows, we use transformations that we call similarity-like. Whenever one deals with the product of matrices, say A_n, one can as well consider matrices $C_n = T_{n+1}^{-1} A_n T_n$ (where the matrix sequence T_n is chosen somehow; we will call matrices T_n affinity-like). Due to the cancellation of intermediate terms the product of matrices A_n equals*

$$\prod_{n=n_1}^{n_2} A_n = \prod_{n=n_1}^{n_2} (T_{n+1} C_n T_n^{-1}) = T_{n_2+1} \left(\prod_{n=n_1}^{n_2} C_n \right) T_{n_1}^{-1}.$$

So the study of the linear difference system with coefficient matrices A_n can be completely reduced to the study of the linear difference system with coefficient matrices C_n.

3. Calculation of the asymptotics in the hyperbolic case.

In this Section, we proceed through several transformations in order to simplify the problem and finally obtain the system which can be treated with Janas-Moszynski Theorem (which is properly adjusted in Appendix). So we divide this Section into four steps. In fact we have already made the "zero" step in the first Section, which is reduction to the smooth matrix system by taking the product of two transfer-matrices. But we do not include this step into Section 3 because the double root problem actually arises only at this stage. Moreover after grouping the transfer-matrices by pairs one does not need any inverse transformation to make in order to obtain the answer.

Step 1: Reduction of the meaningful part of M_n to the transfer-matrix form.

We used the matrix form of writing the spectral equation to settle the problem of periodically modulated coefficients by grouping transfer-matrices in pairs. But now it is simpler to consider a smooth three-term recurrence relation which is equivalent to the system. The transfer-matrix corresponding to the three-term recurrence relation should have entries 0 and 1 in the upper raw. In this step, we find the proper transformation which makes the coefficient matrix of the system resemble transfer-matrix. This similarity-like transformation is generated by the matrix sequence T_n of the form

$$T_n := (-1)^n \left[\begin{pmatrix} 1 & -b \\ 1 & 0 \end{pmatrix} + \frac{\lambda}{(2n)^\alpha} \begin{pmatrix} b+\frac{1}{2b} & 0 \\ \frac{1}{2b} & -\frac{1}{2} \end{pmatrix} + \frac{\alpha}{2n} \begin{pmatrix} 0 & 0 \\ -1 & b \end{pmatrix} \right].$$

The similarity-like transformation

$$N_n := T_{n+1} M_n T_n^{-1} \tag{5}$$

gives

$$N_n = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \frac{b\lambda}{(2n)^\alpha} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\alpha}{n} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + O\left(\frac{1}{n^{2\alpha}}\right).$$

The meaningful part (or the part which we expect bo be meaningful, to be precise) the sum of first few orders is now the matrix with the upper raw of the form 0, 1. The exact form of the matrices T_n can be determined from this requirement (but the answer is not unique of course). Indeed, try to find T_n in the form

$$T_n = (-1)^n \left[\begin{pmatrix} 1 & -b \\ 1 & 0 \end{pmatrix} + \frac{\lambda}{(2n)^\alpha} T^{(1)} + \frac{\alpha}{2n} T^{(2)} \right], \tag{6}$$

with unknown matrices $T^{(1)}$ and $T^{(2)}$ independent of n. Denote $T := \begin{pmatrix} 1 & -b \\ 1 & 0 \end{pmatrix}$ which is chosen in order to satisfy

$$-T \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} =: N.$$

By substitution of T_n in the form (6) into the relation $T_{n+1}M_n = N_n T_n$ we obtain, looking at the terms of orders $\frac{1}{n^\alpha}$ and $\frac{1}{n}$, the following linear conditions on the matrices $T^{(1)}$ and $T^{(2)}$:

$$[T^{(1)}T^{-1}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}] = T \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} T^{-1} + \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$$
$$[T^{(2)}T^{-1}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}] = I + \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}.$$

By stars we denote matrix entries which values are of no interest for us. Therefore the problem may be reduced to the following one. To prove for the commutator equation

$$[X, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

that for any given values α and β there exist unique values γ and δ and 2×2 matrix X (obviously not unique) which satisfy the equation. Multiplying the equality on the left by the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and on the right by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (the inverse one) we obtain another form of the commutator equation:

$$\begin{aligned} [Y, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}] &= [Y, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] \\ &= \begin{pmatrix} -y_3 & y_1 - y_4 \\ 0 & y_3 \end{pmatrix} = \begin{pmatrix} \alpha + \beta & \beta \\ \gamma + \delta - \alpha - \beta & \delta - \beta \end{pmatrix}. \end{aligned}$$

We denoted $Y := \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} X \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. It follows now that $\delta = -\alpha$ and $\gamma = 2\alpha + \beta$ are uniquely determined by α and β and the matrix Y exists and is unique up to $c_1 I + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Therefore one can reduce the original system (4) to the new one, where coefficient matrices M_n are replaced by N_n. Note that the exact form of the matrices $T^{(1)}$ and $T^{(2)}$ is not essential for the result of our transformation. We need only to prove the existence of such matrices.

Step 2: Reduction of the main term to the identity matrix.

The structure of the main part of the system allows to write

$$N_n = \begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} + O\left(\frac{1}{n^{2\alpha}}\right),$$

where the error term is a 2×2 matrix which norm is $O(n^{-2\alpha})$ and the matrix entries $F_1(n)$ and $F_1(n)$ are (denote $B := \sqrt{\frac{b\lambda}{2^\alpha}}$)

$$F_1(n) = -2 - \frac{B^2}{n^\alpha} + \frac{\alpha}{n}, \; F_2(n) = 1 - \frac{\alpha}{n}.$$

Our goal is to use the fact that the system

$$\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} \begin{pmatrix} v_n \\ w_n \end{pmatrix}$$

where the remainder is omitted is *equivalent* to the three-term recurrence relation

$$u_{n+1} + F_1(n)u_n + F_2(n)u_{n-1} = 0. \tag{7}$$

The latter has two "approximate solutions" of the form

$$z_n^{\pm} = n^{\gamma} e^{\pm A n^{\delta}},$$

with $\gamma = -\frac{\alpha}{4}$, $\delta = 1 - \frac{\alpha}{2}$, $A = \frac{B}{\delta}$. This means that

$$z_{n+1}^{\pm} + F_1(n) z_n^{\pm} + F_2(n) z_{n-1}^{\pm} = O(n^{-2\alpha}) z_n^{\pm} \tag{8}$$

and can be verified by a direct calculation. The form of these "approximate solutions" can be obtained, for instance, by the analogy to the WKB method (see [10] for that type of argument). Having this structure of the "solution" one can determine unknown values of γ, δ and A from the condition of cancellation of all decreasing terms in (8) up to the order $O(n^{-2\alpha})$. The equation (8) implies that

(9)
$$\begin{pmatrix} z_n^{\pm} \\ z_{n+1}^{\pm} \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & O(n^{-2\alpha}) \end{pmatrix} \right) \begin{pmatrix} z_{n-1}^{\pm} \\ z_n^{\pm} \end{pmatrix}$$
$$= (N_n + O(n^{-2\alpha})) \begin{pmatrix} z_{n-1}^{\pm} \\ z_n^{\pm} \end{pmatrix}.$$

It is useful now to write the last vector equality in a matrix form. Denote

$$S_n = \begin{pmatrix} z_{n-1}^{-} & z_{n-1}^{+} \\ z_n^{-} & z_n^{+} \end{pmatrix},$$

then combining (9) for both signs one has

$$S_{n+1} = (N_n + O(n^{-2\alpha})) S_n.$$

and hence

$$S_{n+1}^{-1} N_n S_n = I + S_{n+1}^{-1} O(n^{-2\alpha}) S_n.$$

This actually follows *only* from the fact that $z_n^{\pm}$ are "approximate solutions" of the recurrence relation (7), i.e. from (8).

Substituting the expression for $z_n^{\pm} = n^{\gamma} e^{\pm A n^{\delta}}$ into the term

$$S_{n+1}^{-1} O(n^{-2\alpha}) S_n = \frac{1}{\det S_{n+1}} \begin{pmatrix} z_n^{+} z_n^{-} O(n^{-2\alpha}) & z_n^{+2} O(n^{-2\alpha}) \\ z_n^{-2} O(n^{-2\alpha}) & z_n^{+} z_n^{-} O(n^{-2\alpha}) \end{pmatrix}$$

and using the fact that $\det S_{n+1} \sim 2A\delta n^{-\alpha}$ as $n \to \infty$ (calculations, cf. [5]; we can also find this rate of decay from the fact of modified wronskian persistence), we obtain

$$S_{n+1}^{-1} N_n S_n = I + \frac{1}{n^{3\alpha/2}} \begin{pmatrix} O(1) & e^{2An^\delta} O(1) \\ e^{-2An^\delta} O(1) & O(1) \end{pmatrix} =: K_n. \tag{10}$$

The calculations produced enable us to reduce the original system to the system with coefficient matrix K_n, which "main term" is the identity matrix. The problem is that it contains exponentially increasing anti-diagonal terms. In the next step we show how to overcome this difficulty due to the symmetric cancellation of both anti-diagonal entries.

Remark 2. *The calculations above demonstrate a simple "geometrical" approach based on the reduction to the tree-term recurrence relation* (7)*. Another approach (a simplification of calculations from* [10]*) for the reduction to the matrix form* (10) *may be rather straightforward where the "geometrical construction" is replaced by an explicit calculation. Since* $\det S_{n+1} = z_n^- z_{n+1}^+ - z_n^+ z_{n+1}^-$*, the explicit substitution of matrices* S_n *gives*

$$S_{n+1}^{-1} \begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} S_n = \frac{1}{z_n^- z_{n+1}^+ - z_n^+ z_{n+1}^-}$$
$$\times \begin{pmatrix} z_n^- z_{n+1}^+ + z_n^+ (F_1 z_n^- + F_2 z_{n-1}^-) & z_n^+ (z_{n+1}^+ + F_1 z_n^+ + F_2 z_{n-1}^+) \\ -z_n^- (z_{n+1}^- + F_1 z_n^- + F_2 z_{n-1}^-) & -z_n^+ z_{n+1}^- - z_n^- (F_1 z_n^+ + F_2 z_{n-1}^+) \end{pmatrix}.$$

After adding and subtracting terms $z_n^+ z_{n+1}^-$ *in the upper-left entry and* $z_n^- z_{n+1}^+$ *in the lower-right entry for extracting the determinant, the last expression immediately reduces to the following one*

$$I + \frac{1}{z_n^- z_{n+1}^+ - z_n^+ z_{n+1}^-}$$
$$\times \begin{pmatrix} z_n^+ (z_{n+1}^- + F_1 z_n^- + F_2 z_{n-1}^-) & z_n^+ (z_{n+1}^+ + F_1 z_n^+ + F_2 z_{n-1}^+) \\ -z_n^- (z_{n+1}^- + F_1 z_n^- + F_2 z_{n-1}^-) & -z_n^- (z_{n+1}^- + F_1 z_n^+ + F_2 z_{n-1}^+) \end{pmatrix}.$$

It has a remarkable property that the expressions $z_{n+1}^\pm + F_1 z_n^\pm + F_2 z_{n-1}^\pm$ *appear in each matrix entry of the second term. Now substituting* (8) *and taking into consideration that* $\det S_{n+1} \sim 2A\delta n^{-\alpha}$ *we run at the same expression as in* (10)*:*

$$I + \frac{1}{\det S_{n+1}} \begin{pmatrix} z_n^+ z_n^- O(n^{-2\alpha}) & z_n^{+2} O(n^{-2\alpha}) \\ z_n^{-2} O(n^{-2\alpha}) & z_n^+ z_n^- O(n^{-2\alpha}) \end{pmatrix}$$
$$= I + \frac{1}{n^{3\alpha/2}} \begin{pmatrix} O(1) & e^{2An^\delta} O(1) \\ e^{-2An^\delta} O(1) & O(1) \end{pmatrix}.$$

The problem now is the growing exponent in the upper-right entry of the matrix, which leaves no chance to call the remainder small in any sense. But this exponent can be compensated by the decaying one in the lower-left entry as we show below.

Step 3: Elimination of exponentially increasing off-diagonal terms.

In order to produce the elimination we make yet another similarity-like transformation with the affinity-like matrix

$$X_n = \begin{pmatrix} e^{2An^\delta} & 0 \\ 0 & 1 \end{pmatrix},$$

which yields

$$L_n := X_{n+1}^{-1} K_n X_n = \begin{pmatrix} e^{2A(n^\delta-(n+1)^\delta)} & 0 \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{n^{3\alpha/2}}\right) \tag{11}$$
$$= \begin{pmatrix} 1 - \frac{2A\delta}{n^{\alpha/2}} + \frac{(2A\delta)^2}{2n^\alpha} & 0 \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{n^{3\alpha/2}}\right).$$

As a result of all the three transformations the original system is reduced to a new one with coefficient matrices L_n for which Janas-Moszynski Theorem [9] (Theorem 2, Appendix) is applicable. Take $p_n = \frac{2A\delta}{n^{\alpha/2}}$, $V_n \equiv V = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and let $R_n = O(n^{-3\alpha/2})$ be the matrix remainder which belongs to l^1 for $\alpha > 2/3$. The Theorem asserts that the system with coefficient matrices L_n has a basis of solutions of the form $e^{-2An^\delta}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)\right)$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)$.

Remark 3. *Note that we need to apply Theorem 2 only to ignore the remainder term of order $O(n^{-3\alpha/2})$. Without it, the system generated by matrix* (11) *obviously has two exact solutions $e^{-2An^\delta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. However this fact is completely not obvious since we cannot apply Levinson Theorem* [13] *due to coincidence of eigenvalues of the limit matrix $\lim\limits_{n\to\infty} L_n$.*

Step 4: Returning to the original system.

Returning to the system with coefficient matrices M_n we recall steps 1, 2 and 3. We should take into account the affinity-like matrices that stay in front of the product of matrices L_n. Since $M_k = T_{k+1}^{-1} S_{k+1} X_{k+1} L_k X_k^{-1} S_k^{-1} T_k$

we have

$$\begin{aligned}
\begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} &= \left(\prod_{k=2}^{n} M_k\right)\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \\
&= (T_{n+1}^{-1}S_{n+1}X_{n+1})\left(\prod_{k=2}^{n} L_k\right) X_2^{-1}S_2^{-1}T_2\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \\
&= (-1)^{n+1}\begin{pmatrix} z_n^+(1+o(1)) & z_n^+(1+o(1)) \\ -\frac{A\delta z_n^+}{bn^{\alpha/2}}(1+o(1)) & \frac{A\delta z_n^+}{bn^{\alpha/2}}(1+o(1)) \end{pmatrix} \\
&\qquad\qquad \times \left(\prod_{k=2}^{n} L_k\right) X_2^{-1}S_2^{-1}T_2\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}.
\end{aligned}$$

The vector $X_2^{-1}S_2^{-1}T_2\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$ is arbitrary provided $\det(X_2^{-1}S_2^{-1}T_2) \neq 0$. Note that we can put $\det T_2 \neq 0$ without loss of generality. Otherwise we can analyse the asymptotics of the solution starting from some $n_0 > 1$ ($\lim\limits_{n\to\infty} \det T_n = b \neq 0$). So the vector $\left(\prod\limits_{k=2}^{n} L_k\right) X_2^{-1}S_2^{-1}T_2\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$ is an arbitrary solution to the difference system with the coefficient matrices L_n, which has a basis of solutions $e^{-2An^\delta}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)\right)$ and $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right)$. Therefore the system with coefficient matrices M_n has a basis of solutions of the form $(-1)^n\begin{pmatrix} z_n^\pm(1+o(1)) \\ \pm\frac{A\delta z_n^\pm}{bn^{\alpha/2}}(1+o(1)) \end{pmatrix}$. We have proved the following Theorem.

Theorem 1. *For any $\lambda > 0$ spectral equation* (1) *has a basis of solutions u_n^+ and u_n^- with asymptotics as $n \to \infty$ of even components*

$$u_{2n}^\pm \sim (-1)^n n^{-\frac{\alpha}{4}} \exp\left(\pm\sqrt{\frac{b\lambda}{2^\alpha}}\frac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right)$$

and asymptotics of odd components

$$u_{2n+1}^\pm \sim \pm\sqrt{\frac{\lambda}{2^\alpha b}}\left(1-\frac{\alpha}{2}\right)(-1)^n n^{-\frac{3\alpha}{4}} \exp\left(\pm\sqrt{\frac{b\lambda}{2^\alpha}}\frac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right).$$

Regarding the operator J this means (due to the subordinacy theory of Gilbert and Pearson [6] [12]), that the spectrum of the operator on the positive semiaxis is of pure point type. Moreover the solution u_n^- is the eigenvector of J provided $\lambda \in \sigma_p(J)$.

Note that the asymptotics of $u_n^{\pm}$ as $n \to \infty$ for $\lambda > 0$ formally coincide with the ones obtained in [5] for $\lambda < 0$ as it was mentioned above.

4. Appendix: Janas-Moszynski Theorem revisited.

Now we turn to a Theorem from [9], which was formulated there not in its full generality. The existence of the principal ("smaller") solution for some system was proved, which used the complicated technique, but the existence of the second ("larger") solution was not stated. In what follows we present a proof of this fact.

Remark 4. *For the sequences $\{a_n\}$, to avoid any inessential problems related to the sequence element vanishing, we assume the notation for the product $\prod\limits_{n=1}^{N} a_n$ meaning $\prod\limits_{n:\, a_n \neq 0} a_n$. We also use the notation l^1 and D^1 for the matrix sequences, i.e. the sequence of matrices $\{M_n\}_{n=1}^{\infty}$ belongs to*

- *l^1, iff $\sum\limits_{n=1}^{\infty} \|M_n\| < \infty$,*
- *D^1, iff $\sum\limits_{n=1}^{\infty} \|M_{n+1} - M_n\| < \infty$.*

Consider a linear difference system

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = (I + p_n V_n + R_n) \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{12}$$

supposing it non-degenerate, i.e. $\det(I + p_n V_n + R_n) \neq 0$ for every n (this means that the system has two linearly independent solutions).

Theorem 2. *(see* [9] *where the the essential part of the Theorem has been proved).*

Let

- *$p_n \to 0$ be a* positive *sequence with $\sum\limits_{n=1}^{\infty} p_n = +\infty$.*
- *$\{V_n\} \in D^1$ be a* real *2×2 matrix sequence with discr$V \neq 0$ ($V := \lim\limits_{n\to\infty} V_n$).*
- *$\{R_n\} \in l^1$ be a 2×2 matrix sequence.*

Then the system (12) *has a basis of solutions $\overrightarrow{u}_n^{(1)}$ and $\overrightarrow{u}_n^{(2)}$ with the following asymptotics as $n \to \infty$:*

$$\overrightarrow{u}_n^{(1,2)} = \left(\prod_{k=1}^{n} [1 + p_k \mu_{1,2}(k)] \right) (\overrightarrow{e}_{1,2} + o(1)),$$

where μ_1 and μ_2 (Re $\mu_1 \leq$ Re μ_2) are the eigenvalues of the matrix V, $\overrightarrow{e}_1$ and $\overrightarrow{e}_2$ are the corresponding eigenvectors and $\mu_1(n)$ and $\mu_2(n)$ are the eigenvalues of the matrix V_n chosen in such a way that $\mu_1(n) \to \mu_1$ and $\mu_2(n) \to \mu_2$ as $n \to \infty$.

Proof. The elliptic case discr$V < 0$ is a special case of Janas-Moszynski Theorem proved in [9]. In the hyperbolic case discr$V > 0$, the existence of the "smaller" solution is also guaranteed by that Theorem. So we need to prove only the existence of the second ("larger") solution (the solution corresponding to the eigenvalue μ_2 with the larger real part) in that case. We emphasize that we do not present a new proof of the result from [9], but we only add one extra (and rather simple) statement to it.

We reduce our situation to its simpler subcase and first prove the Theorem for a system of a special form

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \left(I - \begin{pmatrix} q_n & 0 \\ 0 & 0 \end{pmatrix} + R_n \right) \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{13}$$

The meaningful part of this system is diagonal and the larger eigenvalue of the corresponding matrix V is exactly zero. In what follows we will use the following statement (its proof can be found in the very end of the Appendix).

Lemma 1. *Let*

- $q_n \to 0$ *be such a* positive *sequence that* $\sum\limits_{n=1}^{\infty} q_n = +\infty$.
- $\{R_n\} \in l^1$ *be a* 2×2 *matrix sequence.*

Then there exists a solution of the system (13)*of the form*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)$$

as $n \to \infty$.

The reduction to the subcase considered by this Lemma can be done using the D^1-diagonalizability of every D^1 matrix sequence with invertible limit asserted by the following Proposition from [8] (also [9, Lemma 1.3], this is a discrete version of the result from [3]).

Proposition 1. *Let* $\{V_n\}$ *be such a* 2×2 *complex matrix sequence that* $\{V_n\} \in D^1$ *and discr*$\left(\lim\limits_{n\to\infty} V_n\right) \neq 0$. *Then there exists such a matrix sequence* $\{T_n\} \in D^1$ with invertible limit *that for sufficiently large values of* n

$$V_n = T_n \begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n) \end{pmatrix} T_n^{-1}.$$

Without loss of generality we may consider all the matrices $\{V_n\}$ to be diagonalizable. General case can be reduced to that one by a proper correction of the sequence R_n for finitely many values of n only. In what follows we suppose for the notation convenience that all the matrices V_n are diagonalizable. Following the standard strategy ([3], [9]) we reduce the original system (12) to its special form (13).

By Proposition 1, there exist a sequence of matrices $\{T_n\} \in D^1$ which diagonalize V_n for sufficiently large n. Explicit calculation shows:

$$
\begin{aligned}
&T_{n+1}^{-1}(I + p_n V_n + R_n)T_n \\
&\qquad = (T_{n+1}^{-1}T_n)\left(I + p_n\begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n)\end{pmatrix}\right) + T_{n+1}^{-1}R_nT_n \\
&\qquad\qquad = I + p_n\begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n)\end{pmatrix} + Q_n,
\end{aligned}
$$

where

$$
Q_n = T_{n+1}^{-1}(T_n - T_{n+1})\left[I + p_n\begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n)\end{pmatrix}\right] + T_{n+1}^{-1}R_nT_n.
$$

belongs to l^1 because the sequence $\{T_n\} \in D^1$ and the limit $\lim\limits_{n\to\infty} T_n$ is invertible (hence norms $\|T_n^{-1}\|$ are bounded). Therefore the chronological product equals

$$
\begin{aligned}
&\prod_{k=1}^{n}(I + p_kV_k + R_k) \\
&\qquad = T_{n+1}\left(\prod_{k=1}^{n}\left[I + p_k\begin{pmatrix} \mu_1(k) & 0 \\ 0 & \mu_2(k)\end{pmatrix} + Q_k\right]\right)T_1^{-1} \\
&\qquad = \left(\prod_{k=1}^{n}(1 + p_k\mu_2(k))\right)T_{n+1}\left(\prod_{k=1}^{n}\left[I - \begin{pmatrix} q_k & 0 \\ 0 & 0\end{pmatrix} + P_k\right]\right)T_1,
\end{aligned}
$$

where we used the following notations

$$
q_n := \frac{p_n(\mu_2(n) - \mu_1(n))}{1 + p_n\mu_2(n)} \sim p_n(\mu_2 - \mu_1),\ \ P_n := \frac{1}{1 + p_n\mu_2(n)}Q_n.
$$

Since we consider the hyperbolic case $\mathrm{discr}V > 0$, the eigenvalues $\mu_1(n), \mu_2(n) = \frac{\mathrm{tr}V_n}{2} \pm \frac{\sqrt{\mathrm{discr}V_n}}{2}$ are real for large values of n. Again without loss of generality (by changing sequence P_n for finitely many values of n) we may suppose the positiveness of $\mu_2(n) - \mu_1(n)$ for all $n = 1, 2, 3, \ldots$. So the conditions of Lemma 1 are satisfied for the system with coefficient matrices $I - \begin{pmatrix} q_n & 0 \\ 0 & 0\end{pmatrix} + P_n$. Therefore the original system (12) has a solution of the form

$$\left(\prod_{k=1}^{n}[1+p_k\mu_2(k)]\right) T_{n+1}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}+o(1)\right] = \left(\prod_{k=1}^{n}[1+p_k\mu_2(k)]\right)(\overrightarrow{e}_2+o(1)),$$

because $T_{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \overrightarrow{e}_2+o(1)$ with some $\overrightarrow{e}_2$, a non-zero eigenvector of the matrix V corresponding to the eigenvalue μ_2. The last fact follows from Proposition 1 immediately. It is clear that this solution plays the role of the solution $\overrightarrow{u}_n^{(2)}$. This completes the proof of the Theorem. □

Proof. of Lemma 1.

One can assume $0 < q_n < 1$ for every n without loss of generality. Introduce the notation for matrix entries $R_n =: \begin{pmatrix} (r_{11})_n & (r_{12})_n \\ (r_{21})_n & (r_{22})_n \end{pmatrix}$. For any natural $n_1 < n_2$

$$\left\|\prod_{n=n_1}^{n_2}\left[I-\begin{pmatrix} q_n & 0 \\ 0 & 0 \end{pmatrix}+R_n\right]\right\| \leq \prod_{n=n_1}^{n_2}\left[\left\|I-\begin{pmatrix} q_n & 0 \\ 0 & 0 \end{pmatrix}\right\|+\|R_n\|\right] \leq \prod_{n=n_1}^{n_2}[1+\|R_n\|] < \infty.$$

Therefore every solution of the system (13) is bounded and there exists such a constant C that for any natural $n_1 < n_2$ and any solution u_n

$$\max\{|u_{n_2}|,|v_{n_2}|\} < C\max\{|u_{n_1}|,|v_{n_1}|\}.$$

Using the variation of parameters method one can rewrite the system (13) in the following way:

$$(14)\quad u_{n+1} = u_{n_0}\prod_{l=n_0}^{n}(1-q_l)+\sum_{k=n_0}^{n}\left[((r_{11})_k u_k+(r_{12})_k v_k)\prod_{l=k+1}^{n}(1-q_l)\right]$$

and

$$(15)\qquad v_{n+1} = v_{n_0}+\sum_{k=n_0}^{n}\left[(r_{21})_k u_k+(r_{22})_k v_k\right],$$

for $n \geq n_0$. Note that for $k = n$ we formally keep the notation $\prod_{l=n+1}^{n}(1-q_l) := \frac{\prod_{l=1}^{n}(1-q_l)}{\prod_{l=1}^{n}(1-q_l)} = 1$. With given values of u_{n_0} and v_{n_0} we can construct the sequence $\left\{\begin{pmatrix} u_n \\ v_n \end{pmatrix}\right\}$ for $n \geq n_0 + 1$ using (14) and (15).

Due to properties of the sequence $\{q_n\}$, the product $\prod_{k=n_0}^{n}(1-q_k)$ converges to zero as $n \to \infty$. From (14) and the boundedness of sequences $\{u_n\}$ and $\{v_n\}$ it follows that $u_n \to 0$ (we can decompose the sum into two parts, $\sum_{k=1}^{n} = \sum_{k=1}^{[n/2]-1} + \sum_{k=[n/2]}^{n}$, and each of them goes to zero as $n \to \infty$). From (15) and uniform boundedness it follows that the sequence v_n has a limit as $n \to \infty$. Choose such a natural n_0 that

$$\sum_{k=n_0}^{\infty}\left(|(r_{21})_k| + |(r_{22})_k|\right) < \frac{1}{2C}$$

and put the initial condition at $n = n_0$

$$\begin{pmatrix} u_{n_0} \\ v_{n_0} \end{pmatrix} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\left|\sum_{k=n_0}^{n}\left[(r_{21})_k u_k + (r_{22})_k v_k\right]\right| < \frac{1}{2}$ and therefore $|v_n| > \frac{1}{2}$ for any $n \geq n_0$. So $\lim_{n\to\infty} v_n \neq 0$, which proves the Lemma. $\square$

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