ON THE CAUCHY PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION WITH STEPLIKE FINITE-GAP INITIAL DATA I. SCHWARZ-TYPE PERTURBATIONS

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Abstract. We solve the Cauchy problem for the Korteweg–de Vries equation with initial conditions which are steplike Schwartz-type perturbations of finite-gap potentials under the assumption that the mutual spectral bands either coincide or are disjoint.

1. Introduction

Since the seminal work of Gardner et al. [22] in 1967 the inverse scattering transform is one of the main tools for solving the Korteweg–de Vries (KdV) equation

$$q_t = -q_{xxx} + 6qq_x \tag{1.1}$$

and numerous articles have been devoted to this subject since then. In particular, the case when the initial condition is asymptotically close to 0 is well understood and we just refer to the monographs by Eckhaus and Van Harten [14], Marchenko [39], Novikov, Manakov, Pitaevskii, and Zakharov [40] or Faddeev and Takhtajan [19]. The same is true for the case of steplike initial conditions which are asymptotically constant (with different constants in different directions), where we refer to Buslaev and Fomin [10], Cohen [11], Chohen and Kappeler [12] and Kappeler [29]. In fact, even the case where the asymptotics are given by some power-like behaviour (including some unbounded initial conditions) were investigated by Bondareva, Kappeler, Perry, Shubin and, Topalov [6], [7], [30]. On the other hand, essentially nothing is known about the Cauchy problem for initial conditions which are asymptotically periodic.The first to consider a periodic background seem to be Kuznetsov and A.V. Mikhaĭlov, [37], who informally treated the Korteweg–de Vries equation with the Weierstraß elliptic function as background solution. The only known results, concerning to the existence of the solution seem to be by Ermakova [17], [18] and Firsova [21] (where the evolution of the scattering data for periodic background was given). However, both works are incomplete from the point of view of a rigorous application of the inverse scattering method. Surprisingly, much more is know about the asymptotical behavior (assuming existence) of such solutions, see for example [1], [3]–[5], [31]–[35], [41]. Finally we mention that in the discrete case (Toda lattice) the same problem was completely solved in [16] (for corresponding long-time asymptotics see [8], [13], [25], [26], [27], [28], [36], [43]).

Our aim in the present paper is to provide a rigorous treatment of the inverse scattering transform for the KdV equation in the case of initial conditions which

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are steplike Schwarz-type perturbations of finite-gap solutions. The reason which makes the periodic case much more difficult are the poles of the Baker–Akhiezer functions which reflect the fact that the underlying hyperelliptic Riemann surface is no longer simply connected. In particular, we include a complete discussion of the problems arising from these poles. In order to keep our presentation within reasonable limits and to be able to focus on the novel features of our approach, we have chosen to limit ourselves to the case of Schwarz-type perturbations and the additional assumption that the mutual spectral bands either coincide or are disjoint. While this last assumption excludes the classical case of steplike constant background, it clearly includes the case of short range perturbations of arbitrary finite-gap solutions. The latter being solved to the best of our knowledge for the first time here.

More precisely, we will prove the following result

Theorem 1.1. *Let $p_\pm(x,t)$ be a real-valued finite-gap solution of the KdV equation corresponding to the initial condition $p_\pm(x) = p_\pm(x,0)$. Suppose that the mutual spectral bands of the one-dimensional Schrödinger operators associated with p_+ and p_- either coincide or are disjoint.*

Let $q(x)$ be a real-valued smooth function such, that (the Schwarz class)

$$\pm\int_0^{\pm\infty}\left|\frac{\partial^n}{\partial x^n}\big(q(x)-p_\pm(x)\big)\right|(1+|x|^m)dx<\infty,\quad \forall m,n\in\mathbb{N}\cup\{0\},\tag{1.2}$$

then there is a unique smooth solution $q(x,t)$ of the KdV equation corresponding to the initial condition $q(x,0) = q(x)$ and satisfying

$$\pm\int_0^{\pm\infty}\left|\frac{\partial^n}{\partial x^n}\big(q(x,t)-p_\pm(x,t)\big)\right|(1+|x|^m)dx<\infty,\quad \forall m,n\in\mathbb{N}\cup\{0\},\tag{1.3}$$

for all $t \in \mathbb{R}$.

We will show how to remove the spectral restriction and how to handle a finite number of moments respectively derivatives in a follow-up publication [15].

2. Some general facts on the KdV flow

Let $q(x,t)$ be a classical solution of the KdV equation, that is, all partial derivatives appearing above exist and are continuous. Moreover, suppose $q(x,t)$ and $q_x(x,t)$ are bounded with respect to x for all $t \in \mathbb{R}_+$.

Introduce the Lax pair [38]

$$L_q(t) = -\partial_x^2 + q(x,t),\tag{2.1}$$

$$P_q(t) = -4\partial_x^3 + 6q(x,t)\partial_x + 3q_x(x,t).\tag{2.2}$$

Note that $L_q(t)$ is self-adjoint on $\mathfrak{D}(L_q(t)) = H^2(\mathbb{R})$ and $P_q(t)$ is skew-adjoint on $\mathfrak{D}(P_q(t)) = H^3(\mathbb{R})$. Moreover, the KdV equation is equivalent to the Lax equation

$$\partial_t L_q(t) = [P_q(t), L_q(t)]$$

on $H^5(\mathbb{R})$.

The following result follows from classical theory of ordinary differential equations.

Lemma 2.1. *Let $c(\lambda, x, t)$ and $s(\lambda, x, t)$ be the solutions of the differential equation $L_q(t)u = \lambda u$ corresponding to the initial conditions $c(\lambda, 0, t) = s_x(\lambda, 0, t) = 1$ and $c_x(\lambda, 0, t) = s(\lambda, 0, t) = 0$.*

Then, $c(\lambda, x, t)$ and $c_x(\lambda, x, t)$ are holomorphic with respect to $\lambda \in \mathbb{C}$ (for fixed x and t) and continuously differentiable with respect to t (provided $q(x, t)$ is). Similarly for $s(\lambda, x, t)$ and $s_x(\lambda, x, t)$.

Next note

Lemma 2.2. *Suppose $q(x, t)$ is three times differentiable with respect to x and once with respect to t. If $L_q(t)u = \lambda u$ holds, then*

$$(L_q(t) - \lambda)(u_t - P_q(t)u) = -(q_t + q_{xxx} - 6qq_x)u \tag{2.3}$$

Proof. Suppose $L_q u = \lambda u$, then we have $P_q u = (2(q + 2\lambda)\partial_x - q_x)u$ and thus

$$(L_q - \lambda)P_q u = (q_{xxx} - 6qq_x)u$$

respectively

$$(L_q - \lambda)u_t = -q_t u$$

which proves the claim. □

Corollary 2.3 ([39], corollary to Lemma 4.1.1'). *Suppose $q(x, t)$ is three times differentiable with respect to x and once with respect to t. The function $q(x, t)$ satisfies the KdV equation* (1.1) *if and only if the operator*

$$\mathcal{A}_q(t) = \partial_t - 2(q(x, t) + 2\lambda)\partial_x + q_x(x, t) \tag{2.4}$$

transforms solutions of equation (2.5) into solutions of the same equation.

Furthermore, we obtain

Lemma 2.4. *Let $q(x, t)$ be a classical solution of the KdV equation* (1.1)*. The system of differential equations*

$$L_q(t)u = \lambda u, \tag{2.5}$$

$$u_t = P_q(t)u \tag{2.6}$$

has a unique solution $u(\lambda, x, t)$ for any given initial conditions $u(\lambda, 0, 0) = a_0(\lambda)$ and $u_x(\lambda, 0, 0) = b_0(\lambda)$. It will be continuous with respect to λ if a_0, b_0 are.

Proof. Write

$$u(\lambda, x, t) = a(\lambda, t)c(\lambda, x, t) + b(\lambda, t)s(\lambda, x, t),$$

then clearly $L_q(t)u = \lambda u$ holds by construction and Lemma 2.2 implies

$$(L_q - \lambda)(u_t - P_q u) = 0.$$

Hence $u_t = P_q u$ will hold if and only if

$$a_t c + a c_t + b_t s + b s_t = a(P_q c) + b(P_q s) = 2(2\lambda + q)(a c_x + b s_x) - q_x(ac + bs).$$

vanishes together with its x derivative at $x = 0$, that is,

$$a_t(\lambda, t) = -a(\lambda, t)q_x(0, t) + b(\lambda, t)(4\lambda + 2q(0, t)),$$
$$b_t(\lambda, t) = b(\lambda, t)q_x(0, t) + a(\lambda, t)\left(2(2\lambda + q(0, t))(q(0, t) - \lambda) - q_{xx}(0, t)\right).$$

This is a system of ordinary differential equations for the unknown functions $a(\lambda, t)$, $b(\lambda, t)$ and hence the claim follows. □

Let $c(\lambda,x,t)+m_\pm(\lambda,t)s(\lambda,x,t)$ be a pair of Weyl solutions for operator $L_q(t)$, where $m_\pm(\lambda,t)$ are the Weyl m-functions associated with L_q.

Lemma 2.5. *The functions*

$$u_\pm(\lambda,x,t)=a_\pm(\lambda,t)\big(c(\lambda,x,t)+m_\pm(\lambda,t)s(\lambda,x,t)\big), \tag{2.7}$$

where

$$a_\pm(\lambda,t)=\exp\left(\int_0^t\Big(2\big(q(0,s)+2\lambda\big)m_\pm(\lambda,s)-q_x(0,s)\Big)ds\right), \tag{2.8}$$

solve (2.5), (2.6).

Proof. Recall that the Lax equation implies existence of a unitary propagator $U(t,s)$ (see, e.g., [42, Thm. X.69]) and thus particular,

$$\partial_t\langle f,(L_q(t)-\lambda)^{-1}g\rangle=\langle f,P_q(t)(L_q(t)-\lambda)^{-1}g\rangle$$

for any $g\in L^2(\mathbb{R})$ and $\lambda\in\mathbb{C}\backslash\sigma(L_q)$. Set

$$h(\lambda,x,t)=(L_q(t)-\lambda)^{-1}g(x).$$

Then choosing g with compact support in $(-\infty,0)$ for the $+$ case (resp., $(0,+\infty)$ for the $-$ case) implies that $h(\lambda,x,t)$ is of the form (2.7) for $x\ge 0$ (resp., $x\le 0$). Moreover, since $h(\lambda,x,t)$ solves (2.5), (2.6) in a neighborhood of $x=0$, the corresponding $a_\pm(\lambda,t)$ and $b_\pm(z,t)=a_\pm(\lambda,t)m_\pm(\lambda,t)$ solve the system of ordinary differential equations from Lemma 2.4. Inserting this into the differential equation for $a_\pm(\lambda,t)$ found in the proof of the previous lemma establishes the claim. □

The next lemma is a straightforward calculation.

Lemma 2.6. *Let u_1, u_2 be two solutions of (2.5), (2.6), then the Wronskian $W(u_1,u_2)$ does neither depend on x nor on t.*

3. Some general facts on finite-gap potentials

Since we want to study the initial value problem for the KdV equation in the class of initial conditions which asymptotically look like (different) finite-gap solutions, we need to recall some necessary background from finite-gap solutions first. For further information we refer to [23],[24], or [39].

Let $L_\pm(t):=L_{p_\pm}(t)$ be two one-dimensional Schrödinger operators associated with two arbitrary quasi-periodic finite-gap solutions $p_\pm(x,t)$ of the KdV equation. We denote by

$$\psi_\pm(\lambda,x,t)=c_\pm(\lambda,x,t)+m_\pm(\lambda,t)s_\pm(\lambda,x,t) \tag{3.1}$$

the corresponding Weyl solutions, normalized according to $\psi_\pm(\lambda,0,t)=1$, as in the previous section.

It is well-known that the spectra of $\sigma_\pm:=\sigma(L_\pm(t))$ are t independent and consist of a finite number, say $r_\pm+1$, bands:

$$\sigma_\pm=[E_0^\pm,E_1^\pm]\cup\cdots\cup[E_{2j-2}^\pm,E_{2j-1}^\pm]\cup\cdots\cup[E_{2r_\pm}^\pm,\infty). \tag{3.2}$$

Then $p_\pm$ are uniquely determined by their associated Dirichlet divisors

$$\Big\{(\mu_1^\pm(t),\sigma_1^\pm(t)),\dots,(\mu_{r_\pm}^\pm(t),\sigma_{r_\pm}^\pm(t))\Big\},$$

where $\mu_j^\pm(t)\in[E_{2j-1}^\pm,E_{2j}^\pm]$ and $\sigma_j^\pm(t)\in\{\pm1\}$.

Let us cut the complex plane along the spectrum $\sigma_\pm$ and denote the upper and lower sides of the cuts by $\sigma_\pm^{\mathrm{u}}$ and $\sigma_\pm^{\mathrm{l}}$. The corresponding points on these cuts will be denoted by λ^{u} and λ^{l}, respectively. In particular, this means

$$f(\lambda^{\mathrm{u}}) := \lim_{\varepsilon\downarrow 0} f(\lambda+\mathrm{i}\varepsilon), \qquad f(\lambda^{\mathrm{l}}) := \lim_{\varepsilon\downarrow 0} f(\lambda-\mathrm{i}\varepsilon), \qquad \lambda\in\sigma_\pm.$$

Set

$$Y_\pm(\lambda) = -\prod_{j=0}^{2r_\pm}(\lambda - E_j^\pm), \tag{3.3}$$

and introduce the functions

$$g_\pm(\lambda,t) = -\frac{\prod_{j=1}^{r_\pm}(\lambda-\mu_j^\pm(t))}{2Y_\pm^{1/2}(\lambda)}, \tag{3.4}$$

where the branch of the square root is chosen such that

$$\frac{1}{\mathrm{i}} g_\pm(\lambda^{\mathrm{u}}) = \mathrm{Im}(g_\pm(\lambda^{\mathrm{u}})) > 0 \quad \text{for} \quad \lambda\in\sigma_\pm. \tag{3.5}$$

The functions $\psi_\pm$ admit two other well-known representations that will be used later on. The first one is

$$\psi_\pm(\lambda,x,t) = u_\pm(\lambda,x,t)\mathrm{e}^{\pm\mathrm{i}\theta_\pm(\lambda)x} \qquad \lambda\in\mathbb{C}\setminus\sigma_\pm \tag{3.6}$$

where $\theta_\pm(\lambda)$ are the quasimoments and the functions $u_\pm(\lambda,x,t)$ are quasiperiodic with respect to x with the same basic frequencies as the potentials $p_\pm(x,t)$. The quasimoments are holomorphic as $\lambda\in\mathbb{C}\setminus\sigma_\pm$ and normalized according to

$$\frac{d\theta_\pm}{d\lambda} > 0 \quad \text{as} \quad \lambda\in\sigma_\pm^{\mathrm{u}}, \qquad \theta_\pm(E_0^\pm)=0. \tag{3.7}$$

This normalization implies (cf. (3.5))

$$\frac{d\theta_\pm}{d\lambda} = \frac{\mathrm{i}\prod_{j=1}^{r_\pm}(\lambda-\zeta_j^\pm)}{Y_\pm^{1/2}(\lambda)}, \qquad \zeta_j^\pm\in(E_{2j-1}^\pm, E_{2j}^\pm), \tag{3.8}$$

and therefore, the quasimoments are real-valued on $\sigma_\pm$. Note, in the case where $p_\pm(x,t)\equiv 0$ we have $\theta_\pm(\lambda)=\sqrt{\lambda}$ and $u_\pm(\lambda,x,t)\equiv 1$.

On the other side, the Weyl solutions possess more complicated properties, for example, they can have poles, as we see from the other representation. Namely, let $\mathbb{P}_\pm$ be the Riemann surfaces, associated with the functions $Y_\pm^{1/2}(\lambda)$ and let $\pi_\pm$ be parameters on these surfaces, corresponding to the spectral parameter λ, where π_+ (resp. π_-) is the parameter on the upper (resp., lower) sheet of $\mathbb{P}_+$ (resp. $\mathbb{P}_-$). Then

$$\psi_\pm(\pi_\pm,x,t) = \exp\left(\int_0^x m_\pm(\pi_\pm,y,t)dy\right), \tag{3.9}$$

where $m_\pm(\pi_\pm,x,t)$ are "shifted" Weyl function. Note, that the Weyl function $m_+(\lambda,t)$ is the branch, corresponding to values of $m_+(\pi_+,0,t)$ and $m_-(\lambda,t) = m_-(\pi_-,0,t)$. Denote the divisor of poles (the Dirichlet divisor) of the shifted Weyl functions by $\sum_{j=1}^{r_\pm}(\mu_j^\pm(x,t),\sigma_j^\pm(x,t))$. Then the functions $\mu_j^\pm(x,t)$ satisfy the system of Dubrovin equations ([23, Lem. 1.37])

$$\frac{\partial\mu_j^\pm(x,t)}{\partial x} = -2\sigma_j^\pm(x,t)Y_{\pm,j}(\mu_j^\pm(x,t),x,t), \tag{3.10}$$

$$\frac{\partial\mu_j^\pm(x,t)}{\partial t} = -4\sigma_j^\pm(x,t)(p_\pm(x,t)+2\mu_j^\pm(x,t))Y_{\pm,j}(\mu_j^\pm(x,t),x,t), \tag{3.11}$$

where

$$Y_{\pm,j}(\lambda,x,t) = \frac{Y_\pm^{1/2}(\lambda)(\lambda-\mu_j^\pm(x,t))}{G_\pm(\lambda,x,t)} \tag{3.12}$$

and

$$G_\pm(\lambda,x,t) = \prod_{j=1}^{r_\pm}(\lambda-\mu_j^\pm(x,t)). \tag{3.13}$$

In (3.11) $p_\pm(x,t)$ have to be replaced by the trace formulas

$$p_\pm(x,t) = \sum_{j=0}^{2r_\pm}E_j^\pm - 2\sum_{j=1}^{r_\pm}\mu_j^\pm(x,t). \tag{3.14}$$

Moreover, the following formula holds ([23, (1.165)])

$$m_\pm(\lambda,x,t) = \frac{H_\pm(\lambda,x,t)\pm Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,x,t)}, \tag{3.15}$$

where

$$H_\pm(\lambda,x,t) = \frac{1}{2}\frac{\partial}{\partial x}G_\pm(\lambda,x,t). \tag{3.16}$$

We will also use

$$\breve{m}_\pm(\lambda,x,t) = \frac{H_\pm(\lambda,x,t)\mp Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,x,t)}, \tag{3.17}$$

to denote the other branches of the Weyl functions on the Riemann surfaces $\mathbb{P}_\pm$, that is, $\breve{m}_\pm(\lambda,x,t) = m_\pm(\pi_\pm^*,x,t)$. In addition,

$$m_\pm(\lambda,t)-\breve{m}_\pm(\lambda,t) = \frac{\pm 2Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,0,t)}. \tag{3.18}$$

Lemma 3.1. *The following asymptotic expansion for large λ is valid*

$$\psi_\pm(\lambda,x,t) = \exp\left(\pm\mathrm{i}\sqrt{\lambda}x + \int_0^x \kappa_\pm(\lambda,y,t)dy\right), \tag{3.19}$$

where

$$\kappa_\pm(\lambda,x,t) = \sum_{k=1}^\infty \frac{\kappa_k^\pm(x,t)}{(\pm 2\mathrm{i}\sqrt{\lambda})^k}, \tag{3.20}$$

with coefficients defined recursively via

$$\kappa_1^\pm(x,t) = p_\pm(x,t), \quad \kappa_{k+1}^\pm(x,t) = -\frac{\partial}{\partial x}\kappa_k^\pm(x,t) - \sum_{m=1}^{k-1}\kappa_{k-m}^\pm(x,t)\kappa_m^\pm(x,t). \tag{3.21}$$

Proof. By (3.15) we conclude that

$$m_\pm(\lambda,x,t) = \pm\mathrm{i}\sqrt{\lambda} + \kappa_\pm(\lambda,x,t),$$

where $\kappa_\pm(\lambda,x,t)$ has an asymptotic expansion of the type (3.20). Inserting this expansion into the Riccati equation

$$\frac{\partial}{\partial x}\kappa_\pm(\lambda,x,t) \pm 2\mathrm{i}\sqrt{\lambda}\kappa_\pm(\lambda,x,t) + \kappa_\pm^2(\lambda,x,t) - p_\pm(x,t) = 0 \tag{3.22}$$

and comparing coefficients shows (3.21). □

As a special case of Lemma 2.5 we obtain

Lemma 3.2. *The functions*

$$\hat\psi_\pm(\lambda,x,t) = \mathrm{e}^{\alpha_\pm(\lambda,t)}\psi_\pm(\lambda,x,t), \tag{3.23}$$

where

$$\alpha_\pm(\lambda,t) := \int_0^t \left(2(p_\pm(0,s)+2\lambda)m_\pm(\lambda,s) - \frac{\partial p_\pm(0,s)}{\partial x}\right) ds, \tag{3.24}$$

satisfy the system of equations

$$L_\pm(t)\hat\psi_\pm = \lambda\hat\psi_\pm, \tag{3.25}$$

$$\frac{\partial\hat\psi_\pm}{\partial t} = P_\pm(t)\hat\psi_\pm. \tag{3.26}$$

We note that ([23, (1.148)])

$$\alpha_\pm(\lambda,t) = \frac{1}{2}\log\left(\frac{G_\pm(\lambda,0,t)}{G_\pm(\lambda,0,0)}\right) \pm 2Y_\pm^{1/2}(\lambda)\int_0^t \frac{p_\pm(0,s)+2\lambda}{G_\pm(\lambda,0,s)}ds \tag{3.27}$$

and corresponding to $\breve m_\pm(\lambda,t)$ we also introduce

$$\begin{aligned}\breve\alpha_\pm(\lambda,t) &:= \int_0^t \left((2p_\pm(0,s)+4\lambda)\breve m_\pm(\lambda,s) - \frac{\partial p_\pm(0,s)}{\partial x}\right) ds\\ &= \frac{1}{2}\log\left(\frac{G_\pm(\lambda,0,t)}{G_\pm(\lambda,0,0)}\right) \mp 2Y_\pm^{1/2}(\lambda)\int_0^t \frac{p_\pm(0,s)+2\lambda}{G_\pm(\lambda,0,s)}ds.\end{aligned} \tag{3.28}$$

Note

$$\overline{\alpha_\pm(\lambda,t)} = \breve\alpha_\pm(\lambda,t), \qquad \lambda\in\sigma_\pm. \tag{3.29}$$

In order to remove the singularities of the functions $\psi_\pm(\lambda,x,t)$ we set

$$\begin{aligned} M_\pm(t) &= \{\mu_j^\pm(t) \mid \mu_j^\pm(t)\in(E_{2j-1},E_{2j}) \text{ and } m_\pm(\lambda,t) \text{ has a simple pole}\},\\ \hat M_\pm(t) &= \{\mu_j^\pm(t) \mid \mu_j^\pm(t)\in\{E_{2j-1},E_{2j}\}\},\end{aligned} \tag{3.30}$$

and introduce the functions

$$\begin{aligned}\delta_\pm(\lambda) &:= \prod_{\mu_j^\pm(t)\in M_\pm(t)}(\lambda-\mu_j^\pm(t)),\\ \hat\delta_\pm(\lambda) &:= \prod_{\mu_j^\pm(t)\in M_\pm(t)}(\lambda-\mu_j^\pm(t)) \prod_{\mu_j^\pm(t)\in\hat M_\pm(t)}\sqrt{\lambda-\mu_j^\pm(t)},\end{aligned} \tag{3.31}$$

where $\prod = 1$ if the index set is empty.

Lemma 3.3. *For each $t \geq 0$ and $\lambda \in \mathbb{C} \setminus \sigma_\pm$ the functions $\alpha_\pm(\lambda, t)$ possess the properties*

$$\exp\big(\alpha_\pm(\lambda,t) + \breve{\alpha}_\pm(\lambda,t)\big) = \frac{G_\pm(\lambda,0,t)}{G_\pm(\lambda,0,0)}. \tag{3.32}$$

$$\exp\big(\alpha_\pm(\lambda,t)\big) = \frac{\hat{\delta}_\pm(\lambda,t)}{\hat{\delta}_\pm(\lambda,0)} f_\pm(\lambda,t), \tag{3.33}$$

where the functions $f_\pm(\lambda,t)$ are holomorphic in $\mathbb{C}\setminus\sigma_\pm$, continuous up to the boundary and $f_\pm(\lambda,t) \neq 0$ for all $\lambda \in \mathbb{C}$.

Furthermore, let $E \in \{E_{2j-1}^\pm, E_{2j}^\pm\}$, then

$$\lim_{\lambda\to E} (\alpha_\pm(\lambda,t) - \breve{\alpha}_\pm(\lambda,t)) = \begin{cases} 0, & \mu_j^\pm(t) \neq E, \mu_j^\pm(0) \neq E, \\ 0, & \mu_j^\pm(t) = E, \mu_j^\pm(0) = E, \\ \mathrm{i}\pi, & \mu_j^\pm(t) = E, \mu_j^\pm(0), \neq E, \\ \mathrm{i}\pi, & \mu_j^\pm(t) \neq E, \mu_j^\pm(0) = E, \end{cases} \pmod{2\pi \mathrm{i}}. \tag{3.34}$$

Proof. To shorten notations let us denote the derivative with respect to t by a dot and the derivative with respect to x by a prime. Equations (3.24) and (3.27) immediately give (3.32) and

$$\alpha_\pm(\lambda,t) - \breve{\alpha}_\pm(\lambda,t) = \pm 4 Y_\pm^{1/2}(\lambda) \int_0^t \frac{p_\pm(0,s) + 2\lambda}{G_\pm(\lambda,s)} ds, \tag{3.35}$$

where we have abbreviated

$$G_\pm(\lambda,t) := G_\pm(\lambda,0,t).$$

This function is well-defined on the set $\mathbb{C} \setminus \cup_{j=1}^{r_\pm} [E_{2j-1}^\pm, E_{2j}^\pm]$, but may have singularities inside gaps. Note, that

$$\alpha_\pm(\lambda,t) - \breve{\alpha}_\pm(\lambda,t) \in \mathbb{R}, \quad \text{for} \quad \lambda \in \mathbb{R} \setminus \sigma_\pm. \tag{3.36}$$

Consider the behavior of this function in the jth gap. By splitting the integral $\int_0^t$ in the definition of $\alpha_\pm(\lambda,t)$ (resp. $\breve{\alpha}_\pm(\lambda,t)$) into a sum of smaller integrals $\int_{t_0}^{t_1}$ it suffices to consider the cases where $\mu_j^\pm(s) \notin \{E_{2j-1}^\pm, E_{2j}^\pm\}$ for $s \in [t_0, t_1)$ or $s \in (t_0, t_1]$. We will only investigate the first case (the other being completely analogous) and assume $t_0 = 0$ without loss of generality. In other words, it suffices to consider the case where $\mu_j^\pm(0) \in (E_{2j-1}^\pm, E_{2j}^\pm)$ and the time $t > 0$ is so small, that $\sigma_j^\pm(s) = \sigma_j^\pm(0)$ for $s \leq t$. Consequently, $\mu_j^\pm(t) \in (E_{2j-1}^\pm, E_{2j}^\pm)$ and there exists an $\varepsilon = \varepsilon(t)$ such that

$$\mu_j^\pm(s) \in (E_{2j-1}^\pm + 2\varepsilon, E_{2j}^\pm - 2\varepsilon), \quad 0 \leq s \leq t. \tag{3.37}$$

Let, for example, the point $\mu_j^\pm(s)$ moves to the right, that is $\mu_j^\pm(0) < \mu_j^\pm(t)$. If $\lambda \notin (\mu_j^\pm(0) - \varepsilon, \mu_j^\pm(t) + \varepsilon)$, then the integral (3.35) is well-defined and by definition (3.3) the first case of (3.34) is fulfilled. Let now

$$\lambda \in (\mu_j^\pm(0) - \varepsilon, \mu_j^\pm(t) + \varepsilon). \tag{3.38}$$

From equation (3.11) we have

$$\dot{\mu}_j^\pm(s) = -\sigma_j^\pm(s) \tilde{Y}_{\pm,j}(\mu_j^\pm(s), s), \tag{3.39}$$

where

$$\tilde{Y}_{\pm,j}(\lambda,s)=4(p_\pm(s)+2\lambda)Y_{\pm,j}(\lambda,0,s) \tag{3.40}$$

and the function $Y_{\pm,j}(\lambda,0,s)$ is defined by formula (3.12). Recall that $\sigma_j^\pm(s)=$ const. Thus

$$\begin{aligned}\int_0^t \frac{\pm 4(p_\pm(s)+2\lambda)Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,s)} &= \pm\int_0^t \frac{\tilde{Y}_{\pm,j}(\lambda,s)}{\lambda-\mu_j^\pm(s)}ds\\ &= \pm\int_0^t \frac{\tilde{Y}_{\pm,j}(\mu_j^\pm(s),s)}{\lambda-\mu_j^\pm(s)}ds \pm \int_0^t \frac{\partial}{\partial\lambda}\tilde{Y}_{\pm,j}(\lambda,s)_{|\lambda=\xi_j^\pm(s)}\,ds,\end{aligned} \tag{3.41}$$

where $\xi_j^\pm(s)\in(E_{2j-1}^\pm+\varepsilon,E_{2j}^\pm-\varepsilon)$, and, therefore $\frac{\partial}{\partial\lambda}\tilde{Y}_{\pm,j}(\lambda,s)$ is bounded here. But

$$\begin{aligned}\pm\int_0^t \frac{\tilde{Y}_{\pm,j}(\mu_j^\pm(s),s)}{\lambda-\mu_j^\pm(s)}ds &= \mp\sigma_j^\pm(0)\int_0^t \frac{\dot{\mu}_j^\pm(s)}{\lambda-\mu_j^\pm(s)}ds\\ &= \pm\sigma_j^\pm(0)\log\frac{\lambda-\mu_j^\pm(t)}{\lambda-\mu_j^\pm(0)}.\end{aligned}$$

Thus, in the case under consideration we have

$$\alpha_\pm(\lambda,t)-\breve{\alpha}_\pm(\lambda,t)=\log\frac{(\lambda-\mu_j^\pm(t))^{\pm\sigma_j^\pm(t)}}{(\lambda-\mu_j^\pm(0))^{\pm\sigma_j^\pm(0)}}+\tilde{f}_\pm(\lambda,\varepsilon), \tag{3.42}$$

where $\tilde{f}_\pm(\lambda,\varepsilon)$ is a smooth function, bounded by virtue of (3.38). Combining this formula with (3.32) we arrive at the following representation:

$$\exp\big(2\alpha_\pm(\lambda,t)\big)=\frac{(\lambda-\mu_j^\pm(t))^{\pm\sigma_j^\pm(t)+1}}{(\lambda-\mu_j^\pm(0))^{\pm\sigma_j^\pm(0)+1}}f_\pm^{(1)}(\lambda,t),\quad f_\pm^{(1)}(\lambda,t)\neq 0, \tag{3.43}$$

which is valid provided (3.37) and (3.38) hold. According to our notations $\mu_j^\pm(s)\in M_\pm(s)$ iff $\pm\sigma_j^\pm(s)=1$. Thus, if $\mu_j^\pm(t)\in M_\pm(t)$ (resp. $\mu_j^\pm(0)\in M_\pm(0)$), then the function $\exp(\alpha_\pm(\lambda,t))$ has a first order zero (resp. pole) at such a point and does not have any other poles or zeros inside the gap $(E_{2j-1}^\pm,E_{2j}^\pm)$. But if $\pm\sigma_j^\pm(t)=-1$ (resp. $\pm\sigma_j^\pm(0)=-1$), then the function $\exp(\alpha_\pm(\lambda,t))$ has no zero (resp. pole) at this point.

Now let us turn to the case $\mu_j^\pm(t)$ or $\mu_j^\pm(0)\in\{E_{2j-1}^\pm,E_{2j}^\pm\}$. Here we cannot use the decomposition (3.41) since the function $\frac{\partial}{\partial\lambda}\tilde{Y}_{\pm,j}(\lambda,s)$ is not bounded at the edges of the spectrum $\sigma_\pm$. Suppose, that $\mu_j^\pm(0)\in(E_{2j-1}^\pm,E_{2j}^\pm)$, the point $\mu_j^\pm(s)$ moves to the right, and the time $t>0$ is such, that $\sigma_j^\pm(s)=\sigma_j^\pm(0)$ for $s<t$ and $\mu_j^\pm(t)=E_{2j}^\pm$. Set $\varepsilon<1/2(\mu_j^\pm(0)-E_{2j-1}^\pm)$ and let λ be such that

$$E_{2j-1}^\pm+\varepsilon<\lambda<E_{2j}^\pm+\varepsilon<E_{2j+1}^\pm.$$

Represent the function $\tilde{Y}_{\pm,j}(\lambda,s)$, defined by (3.40), as

$$\tilde{Y}_{\pm,j}(\lambda,s)=\sqrt{\lambda-E_{2j}^\pm}\;\breve{Y}_{\pm,j}(\lambda,s), \tag{3.44}$$

with

$$\breve{Y}_{\pm,j}(\lambda,s)=\breve{Y}_{\pm,j}(\mu_j^\pm(s),s)+(\lambda-\mu_j^\pm(s))\frac{\partial}{\partial\lambda}\breve{Y}_{\pm,j}(\zeta_j^\pm(s),s) \tag{3.45}$$

where $\frac{\partial}{\partial\lambda}\check{Y}_{\pm,j}$ is evidently bounded. From (3.39) it follows that

$$\check{Y}_{\pm,j}(\mu_j^\pm(s),s) = -\frac{\sigma_j^\pm(0)\dot{\mu}_j^\pm(s)}{\sqrt{\mu_j^\pm(s)-E_{2j}^\pm}}, \quad 0\le s\le t,$$

and

$$\begin{aligned}\int_0^t \frac{\tilde{Y}_{\pm,j}(\lambda,s)}{\lambda-\mu_j^\pm(s)}ds &= -\sigma_j^\pm(0)\sqrt{\lambda-E_{2j}^\pm}\left(\int_0^t \frac{\dot{\mu}_j^\pm(s)}{\sqrt{\mu_j^\pm(s)-E_{2j}^\pm}(\lambda-\mu_j^\pm(s))}ds + f_j^\pm(t,\varepsilon)\right)\\ &= -\sigma_j^\pm\sqrt{E_{2j}^\pm-\lambda}\left(\int_{\mu_j^\pm(0)}^{E_{2j}^\pm}\frac{d\tau}{(\lambda-\tau)\sqrt{E_{2j}^\pm-\tau}} + f_j^\pm(t,\varepsilon)\right) =\\ &= \sigma_j^\pm\sqrt{E_{2j}^\pm-\lambda}\int_{\sqrt{E_{2j}^\pm-\mu_j^\pm(0)}}^0 \frac{2dy}{y^2+\lambda-E_{2j}^\pm} + O\left(\sqrt{\lambda-E_{2j}^\pm}\right). \end{aligned} \tag{3.46}$$

To compute the first summand in (3.46) we will distinguish two cases. First let $\lambda\in\sigma_\pm$, that is, $\lambda>E_{2j}^\pm$. Set $\gamma^2=\lambda-E_{2j}^\pm$. Then the first summand in (3.46) is equal to

$$2\sigma_j^\pm(0)\mathrm{i}\arctan\frac{\sqrt{E_{2j}^\pm-\mu_j^\pm(0)}}{\gamma}\to\sigma_j^\pm(0)\mathrm{i}\pi, \quad \text{as} \quad \lambda\to E_{2j}^\pm, \quad \lambda\in\sigma_\pm.$$

This proves the two lower cases in (3.34). Next, consider the case when $\lambda\in(\mu_j^\pm(0),E_{2j}^\pm)$. Then

$$\begin{aligned}&\sigma_j^\pm(0)\sqrt{E_{2j}^\pm-\lambda}\int_{\sqrt{E_{2j}^\pm-\mu_j^\pm(0)}}^0\frac{2dy}{y^2+\lambda-E_{2j}^\pm} =\\ &= \sigma_j^\pm(0)\left(-\log\frac{\sqrt{E_{2j}^\pm-\mu_j^\pm(0)}-\sqrt{E_{2j}^\pm-\lambda}}{\sqrt{E_{2j}^\pm-\mu_j^\pm(0)}+\sqrt{E_{2j}^\pm-\lambda}}+\log(-1)\right) =\\ &= -\sigma_j^\pm(0)\log\frac{\lambda-\mu_j^\pm(0)}{\left(\sqrt{E_{2j}^\pm-\mu_j^\pm(0)}+\sqrt{E_{2j}^\pm-\lambda}\right)^2}+\sigma_j^\pm(0)\mathrm{i}\pi.\end{aligned} \tag{3.47}$$

If $\lambda\to E_{2j}^\pm$, then the first summand in (3.47) vanishes, and we arrive again at (3.34). If λ is in a small vicinity of $\mu_j^\pm(0)$, then

$$\pm\int_0^t\frac{\tilde{Y}_{\pm,j}(\lambda,s)}{\lambda-\mu_j^\pm(s)}ds = \mp\sigma_j^\pm(0)\log(\lambda-\mu_j^\pm(0))+O(1),$$

that confirm (3.33) for the case under consideration.

□

4. Scattering theory

First we collect some facts from scattering theory for Schrödinger operators with step-like finite-gap potentials (cf. [9]). To shorten notations we omit the dependence on t throughout this section.

Let $L_\pm$ be two Schrödinger operators with real-valued finite-gap potentials, corresponding to the spectra (3.2) and the Dirichlet divisors $\sum_{j=1}^{r^\pm}(\mu_j^\pm, \sigma_j^\pm)$, where $\mu_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm \in \{-1, 1\}$.

Let $q(x)$ be a real-valued smooth function such, that

$$\pm\int_0^{\pm\infty} |q^{(n)}(x) - p_\pm^{(n)}(x)|(1+|x|^m)dx < \infty, \quad \forall m, n \in \mathbb{N}\cup\{0\}. \tag{4.1}$$

The case $m = 2$ and $n = 0$ was rigorously studied in [9]. In this section we point out the necessary modifications for the above case. Let

$$L_q := -\frac{d^2}{dx^2} + q(x), \quad x \in \mathbb{R}, \tag{4.2}$$

be the "perturbed" operator with a potential $q(x)$, satisfying (4.1). The spectrum of L_q consists of a purely absolutely continuous part $\sigma := \sigma_+ \cup \sigma_-$ plus a finite number of eigenvalues situated in the gaps, $\sigma_d \subset \mathbb{R}\setminus\sigma$. We will use the notation $\mathrm{int}(\sigma_\pm)$ for the interior of the spectrum, that is, $\mathrm{int}(\sigma_\pm) := \sigma_\pm \setminus \partial\sigma_\pm$. The set $\sigma^{(2)} := \sigma_+ \cap \sigma_-$ is the spectrum of multiplicity two, and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$ with $\sigma_\pm^{(1)} = \mathrm{clos}(\sigma_\pm \setminus \sigma_\mp)$ is the spectrum multiplicity one.

The Jost solutions of the equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right) y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{4.3}$$

that are asymptotically close to the Weyl solutions of the background operators as $x \to \pm\infty$, can be represented with the help of the transformation operators as

$$\phi_\pm(\lambda, x) = \psi_\pm(\lambda, x) \pm \int_x^{\pm\infty} K_\pm(x, y)\psi_\pm(\lambda, y)dy, \tag{4.4}$$

where $K_\pm(x, y)$ are real-valued functions, that satisfy the integral equations

$$\begin{aligned} K_\pm(x, y) = &-2\int_{\frac{x+y}{2}}^{\pm\infty} \left(q(s) - p_\pm(s)\right) D_\pm(x, s, s, y)ds \\ &\mp 2\int_x^{\pm\infty} ds \int_{y\pm x\mp s}^{y\pm s\mp x} D_\pm(x, s, r, y)K_\pm(s, r)\left(q(s) - p_\pm(s)\right)dr, \quad \pm y > \pm x, \end{aligned} \tag{4.5}$$

where

$$D_\pm(x, y, r, s) = \mp\frac{1}{4}\sum_{E\in\partial\sigma_\pm} \frac{f_\pm(E, x, y)f_\pm(E, r, s)}{\frac{d}{dz}Y_\pm(E)}, \tag{4.6}$$

with

$$f_\pm(E, x, y) = \lim_{z\to E}\left(\prod_{j=1}^{r_\pm}(z - \mu_j^\pm)\right)\psi_\pm(z, x)\breve{\psi}_\pm(z, y). \tag{4.7}$$

In particular,

$$K_\pm(x, x) = \pm\frac{1}{2}\int_x^{\pm\infty}(q(s) - p_\pm(s))ds. \tag{4.8}$$

Since

$$\frac{\partial^{n+l}}{\partial x^l \partial y^n} f_\pm(E, x, y) \in L^\infty(\mathbb{R}\times\mathbb{R}),$$

condition (4.1) and the method of successive approximations imply smoothness of the kernels for the transformation operators and the following estimate

$$\left|\frac{\partial^{n+l}}{\partial x^n \partial y^l} K(x,y)\right| < \frac{C_\pm(n,l,m)}{|x+y|^m}, \quad x,y\to\pm\infty, \quad m,n,l\in\mathbb{N}\cup\{0\}, \tag{4.9}$$

where $C_\pm(n,l,m)$ are positive constants (cf [9]).

Representation (4.4) shows, that the Jost solutions inherit all singularities of the background Weyl m-functions $m_\pm(\lambda)$. Hence we set (recall (3.31))

$$\tilde\phi_\pm(\lambda,x) = \delta_\pm(\lambda)\phi_\pm(\lambda,x) \tag{4.10}$$

such that the functions $\tilde\phi_\pm(\lambda,x)$ have no poles in the interior of the gaps of the spectrum σ. Let

$$\sigma_d = \{\lambda_1,\dots,\lambda_p\}\subset\mathbb{R}\setminus\sigma.$$

be the set of eigenvalues of the operator L_q. For every eigenvalue we introduce the corresponding norming constants

$$\left(\gamma_k^\pm\right)^{-2} = \int_{\mathbb{R}} \tilde\phi_\pm^2(\lambda_k,x)dx. \tag{4.11}$$

Furthermore, introduce the scattering relations

$$T_\mp(\lambda)\phi_\pm(\lambda,x) = \overline{\phi_\mp(\lambda,x)} + R_\mp(\lambda)\phi_\mp(\lambda,x), \quad \lambda\in\sigma_\mp^{\mathrm{u,l}}, \tag{4.12}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda) := \frac{\mathsf{W}(\overline{\phi_\pm(\lambda)},\phi_\pm(\lambda))}{\mathsf{W}(\phi_\mp(\lambda),\phi_\pm(\lambda))}, \qquad R_\pm(\lambda) := -\frac{\mathsf{W}(\phi_\mp(\lambda),\overline{\phi_\pm(\lambda)})}{\mathsf{W}(\phi_\mp(\lambda),\phi_\pm(\lambda))}, \quad \lambda\in\sigma_\pm^{\mathrm{u,l}}, \tag{4.13}$$

and $\mathsf{W}(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the Wronski determinant.

Lemma 4.1. *The scattering data have the following properties:*

I. **(a)** $T_\pm(\lambda^{\mathrm{u}}) = \overline{T_\pm(\lambda^{\mathrm{l}})}$ *for* $\lambda\in\sigma_\pm$.
$R_\pm(\lambda^{\mathrm{u}}) = \overline{R_\pm(\lambda^{\mathrm{l}})}$ *for* $\lambda\in\sigma_\pm$.
(b) $\dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda)$ *for* $\lambda\in\sigma_\pm^{(1)}$.
(c) $1-|R_\pm(\lambda)|^2 = \dfrac{g_\pm(\lambda)}{g_\mp(\lambda)}|T_\pm(\lambda)|^2$ *for* $\lambda\in\sigma^{(2)}$.
(d) $\overline{R_\pm(\lambda)}T_\pm(\lambda) + R_\mp(\lambda)\overline{T_\pm(\lambda)} = 0$ *for* $\lambda\in\sigma^{(2)}$.
(e) $T_\pm(\lambda) = 1 + O\Big(\frac{1}{\sqrt{\lambda}}\Big)$ *for* $\lambda\to\infty$.
(f) $R_\pm(\lambda) = O\Big(\frac{1}{(\sqrt{\lambda})^{n+1}}\Big)$ *for* $\lambda\to\infty$ *and for all* $n\in\mathbb{N}$.

II. *The functions* $T_\pm(\lambda)$ *can be extended as meromorphic functions into the domain* $\mathbb{C}\setminus\sigma$ *and satisfy*

$$\frac{1}{T_+(\lambda)g_+(\lambda)} = \frac{1}{T_-(\lambda)g_-(\lambda)} =: -W(\lambda), \tag{4.14}$$

where $W(\lambda)$ *possesses the following properties:*
(a) *The function* $\tilde W(\lambda) = \delta_+(\lambda)\delta_-(\lambda)W(\lambda)$ *is holomorphic in the domain* $\mathbb{C}\setminus\sigma$, *with simple zeros at the points* λ_k, *where*

$$\left(\frac{d\tilde W}{d\lambda}(\lambda_k)\right)^2 = (\gamma_k^+\gamma_k^-)^{-2}. \tag{4.15}$$

In addition, it satisfies

$$\overline{\tilde{W}(\lambda^{\mathrm{u}})} = \tilde{W}(\lambda^{\mathrm{l}}), \quad \lambda \in \sigma \quad \text{and} \quad \tilde{W}(\lambda) \in \mathbb{R} \quad \text{for} \quad \lambda \in \mathbb{R} \setminus \sigma. \tag{4.16}$$

(b) *The function $\hat{W}(\lambda) = \hat{\delta}_+(\lambda)\hat{\delta}_-(\lambda)W(\lambda)$ is continuous on the set $\mathbb{C} \setminus \sigma$ up to the boundary $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$. Moreover, this function is infinitely many times differentiable with respect to λ on the set $\big(\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}\big) \setminus \partial\sigma$ and continuously differentiable with respect to the local variable $\sqrt{\lambda - E}$ for $E \in \partial\sigma$. It can have zeros on the set $\partial\sigma$ and does not vanish at the other points of the set σ. If $\hat{W}(E) = 0$ as $E \in \partial\sigma$, then $\hat{W}(\lambda) = \sqrt{\lambda - E}(C(E) + o(1))$, $C(E) \neq 0$.*

III. (a) *The reflection coefficients $R_\pm(\lambda)$ are continuously differentiable infinitely many time functions on the sets $\mathrm{int}(\sigma_\pm^{\mathrm{u,l}})$.*

(b) *If $E \in \partial\sigma$ and $\hat{W}(E) \neq 0$ then the functions $R_\pm(\lambda)$ are also continuous at E. Moreover, in this case*

$$R_\pm(E) = \begin{cases} -1 & \text{for } E \notin \hat{M}_\pm, \\ 1 & \text{for } E \in \hat{M}_\pm. \end{cases} \tag{4.17}$$

Proof. For the case $m = 2$ and $n = 0$ this lemma was proven in [9]. In particular, except for the differentiability properties of the scattering data and item **I.(f)** everything follows from Lemma 3.3 in [9].

Differentiability of $\hat{W}(\lambda)$ and $R_\pm(\lambda)$ is a direct consequence of differentiability of the Jost solutions. In fact, since $\frac{\partial^l \psi_\pm(\lambda,y)}{\partial\lambda^l} = O(|y|^l)$ for $\lambda \in \mathrm{int}\,\sigma_\pm$ as $y \to \pm\infty$, equations (4.4), (4.9), and (4.1) imply, that $\phi_\pm(\lambda, x)$ are continuously differentiable infinitely many times with respect to $\lambda \in \mathrm{int}\,\sigma_\pm$ since $\psi_\pm(\lambda, x)$ are. Moreover, note, that at the points $E_j^\pm$ these solutions are continuously differentiable with respect to the local parameter $\sqrt{\lambda - E_j^\pm}$ since this holds for $\psi_\pm(\lambda, x)$. Furthermore, since $\mathrm{Im}\,\theta_\pm(\lambda) > 0$ for $\lambda \in \mathbb{R} \setminus \sigma_\pm$, we infer that $\psi_\pm(\lambda, y)$ are exponentially decaying together with all derivatives as $y \to \pm\infty$ if $\lambda \in \mathbb{R} \setminus \sigma_\pm$.

It remains to show **I.(f)**. To this end, represent the Jost solutions in the form

$$\phi_\pm(\lambda, x) = \psi_\pm(\lambda, x) \exp\left(-\int_x^{\pm\infty} \tilde{\kappa}_\pm(\lambda, y) dy\right), \tag{4.18}$$

where

$$\tilde{\kappa}(\lambda, x) = \sum_{k=1}^{\infty} \frac{\tilde{\kappa}_k^\pm(x)}{(\pm 2\mathrm{i}\sqrt{\lambda})^k}. \tag{4.19}$$

To derive a differential equation for $\tilde{\kappa}_\pm(\lambda, x)$ we substitute (4.18) into (4.3) and use (3.19) and (3.22). This yields the differential equations

$$\frac{\partial}{\partial x}\tilde{\kappa}_\pm(\lambda, x) + \tilde{\kappa}_\pm^2(\lambda, x) \pm 2(\mathrm{i}\sqrt{\lambda} + \kappa_\pm(\lambda, x))\tilde{\kappa}_\pm(\lambda, x) + p_\pm(x) - q(x) = 0, \tag{4.20}$$

from which we obtain the recurrence formulas

$$\tilde{\kappa}_1^\pm(x) = q(x) - p_\pm(x), \quad \tilde{\kappa}_{k+1}^\pm(x) = -\frac{\partial}{\partial x}\tilde{\kappa}_k^\pm(x) - \sum_{m=1}^{k-1} \tilde{\kappa}_{k-m}^\pm(x)(\tilde{\kappa}_m^\pm(x) + 2\kappa_m^\pm(x)). \tag{4.21}$$

Using (4.13) we now derive an asymptotic formula for $R_+(\lambda)$ (for R_- the considerations are analogous). By (4.18) and (4.19)
(4.22)
$$\mathsf{W}(\phi_-(\lambda),\phi_+(\lambda)) = \phi_-(\lambda,0)\phi_+(\lambda,0)\left(2\mathrm{i}\sqrt{\lambda}+O\left(\frac{1}{\sqrt{\lambda}}\right)\right) = 2\mathrm{i}\sqrt{\lambda}(1+o(1))$$

and

$$\mathsf{W}(\phi_-(\lambda),\overline{\phi_+(\lambda)}) = \phi_-(\lambda,0)\overline{\phi_+(\lambda,0)}\left(\overline{y_+(\lambda,0)}-y_-(\lambda,0)\right), \tag{4.23}$$

where we have set $y_\pm(\lambda,x) := \tilde\kappa_\pm(\lambda,x)+\kappa_\pm(\lambda,x)$. Equations (3.22) and (4.20) imply

$$\frac{\partial}{\partial x}y_\pm(\lambda,x)\pm 2\mathrm{i}\sqrt{\lambda}y_\pm(\lambda,x)+y_\pm^2(\lambda,x)-q(x)=0. \tag{4.24}$$

Therefore, the functions $\tilde y_+(\lambda,x) := \overline{y_+(\lambda,x)}$ and $\tilde y_-(\lambda,x) := y_-(\lambda,x)$ satisfy one and the same equation. Moreover, $\kappa_1^\pm(x)+\tilde\kappa_1^\pm(x) = q(x)$. Hence, since $q(x)$ is smooth, the functions $\tilde y_\pm$ admit asymptotic expansions

$$\tilde y_\pm(\lambda,x) = \sum_{k=1}^\infty \frac{\tilde y_k^\pm(x)}{(-2\mathrm{i}\sqrt{\lambda})^k},$$

where $\tilde y_k^+(x)$ and $\tilde y_k^-(x)$ satisfy the same recurrence equations

$$\tilde y_1^\pm(x) = q(x), \quad \tilde y_{k+1}^\pm(x) = -\frac{\partial}{\partial x}\tilde y_k^\pm(x) - \sum_{l=1}^{k-1}\tilde y_{k-l}^\pm(x)\tilde y_l^\pm(x). \tag{4.25}$$

Therefore,

$$\overline{y_+(\lambda,0)} - y_-(\lambda,0) = O(\lambda^{-n/2})$$

for $\lambda\to\infty$ and for all $n\in\mathbb{N}$ and the same is true for $R_+(\lambda)$. □

Next we want to show how these data can be used to reconstruct the potential $q(x)$. To this end, we introduce the set of scattering data

$$\begin{aligned}\mathcal{S} = \Big\{ & R_+(\lambda),\ T_+(\lambda),\ \lambda\in\sigma_+^{\mathrm{u,l}};\ R_-(\lambda),\ T_-(\lambda),\ \lambda\in\sigma_-^{\mathrm{u,l}};\\ & \lambda_1,\dots,\lambda_p\in\mathbb{R}\setminus\sigma,\ \gamma_1^\pm,\dots,\gamma_p^\pm\in\mathbb{R}_+\Big\}\end{aligned} \tag{4.26}$$

and construct the associated Gelfand-Levitan-Marchenko (GLM) equations.

Lemma 4.2. *The kernels $K_\pm(x,y)$ of the transformation operators satisfy the Gelfand-Levitan-Marchenko equations*

$$K_\pm(x,y)+F_\pm(x,y)\pm\int_x^{\pm\infty}K_\pm(x,s)F_\pm(s,y)ds=0, \quad \pm y>\pm x, \tag{4.27}$$

where [1]

$$F_\pm(x,y)=\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm}R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\pm(\lambda)d\lambda+ \\ +\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\lambda)|^2\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\mp(\lambda)d\lambda \\ +\sum_{k=1}^{p}(\gamma_k^\pm)^2\tilde\psi_\pm(\lambda_k,x)\tilde\psi_\pm(\lambda_k,y). \tag{4.28}$$

IV. *The functions $F_\pm(x,y)$ are differentiable infinitely many times with respect to both variables and satisfy*

$$\left|\frac{\partial^{l+n}}{\partial x^l\partial y^n}F_\pm(x,y)\right|\le\frac{C_\pm(m,n,l)}{|x+y|^m}\quad \text{as}\quad x,y\to\pm\infty,\quad m,l,n=0,1,2,\dots \tag{4.29}$$

Proof. Formulas (4.27) and (4.28) are obtained in [9], estimate (4.29) follows directly from (4.27) and (4.9). □

In fact properties **I**–**IV** from above are characteristic for the scattering data, that is

Theorem 4.3 (characterization, [9])**.** *Properties* **I**–**IV** *are necessary and sufficient for a set $\mathcal{S}$ to be the set of scattering data for operator L with a potential $q(x)$ from the class* (4.1).

In addition, we will now describe a procedure of solving of the inverse scattering problem.

Let $L_\pm$ be two one-dimensional finite-gap Schrödinger operators associated with the potentials $p_\pm(x)$. Let $\mathcal{S}$ be given scattering data (4.26) satisfying **I**–**IV** and define corresponding kernels $F_\pm(x,y)$ via (4.28). As it shown in [9], condition ***IV*** the GLM equations (4.27) have unique smooth real-valued solutions $K_\pm(x,y)$, satisfying estimate of type (4.9), possibly with some other constants $C_\pm$, than in (4.29). In particular,

$$\pm\int_0^{\pm\infty}(1+|x|^m)\left|\frac{d^n}{dx^n}K_\pm(x,x)\right|dx<\infty,\qquad \forall m,n\in\mathbb{N}. \tag{4.30}$$

Now introduce the functions

$$q_\pm(x)=\mp 2\frac{d}{dx}K_\pm(x,x)+p_\pm(x),\quad x\in\mathbb{R} \tag{4.31}$$

and note that the estimate (4.30) reads

$$\pm\int_0^{\pm\infty}|q_\pm^{(n)}(x)-p_\pm^{(n)}(x)|(1+|x|^m)dx<\infty,\quad \forall n,m\in\mathbb{N}\cup\{0\}. \tag{4.32}$$

Moreover, define functions $\phi_\pm(\lambda,x)$ by formula (4.4), where $K_\pm(x,y)$ are the solutions of (4.27). Then these functions solve the equations

$$\left(-\frac{d^2}{dx^2}+q_\pm(x)\right)\phi_\pm(\lambda,x)=\lambda\phi_\pm(\lambda,x). \tag{4.33}$$

[1] Here we have used the notation

$$\oint_{\sigma_\pm}f(\lambda)d\lambda:=\int_{\sigma_\pm^{\mathrm{u}}}f(\lambda)d\lambda-\int_{\sigma_\pm^{\mathrm{l}}}f(\lambda)d\lambda.$$

.

The only remaining difficulty is to show that in fact $q_-(x) = q_+(x)$:

Theorem 4.4 ([9])**.** *Let the scattering data $\mathcal{S}$, defined as in* (4.26)*, satisfy the properties* **I–IV**. *Then the functions $q_\pm(x)$, defined by* (4.31) *coincide, $q_-(x) \equiv q_+(x) =: q(x)$. Moreover, the data $\mathcal{S}$ are the scattering data for the Schrödinger operator with potential $q(x)$ from the class* (4.1)*.*

5. The inverse scattering transform

As our next step we show how to use the solution of the inverse scattering problem found in the previous section to give a formal scheme for solving the initial-value problem for the KdV equation with initial data from the class (4.1).

Suppose first that our initial-value problem has a solution $q(x,t)$ satisfying (4.1) for each $t > 0$. Then all considerations from the previous section apply to the operator $L_q(t)$ if we consider t as an additional parameter. In particular, there are time-dependent transformation operators with kernels $K_\pm(x,y,t)$ satisfying the estimates

$$\left|\frac{\partial^{l+n}}{\partial x^l \partial y^n} K_\pm(x,y,t)\right| \le \frac{C_\pm(m,n,l,t)}{|x+y|^m}, \quad x,y \to \pm\infty, \quad l,n,m = 0,1,2,\dots. \tag{5.1}$$

and

$$\left|\frac{\partial^{n+l+1}}{\partial x^n \partial y^l \partial t} K_\pm(x,y,t)\right| \le \frac{C_\pm(m,n,l,t)}{|x+y|^m}, \quad x,y \to \pm\infty, \quad l,n,m = 0,1,2,\dots. \tag{5.2}$$

These estimates follows from the fact that the kernels $D_\pm(x,y,s,r,t)$ of the time-dependent equations (4.5) are smooth with respect to all variables, and each partial derivative is uniformly bounded with respect to $x,y,s,r,t \in \mathbb{R}$. Consequently, the Jost solutions

$$\phi_\pm(\lambda,x,t) = \psi_\pm(\lambda,x,t) \pm \int_x^{\pm\infty} K_\pm(x,y,t)\psi_\pm(\lambda,y,t)dy, \tag{5.3}$$

are also differentiable with respect to t and satisfy

$$\frac{\partial}{\partial t}\phi_\pm(\lambda,x,t) = \frac{\partial}{\partial t}\psi_\pm(\lambda,x,t)(1+o(1)) \qquad \text{as } x \to \pm\infty, \tag{5.4}$$

$$\frac{\partial^n}{\partial x^n}\phi_\pm(\lambda,x,t) = \frac{\partial^n}{\partial x^n}\psi_\pm(\lambda,x,t)(1+o(1)) \qquad \text{as } x \to \pm\infty. \tag{5.5}$$

By Lemma 2.5 we know that there is some $a_\pm(\lambda,t)$ such that $a_\pm(\lambda,t)\phi_\pm(\lambda,x,t)$ solves (2.6). Since $a_\pm(\lambda,t)$ is independent of x we can find it by letting $x \to \pm\infty$ and using the asymptotics (5.4) and (5.5) we see that this factor is the same as for $\psi_\pm(\lambda,x,t)$ (cf. Lemma 3.2).

Lemma 5.1. *Let $\alpha(\lambda,t)$ be defined by* (3.24) *and let $q(x,t)$ be a solution of the KdV equation satisfying* (4.1)*. Then the functions*

$$\hat\phi_\pm(\lambda,x,t) = \mathrm{e}^{\alpha_\pm(\lambda,t)}\phi_\pm(\lambda,x,t) \tag{5.6}$$

solve the system (2.5)*,* (2.6)*.*

Before we proceed further we note that equation (3.33) implies

Corollary 5.2. *The function $\hat\phi_\pm(\lambda,x,t)$, defined by formula* (5.6)*, have simple poles on the set $M_\pm(0)$, square root singularities on the set $\hat M_\pm(0)$, and no other singularities.*

Next, consider the time-dependent scattering relations

$$T_{\mp}(\lambda,t)\phi_{\pm}(\lambda,x,t)=\overline{\phi_{\mp}(\lambda,x,t)}+R_{\mp}(\lambda,t)\phi_{\mp}(\lambda,x,t),\quad \lambda\in\sigma_{\mp}^{\mathrm{u,l}}. \tag{5.7}$$

Then, using the previous lemma in combination with Lemma 2.6 to evaluate (4.13) we infer

Lemma 5.3. *Let $q(x,t)$ be a solution of the KdV equation satisfying* (4.1)*. Then $\lambda_k(t)=\lambda_k(0)\equiv\lambda_k$;*

$$R_{\pm}(\lambda,t)=R_{\pm}(\lambda,0)\mathrm{e}^{\alpha_{\pm}(\lambda,t)-\breve{\alpha}_{\pm}(\lambda,t)},\quad \lambda\in\sigma_{\pm}, \tag{5.8}$$

$$T_{\mp}(\lambda,t)=T_{\mp}(\lambda,0)\mathrm{e}^{\alpha_{\pm}(\lambda,t)-\breve{\alpha}_{\mp}(\lambda,t)},\quad \lambda\in\mathbb{C}, \tag{5.9}$$

$$\big(\gamma_k^{\pm}(t)\big)^2=\big(\gamma_k^{\pm}(0)\big)^2\,\frac{\delta_{\pm}^2(\lambda_k,0)}{\delta_{\pm}^2(\lambda_k,t)}\,\mathrm{e}^{2\alpha_{\pm}(\lambda_k,t)}, \tag{5.10}$$

where $\alpha_{\pm}(\lambda,t)$, $\breve{\alpha}_{\pm}(\lambda,t)$, $\delta_{\pm}(\lambda,t)$ are defined in (3.24), (3.28), (3.31)*, respectively.*

Proof. First of all set $\hat{W}(\lambda,t)=\hat{\delta}_+(\lambda,t)\hat{\delta}_-(\lambda,t)W(\lambda,t)$ (recall (4.14)). Then, since $\mathsf{W}(\hat{\phi}_-(\lambda,t),\hat{\phi}_+(\lambda,t))$ does not depend on t by Lemma 2.6, we obtain from (4.14)

$$f(\lambda,t)\hat{W}(\lambda,t)=\hat{W}(\lambda,0),\quad f(\lambda,t)=f_-(\lambda,t)f_+(\lambda,t)\neq 0. \tag{5.11}$$

This implies, that the discrete spectrum of the operator $L(t)$, which is the set of zeros of the Wronskian $\hat{W}(\lambda,t)$ on the set $\mathbb{R}\setminus\sigma$, does not depend on t.

Similarly, using constancy of the Wronskian when $\phi_{\pm}(\lambda,x,t)$ are replaced by $\hat{\phi}_{\pm}(\lambda,x,t)$ to evaluate (4.13) we obtain (5.8) and (5.9). To obtain (5.10) we set $\check{\phi}(\lambda,x,t)=\delta_{\pm}(\lambda,0)\hat{\phi}_{\pm}(\lambda,x,t)$ (which is continuous near λ_k) and compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\check{\phi}_{\pm}(\lambda_k,x,t)^2dx&=2\int_{\mathbb{R}}\check{\phi}_{\pm}(\lambda_k,x,t)\partial_t\check{\phi}_{\pm}(\lambda_k,x,t)dx\\&=\int_{\mathbb{R}}\check{\phi}_{\pm}(\lambda_k,x,t)P_q(t)\check{\phi}_{\pm}(\lambda_k,x,t)dx=0,\end{aligned}$$

since P_q is skew-adjoint and $\check{\phi}_{\pm}(\lambda_k,x,t)$ is real-valued. Note that interchanging differentiation and integration is permissible by the dominated convergence theorem (recall that the quasimoments $\theta_{\pm}(\lambda)$ are independent of t). Thus, (4.10) and (4.11) imply

$$\frac{d}{dt}\frac{\delta_{\pm}(\lambda_k,0)\ \mathrm{e}^{\alpha_{\pm}(\lambda_k,t)}}{\delta_{\pm}(\lambda_k,t)\ \gamma_k^{\pm}(t)}=0,$$

which finishes the proof. □

Hence the solution $q(x,t)$ can be computed from the time-dependent scattering data as follows. Construct one of the functions $F_+(x,y,t)$ or $F_-(x,y,t)$ via

$$\begin{aligned}F_{\pm}(x,y,t)=&\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_{\pm}}R_{\pm}(\lambda,t)\psi_{\pm}(\lambda,x,t)\psi_{\pm}(\lambda,y,t)g_{\pm}(\lambda,t)d\lambda+\\&+\frac{1}{2\pi\mathrm{i}}\int_{\sigma_{\mp}^{(1),\mathrm{u}}}|T_{\mp}(\lambda,t)|^2\psi_{\pm}(\lambda,x,t)\psi_{\pm}(\lambda,y,t)g_{\mp}(\lambda,t)d\lambda\\&+\sum_{k=1}^{p}(\gamma_k^{\pm}(t))^2\tilde{\psi}_{\pm}(\lambda_k,x,t)\tilde{\psi}_{\pm}(\lambda_k,y,t).\end{aligned} \tag{5.12}$$

Solve the corresponding GLM equation

$$K_\pm(x,y,t) + F_\pm(x,y,t) \pm \int_x^{\pm\infty} K_\pm(x,s,t)F_\pm(s,y,t)ds = 0, \quad \pm y > \pm x, \tag{5.13}$$

and obtain the solution by

$$q(x,t) = \mp 2\,\frac{d}{dx}K_\pm(x,x,t) + p_\pm(x,t), \quad x\in\mathbb{R}. \tag{5.14}$$

Theorem 4.4 guarantees, that both formulas give one and the same solution.

Up to now we have assumed that $q(x,t)$ is a solution the KdV equation satisfying (4.1). Now we can get rid of this assumption. We will proceed as follows. Suppose the initial condition $q(x)$ satisfies (4.1) with some finite-gap potential $p_\pm(x)$. Consider the corresponding scattering data $\mathcal{S} = \mathcal{S}(0)$ which obey conditions **I**–**IV**. Let $p_\pm(x,t)$ be the finite-gap solution of the KdV equation with initial condition $p_\pm(x)$ and let $m_\pm(\lambda,t)$, $\psi_\pm(\lambda,x,t)$, and $\alpha_\pm(\lambda,t)$ be the corresponding quantities as in Section 3.

Introduce the set of scattering data $\mathcal{S}(t)$, where $R_\pm(\lambda,t)$, $T_\pm(\lambda,t)$ and $\gamma_k^\pm(t)$ are defined by formulas (5.8)–(5.10). In the next section we prove, that these data satisfies conditions **I**–**III**, and the functions $F_\pm(x,y,t)$, defined via (5.12), satisfy **IV** under the assumption that the respective bands of the spectra $\sigma_\pm$ either coincide or otherwise do not intersect at all, that is

$$\sigma^{(2)}\cap\sigma_\pm^{(1)} = \emptyset \quad\text{and}\quad \sigma_+^{(1)}\cap\sigma_-^{(1)} = \emptyset. \tag{5.15}$$

The typical situation is depicted in Figure 1.

FIGURE 1. Typical mutual locations of σ_- and σ_+.

Then Theorem 5.3 from [9] ensures the unique solvability for each of the GLM equations (5.13)with the solutions $K_\pm(x,y,t)$ that satisfy the estimate of type (5.1). Moreover, since $F_\pm(x,y,t)$ are differentiable with respect to t with (4.29) valid for this derivative, then (4.27) implies (5.2). Consequently, the function $q(x,t)$, defined by formula (5.14), has a continuous derivative with respect to t and satisfies

$$\pm\int_0^{\pm\infty}\left|\frac{\partial^n}{\partial x^n}\big(q(x,t)-p_\pm(x,t)\big)\right|(1+|x|^m)dx<\infty, \tag{5.16}$$

and

$$\pm\int_0^{\pm\infty}\left|\frac{\partial}{\partial t}\big(q(x,t)-p_\pm(x,t)\big)\right|(1+|x|^m)dx<\infty. \tag{5.17}$$

Moreover, the functions $\phi_\pm(\lambda,x,t)$, defined via (5.3), solve equation (2.5) with $q(x,t)$, defined by (5.14). To prove, that this $q(x,t)$ solves the KdV equation, we will apply Corollary 2.3 as follows.

Since $\phi_+(\lambda,x,t)$ and $\phi_-(\lambda,x,t)$ are independent for all $\lambda\in\mathbb{C}$ but a finite number of values, it is sufficient to check that both functions $(\mathcal{A}_q\phi_\pm)(\lambda,x,t)$ solve (2.5), where $\mathcal{A}_q$ is defined by (2.4) with $q(x,t)$ from (5.14). But due to (5.3) and the estimates (5.1), (5.2) we have (5.4) and (5.5). Together with Lemma 3.2 and with the evident equalities

$$(\mathcal{A}_{p_\pm}\psi_\pm)(\lambda,x,t) = -\frac{\partial\alpha_\pm(\lambda,t)}{\partial t}\psi_\pm(\lambda,x,t), \tag{5.18}$$

it implies, that one should show, that

$$(\mathcal{A}_q\phi_\pm)(\lambda,x,t) = \beta_\pm(\lambda,t)\phi_\pm(\lambda,x,t), \tag{5.19}$$

for some $\beta_\pm(\lambda,t)$. Letting $x\to\pm\infty$ and comparing with (5.18) then shows

$$\beta_\pm(\lambda,t) = -\frac{\partial\alpha_\pm(\lambda,t)}{\partial t} = -2(p_\pm(0,t)+2\lambda)m_\pm(\lambda,t) + \frac{\partial p_\pm(0,t)}{\partial x}. \tag{5.20}$$

Finally, as already pointed out before, (5.19) is equivalent to the KdV equation for $q(x,t)$ by Corollary 2.3. Equality (5.19) will be proved in the next section.

6. Justification of the inverse scattering transform

Our first task is to check, that if $\mathcal{S}(0)$ satisfies **I–III**, then the time-dependent scattering data $\mathcal{S}(t)$, defined by (5.8)–(5.10) satisfy the same conditions (with $g_\pm(\lambda)=g_\pm(\lambda,t)$). Properties **I**, **(a)–(f)** are straightforward to check. Using

$$g_\pm(\lambda,t) = g_\pm(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)+\breve{\alpha}_\pm(\lambda,t)}, \tag{6.1}$$

which follows from (3.4) and (3.32), we see that $W(\lambda,t)$ defined as in (4.14) satisfies

$$W(\lambda,t) = W(\lambda,0)\mathrm{e}^{-\alpha_-(\lambda,t)-\alpha_+(\lambda,t)}. \tag{6.2}$$

Hence Lemma 3.3 implies that properties **II**, **(a)** and **(b)** hold.

Property **III**, **(a)** is evident, and property **III**, **(b)** follows from (3.34). In summary,

Lemma 6.1. *Let the set $\mathcal{S}(0)$ satisfy properties* **I**–**III** *and let the set $\mathcal{S}(t)$ be defined by* (5.8)–(5.10)*. Then the set $\mathcal{S}(t)$ satisfies* **I**–**III** *with $g_\pm(\lambda,t)$ defined by* (6.1)*.*

Now substitute formulas (5.8)–(5.10), (3.23), (3.32), and (6.1) into (5.12), then we obtain the following representation for the kernels of GLM equations

$$\begin{aligned} F_\pm(x,y,t) = &\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm} R_\pm(\lambda,0)\,\hat{\psi}_\pm(\lambda,x,t)\hat{\psi}_\pm(\lambda,y,t)g_\pm(\lambda,0)d\lambda \\ &+\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\lambda,0)|^2\hat{\psi}_\pm(\lambda,x,t)\hat{\psi}_\pm(\lambda,y,t)g_\mp(\lambda,0)d\lambda \\ &+\sum_{k=1}^p(\gamma_k^\pm(0))^2\tilde{\psi}_\pm(\lambda_k,x,t)\tilde{\psi}_\pm(\lambda_k,y,t), \end{aligned} \tag{6.3}$$

where the functions

$$\tilde{\psi}_\pm(\lambda,x,t) := \delta_\pm(\lambda,0)\hat{\psi}_\pm(\lambda,x,t) \tag{6.4}$$

are well-defined (bounded, continuous) for $\lambda\in\mathbb{C}\setminus\sigma_\pm$. Recall that the functions $\hat{\psi}_\pm(\lambda,x,t)$ inherit all singularities from the functions $\psi_\pm(\lambda,x,0)$, that is, they have simple poles on the set $M_\pm(0)$, square-root singularities on the set $\hat{M}_\pm(0)$, and no other singularities. Therefore, formula (6.3) consists of three well-defined summands, the singularities of the integrands are integrable (cf. [9, Sect. 5]), and it remains to verify **IV**.

Due to our assumption (5.15) the second and third summands in (6.3) (or (5.12)) satisfies **IV** for all m and n, and hence we only need to investigate the first summand in (5.12). To this end, we use (3.6)–(3.8) to obtain the representation

$$\begin{aligned} F_{\pm,R}(x,y,t) &:= 2\,\mathrm{Re}\int_{\sigma_\pm^{\mathrm{u}}} R_\pm(\lambda,t)\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)\frac{g_\pm(\lambda,t)}{2\pi\mathrm{i}}d\lambda \\ &= \mathrm{Re}\int_0^\infty \mathrm{e}^{\pm\mathrm{i}(x+y)\theta_\pm}\rho_\pm(\theta_\pm,x,y,t)d\theta_\pm, \end{aligned} \tag{6.5}$$

where

$$\rho_\pm(\theta_\pm,x,y,t) := \frac{1}{2\pi}\Psi_\pm(\theta_\pm,x,y,t)\mathrm{e}^{\alpha_\pm(\lambda,t)-\breve\alpha_\pm(\lambda,t)}R_\pm(\lambda,0), \tag{6.6}$$

$$\Psi_\pm(\theta_\pm,x,y,t) := u_\pm(\lambda,x,t)u_\pm(\lambda,y,t)\prod_{j=1}^{r_\pm}\frac{\lambda-\mu_j^\pm(t)}{\lambda-\zeta_j^\pm}, \tag{6.7}$$

and $\lambda=\lambda(\theta_\pm)$. We will integrate (6.5) by parts infinitely many times. Since the integrand is not continuous for $\theta_\pm\in[0,\infty)$, we regard this integral as

$$F_{\pm,R}(x,y,t) = \mathrm{Re}\sum_{k=0}^{r_\pm}\int_{\theta_\pm(E_{2k}^\pm)}^{\theta_\pm(E_{2k+1}^\pm)} \mathrm{e}^{\pm\mathrm{i}(x+y)\theta}\rho_\pm(\theta,x,y,t)d\theta, \tag{6.8}$$

where we set $E_{2r_\pm+1}^\pm=+\infty$ for notational convenience. Then the boundary terms during integration by parts will be

$$\mathrm{Re}\lim_{\lambda\to E}\frac{e^{\pm\mathrm{i}\theta_\pm(E)(x+y)}\frac{\partial^s\rho_\pm(\lambda(\theta_\pm),x,y,t)}{\partial\theta_\pm^s}}{\big(\mathrm{i}(x+y)\big)^{s+1}}, \quad s=0,1,\dots,\ E\in\partial\sigma_\pm, \tag{6.9}$$

and we will prove that they vanish.

Lemma 6.2. *The following limits exists and take real or pure imaginary values:*

$$\lim_{\lambda\to E,\,\lambda\in\sigma_\pm}\frac{d^s}{d\theta_\pm^s}R_\pm(\lambda(\theta_\pm),0)\in\mathrm{i}^s\,\mathbb{R}, \quad s=0,1,\dots,\ E\in\partial\sigma_\pm; \tag{6.10}$$

$$e^{\pm\mathrm{i}\,\theta_\pm(E)\,(x+y)}\lim_{\lambda\to E}\frac{\partial^s}{\partial\theta_\pm^s}\Psi_\pm(\theta_\pm,x,y,t)\in\mathrm{i}^s\mathbb{R}, \quad s=0,1,\dots,\ E\in\partial\sigma_\pm; \tag{6.11}$$

$$\lim_{\lambda\to E}\frac{\partial^s}{\partial\theta_\pm^s}\exp\{\alpha_\pm(\lambda,t)-\breve\alpha_\pm(\lambda,t)\}\in\mathrm{i}^s\mathbb{R}, \quad s=0,1,\dots,\ E\in\partial\sigma_\pm. \tag{6.12}$$

Proof. The proof is the same for $+$ and $-$ cases, we will give it for $+$ case and omit the sign $+$ in notations, except of notation for spectrum σ_+.

Let ε be a positive value smaller than the minimal length of all bands in σ_+ and abbreviate

$$\mathcal{O}(E)=(E-\varepsilon,E+\varepsilon)\cap\sigma_+.$$

Let

$$\mathcal{F}(E)=C^\infty(\mathcal{O}(E),\mathbb{R})$$

be the class of all functions $f(\lambda)$ which are smooth and real-valued on $\mathcal{O}(E)$ and let

$$\mathcal{G}(E)=\{f_1(\lambda)+\mathrm{i}\frac{d\lambda}{d\theta}f_2(\lambda)\,|\,f_1,f_2\in\mathcal{F}(E)\}.$$

From (3.8) we see that $\frac{d\lambda}{d\theta}$ is a real-valued and bounded function on the set σ_+ and $\frac{d\lambda}{d\theta}(E) = 0$. This function is smooth with respect to θ on the set $\mathcal{O}(E)$. From (3.7) we conclude, that

$$\frac{d^2\lambda}{d\theta^2} = \frac{d}{d\lambda}\left(\frac{\mathrm{i}\,Y^{1/2}(\lambda)}{\prod(\lambda-\zeta_j)}\right)\frac{\mathrm{i}\,Y^{1/2}(\lambda)}{\prod(\lambda-\zeta_j)} \in \mathcal{F}(E) \text{ and } \left(\frac{d\lambda}{d\theta}\right)^2 \in \mathcal{F}(E) \tag{6.13}$$

In particular, the last one implies that $\mathcal{G}(E)$ is an algebra. Moreover, these two equalities imply

$$\frac{d^{2k}\lambda}{d\theta^{2k}}(E) \in \mathbb{R}, \quad \frac{d^{2k+1}\lambda}{d\theta^{2k+1}}(E) = 0. \tag{6.14}$$

Now let

$$g(\lambda) = f_1(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta}\, f_2(\lambda) \in \mathcal{G}(E), \tag{6.15}$$

then (6.13) shows that

$$\frac{dg(\lambda)}{d\theta} = \mathrm{i}\left(\frac{df_2}{d\lambda}\left(\frac{d\lambda}{d\theta}\right)^2 + f_2\frac{d^2\lambda}{d\theta^2} - \mathrm{i}\frac{df_1}{d\lambda}\frac{d\lambda}{d\theta}\right) \in \mathrm{i}\mathcal{G}(E). \tag{6.16}$$

Hence (6.15) and (6.16) imply

$$\frac{d^s g}{d\theta^s}(E) \in \mathrm{i}^s\mathbb{R}, \quad s = 0, 1, \dots, \tag{6.17}$$

where the values are to be understood as limits at E from within the spectrum. In particular, for any $f(\lambda) \in \mathcal{F}(E)$,

$$\frac{d^{2k} f}{d\theta^{2k}}(E) \in \mathbb{R}, \quad \frac{d^{2k+1} f}{d\theta^{2k+1}}(E) = 0, \quad k = 0, 1, \dots. \tag{6.18}$$

The idea of the proof of (6.10) and (6.11) is to write $R(\lambda, 0)$ and

$$\hat{\Psi}(\theta, x, y, t) := \psi(\lambda, x, t)\psi(\lambda, y, t)\prod_{j=1}^{r}\frac{\lambda - \mu_j(t)}{\lambda - \zeta_j} \tag{6.19}$$

in the form (6.15). We start with $\hat{\Psi}(\theta)$ (where x, y, t play the role of parameters). From (3.16), (3.10), (3.12), and (3.13) we see, that the function $\frac{H(\lambda,0,t)}{G(\lambda,0,t)}$ is a holomorphic function in a vicinity of E even if $\mu_j(t) = E$. Thus,

$$\frac{H(\lambda, 0, t)}{G(\lambda, 0, t)} \in \mathcal{F}(E). \tag{6.20}$$

Since $\zeta_j \in (E_{2j-1}, E_{2j})$, then $\prod(\lambda - \zeta_j)^{-1} \in \mathcal{F}$. Also $s(\lambda, x, t)$, $c(\lambda, x, t) \in \mathcal{F}(E)$. Using in (6.19) the representations (3.1), (3.15), and (3.16) we conclude that the function $\hat{\Psi}(\theta, x, y, t)$ admits a representation of the type (6.15). Therefore

$$\lim_{\lambda\to E}\frac{\partial^s}{\partial\theta^s}\hat{\Psi}(\theta, x, y, t) \in \mathrm{i}^s\mathbb{R}, \quad s = 0, 1, \dots. \tag{6.21}$$

Note that in this formula it is in fact irrelevant from what side the limit is taken.

Now consider the function $\Psi(\lambda, x, y, t)$ defined by formula (6.7). As is known (cf.[2], [23]) for each t and λ this function is a quasiperiodic bounded function with respect to x and y. Therefore, if its derivatives with respect to the quasimomentum variable exist, then they will be bounded with respect to x and y. Taking into account (6.21) we obtain

$$\lim_{\lambda\to E}\frac{\partial^s}{\partial\theta^s}\Psi(\theta,x,y,t)=U_s(E,x,y,t)\mathrm{e}^{-\mathrm{i}\theta(E)(x+y)},$$

where $U_s(E,x,y,t)\in \mathrm{i}^s\mathbb{R}$, $s=0,1,\dots$, are functions which are bounded with respect to $x,y\in\mathbb{R}$ for each t. This proves (6.7). Note that $\mathrm{e}^{-\mathrm{i}\theta(E)(x+y)}$ has modulus one, but it is in general not real-valued.

To prove (6.10) we will distinguish the resonant and nonresonant cases. We start with nonresonant case $\hat W(E,t)\neq 0$ (cf. **II, (b)** and note that by (5.11) this is independent of t).

Suppose, that $E\in\partial\sigma_+\cap\partial\sigma^{(2)}$ is a left edge of the spectrum σ, that is,

$$E=E_{2j}^+=E_{2k}^-. \tag{6.22}$$

Consider the reflection coefficient $R_+(\lambda,0)$, defined by formula (4.13) and let $\theta:=\theta_+$. Suppose, that $\mu_j^+(0)\neq E$, $\mu_k^-(0)\neq E$. Then from (3.1), (6.20), (4.4), (4.9),(4.1), and (3.8) we see, that the Jost solution $\phi_+(\lambda,x)$ plus its derivative $\frac{\partial}{\partial x}\phi_+(\lambda,x)$ is in $\mathcal{G}(E)$. Moreover, by (3.8) and (6.22)

$$\frac{d\theta_+}{d\theta_-}=\frac{d\theta_+}{d\lambda}\frac{d\lambda}{d\theta_-}\in\frac{\sqrt{(\lambda-E_{2k}^-)(\lambda-E_{2k+1}^-)}}{\sqrt{(\lambda-E_{2j}^-)(\lambda-E_{2j+1}^-)}}\mathcal{F}(E)=\mathcal{F}(E)).$$

Therefore, the same is true for $\phi_-(\lambda,x)$ and hence we also have

$$\mathsf{W}(\phi_-(\lambda),\phi_+(\lambda)),\ \mathsf{W}(\phi_-(\lambda),\overline{\phi_+(\lambda)})\in\mathcal{G}(E)$$

Since $\mathsf{W}(\phi_-,\,\phi_+)(E)\neq 0$ we conclude $R_+(\lambda,0)\in\mathcal{G}(E)$ and (6.10) is proven in this case.

If $\mu_j^+(0)\neq E$, but $\mu_k^-(0)=E$ replace $\phi_-(\lambda,x)$ by by

$$\phi_-^{(1)}(\lambda,x):=\mathrm{i}\frac{d\lambda}{d\theta}\,\phi_-(\lambda,x)$$

which is in $\mathcal{G}(E)$ and proceed as before (observe that the extra factor cancels in the definition of $R_+(\lambda,0)$. The cases $\mu_j^+(0)=E$, $\mu_k^-(0)\neq E$ and $\mu_j^+(0)=\mu_k^-(0)=E$ can be handled similarly.

In the nonresonant case, when $E\in\partial\sigma_+^{(1)}\cap\partial\sigma$ the consideration are even simpler, because in this case (cf. (4.10)) $\tilde\phi_-(\lambda,x)\in\mathcal{F}(E)$. We assume $\mu_j^+(0)\neq E$, if $\mu_j^+(0)=E$ one only needs to replace $\phi_+(\lambda)$ by $\phi_-^{(1)}(\lambda)$ as pointed out before. Thus

$$R_+(\lambda,0)=\frac{f_1(\lambda)+\mathrm{i}\frac{d\lambda}{d\theta}f_2(\lambda)}{f_3(\lambda)+\mathrm{i}\frac{d\lambda}{d\theta}f_4(\lambda)},\qquad \text{where } f_i(\lambda)\in\mathcal{F}(E),\ i=1,2,3,4. \tag{6.23}$$

This finishes the proof of formula (6.10) in the nonresonant case, because in this case we have $f_3(E)\neq 0$ and, therefore $R_+(\lambda,0)\in\mathcal{G}(E)$.

In the resonance case we have $\hat W(E)=0$, but $\frac{d\hat W}{d\theta}(E)\neq 0$ (cf. **II, (b)**). Hence we have (6.23) with $f_1(E)=f_3(E)=0$ and $f_4(E)\neq 0$. Let us show, that the derivative of the right-hand side of (6.23) satisfies

$$\frac{d}{d\theta}\frac{f_1(\lambda)+\mathrm{i}\frac{d\lambda}{d\theta}f_2(\lambda)}{f_3(\lambda)+\mathrm{i}\frac{d\lambda}{d\theta}f_4(\lambda)}\in\mathrm{i}\mathcal{G}(E). \tag{6.24}$$

Namely, denote by dot the derivative with respect to θ and by prime - with respect to λ. Then

$$\frac{d}{d\theta}\frac{g_1(\lambda)+\mathrm{i}\dot\lambda g_2(\lambda)}{g_3(\lambda)+\mathrm{i}\dot\lambda g_4(\lambda)}=\mathrm{i}\left(\ddot\lambda(g_2g_3-g_4g_1)+(\dot\lambda)^2(g_1'g_4-g_3'g_2+g_2'g_3-g_4'g_1)+\right.$$
$$\left.+\mathrm{i}\dot\lambda\left(g_3'g_1-g_1'g_3+(\dot\lambda)^2(g_2'g_4-g_4'g_2)\right)\right)\left(-(\dot\lambda)^2\,g_4^2+g_3^2+\mathrm{i}\dot\lambda(2g_4g_2)\right)^{-1}.$$

Functions g_1, g_3 and $(\dot\lambda)^2$ have zeros of the first order with respect to λ at the point E and $g_4(E)\ddot\lambda(E)\neq 0$. It means, that we can divide nominator and denominator in the r.h.s. of the last equality and using (6.13) arrive at (6.24) and we obtain (6.10) for $s\geq 1$. To prove the remaining case $s=0$ we have to check that $R_+(E,0)\in\mathbb{R}$ in the resonance case. Since the nominator and denominator in (6.23) vanishes,

$$\lim_{\lambda\to E}R_+(\lambda,0)=\lim_{\lambda\to E}\frac{(f_1'+\mathrm{i}f_2)\dot\lambda+\mathrm{i}\ddot\lambda f_2}{(f_3'+\mathrm{i}f_4)\dot\lambda+\mathrm{i}\ddot\lambda f_4}=\frac{f_2(E)}{f_4(E)}\in\mathbb{R}.$$

this completes the proof of (6.10).

To prove (6.12) we use the same approach. Again the prove will be done for the $+$ case. From (3.34) it follows, that

$$\lim_{\lambda\to E}\exp\left(\alpha_+(\lambda,t)-\overline{\alpha_+}(\lambda,t)\right)\in\mathbb{R},$$

therefore it suffices to show that for

$$h(\lambda):=(\alpha_+(\lambda,t)-\overline{\alpha_+}(\lambda,t))$$

the derivative $\dot h(\lambda)=\frac{dh}{d\theta}$ satisfies

$$\dot h(\lambda)=\mathrm{i}f(\lambda),\quad f(\lambda)\in\mathcal{F}(E). \tag{6.25}$$

To simplify notations, we will omit sign $+$ until the end of this lemma.

Suppose first, that

$$\mu_j(t)\neq E=E_{2j},\quad \mu_j(0)\neq E \tag{6.26}$$

Let $0<t_1<...<t_N<t$ be the set of points, where $\mu_j(t_k)=E$. Choose $\delta>0$ so small, that

$$\mu_j(E\pm\delta)>\max\{\mu_j(0),\mu_j(t),(E_{2j-1}+E)/2\}.$$

Denote

$$\Delta=[0,t]\setminus\cup_{k=1}^N(t_k-\delta,t_k+\delta).$$

Let $\lambda>E$ be a point in the spectrum, close to E. Then for $s\in\Delta$ $|\mu_j(s)-\lambda|>const(E)>0$ we have (see (3.35))

$$4Y^{1/2}(\lambda)\int_\Delta\frac{p_\pm(0,s)+2\lambda}{G_\pm(\lambda,s)}ds=\mathrm{i}\dot\lambda f_1(\lambda),\quad f_1\in\mathcal{F}(E). \tag{6.27}$$

On the remaining set we use the representations (3.44) and (3.45). Proceeding as in (3.46) we obtain

$$4Y^{1/2}(\lambda)\int_{t_k-\delta}^{t_k}\frac{p_+(0,s)+2\lambda}{H_+(\lambda,s)}ds=\sigma_j\mathrm{i}\left(\arctan\frac{\sqrt{E-\mu_j(t_k-\delta)}}{\sqrt{\lambda-E}}+\pi\right)+$$

$$+\sqrt{\lambda-E}\int_{t_k-\delta}^{t_k}\frac{\partial}{\partial\lambda}\breve G_j(\xi_j(s,\lambda),s)ds,\quad \sigma_j\in\{-1,1\}, \tag{6.28}$$

where $\xi(\lambda,s)\in\mathcal{F}(E)$ such that $\mu_j(t_k-\delta)\le\xi(\lambda,s)\le\lambda$ for $t_k-\delta\le s\le t_k$. Furthermore, note that the function

$$\breve{G}(\xi,s)=\frac{Y^{1/2}(\xi)}{\sqrt{\xi-E}\prod_{l\neq j}(\xi-\mu_l)}$$

is smooth with respect to ξ in the domain $\mu_j(t_k-\delta)\le\xi\le\lambda$ and takes pure imaginary values there. Namely,

$$\begin{array}{llll} Y^{1/2}(\xi)\in\mathrm{i}\mathbb{R}, & \sqrt{\xi-E}\in\mathbb{R} & \text{for} & E\le\xi\le\lambda,\\ Y^{1/2}(\xi)\in\mathbb{R}, & \sqrt{\xi-E}\in\mathrm{i}\mathbb{R} & \text{for} & \mu_j(t_k-\delta)\le\xi\le E.\end{array}$$

Thus,

$$\frac{\partial^s\breve{G}(\xi,s)}{\partial\xi^s}\in\mathrm{i}\mathbb{R}\quad\text{for}\quad\mu_j(t_k-\delta)\le\xi\le\lambda,\quad s=0,1,\dots. \tag{6.29}$$

The same considerations show

$$\sqrt{\lambda-E}=\dot\lambda f_2(\lambda)\quad\text{where}\quad f_2(\lambda)\in\mathcal{F}(E),\ f(E)\neq0. \tag{6.30}$$

Combining this with (6.29) we obtain

$$\sqrt{\lambda-E}\int_{t_k-\delta}^{t_k}\frac{\partial}{\partial\lambda}\breve{G}_j(\xi_j(s,\lambda),s)ds=\mathrm{i}\dot\lambda\, f_3(\lambda),\quad f_3(\lambda)\in\mathcal{F}(E).$$

Thus

$$\frac{d}{d\theta}\left(\sqrt{\lambda-E}\int_{t_k-\delta}^{t_k}\breve{G}_j'(\xi_j(s,\lambda),s)ds\right)=\mathrm{i}f_4(\lambda),\quad f_4(\lambda)\in\mathcal{F}(E). \tag{6.31}$$

Using (6.30) one can also represent the argument under arctan in the first summand of (6.28) as $\frac{f_5(\lambda)}{\dot\lambda}$, where $f_5(\lambda)\in\mathcal{F}(E)$ and $f_5(E)\neq0$. Therefore,

$$\frac{d}{d\theta}\left(\arctan\frac{\sqrt{E-\mu_j(t_k-\delta)}}{\sqrt{\lambda-E}}+\pi\right)\sigma_j\mathrm{i}=\sigma_j\mathrm{i}\frac{f_5'(\dot\lambda)^2-\ddot\lambda\, f_5}{(\dot\lambda)^2+f_5^2}\in\mathrm{i}\mathcal{F}(E). \tag{6.32}$$

The same will be valid for the interval $(t_k,t_k+\delta)$. Combining (6.27), (6.31), and (6.32) we obtain (6.25). These considerations also show that the restriction (6.26) is unessential. □

Our next goal is to prove formula (5.19). Since for any solution of the equation $L_v(t)u=\lambda u$ the equality $\mathcal{A}_v u=u_t-P_v(t)u$ is valid, it suffices to prove the following

Lemma 6.3. *Let $K_\pm(x,y,t)$ be the solutions of the GLM equations* (5.13) *with the kernels* (5.12)*, corresponding to the scattering data* (5.8)–(5.10)*. Let the functions $\phi_\pm(\lambda,x,t)$ be defined by* (5.3) *and let $q(x,t)$ be defined by* (5.14)*. Then $\phi_\pm(\lambda,x,t)$ satisfy*

$$\Big(\frac{\partial}{\partial t}-P_q(t)\Big)\phi_\pm(\lambda,x,t)=\beta_\pm(\lambda,t)\phi_\pm(\lambda,x,t), \tag{6.33}$$

where $\beta_\pm(\lambda,t)$ is defined by (5.20)*.*

Proof. As before we prove this lemma only for the + case. To simplify notations, set $P=P_q(t)$, $P_0=P_+(t)$, $\phi=\phi_+(\lambda,x,t)$, $\psi=\psi_+(\lambda,x,t)$, $p=p_+$,

$$(\mathcal{K}f)(x,t)=\int_x^{+\infty}K_+(x,y,t)f(y,t)dy$$

$$(\dot{\mathcal{K}}f)(x,t)=\int_x^{+\infty}\frac{\partial}{\partial t}K_+(x,y,t)f(y,t)dy, \tag{6.34}$$

and denote by a dot the derivative with respect to t and by a prime the derivative with respect to spatial variables. Moreover, we will omit the variable t whenever it is possible and use the notations

$$D_{x^l y^m}(x)=\left(\frac{\partial^l}{\partial x^l}+\frac{\partial^m}{\partial y^m}\right)D(x,y)|_{y=x}, D_{x^0y^0}(x)=D(x).$$

Since $\dot\psi-P_0\psi=\beta\psi$, then

$$\dot\phi-P\phi=\beta\phi+(P_0-P)\psi+\dot{\mathcal{K}}\psi+\mathcal{K}P_0\psi-P\mathcal{K}\psi. \tag{6.35}$$

Differentiating the last term and integrating by parts gives

$$\begin{aligned}(P\mathcal{K}\psi)(x)=&\{-2(q'(x)-p'(x))+4K_{xy}(x)+8K_{x^2}(x)-6q(x)K(x)\}\,\psi(x)\\ &-\{4(q(x)-p(x))-4K_x(x)\}\,\psi'(x)+4K(x)\psi''(x)+\\ &+\int_x^\infty\left(-4K_{x^3}(x,y)+6q(x)K_x(x,y)+3q'(x)K(x,y)\right)\psi(y)dy,\end{aligned} \tag{6.36}$$

and

$$\begin{aligned}\left(\mathcal{K}P_0\psi\right)(x)=&\big(4K_{y^2}(x)-6K(x)p(x)\big)\,\psi(x)-4K_y(x)\psi'(x)+4K(x)\psi''(x)\\ &+\int_x^\infty\big(4K_{y^3}(x,y)-6K_y(x,y)p(y)-3K(x,y)p'(y)\big)\,\psi(y)dy.\end{aligned} \tag{6.37}$$

Besides,

$$(P-P_0)\psi(x)=6(q(x)-p(x))\psi'(x)+3(q'(x)-p'(x))\psi(x). \tag{6.38}$$

Combining (6.34)–(6.38) and taking into account the formula (cf. [20])

$$-K_{xx}(x,y)+q(x)K(x,y)=-K_{yy}(x,y)+p(y)K(x,y), \tag{6.39}$$

where we put $x=y$, we arrive at the representation

$$(\dot\phi-P\phi-\beta\phi)(x)=A(x)\psi(x)+B(x)\psi'(x)+\int_x^\infty(\tau^{xy}K(x,y))\psi(y)dy=0, \tag{6.40}$$

where

$$\begin{aligned}A(x)&=p'(x)-q'(x)-2K_{x^2}(x)-4K_{xy}(x)-2K_{y^2}(x),\\ B(x)&=2(p(x)-q(x))-4(K_x(x)+K_y(x)),\end{aligned}$$

and

$$\tau^{xy}:=\frac{\partial}{\partial t}+\tau_q^x+\tau_p^y,\quad \tau_q^x:=4\frac{\partial^3}{\partial x^3}-6q(x)\frac{\partial}{\partial x}-3q'(x). \tag{6.41}$$

But according to (5.14)

$$p(x)-q(x)=2K_x(x)+2K_y(x),\quad p'(x)-q'(x)=2K_{x^2}(x)+4K_{xy}(x)+2K_{y^2}(x), \tag{6.42}$$

and therefore, $A(x)=B(x)=0$. Thus, to prove (6.33) one has to check, that

$$\begin{aligned}D(x,y):=&\tau^{xy}K(x,y)=K_t(x,y)+4K_{y^3}(x,y)+4K_{x^3}(x,y)-6q(x)K_x(x,y)\\ &-6p(y)K_y(x,y)-3q'(x)K(x,y)-3p'(y)K(x,y)\equiv 0.\end{aligned} \tag{6.43}$$

To this end, let us derive an equation for the function $F = F_+(x, y, t)$, defined by formula (5.12). This function can be represented (see (6.3)) as

$$F(x, y, t) = \int_{\mathbb{R}} \hat{\psi}(\lambda, x, t)\hat{\psi}(\lambda, y, t)d\rho(\lambda),$$

where the measure

$$\begin{aligned} d\rho(\lambda) =& \Big(\frac{1}{\pi \mathrm{i}} R_+(\lambda, 0) g_+(\lambda, 0)\chi_{\sigma_+^u}(\lambda) + \frac{1}{2\pi \mathrm{i}}|T_-(\lambda, 0)|^2 g_-(\lambda, 0)\chi_{\sigma_-^{(1)}}(\lambda) \\ &+ \sum_k (\gamma_k^+)^2(0)\delta(\lambda - \lambda_k)\delta_+(\lambda_k, 0)^2\Big)d\lambda \end{aligned}$$

does not depend on t. Using (3.26) we conclude, that

$$\tau_0^{xy} F(x, y) = 0, \quad \tau_0^{xy} = \frac{\partial}{\partial t} + \tau_p^x + \tau_p^y. \tag{6.44}$$

Now set $V(x) = q(x) - p(x)$ and apply the operator τ^{xy} to the GLM equation (5.13). Taking into account (6.43), (6.44) and the equality

$$\tau^{xy} - \tau_0^{xy} = -6V(x)\frac{\partial}{\partial x} - 3V'(x)$$

we obtain

$$\begin{aligned} D(x, y) =& \int_x^\infty \left\{K(x, s)\tau_p^s\left[F(s, y)\right] - K_t(x, s)F(s, y)\right\} ds \\ &- \tau_q^x\left[\int_x^\infty K(x, s)F(s, y)ds\right] + 6V(x)F_x(x, y) + 3V'(x)F(x, y), \end{aligned}$$

or

$$D(x, y) + \int_x^\infty D(x, s)F(s, y)ds = r(x, y), \tag{6.45}$$

where

$$\begin{aligned} r(x, y) =& \int_x^\infty \left\{\tau_p^s\left[K(x, s)\right]F(s, y) + K(x, s)\tau_p^s\left[F(s, y)\right]\right\} ds+ \\ &+ \int_x^\infty \tau_q^x\left[K(x, s)\right]F(s, y)ds - \tau_q^x\left[\int_x^\infty K(x, s)F(s, y)ds\right]+ \\ &+6V(x)F_x(x, y) + 3V'(x)F(x, y). \end{aligned} \tag{6.46}$$

It is proved in [9], that the equation $D(x, y) + \int_x^\infty D(x, s)F(s, y)ds = 0$, where x plays the role of a parameter, has only the trivial solution in the space $L^1(x, \infty)$. Since the function $D(x, \cdot)$ evidently belongs to this space, then to prove (6.43) is to prove, that $r(x, y) = 0$.

Taking into account, that $V(x) = -2\frac{d}{dx}K(x, x)$, direct computations imply

$$\begin{aligned} &\int_x^\infty \tau_q^x\left[K(x, s)\right]F(s, y)ds - \tau_q^x\left[\int_x^\infty K(x, s)F(s, y)ds\right]+ \\ &+6V(x)F_x(x, y) + 3V'(x)F(x, y) = 4K(x, x)F_{x^2}(x, y)+ \\ &+4K_x(x, x)F_x(x, y) + 8K_{x^2}(x, x)F(x, y) + 4K_{xy}(x, x)F(x, y)+ \\ &+V'(x)F(x, y) + 2V(x)F_x(x, y) - 6q(x)K(x, x)F(x, y). \end{aligned} \tag{6.47}$$

From the other side, integration by parts gives

$$\int_x^\infty \left\{\tau_p^s\left[K(x, s)\right]F(s, y) + K(x, s)\tau_p^s\left[F(s, y)\right]\right\} ds = \tag{6.48}$$

$$= -4\left\{K_{s^2}(x,s)F(x,y) + K(x,x)F_{s^2}(s,y) - K_s(x,s)F_s(s,y)\right\}|_{s=x} + \\ +6p(x)K(x,x)F(x,y).$$

Substituting last to formulas to (6.46) gives

$$r(x,y) = F_x(x,y)(4K_x(x,x) + 4K_y(x,x) + 2V(x)) + \\ +F(x,y)\left(-6V(x)K(x,x) + 8K_{x^2}(x,x) + 4K_{xy}(x,x) - 4K_{y^2}(x,x) + V'(x)\right).$$

Taking into account (6.42) we obtain

$$r(x,y) = F(x,y)\left(-6V(x)K(x,x) + 6K_{x^2}(x,x) - 6K_{y^2}(x,x)\right),$$

and (6.39) implies $r(x,y) = 0$. □

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