Taming Nonrenormalizability

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Abstract

Unlike asymptotically free quantum field theories, asymptotically nonfree theories have not been well served by conventional perturbation analysis. In contrast, the goal of a proper renormalization scheme for perturbatively nonrenormalizable scalar quantum field theories should be to neutralize the source of the divergences. Achieving this goal requires an entirely different kind of counterterm than those typically employed, and, as we shall argue, including the appropriate counterterm as part of a pseudofree model about which to expand leads to an alternative perturbative formulation that is term-by-term finite.

Introduction

Nonrenormalizable quantum field models, such as φ_n^4, with a spacetime dimension $n \geq 5$, require the introduction of nonclassical, nontrivial counterterms [1]. A conventional regularized perturbation theory is *re-active* in the sense that it cancels divergences one-by-one as they are encountered in a perturbation analysis. Rather than following this traditional procedure, suppose one could find a *pro-active* procedure that removes the *cause* of the

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divergences in the first place. In a nutshell, that is our procedure; namely, we choose our counterterm to remove the *cause of the divergences at the outset*, rather than adopt a series of counterterms that deal with divergences as they arise.

Initially, we choose an n-dimensional, periodic, hypercubic, Euclidean spacetime lattice with a lattice spacing a, L lattice points on each side, and lattice points labeled by multi-integers $k = (k_0, k_1, \ldots, k_s) \in \mathbb{Z}^n$, where $s = n-1$ is the spatial dimension, and k_0 refers to a future time direction. The lattice-regularized functional integral for the Schwinger function generating functional is given by

$$\begin{aligned} S(h) &\equiv M \int e^{Z^{-1/2}\Sigma_k h_k \phi_k a^n/\hbar \,-\, I_n(\phi,a,N)/\hbar \,-\, C(\phi,a,\hbar)/\hbar}\, \Pi_k d\phi_k \\ &\equiv \langle e^{Z^{-1/2}\Sigma_k h_k \phi_k a^n/\hbar} \rangle \,, \end{aligned} \tag{1}$$

where $\{h_k\}$ determines an appropriate test sequence, and the normalization factor M ensures that $S(0) = 1$. The continuum limit is taken in two steps: (i) The number of lattice sites on an edge $L \rightarrow \infty$ and the lattice spacing $a \rightarrow 0$ so that La remains constant and finite. Thus the spacetime volume $V = (La)^n$ as well as the spatial volume (at fixed Euclidean time) $V' = (La)^s$ are both finite; (ii) The final step involves $V \rightarrow \infty$ and $V' \rightarrow \infty$. In this article we focus on just the first step in the continuum limit and assume that both V and V' are sufficiently large. Notationally, we also introduce $N = L^n$ and $N' = L^s$, and note that sums (and products) such as $\Sigma_k(\Pi_k)$ are over all spacetime, while $\Sigma'_k(\Pi'_k)$ are over all space alone at some fixed k_0.

In (1), Z denotes the field strength renormalization factor and $I_n(\phi, a, N)$ is the naive lattice action,

$$I_n(\phi,a,N) \equiv \tfrac{1}{2}\textstyle\sum_k\sum_{k^*} (\phi_{k^*} - \phi_k)^2\, a^{n-2} + \tfrac{1}{2}m_0^2 \sum_k \phi_k^2\, a^n + \lambda_0 \sum_k \phi_k^4\, a^n \,, \tag{2}$$

where k^* denotes any one of the n nearest neighbors to k in the positive sense, i.e., $k^* \in \{\, (k_0+1, k_1, \ldots, k_s), \ldots, (k_0, k_1, \ldots, k_s+1)\,\}$. Also in (1) the term

$$C(\phi,a,\hbar) \equiv \tfrac{1}{2}\hbar^2 \textstyle\sum_k \mathcal{F}_k(\phi)\, a^n \tag{3}$$

represents the still-to-be-chosen counterterm.

Assuming that the spacetime volume $V < \infty$, it is clear that full spacetime averages such as $\langle [\sum_k \phi_k^r a^n]^p \rangle$ are finite, for all positive integers r and p, provided that all the corresponding sharp-time, spatial averages $\langle[\sum'_k \phi^r_k a^s]^p\rangle$ are finite; for a proof, see [2]. In turn, the latter expression can be represented as

$$\langle[\sum'_k \phi^r_k a^s]^p\rangle = \int [\sum'_k \phi^r_k a^s]^p \, \Psi(\phi)^2 \, \Pi'_k d\phi_k \; , \tag{4}$$

where $\Psi(\phi)$ denotes the ground state of the system.

Choice of Counterterm

As an idealized example of a Gaussian ground-state distribution characteristic of a typical free theory, let $\Psi_G(\phi)^2 \equiv R \exp[-A\Sigma'_k \phi^2_k a^s]$ and, for integral $p \geq 0$, consider the integrals

$$I_p(A) \equiv R \int [\Sigma'_k \phi^2_k a^s]^p \, e^{-A\Sigma'_k \phi^2_k a^s} \, \Pi'_k d\phi_k \; . \tag{5}$$

Such integrals can be evaluated exactly, but we prefer to study them in an approximate sense by steepest descent methods. To that end we introduce hyper-spherical coordinates [3] defined by

$$\phi_k \equiv \kappa \eta_k \;\; , \quad \Sigma'_k \phi^2_k = \kappa^2 \;\; , \quad \Sigma'_k \eta^2_k = 1 \;\; , \quad 0 \leq \kappa < \infty \;\; , \quad -1 \leq \eta_k \leq 1 \; , \tag{6}$$

and it follows that

$$I_p(A) = 2\,R \int \kappa^{2p} \, a^{sp} e^{-A\kappa^2 a^s} \, \kappa^{(N'-1)} d\kappa \, \delta(1 - \Sigma'_k \eta^2_k) \, \Pi'_k d\eta_k \; . \tag{7}$$

A steepest descent argument leads to

$$I_p(A) = O((N'/A)^p) \, I_0(A) \; , \tag{8}$$

and a perturbation series for $I_1(A)$ about $I_1(1)$ is given by

$$I_1(A) = I_1(1) - \Delta \, I_2(1) + \tfrac{1}{2}\Delta^2 \, I_3(1) - \cdots \; , \tag{9}$$

where $\Delta = A - 1$. As $N' \to \infty$, such a series has higher-order, term-by-term divergences because the support of the ground-state distribution is concentrated on *disjoint sets for distinct A values* due, essentially, to the factor $\kappa^{(N'-1)}$ in the integrand. Our goal is to introduce a counterterm that effectively cancels the factor $\kappa^{(N'-1)}$, and this can be accomplished, loosely speaking, by choosing an idealized example of a *pseudofree model* about which to expand with a ground-state distribution such that

$$\Psi_I(\phi)^2 \propto \kappa^{-(N'-1)}\, e^{-A\Sigma'_k \phi_k^2 a^s} \ . \tag{10}$$

Observe that the use of the distribution $\Psi_I(\phi)^2$ in place of $\Psi_G(\phi)^2$ above leads to a series analogous to (9) that is term-by-term finite.

More precisely, to deal with realistic models, we focus on a ground state for the pseudofree *(pf)* model given by

$$\Psi_{pf}(\phi) = K\, \frac{e^{-\Sigma'_{k,l}\phi_k A_{k-l}\phi_l a^{2s}/2\hbar \,-\, W(\phi\, a^{(s-1)/2}/\hbar^{1/2})/2}}{\Pi'_k[\Sigma'_l J_{k,l}\phi_l^2]^{(N'-1)/4N'}} \ ; \tag{11}$$

we discuss the constants A_{k-l} and $J_{k,l}$ and the function W below. This form for the ground state is ensured if we define the pseudofree theory – the theory about which a perturbation expansion is to take place – as

$$S_{pf}(h) = M_{pf}\int e^{-\frac{1}{2}\sum_k\sum_{k^*}(\phi_{k^*}-\phi_k)^2 a^{n-2}/\hbar \,-\, \frac{1}{2}\hbar\sum_k \mathcal{F}_k(\phi)\, a^n}\, \Pi_k\, d\phi_k \ , \tag{12}$$

and choose $\mathcal{F}_k(\phi)$ to yield the denominator in (11). Based on the lattice Hamiltonian for the pseudofree model,

$$\begin{aligned}\mathcal{H}_{pf} = &-\tfrac{1}{2}\hbar^2\, a^{-s}{\textstyle\sum_k}'\tfrac{\partial^2}{\partial\phi_k^2} + \tfrac{1}{2}{\textstyle\sum'_k\sum'_{k^*}}(\phi_{k^*}-\phi_k)^2 a^{s-2}\\ &+\tfrac{1}{2}\hbar^2{\textstyle\sum_k}'\mathcal{F}_k(\phi)\, a^s - E_0 \ ,\end{aligned} \tag{13}$$

this connection leads to

$$\begin{aligned}\mathcal{F}_k(\phi) \equiv &\ \frac{1}{4}\Big(\frac{N'-1}{N'}\Big)^2 a^{-2s}{\textstyle\sum'_{r,t}}\frac{J_{r,k}\,J_{t,k}\,\phi_k^2}{[\Sigma'_l J_{r,l}\,\phi_l^2]\,[\Sigma'_m J_{t,m}\,\phi_m^2]}\\ &-\frac{1}{2}\Big(\frac{N'-1}{N'}\Big) a^{-2s}{\textstyle\sum'_t}\frac{J_{t,k}}{[\Sigma'_m J_{t,m}\,\phi_m^2]}\\ &+\Big(\frac{N'-1}{N'}\Big) a^{-2s}{\textstyle\sum'_t}\frac{J_{t,k}^2\,\phi_k^2}{[\Sigma'_m J_{t,m}\,\phi_m^2]^2} \ .\end{aligned} \tag{14}$$

Irrespective of the choice for $J_{k,l}$, we note that: (i) the denominator in the expression for the pseudofree ground state specifically leads to the counterterm in the Hamiltonian; (ii) the term in the exponent quadratic in ϕ is chosen to yield the spatial-gradient term in the Hamiltonian (and possibly part of E_0), and this requires that $A_{k-l} \propto a^{-(s+1)}$; and (iii) the unspecified term W ensures that no additional terms (other than the rest of E_0) appear in the Hamiltonian. The functional form of the argument in W follows from the manner in which both $\hbar$ and a appear in the Hamiltonian. In addition, note that the quadratic and denominator terms in $\Psi_{pf}(\phi)$ are correct for very large and very small field values, respectively; hence W is relatively most effective for intermediate field values.

The choice $J_{k,l} = \delta_{k,l}$ leads to a local covariant potential for which $\mathcal{F}_k(\phi) \propto 1/\phi_k^2$, but it also gives rise to an unsuitable form for the ground-state distribution with incipient normalization divergences at $\phi_k = 0$, for each k, as $N' \to \infty$. To eliminate that feature, we choose the factors $J_{k,l}$ to provide a minimally regularized, lattice-symmetric, local spatial averaging in the form

$$J_{k,l} \equiv \frac{1}{2s+1} \delta_{k,l \in \{k \cup k_{nn}\}} \ , \tag{15}$$

where $\delta_{k,l}$ is a Kronecker delta. This notation means that an equal weight of $1/(2s+1)$ is given to the $2s+1$ points in the set composed of k and its $2s$ nearest neighbors in the spatial sense only; $J_{k,l} = 0$ for all other points in that spatial slice. [Specifically, we define $J_{k,l} = 1/(2s+1)$ for the points $l = k = (k_0, k_1, k_2, \ldots, k_s)$, $l = (k_0, k_1 \pm 1, k_2, \ldots, k_s)$, $l = (k_0, k_1, k_2 \pm 1, \ldots, k_s)$,$\ldots$, $l = (k_0, k_1, k_2, \ldots, k_s \pm 1)$.] This definition implies that $\Sigma'_l J_{k,l} = 1$.

In the continuum limit, it is important to observe that the form of the counterterm given by (14) leads to a local covariant potential.

The Continuum Limit, and Term-by-term Finiteness of a Perturbation Analysis

Before focusing on the limit $a \to 0$ and $L \to \infty$, we note several important facts about ground-state averages of the direction field variables $\{\eta_k\}$. First, we assume that such averages have two important symmetries: (i) averages of an odd number of η_k variables vanish, i.e.,

$$\langle \eta_{k_1} \cdots \eta_{k_{2p+1}} \rangle = 0 \ , \tag{16}$$

and (ii) such averages are invariant under any spacetime translation, i.e.,

$$\langle \eta_{k_1} \cdots \eta_{k_{2p}} \rangle = \langle \eta_{k_1+l} \cdots \eta_{k_{2p}+l} \rangle \tag{17}$$

for any $l \in \mathbb{Z}^n$ due to a similar translational invariance of the lattice Hamiltonian. Second, we note that for any ground-state distribution, it is necessary that $\langle \eta_k^2 \rangle = 1/N'$ for the simple reason that $\Sigma'_k \eta_k^2 = 1$. Hence, $|\langle \eta_k \eta_l \rangle| \le 1/N'$ as follows from the Schwarz inequality. Since $\langle [\Sigma'_k \eta_k^2]^2 \rangle = 1$, it follows that $\langle \eta_k^2 \eta_l^2 \rangle = O(1/N'^2)$. Similar arguments show that for any ground-state distribution

$$\langle \eta_{k_1} \cdots \eta_{k_{2p}} \rangle = O(1/N'^p) \;, \tag{18}$$

which will be useful in the sequel.

Field strength renormalization

For $\{h_k\}$ a suitable spatial test sequence, we insist that expressions such as

$$\int Z^{-p} \, [\Sigma'_k h_k \, \phi_k \, a^s]^{2p} \, \Psi_{pf}(\phi)^2 \, \Pi'_k \, d\phi_k \tag{19}$$

are finite in the continuum limit. Due to the intermediate field relevance of the factor W in the pseudofree ground state, an approximate evaluation of the integral (19) will be adequate for our purposes. Thus, we are led to consider

$$K \int Z^{-p} \, [\Sigma'_k h_k \, \phi_k \, a^s]^{2p} \, \frac{e^{-\Sigma'_{k,l} \phi_k \, A_{k-l} \, \phi_l \, a^{2s}/\hbar \, - \, W}}{\Pi'_k [\Sigma'_l J_{k,l} \phi_l^2]^{(N'-1)/2N'}} \, \Pi'_k \, d\phi_k \tag{20}$$

$$\simeq 2 K_0 \int Z^{-p} \, \kappa^{2p} \, [\Sigma'_k h_k \, \eta_k \, a^s]^{2p} \, \frac{e^{-\kappa^2 \, \Sigma'_{k,l} \eta_k \, A_{k-l} \, \eta_l \, a^{2s}/\hbar}}{\Pi'_k [\Sigma'_l J_{k,l} \, \eta_l^2]^{(N'-1)/2N'}} \, d\kappa \, \delta(1 - \Sigma'_k \eta_k^2) \, \Pi'_k \, d\eta_k \;,$$

where K_0 is the normalization factor when W is dropped. Our goal is to use this integral to determine a value for the field strength renormalization constant Z. To estimate this integral we first replace two factors with η variables by their appropriate averages. In particular, the quadratic expression in the exponent is estimated by

$$\kappa^2 \, \Sigma'_{k,l} \eta_k \, A_{k-l} \eta_l \, a^{2s} \simeq \kappa^2 \, \Sigma'_{k,l} \, N'^{\,-1} A_{k-l} \, a^{2s} \propto \kappa^2 \, N' \, a^{2s} \, a^{-(s+1)} \;, \tag{21}$$

and the expression in the integrand is estimated by

$$[\Sigma'_k h_k \eta_k \, a^s]^{2p} \simeq N'^{\,-p} \, [\Sigma'_k h_k \, a^s]^{2p} \; . \tag{22}$$

The integral over κ is then estimated by first rescaling the variable $\kappa^2 \to \kappa^2/(N' a^{s-1}/\hbar)$, which then leads to an overall integral estimate proportional to

$$Z^{-p} \, [N' a^{s-1}]^{\,-p} \, N'^{-p} \, [\Sigma'_k h_k \, a^s]^{2p} \; ; \tag{23}$$

at this point, all factors of a are now outside the integral. For this result to be meaningful in the continuum limit, we are led to choose $Z = N'^{\,-2} a^{-(s-1)}$. However, Z must be dimensionless, so we introduce a fixed positive quantity q with dimensions of an inverse length, which allows us to set

$$Z = N'^{\,-2} (qa)^{-(s-1)} \; . \tag{24}$$

Mass and coupling constant renormalization

A power series expansion of the mass and coupling constant terms lead to the expressions $\langle [m_0^2 \, \Sigma_k \phi_k^2 a^n]^p \rangle$ and $\langle [\lambda_0 \, \Sigma_k \phi_k^4 a^n]^p \rangle$ for $p \geq 1$, which we treat together as part of the larger family governed by $\langle [g_{0,r} \, \Sigma_k \phi_k^{2r} a^n]^p \rangle$ for integral $r \geq 1$. Thus we consider

$$K \int [g_{0,r} \Sigma'_k \phi_k^{2r} \, a^s]^p \, \frac{e^{-\Sigma'_{k,l} \phi_k \, A_{k-l} \, \phi_l \, a^{2s}/\hbar \, - \, W}}{\Pi'_k [\Sigma'_l J_{k,l} \phi_l^2]^{(N'-1)/2N'}} \, \Pi'_k \, d\phi_k \tag{25}$$

$$\simeq 2 K_0 \int g_{0,r}^p \, \kappa^{2rp} \, [\Sigma'_k \eta_k^{2r} \, a^s]^p \, \frac{e^{-\kappa^2 \, \Sigma'_{k,l} \eta_k \, A_{k-l} \, \eta_l \, a^{2s}/\hbar}}{\Pi'_k [\Sigma'_l J_{k,l} \eta_l^2]^{(N'-1)/2N'}} \, d\kappa \, \delta(1 - \Sigma'_k \eta_k^2) \, \Pi'_k \, d\eta_k \; .$$

The quadratic exponent is again estimated as

$$\kappa^2 \, \Sigma'_{k,l} \eta_k \, A_{k-l} \, \eta_l \, a^{2s} \propto \kappa^2 \, N' a^{2s} a^{-(s+1)} \; , \tag{26}$$

while the integrand factor

$$[\Sigma'_k \eta_k^{2r}]^p \simeq N'^p \, N'^{-rp} \; . \tag{27}$$

The same transformation of variables used above precedes the integral over κ, and the result is an integral, no longer depending on a, that is proportional to

$$g_{0,r}^p \, N'^{-(r-1)p} \, a^{sp} / N'^{rp} \, a^{(s-1)rp} \; . \tag{28}$$

To have an acceptable continuum limit, it suffices that

$$g_{0,r} = N'^{(2r-1)}\,(qa)^{(s-1)r-s}\,g_r\ , \tag{29}$$

where g_r may be called the physical coupling factor. Moreover, it is noteworthy that $Z^r\,g_{0,r} = [N'(qa)^s]^{-1}\,g_r$, for all values of r, which for a finite spatial volume $V' = N'a^s$ leads to a finite nonzero result for $Z^r\,g_{0,r}$. It should not be a surprise that there are no divergences for all such interactions because the source of all divergences has been neutralized!

We may specialize the general result established above to the two cases of interest to us. Namely, when $r = 1$ this last relation implies that $m_0^2 = N'(qa)^{-1}\,m^2$, while when $r = 2$, it follows that $\lambda_0 = N'^3(qa)^{s-2}\,\lambda$. In these cases it also follows that $Zm_0^2 = [N'(qa)^s]^{-1}m^2$ and $Z^2\lambda_0 = [N'(qa)^s]^{-1}\,\lambda$, which for a finite spatial volume $V' = N'a^s$ leads to a finite nonzero results for Zm_0^2 and $Z^2\lambda_0$, respectively.

Conclusion

For scalar nonrenormalizable quantum field models, we have shown that the choice of a nonconventional counterterm, but one that is still nonclassical, leads to a formulation for which a perturbation analysis of both the mass term and the nonlinear interaction term, expanded about the appropriate pseudofree model, are term-by-term finite. Alternative insight into such models may possibly be obtained by Monte Carlo studies of the full, nonperturbative model including the special counterterm; for a preliminary discussion of such an approach, see [4].

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