DISPERSION FOR SCHRÖDINGER EQUATION WITH PERIODIC POTENTIAL IN 1D

Scipio Cuccagna

Abstract. We extend a result on dispersion for solutions of the linear Schrödinger equation, proved by Firsova for operators with finitely many energy bands only, to the case of smooth potentials in 1D with infinitely many bands. The proof consists in an application of the method of stationary phase. Estimates for the phases, essentially the band functions, follow from work by Korotyaev. Most of the paper is devoted to bounds for the Bloch functions. For these bounds we need a detailed analysis of the quasimomentum function and the uniformization of the inverse of the quasimomentum function.

§1 Introduction

We consider an operator $H_0 = -\frac{d^2}{dx^2} + P(x)$ with $P(x)$ a smooth periodic potential of period 1. We will prove:

Main Theorem. *There is a $C > 0$ such that for any $p \geq 2$ and $t > 0$ we have* $\| e^{itH_0} \colon L^{\frac{p}{p-1}}(\mathbb{R}) \to L^p(\mathbb{R}) \| \leq C \max \left\{ t^{-\frac{1}{2}}, t^{-\frac{1}{3}} \right\}^{(1-\frac{2}{p})}$.

We recall that the spectrum $\Sigma(H_0)$ is a union of closed intervals (bands) and that the theorem in the case of finitely many bands is in Firsova [F1], for alternative proofs in the two bands case see [Cai,Cu]. Here we will consider the case when $\Sigma(H_0)$ is a union of infinitely many bands. Just to simplify the notation, we will formulate the proof only in the generic case when all the gaps are nonempty. This generic case contains all the essential difficulties, and a proof for cases when some gaps are empty goes through similarly, only with more complicated notation. We are motivated by nonlinear problems, see [Cu]. Indeed dispersion for linear operators is central to nonlinear equations, see for instance [Str]. For $P \equiv 0$ we have $e^{-itd^2/dx^2}(x,y) = (4\pi it)^{-\frac{1}{2}} e^{-i(x-y)^2/4t}$. For no nonconstant $P(x)$ a similar explicit formula seems to be available. We outline the proof, with terminology and formulas introduced rigourously later. The proof of the Main Theorem reduces to a pointwise bound on the kernel $e^{itH_0}(x,y)$. The kernel is expressed by means of the distorted Fourier transform, written using Bloch functions and is a sum of oscillatory integrals, one term per energy band. In each integral the phase involves the band functions, which express energy E with respect to quasimomentum k. k varies in $\mathbb{C}$ cut along some slits which correspond to the gaps. We derive various estimates for the derivatives of the band function $E(k)$ from the representation of the quasimomentum function $k = k(w)$, $w = \sqrt{E}$, exploiting formulas in Korotyaev [K1]. We have essentially $E(k) = k^2$, except for very narrow regions near the edges of the spectral bands where, in particular, the third derivative of $E(k)$ is very large. In this region very close the edges our bounds on the Bloch functions are large, but in the stationary phase formula are more than offset by the very small upper bound for the inverse of the third derivative of $E(k)$. Away from the edges, the contribution to $e^{itH_0}(x, y)$ is essentially the same of the constant coefficients case and there are good bounds for the Bloch functions. There is an intermediate region, close but not very close, in a relative sense, to the edges, where our bounds for the Bloch functions are large and where the third derivative of $E(k)$ is not large. Yet, combining the narrowness of this region with good enough bounds for the Bloch functions, we control the corresponding contribution to $e^{itH_0}(x, y)$. Since estimates for $E(k)$ follow by fairly direct elaboration on material by Korotyaev [K1], our most serious effort is for bounds on the Bloch functions and their first derivative in k. It is probably because of these that [F1] considers finitely many bands only. The Bloch functions can be expressed as a linear combination of a nice fundamental set of solutions of the equation $H_0 u = Eu$. The coefficient in this linear combination is the Weyl-Titchmarsh function, which near the edges of the bands of higher energy is hard to bound, because it is the ratio of two very small quantities. In the finite bands case, Firsova [F1] uses the fact that the Bloch functions are analytic on the uniformization of the function $w(k)$. Since there are only finitely many bands, this gives a uniform bound. Bounds for the derivatives follow from the Cauchy integral formula. In the case of infinitely many bands, near each edge the Bloch functions are bounded, but there is no obvious uniform bound over an infinite number of bands. Furthermore, the distance of the edges from the boundary of the domain of analyticity goes to 0 as we take higher energies. Hence, for large energy, the Cauchy integral formula gives bad estimates on the k derivative of the Bloch functions, near the edges of the band. This problem for the derivative is the main source of trouble in the paper. Taking Fourier expansion, if one differentiates in k one gets a small divisors problem, with small divisors for modes $n = 0$ and for $\pm n\pi + k \approx 0$. Is seems convenient to use the Cauchy integral formula with bounds for complex k. The Bloch functions are of the form $e^{\pm ikx} m_\pm(x, k)$. Thanks to estimates on a nice pair of fundamental solutions of $H_0 u = Eu$ and relations between the various terms in the Weyl-Titchmarsh function and a certain normalization term denoted by $N(k)$, we see that for large energies in the Fourier series expansion $m_\pm(x, k) = \sum \widehat{m}_\pm(n, k) e^{2\pi inx}$ most terms are small compared to $\widehat{m}_\pm(0, k)$ and $\widehat{m}_\pm(\pm n, k)$ with $|\pi n + k| \ll 1$ and $\|m_\pm(\cdot, k)\|_{L^\infty} \approx \|m_\pm(\cdot, k)\|_{L^2}$. For

$k \in \mathbb{R}$ the Bloch functions are normalized, $\|m_\pm(\cdot,k)\|_{L^2} = 1$. However for $k \notin \mathbb{R}$ and near the boundary of the domain of analyticity, that is near the slits, it is problematic to bound $\|m_\pm(\cdot,k)\|_{L^2}$. In correspondence to the interior of the band $|\widehat{m}_\pm(0,k)| \gg |\widehat{m}_\pm(\mp n,k)|$. Using the normalization of the Bloch functions we get

$$1 \approx \widehat{m}_+(0,k)\widehat{m}_-(0,k) + \widehat{m}_+(-n,k)\widehat{m}_-(n,k), \tag{1.1}$$

so $\|m_+(\cdot,k)\|_{L^2}\|m_-(\cdot,k)\|_{L^2} \lesssim 1$. Near the slits however $|\widehat{m}_\pm(0,k)| \approx |\widehat{m}_\pm(\mp n,k)|$ and in the right in (1.1) we could have a cancelation. However from some explicit formula and thanks to (9.1) we get $\widehat{m}_+(0,k)\widehat{m}_-(0,k) - \widehat{m}_+(-n,k)\widehat{m}_-(n,k) \approx \dot{E}/(2k)$. By the Schwartz Christoffel formula we get a good bound for this quantity for k close enough to the real axis. Hence for k close enough to the real axis in (1.1) there is no cancelation and we get a uniform bound on $\|m_+(\cdot,k)\|_{L^2}\|m_-(\cdot,k)\|_{L^2}$. The argument shows however that near the extremes of the slits the two terms in (1.1) are in fact unbounded. Away from the edges of the bands we improve the estimates significantly thanks to more information on $E(k)$.

One can relax significantly the regularity requirements on $P(x)$ maintaining the proof. The proof goes from §4 to §10. In particular, the estimates on the Bloch functions are in §9 and §10.

Here the spectrum is denoted by $\Sigma(H_0)$, with the usual notation $\sigma(H_0)$ reserved for something else. In a statement or in a proof, notation $a_n \lesssim b_n$ means that there is a fixed constant $C > 0$ independent of n, with $a_n \le C b_n$. If we write $a_n \ll b_n$ we mean $a_n \le C b_n$ for a very small fixed constant $C > 0$. If we write $a_n \approx b_n$ we mean $(1/C)b_n \le a_n \le C b_n$ for a fixed constant $C > 0$ independent of n. If we write $a_n = O(b_n)$ (resp. $a_n = o(b_n)$) we mean $a_n \lesssim b_n$ (resp. $a_n \ll b_n$). For $p \in [1,\infty]$, by $\|f\|_p$ we mean the usual L^p norm of $f(x)$, where the x varies in a set indicated in the context.

§2 Band function, Bloch function, quasimomentum and uniformization

The spectrum is of the form $\Sigma(H_0) = \cup_{n=0}^\infty \Sigma_n$, with each two compact intervals Σ_n and Σ_{n+1} separated by an open gap G_n. We assume $\inf \Sigma_0 = 0$. We set now $\sigma = \cup_{n=-\infty}^\infty \sigma_n$, with each two compact intervals σ_n and σ_{n+1} separated by an open gap g_n, with g_0 empty, and with $\sigma_{-n} = -\sigma_n$ and $\sigma_n^2 = \Sigma_n$. We will assume each g_n non empty for $n \neq 0$. We recall now the following standard result, see [Ea] ch. 4:

Theorem 2.1. *Let $P(x)$ be smooth. Set $\sigma_n = [a_n^+, a_{n+1}^-]$ and $g_n =]a_n^-, a_n^+[$. Then there exist a strictly increasing sequence $\{\ell_n \in \mathbb{Z}\}_{n\in\mathbb{Z}}$ and a fixed constant C such that*

$$|a_n^- - \ell_n\pi| + |a_n^+ - \ell_n\pi| \le C\langle \ell_n\rangle^{-1}.$$

For any N there exists a fixed constant C_N such that the length $|g_n|$ of the gap g_n is $|g_n| \le C_N \langle \ell_n\rangle^{-N}$.

To simplify notation we will assume in the rest of the paper that $\ell_n \equiv n$, the case when all spectral gaps, from a certain one on, are not empty, which is generic, but essentially the same proof goes through in general. For any $w \in \mathbb{C}_+$ (the open upper half plane) we consider the fundamental solutions $\theta(x, w)$ and $\varphi(x, w)$ of $H_0 u = w^2 u$ which satisfy the initial conditions

$$\varphi(0, w) = \theta'(0, w) = 0\,, \quad \varphi'(0, w) = \theta(0, w) = 1. \tag{2.1}$$

The Floquet determinant $D(w)$ is defined by $2D(w) = \varphi'(w) + \theta(w)$ where $\varphi'(w) = \varphi'(1, w)$ and $\theta(w) = \theta(1, w)$. For any $w \in \mathbb{C}_+$ there is a unique $k \in \mathbb{C}_+$, called quasimomentum, and a unique choice of constants $m^{\pm}(w)$ such that the functions

$$\tilde{\phi}_{\pm}(x, w) = \theta(x, w) + m^{\pm}(w)\varphi(x, w) \tag{2.2}$$

are of the form $\tilde{\phi}_{\pm}(x, w) = e^{\pm ikx}\xi_{\pm}(x, w)$ with $\xi_{\pm}(x, w)$ periodic of period 1 in x. We have

$$m^{\pm}(w) = \frac{\varphi'(w) - \theta(w)}{2\varphi(w)} \pm i\frac{\sin k}{\varphi(w)}. \tag{2.3}$$

We have the relation $D(w) = \cos k$. The correspondence between w and the corresponding quasimomentum k is a conformal mapping between $\mathbb{C}_+$ and a "comb" K, that is a set $K = \mathbb{C}_+ \setminus \cup_{n\neq 0} [n\pi, n\pi + ih_n]$ where the $[n\pi, n\pi + ih_n]$ are vertical slits with $h_n \geq 0$. In particular, $|g_n| \leq 2h_n \leq (1 + Cn^{-2})|g_n|$ for a fixed C, see Theorem 1.2 [KK]. Now we will use that all gaps are nonempty, but the following standard discussion extends easily. The map $k(w)$ is called quasimomentum map and extends into a continuous map in $\overline{\mathbb{C}_+}$ with $k(\sigma_n) = [n\pi, (n + 1)\pi]$, with $k(w)$ a one to one and onto map between σ_n and $[n\pi, (n + 1)\pi]$, and with $k(g_n) =]n\pi, n\pi + ih_n]$. We have $k(-\bar{w}) = -\overline{k(w)}$. By the Schwartz reflection principle, $k(w)$ extends into a conformal map from $\mathbb{C} \setminus \cup_{n\neq 0} \overline{g_n}$ into $\mathcal{K} = \mathbb{C} \setminus \cup_n \gamma_n$ with $\gamma_n = [n\pi - ih_n, n\pi + ih_n]$. So we have $k(\bar{w}) = \overline{k(w)}$ and $k(w) = -k(-w)$. Hence also $w(\bar{k}) = \overline{w(k)}$ and $w(k) = -w(-k)$. Then for $|t| < h_n$ we have $w(n\pi + it \pm 0) = -\overline{w(-n\pi + it \mp 0)} = -w(-n\pi + it \mp 0)$. This means that the band function $E(k) = w^2(k)$ extends in an analytic map with values in $\mathbb{C}$ and with domain the Riemann surface $\mathcal{R}$ obtained identifying $n\pi + it \pm 0$ and $-n\pi + it \mp 0$ for each n and for each $|t| < h_n$.

For $w \in \mathbb{C}^+$ we have introduced $\tilde{\phi}_{\pm}(x, w) = e^{\pm ik(w)x}\xi_{\pm}(x, w)$ with $\xi_{\pm}(x, w)$ periodic. These functions extend by continuity to $w \in \mathbb{R}$. For $w \in \mathbb{C}^+$, we have by the properties of $k(w)$ and by the definition of $\overline{\tilde{\phi}_{\pm}(x, w)}$,

$$\overline{\tilde{\phi}_{\pm}(x, w)} = e^{\mp i\overline{k(w)}x}\overline{\xi_{\pm}(x, w)} = e^{\pm ik(-\overline{w})x}\overline{\xi_{\pm}(x, w)} = \tilde{\phi}_{\pm}(x, -\overline{w}) \tag{2.4}$$

and, for $w \in \sigma$, we have, for the same reasons,

$$\overline{\tilde{\phi}_\pm(x,w)} = e^{\mp ik(\overline{w})x}\overline{\xi_\pm(x,w)} = e^{\mp ik(\overline{w})x}\xi_\mp(x,\overline{w}) = \tilde{\phi}_\mp(x,\overline{w}), \tag{2.5}$$

$$\tilde{\phi}_\pm(x,w) = e^{\pm ik(w)x}\xi_\pm(x,w) = e^{\mp ik(-w)x}\xi_\pm(x,w) = \tilde{\phi}_\mp(x,-w). \tag{2.6}$$

By (2.6) the function $\tilde{\phi}_\pm(x,w)$ can be extended across σ into analytic functions in $w \in \mathbb{C}\setminus \cup_{n\neq 0}\overline{g_n}$ setting $\tilde{\phi}_\pm(x,w) = \tilde{\phi}_\mp(x,-w)$. It is elementary to see that (2.5) and (2.6) are now true for any $k \in \mathcal{K} = \mathbb{C}\setminus \cup_{n\neq 0}\gamma_n$. Set now

$$N^2(w) = \int_0^1 \tilde{\phi}_+(x,w)\tilde{\phi}_-(x,w)dx.$$

By (2.5) we have $N^2(w) = \int_0^1 \left|\tilde{\phi}_\pm(x,w)\right|^2 dx > 0$ for $w \in \sigma$ (so that we define $N(w) > 0$ for $w \in \sigma$). $N^2(w)$ is well defined and analytic in $\mathbb{C} - \cup_{\neq 0} g_n$. We have $N^2(w) = N^2(-w) = \overline{N^2(\overline{w})}$, the first equality by (2.6) and the second by (2.5). From formula (1.4) [F2] we have $D'(w) = -4w\varphi(w)N^2(w)$, for a sketch of proof see §3 [Cu]. From $D(w) = \cos k$ we get $D'(w) = -\frac{dk}{dw}\sin k$. Since $k(w)$ is a conformal map, $\frac{dk}{dw} \neq 0$ for $\Im w > 0$ and so for $k \in K$. Hence $N^2(w) \neq 0$ for any $w \in \mathbb{C}\setminus \cup_{n\neq 0}\overline{g_n}$. We set now

$$e^{ik(x-y)}m_+^0(x,w)m_-^0(y,w) = \frac{\tilde{\phi}_+(x,w)\tilde{\phi}_-(y,w)}{N^2(w)}. \tag{2.7}$$

We express $w = w(k)$ for $k \in K$ and with an abuse of notation we write $m_\pm^0(x,k)$ for $m_\pm^0(x,w(k))$. Then the product $m_+^0(x,k)m_-^0(y,k)$ extends analytically for $k \in \mathcal{K}$ and to $\mathcal{R}$. Fix a square root N of N^2 with $N(k) > 0$ for $k \in \mathbb{R}$, ignoring problems of monodromy (here we need the product $m_+^0 m_-^0$ to be analytic). We set $m_\pm^0(x,k) = \xi_\pm^0(x,k)/N(k)$.

§3 Fourier transform

From Theorem XIII.98 [RS] it is possible to conclude:

Lemma 3.1. *Let $N(k) = \sqrt{N^2(k)} > 0$ for $k \in \mathbb{R}$. Set $\phi_\pm(y,k) = \widetilde{\phi}_\pm(y,k)/N(k)$ and $\hat{f}(k) = \int_{\mathbb{R}} \phi_+(y,k)f(y)dy$. Then:*

$$\int_{\mathbb{R}} |f(y)|^2 dy = \int_{\mathbb{R}} |\hat{f}(k)|^2 dk \tag{a}$$

$$f(x) = \int_{\mathbb{R}} \phi_-(x,k)\hat{f}(k)dk \tag{b}$$

$$\widehat{H_0 f}(k) = E(k)\hat{f}(k). \tag{c}$$

Lemma 3.1 implies:

Lemma 3.2. *We have* $e^{itH_0}(x,y) = K(t,x,y) = \sum_{n\in\mathbb{Z}} K^n(t,x,y)$ *with*

$$K^n(t,x,y) = \int_{n\pi}^{(n+1)\pi} e^{i(tE(k)-(x-y)k)} m_-^0(x,k) m_+^0(y,k)dk. \tag{3.1}$$

Then the Main Theorem follows from:

Theorem 3.3. *There is* C *fixed such that* $|K(t,x,y)| \le C \max\{t^{-\frac{1}{3}}, t^{-\frac{1}{2}}\}$.

Theorem 3.3 follows by the method of stationary phase. Notice that the phase in $K^n(t,x,y)$ satisfies the following result by Korotyaev [K2]:

Theorem 3.4. *Consider* $E(k)$ *for* $k \in [\pi n, \pi(n+1)]$. *Then* $E'(k) = 0$ *for* $k = n\pi, (n+1)\pi$ *and* $E'(k) > 0$ *in* $]\pi n, \pi(n+1)[$ *for* $n \ge 0$ *(*$E(k)$ *is even). In* $[\pi n, \pi(n+1)]$ *the equation* $E''(k) = 0$ *admits exactly one solution* k_n. *We have* $k_n \in]\pi n, \pi(n+1)[$ *and* $E'''(k_n) \ne 0$.

Naively the proof of Theorem 3.3 would go as follows. Theorem 3.4 and estimates on $m_\pm^0(x,k)$ for $k \in [\pi n, \pi(n+1)]$ lead to an estimate $|K^n(t,x,y)| \le D_n \langle t\rangle^{-\frac{1}{3}}$ thanks to the method of stationary phase, which we quote from p. 334 [Ste]:

Lemma 3.5. *Suppose* $\phi(x)$ *is real valued and smooth in* $[a,b]$ *with* $|\phi^{(m)}(x)| \ge c_m > 0$ *in* $]a,b[$ *for* $m \ge 1$. *For* $m = 1$ *assume furthermore that* $\phi'(x)$ *is monotonic in* $]a,b[$. *Then we have for* $C_m = 5 \cdot 2^{m-1} - 2$:

$$\left| \int_a^b e^{i\mu\phi(x)} \psi(x) dx \right| \le C_m (c_m \mu)^{-\frac{1}{m}} \left[\min\{|\psi(a)|, |\psi(b)|\} + \int_a^b |\psi'(x)| dx \right].$$

Since we need to add up over all the K^n we have to control the constants D_n. In the next section we state a list of estimates on the band function $E(k)$ and on the Bloch functions $\phi_\pm(x,k)$ which are sufficient to the purpose of the present paper and then prove Theorem 3.3. In the subsequent sections we prove the estimates on $E(k)$ and $\phi_\pm(x,k)$.

§4 Proof of Theorem 3.3

We state estimates for the first three derivatives of $E(k)$. These are proved later using material in Korotyaev [K1]. Notice that $E(k)$ is even in k, so we consider only $k \ge 0$. We start with the first derivative $\dot{E}(k)$:

Lemma 4.1. *For all $k \ge 0$ we have $\dot{E}(k) \ge 0$. $\dot{E}(k) = 0$ implies $k = n\pi$ for some $n \in \mathbb{Z}$. There are fixed constants $C > 0$ and $c > 0$ such that for any n we have*

$$\frac{k}{C}\frac{\sqrt{w - a_n^+}}{\sqrt{|g_n|}} \le \dot{E}(k) \le Ck\frac{\sqrt{w - a_n^+}}{\sqrt{|g_n|}} \quad \text{for} \quad a_n^+ \le w \le a_n^+ + c|g_n|$$

$$|\dot{E}(k) - 2k| \le \frac{C}{\langle k \rangle} \quad \text{for} \quad a_n^+ + c|g_n| \le w \le a_{n+1}^- - c|g_{n+1}|$$

$$\frac{k}{C}\frac{\sqrt{a_{n+1}^- - w}}{\sqrt{|g_{n+1}|}} \le \dot{E}(k) \le kC\frac{\sqrt{a_{n+1}^- - w}}{\sqrt{|g_{n+1}|}} \quad \text{for} \quad a_{n+1}^- - c|g_{n+1}| \le w \le a_{n+1}^-.$$

Now we consider the second derivative $\ddot{E}(k)$:

Lemma 4.2. *There are fixed constants $C > 0$, $C_1 > C_2 > 0$ and $c > 0$ such that for any n and any $k \in [n\pi, (n+1)\pi]$, that is for any $w \in [a_n^+, a_{n+1}^-]$, we have:*

$$w \le a_n^+ + c|g_n| \Rightarrow \big|\ddot{E} - \frac{n}{|g_n|}\big| \le C$$

$$w \ge a_{n+1}^- - c|g_{n+1}| \Rightarrow \big|\ddot{E} + \frac{n+1}{|g_{n+1}|}\big| \le C$$

$$a_n^+ + c|g_n| \le w \le a_{n+1}^- - C_1|n+1|^{\frac{1}{3}}|g_{n+1}|^{\frac{2}{3}} \Rightarrow \ddot{E} \approx \frac{1}{2} + \frac{n|g_n|^2}{|w - a_n^+|^3}.$$

Finally we have (see Lemma 7.5):

Lemma 4.3. *There are $c > 0$ and $c_1 > 0$ such that $\forall\ \alpha \in (1/2, 1)$ and for any $w \in \sigma_n \cup \sigma_{n-1}$ with $c|g_n| \le |w - a_n^\pm| \le |g_n|^\alpha$ we have $|\ddot{E}| \ge c_1|n|\,|g_n|^{4(\frac{1}{2}-\alpha)}$.*

Next we need estimates for the Bloch functions. First of all we have, see in §9:

Lemma 4.4. *There are fixed constants $C > 0$, $C_3 > 0$, $\delta > 0$, $\Gamma > 0$ and $c > 0$ such that for all x, all n we have :*

(1) $\forall\, w \in [a_n^+ + C_3 n^5|g_n|, a_{n+1}^- - C_3(n+1)^5|g_{n+1}|]$ *we have*

$$\big|m_+^0(x,k)m_-^0(y,k) - 1\big| \le C\langle k \rangle^{-1};$$

(2) for all $k = p + iq$ with $k \in \mathcal{K}$, $\pi n < p < \pi(n+1)$, k in $\{k : |q| < \delta|g_n|\} \cup \{k : |\pi n - p| > \Gamma|g_n|\} \cup \{k : |\pi(n+1) - p| > \Gamma|g_{n+1}|\}$ and $|q| \le 1$ we have

$$\big|m_+^0(x,k)m_-^0(y,k)\big| \le C.$$

The proof of (1) Lemma 4.4 is elementary and that of (2) if $k \in \mathbb{R}$, that is the case near the edges not covered by (1), is relatively easy. We will state later more estimates for $k \notin \mathbb{R}$, needed to bound $\dot{m}_\pm^0(x,k) = \partial_k m_\pm^0(x,k)$. We have:

Lemma 4.5. *There are fixed constants $C>0$ and $C_4>0$, with $C_4<C_2$, C_2 the constant in Lemma 4.3, such that for all x, all n and for $v=0$ we have :*

for $a_n^+ + |g_n|^{\frac14} \le u \le \frac{a_n^+ + a_{n+1}^-}{2}$, there is a C such that for the corresponding $k=p+i0$ we have $\left|\partial_k(m_-^0(x,k)m_+^0(y,k))\right| \le \frac{C}{k|k-\pi n|}$;

if $\frac{a_n^+ + a_{n+1}^-}{2} \le u \le a_{n+1}^- - |g_{n+1}|^{\frac35}$ then $\left|\partial_k(m_-^0(x,k)m_+^0(y,k))\right| \le \frac{C}{k|k-\pi(n+1)|}$;

for all w in the remaining part of $[a_n^+, a_{n+1}^-]$ we have for a fixed C and with $m=n$ (resp. $m=n+1$) near a_n^+ (resp. a_{n+1}^-)

$$\left|\partial_k(m_-^0(x,k)m_+^0(y,k))\right| \le C\left(|k-\pi m|+|g_m|\right)^{-1}.$$

We assume now the above lemmas and go ahead with Theorem 3.3.

Proof of Theorem 3.3. Recall $K^n(t,x,y)$ given by (3.1). By the discussion in §3, the only interesting case is when $|n|\gg 1$. Here we will sum over $n\gg 1$, the proof of the $n\ll -1$ being similar. We fix a constant $c\gg 1$. For $n\gg 1$ we partition

$$\begin{aligned}&\left[a_n^+, a_{n+1}^-\right] = \left[a_n^+, a_n^+ + c|g_n|\right] \cup \left[a_n^+ + c|g_n|, a_n^+ + |g_n|^{\frac14}\right] \cup \\ &\left[a_n^+ + |g_n|^{\frac14}, a_{n+1}^- - |g_{n+1}|^{\frac35}\right] \cup \left[a_{n+1}^- - |g_{n+1}|^{\frac35}, a_{n+1}^- - c|g_{n+1}|\right] \\ &\cup \left[a_{n+1}^- - c|g_{n+1}|, a_{n+1}^-\right].\end{aligned}$$

Here $c\gg 1$ so that for $w\in[a_n^+ + c|g_n|, a_{n+1}^- - c|g_{n+1}|]$ we have $\dot w \approx 1$. For $w\in[a_n^+ + c|g_n|^{\frac14}, a_{n+1}^- - |g_{n+1}|^{\frac35}|]$ we have $\ddot E\approx 1$. We introduce a smooth, even, compactly supported cutoff $\chi_0(t)\in[0,1]$ with $\chi_0\equiv 1$ near 0 and $\chi_0\equiv 0$ for $t\ge 2/3$. Set $\chi_1 = 1-\chi_0$. We split each $K^n=\sum_1^5 K_\ell^n$ partitioning the identity in $\sigma_n=[a_n^+, a_{n+1}^-]$

$$\begin{aligned}&1_{\sigma_n}(w) = \chi_0\Big(\frac{w-a_n^+}{c|g_n|}\Big) + \chi_1\Big(\frac{w-a_n^+}{c|g_n|}\Big)\chi_0\Big(\frac{w-a_n^+}{|g_n|^{\frac14}}\Big) + \\ &+\chi_1\Big(\frac{w-a_n^+}{|g_n|^{\frac14}}\Big)\chi_1\Big(\frac{a_{n+1}^- - w}{|g_{n+1}|^{\frac35}}\Big) + \chi_0\Big(\frac{a_{n+1}^- - w}{|g_{n+1}|^{\frac35}}\Big)\chi_1\Big(\frac{a_{n+1}^- - w}{c|g_{n+1}|}\Big) + \chi_0\Big(\frac{a_{n+1}^- - w}{c|g_{n+1}|}\Big).\end{aligned}$$

We bound one by one the K_ℓ^n.

Claim $\ell=1$ and $\ell=5$. *For any $\epsilon>0$ there is a fixed C_ϵ such that $|K_1^n|\le C_\epsilon t^{-\frac12}|g_n|^{\frac12-\epsilon}$ and $|K_5^n|\le C_\epsilon t^{-\frac12}|g_{n+1}|^{\frac12-\epsilon}$.*

Proof of $\ell=1$. By Lemma 3.5, by $\ddot E \ge \frac{|n|}{2|g_n|}$, Lemma 4.1, and by Lemmas 4.4 and 4.5,

$$|K_1^n(t,x,y)| \le \frac{C\sqrt{|g_n|}}{\sqrt{|n|t}} \int_{a_n^+}^{a_n^+ + c|g_n|} \frac{\frac{dk}{dw}dw}{|k-\pi n| + |g_n|}.$$

By Lemma 4.1, $\frac{dk}{dw} \approx \frac{\sqrt{|g_n|}}{\sqrt{w-a_n^+}}$ and $|k-\pi n| \approx \sqrt{|g_n|}\sqrt{w-a_n^+}$. Hence

$$|K_1^n(t,x,y)| \le \frac{C}{\sqrt{|n|t}} \int_{a_n^+}^{a_n^+ + c|g_n|} \frac{dw}{\sqrt{w-a_n^+}} \le \frac{C_1\sqrt{|g_n|}}{\sqrt{|n|t}}.$$

With a similar argument we get the estimate for K_5.

Claim $\ell = 2$. *There is $C > 0$ such that $|K_2^n| \le C \min\{t^{-\frac{1}{2}} \log(1/|g_n|), |g_n|^{\frac{1}{4}}\}$.*

Proof. For $k \in [a_n^+ + c|g_n|, a_n^+ + |g_n|^{\frac{1}{4}}]$ we have $|\ddot{E}| \gtrsim 1$ by Lemma 4.2 and $\frac{dk}{dw} \approx 1$ and $w - a_n^+ \approx k - \pi n$ by Lemma 4.1. Hence $|K_2^n(t,x,y)| \le$

$$\le \frac{C_1}{\sqrt{t}} \int_{a_n^+ + c|g_n|}^{a_n^+ + |g_n|^{\frac{1}{4}}} \frac{\frac{dk}{dw}dw}{|k-\pi n| + |g_n|} \le Ct^{-\frac{1}{2}} \int_{a_n^+ + c|g_n|}^{a_n^+ + |g_n|^{\frac{1}{4}}} \frac{dw}{w - a_n^+} \le At^{-\frac{1}{2}} \log \frac{1}{|g_n|}.$$

On the other hand, taking absolute value in the integral defining K_2^n, thanks to Lemma 4.4 we get $|K_2^n| \le C|g_n|^{\frac{1}{4}}$.

Claim $\ell = 4$. *There is C such that $|K_4^n| \le Ct^{-\frac{1}{3}}|g_{n+1}|^{\frac{1}{31}}$.*

Proof. K_4^n is an integral on an interval where $|\dddot{E}| \gtrsim |n+1||g_{n+1}|^{-\frac{1}{10}}$ by Lemma 4.3. $\frac{dk}{dw} \approx 1$ and $a_{n+1}^- - w \approx \pi(n+1) - k$ by Lemma 4.1. So, by Lemmas 3.5, 4.4 and 4.5, we get a contribution bounded by

$$t^{-\frac{1}{3}}|n+1|^{\frac{1}{3}}|g_{n+1}|^{\frac{1}{30}} \int_{a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}}^{a_{n+1}^- - c|g_{n+1}|} \frac{dw}{a_{n+1}^- - w} \lesssim t^{-\frac{1}{3}}|n+1|^{\frac{1}{3}}|g_{n+1}|^{\frac{1}{30}} \log \frac{1}{|g_{n+1}|}.$$

We get $\sum_n \sum_{\ell \ne 3, \ell=1}^{5} |K_\ell^n(t,x,y)| \le C \max\{t^{-\frac{1}{3}}, t^{-\frac{1}{2}}\}$ by the above claims and by Lemma 2.1. Finally we consider K_3^n. Set $K_3 = \sum_n K_3^n$.

Lemma 4.6. *There is a fixed C such that $|K_3(t,x,y)| \le C \max\{t^{-\frac{1}{2}}, t^{-\frac{1}{3}}\}$.*

Proof. Set $\chi_{int}(k) = \sum_n \chi_1(\frac{w-a_n^+}{|g_n|^{\frac{1}{4}}})\chi_1(\frac{a_{n+1}^- - w}{|g_{n+1}|^{\frac{3}{5}}})$. Then

$$K_3(t,x,y) = \int_{\mathbb{R}} e^{i(tE(k)-(x-y)k)} \chi_{int}(k) m_-^0(x,k) m_+^0(y,k) dk.$$

We will use the following lemma:

Lemma 4.7. *In the support of χ_{int} we have $w \approx k$, $\dot{w} = 1 + O(k^{-2})$, $\ddot{w} = O(k^{-3})$. Furthermore, we can extend w from the support of χ_{int} to the whole of $\mathbb{R}$ so that the extension (which we denote again with w) satisfies the same relations*

$w \approx k$ by Lemma 5.2, $\dot{w} = 1 + O(k^{-2})$ by Lemma 7.1, $\ddot{w} = O(k^{-3})$ by Lemma 7.4 and the last statement is Lemma 7.6. Now assume Lemma 4.7. We have $E = w^2$, $\dot{E} = 2w\dot{w}$ and $\ddot{E} = 2\dot{w}^2 - 2w\ddot{w}$. So by Lemma 4.7 we can extend $E(k)$ from the support of χ_{int} into a function defined on all $\mathbb{R}$ convex with $\dot{E} \approx k$, $\ddot{E} \approx 1$. We express

$$m_-^0(x,k)m_+^0(y,k) = \big(m_-^0(x,k)m_+^0(y,k) - 1\big) + 1. \tag{1}$$

Let k_0 be the unique solution of $\dot{\Phi}(k) = 0$. Then set $q^2/2 = \Phi(k) - \Phi(k_0) = \frac{1}{2}\ddot{\Phi}(\tilde{k})(k-k_0)^2$. Since $1/C \le \ddot{\Phi} \le C$ then $\frac{1}{\sqrt{C}} \le \frac{q}{k-k_0} \le \sqrt{C}$. From $q\dot{q} = \dot{\Phi}(k) = \ddot{\Phi}(k_1)(k-k_0)$ we conclude that for some C we have $1/C \le \dot{q} \le C$. Now we insert (1) in the definition of K_3 obtaining $K_3 = H_1 + H_2$ with

$$H_1(t,x,y) = e^{i(tE(k_0)-(x-y)k_0)} \int_{\mathbb{R}} e^{itq^2}\chi_{int}(k)\left(m_-^0(x,k)m_+^0(y,k) - 1\right)\frac{dk}{dq}dq$$
$$H_2(t,x,y) = e^{i(tE(k_0)-(x-y)k_0)} \int_{\mathbb{R}} e^{itq^2}\chi_{int}(k)\frac{dk}{dq}dq.$$

By $\|\widehat{\rho}\|_1 \le \tilde{C}_\varepsilon \|\rho\|_{H^{\frac{1}{2}+\varepsilon}}$, in what follows we can use

$$\Big|\int_{\mathbb{R}} e^{itq^2}\rho(q)dq\Big| \le C_\varepsilon t^{-\frac{1}{2}}\|\rho\|_{H^{\frac{1}{2}+\varepsilon}}. \tag{4.1}$$

Lemma 4.8. *For a fixed C we have $\big|H_2(t,x,y)\big| \le Ct^{-\frac{1}{2}}$.*

Proof. We write $\chi_{int}(k) = 1 - (1-\chi_{int}(k))$. Let $\zeta(k)$ be either 1 or $-(1-\chi_{int}(k))$. For $\chi(t)$ a cutoff supported near $t = 0$, we insert the partition of unity $\chi(k-k_0) + (1-\chi(k-k_0))$inside $\int e^{i(tE(k)-(x-y)k)}\zeta(k)dk$. Then by $\|\zeta(k)\|_\infty + \|\zeta'(k)\|_1 \lesssim 1$ and by Lemmas 3.5 and 4.2, for a fixed C

$$\Big|\int e^{i(tE(k)-(x-y)k)}\zeta(k)\chi(k-k_0)dk\Big| \le Ct^{-\frac{1}{2}}.$$

Next we want to bound

$$\Big|\int e^{it\Phi(k)}\zeta(k)(1-\chi(k-k_0))dk\Big| = \Big|\int e^{i\frac{t}{2}q^2}\zeta(k)(1-\chi(k-k_0))\frac{dk}{dq}dq\Big|. \tag{2}$$

Let first $\zeta(k) \equiv 1$. Then $\int e^{i\frac{t}{2}q^2}\zeta(k)(1-\chi(k-k_0))\frac{dk}{dq}dq =$

$$= \int e^{i\frac{t}{2}q^2}(1-\chi(k-k_0))dq - \int e^{i\frac{t}{2}q^2}(1-\chi(k-k_0))\left(1-\frac{dk}{dq}\right)dq.$$

By standard arguments the first term in $O(t^{-\frac{1}{2}})$. By (4.1), with $\varepsilon = 1/2$, the second term is also $O(t^{-\frac{1}{2}})$ thanks to the following lemma:

Lemma 4.9. *$\forall\, a>0\ \exists$ a fixed constant C_a such that $\|1-\frac{dk}{dq}\|_{H^1(\{|q|\ge a\}}\le C_a$.*

Proof. We set $\dot q=\frac{dq}{dk}$. We have $2q\dot q=\dot E(k)-\dot E(k_0)=2(k-k_0)+O(\langle k\rangle^{-1})+O(\langle k_0\rangle^{-1})$ by Lemma 4.1. Hence after integration

$$\frac{q^2}{(k-k_0)^2}=1+\frac{O(\langle k_0\rangle^{-1})}{k-k_0}+\frac{O(\log(\langle k/k_0\rangle))}{(k-k_0)^2}. \tag{3}$$

We know from $q\approx k-k_0$ that (3) is uniformly bounded for $|k-k_0|\le 1$. Hence we conclude

$$1-\frac{q}{k-k_0}=\left(1+\frac{q}{k-k_0}\right)^{-1}\left(1-\frac{q^2}{(k-k_0)^2}\right)$$

is in $L^2(\mathbb{R})$ with norm independent from k_0. By $q\dot q=k-k_0+O(\langle k\rangle^{-1})+O(\langle k_0\rangle^{-1})$,

$$\dot q=1+\left(1-\frac{q}{k-k_0}\right)\dot q+\frac{O(\langle k\rangle^{-1})+O(\langle k_0\rangle^{-1})}{k-k_0}.$$

Since we know $\dot q$ is uniformly bounded, we conclude that for a fixed constant C we have $\|1-\dot q\|_2\le C$. Next, since in $\ddot q=\frac{\ddot E-\dot q^2}{q}$ the numerator is bounded, we see that $\ddot q\in L^2(\{|q|\ge 1\})$. By $\frac{d^2k}{dq^2}=-(\dot q)^{-3}\ddot q$ we obtain the desired result.

To complete the proof of Lemma 4.8, we have to show that (2) is $O(t^{-\frac12})$ when $\zeta(k)=-(1-\chi_{int}(k))$. We have $\|\zeta\|_1\lesssim 1$, so for $t\lesssim 1$ we have (2) $=O(1)$. We suppose now $t\gg 1$. In (2) split $\int_{\mathbb{R}}=\int_{|k|\le\sqrt t}+\int_{|k|\ge\sqrt t}$. By Lemma 2.1, for any N $\int_{|k|\ge\sqrt t}=O(\sum_{|n|\gtrsim\sqrt t}|g_n|^{\frac14})\ll t^{-N}$. We have

$$\int_{|k|\le\sqrt t}e^{it\Phi(k)}\zeta(k)(1-\chi(k-k_0))dk=O(t^{-1}\sum_{|n|\lesssim\sqrt t}(1+1))\approx t^{-\frac12},$$

where the terms 1 are of the form $|g_n|^{-\frac14}\int_{a_n^+}^{a_n^++|g_n|^{\frac14}}dk$ with $|g_n|^{-\frac14}\gtrsim|\zeta'|$ in $[a_n^+,a_n^++|g_n|^{\frac14}]$ and $|g_{n+1}|^{-\frac35}\int_{a_{n+1}^--|g_{n+1}|^{\frac35}}^{a_{n+1}^-}dk$ with $|g_{n+1}|^{-\frac35}\gtrsim|\zeta'|$ in $[a_{n+1}^--|g_{n+1}|^{\frac35},a_{n+1}^-]$.

Lemma 4.10. *There is a fixed C such that $|H_1(t,x,y)|\le C\max\{t^{-\frac12},t^{-\frac13}\}$.*

Proof. We split $H_1(t,x,y)=H_{11}(t,x,y)+H_{12}(t,x,y)$ with $H_{11}(t,x,y)$ defined inserting the additional factor $\widetilde\chi_{int}(k)=\sum_n\chi_1\left(\frac{k-\pi n}{n^{\varepsilon-\frac12}}\right)\chi_1\left(\frac{(n+1)\pi-k}{(n+1)^{\varepsilon-\frac12}}\right)$ in the definition $H_1(t,x,y)$, and with $H_{12}(t,x,y)$ defined inserting $\widetilde\chi_{edge}=1-\widetilde\chi_{int}$. The integrals defining $H_{11}(t,x,y)$ are supported in $\pi n+cn^{\varepsilon-\frac12}\le k\le(n+1)\pi-c(n+1)^{\varepsilon-\frac12}$, that is in the interior of the bands, while the integrals defining $H_{12}(t,x,y)$ are supported near the edges.

Claim. *There is a fixed C such that $|H_{11}(t,x,y)| \le Ct^{-\frac12}$.*

Proof. We have $\chi_{int}\widetilde{\chi}_{int} = \widetilde{\chi}_{int}$ since $\widetilde{\chi}_{int}$ is the characteristic function of the union of $\pi n + cn^{\varepsilon-\frac12} \le k \le (n+1)\pi - c(n+1)^{\varepsilon-\frac12}$ smoothed and χ_{int} is the characteristic function of the union of $\pi n + |g_n|^{\frac14} \le k \le (n+1)\pi - |g_{n+1}|^{\frac35}$ smoothed. We split $H_{11}(t,x,y)$ in two pieces. For the first piece we have for $\chi(t)$ a cutoff supported near 0, by Lemma 3.5, by $\|\widetilde{\chi}_{int}\|_\infty + \|\widetilde{\chi}'_{int}\chi(k-k_0)\|_1 \lesssim 1$, $m^0_-(x,k)m^0_+(y,k) - 1 = O(k^{-1})$ and $\widetilde{\chi}_{int}(k)\partial_k\left(m^0_-(x,k)m^0_+(y,k)\right) = O(k^{-\frac12-\varepsilon})$,

$$|\int e^{it\Phi(k)}\widetilde{\chi}_{int}(k)\chi(k-k_0)\left(m^0_-(x,k)m^0_+(y,k)-1\right)dk| \le Ct^{-\frac12}.$$

Next we consider

$$\int e^{itq^2/2}\widetilde{\chi}_{int}(k)(1-\chi(k-k_0))\left(m^0_-(x,k)m^0_+(y,k)-1\right)\frac{dk}{dq}dq. \tag{1}$$

By Lemmas 4.5 and 4.9

$$\|\widetilde{\chi}_{int}(k)(1-\chi(k-k_0))\left(m^0_-(x,k)m^0_+(y,k)-1\right)\frac{dk}{dq}\|_{H^1} \le C\|\{\langle n\rangle^{-\frac12-\varepsilon}\}\|_{l^2(\mathbb{N})}.$$

We can apply (4.1) and bound (1) by $Ct^{-\frac12}$.

We consider $H_{12}(t,x,y) = \sum_n H^n_{12}(t,x,y)$, $H^n_{12}(t,x,y) = \int_{n\pi}^{(n+1)\pi} e^{it\Phi(k)}f(k)dk$

with $f(k) = \Psi_n(k)\left(m^0_-(x,k)m^0_+(y,k)-1\right)$ where

$$\Psi_n(k) = \chi_1\Big(\frac{w-a^+_n}{|g_n|^{\frac14}}\Big)\chi_1\Big(\frac{a^-_{n+1}-w}{|g_{n+1}|^{\frac35}}\Big)\chi_0\left(\frac{k-\pi n}{n^{\varepsilon-\frac12}}\right)\chi_0\left(\frac{(n+1)\pi-k}{(n+1)^{\varepsilon-\frac12}}\right)$$

and with $\Phi(k) = E(k) - E(k_0) - t^{-1}(x-y)(k-k_0)$. Observe that $\Psi_n(k) = \Psi_{n1}(k) + \Psi_{n2}(k)$ with $\Psi_{n1}(k)$ supported in $|g_n|^{\frac14} \lesssim k - \pi n \lesssim n^{\varepsilon-\frac12}$ and with $\Psi_{n2}(k)$ supported in $|g_{n+1}|^{\frac35} \gtrsim \pi(n+1) - k \gtrsim (n+1)^{\varepsilon-\frac12}$. Correspondingly write $f = f_1 + f_2$ and $H^n_{12} = H^{n1}_{12} + H^{n2}_{12}$.

Lemma 4.11. *For a fixed C and for $j=1,2$: $|H^{nj}_{12}(t,x,y)| \le C\langle t\rangle^{-\frac12}|\log t|^2$.*

Proof. We focus on H^{n1}_{12}, the proof for H^{n2}_{12} being almost the same. We have

$$H^{n1}_{12}(t,x,y) = \int_{n\pi}^{(n+\frac12)\pi} F'(k)f_1(k)dk \text{ with } F(k) = \int_{n\pi}^{k} e^{it\Phi(k')}dk'.$$

For H_{12}^{n2} the proof is the same but with $F(k) = \int_{(n+1)\pi}^{k} e^{it\Phi(k')}dk'$. We get

$$\begin{aligned}
&H_{12}^{n1}(t,x,y) = -H_{121}^{n1}(t,x,y) - H_{122}^{n1}(t,x,y) \text{ with}\\
&H_{121}^{n1}(t,x,y) = \int_{n\pi}^{(n+\frac{1}{2})\pi} \left(\int_{n\pi}^{k} e^{it\Phi(k')}dk' \right) \Psi'_{n1}(k) \left(m_-^0(x,k) m_+^0(y,k) - 1 \right) dk\\
&H_{122}^{n1}(t,x,y) = \int_{n\pi}^{(n+\frac{1}{2})\pi} \left(\int_{n\pi}^{k} e^{it\Phi(k')}dk' \right) \Psi_{n1}(k) \partial_k \left(m_-^0(x,k) m_+^0(y,k) - 1 \right) dk.
\end{aligned}$$

Claim. *For $|k_0 - n\pi| \geq 2\pi$ for a fixed C we have*

$$|H_{121}^{n1}(t,x,y)| \leq C \langle t(\pi n - k_0) \rangle^{-1} \langle n \rangle^{-1}.$$

Proof. Indeed we have $|F(k)| \leq C\langle t(\pi n - k_0)\rangle^{-1}$, $|m_-^0(x,k)m_+^0(y,k) - 1| \leq C\langle k\rangle^{-1}$ and $\|\Psi'_{n1}(k)\|_{L^1(\pi n, \pi(n+1))} \leq C$. In a similar fashion we obtain

Claim. *For $|k_0 - n\pi| \leq 2\pi$ for a fixed C we have*

$$|H_{121}^{n1}(t,x,y)| \leq C \langle t \rangle^{-\frac{1}{2}} \langle n \rangle^{-1}.$$

Next we use that there is a fixed C such that for any x_0 and any $t > 0$,

$$\int_{|x-x_0|\geq 1} \frac{dx}{\langle x \rangle \langle t(x - x_0) \rangle} \leq C \min \left\{ t^{-1}, |\log t| \right\}$$

to conclude that for a fixed C

$$\sum_n |H_{121}^{n1}(t,x,y)| \leq C \min \left\{ \langle t \rangle^{-\frac{1}{2}}, |\log t| \right\}.$$

We now consider $H_{122}^{n1}(t,x,y)$. We start by assuming $|k_0 - n\pi| \geq 2\pi$. Then notice that for a fixed C

$$\left| \int_{n\pi}^{k} e^{it\Phi(k')}dk' \right| \leq \min\{ C\langle t(\pi n - k_0)\rangle^{-1}, |k - \pi n| \}.$$

Next we split

$$H_{122}^{n}(t,x,y) = \int_{n\pi}^{n\pi + \langle t(\pi n - k_0)\rangle^{-1}} \cdots + \int_{n\pi + \langle t(\pi n - k_0)\rangle^{-1}}^{(n+\frac{1}{2})\pi} \cdots$$

But now

$$\Big|\int_{n\pi}^{n\pi+\langle t(\pi n-k_0)\rangle^{-1}}\dots\Big|\le C\int_{n\pi}^{n\pi+\langle t(\pi n-k_0)\rangle^{-1}}|k-\pi n||k-\pi n|^{-1}\langle n\rangle^{-1}$$

and

$$\Big|\int_{n\pi+\langle t(\pi n-k_0)\rangle^{-1}}^{(n+\frac12)\pi}\dots\Big|\le\int_{n\pi+\langle t(\pi n-k_0)\rangle^{-1}}^{(n+\frac12)\pi}C\langle t(\pi n-k_0)\rangle^{-1}|k-\pi n|^{-1}\langle n\rangle^{-1}$$

and so

$$|H_{122}^n(t,x,y)|\le C\langle n\rangle^{-1}\langle t(\pi n-k_0)\rangle^{-1}\log\left(\langle t(\pi n-k_0)\rangle\right).$$

Similarly for $|k_0-n\pi|\le 2\pi$ we get

$$|H_{122}^n(t,x,y)|\le C\langle t\rangle^{-\frac12}\log\left(\langle t\rangle\right).$$

We use that there is a fixed C such that for any x_0 and any $t>0$,

$$\int_{|x-x_0|\ge1}\frac{\log\left(\langle t(x-x_0)\rangle\right)dx}{\langle x\rangle\langle t(x-x_0)\rangle}\le C\min\left\{t^{-1}|\log t|,|\log t|^2\right\}$$

to conclude that for a fixed C

$$\sum_n|H_{122}^{n1}(t,x,y)|\le C\langle t\rangle^{-\frac12}|\log t|^2.$$

§5 Asymptotic expansion for $w-k$

We consider $w=u+iv$ and $k=p+iq$. We set $Q_\ell=\frac1\pi\int_{\mathbb R}u^\ell q(u)\,du$. Then $Q_{2\ell+1}=0$. In particular, see [KK] p. 601, we have $Q_0=\frac12\int_0^1P(t)\,dt$ and $Q_2=\frac18\int_0^1P^2(t)\,dt$.

For $u\in\sigma$ we have $q(u)=0$. For $u\in g_n$ we have formula (4.12) [K1]:

$$q(u)=\sqrt{(u-a_n^-)(a_n^+-u)}\left(1+\frac1\pi\sum_{m\ne n}\int_{g_m}\frac{q(t)\,dt}{|t-u|\sqrt{(t-a_n^-)(t-a_n^+)}}\right).\tag{5.1}$$

By Lemma 2.1 there is a fixed constant $C>0$ such that for $|\sigma_m|$ the length of σ_m we have $|\sigma_m|\ge C$ for all m. For $u\in g_n$ and $C_0=1+\frac{Q_0}{\min\{|\sigma_m|:m\in\mathbb Z\}}$ by [K1] p.16

$$\sqrt{(u-a_n^-)(a_n^+-u)}\le q(u)\le C_0\sqrt{(u-a_n^-)(a_n^+-u)}.\tag{5.2}$$

Lemma 5.1. *1) $\forall\ N$ there is a $C_N > 0$ such that $|q(u)| \le C_N \langle u\rangle^{-N}$ $\forall\ u \in \mathbb{R}$.*
2) The distributional derivative $q'(u)$ satisfies $\langle u\rangle^N q' \in L^r(\mathbb{R})$ for any $1 \le r < 2$ and any N.

Proof. We have $0 \le q(u) \le C_0|g_n| \le C_N\langle u\rangle^{-N}$ for $u \in g_n$ by (5.2) and Lemma 2.1. Turning to the second claim, by (5.1) the pointwise derivative $q'(u)$ is well defined except at the points $a_n^{\pm}$ for $n \in \mathbb{Z}$. Obviously $q'(u) = 0$ for any $a_n^+ < u < a_{n+1}^-$. For $a_n^- < u < a_n^+$ we differentiate (5.1) and using the fact that inside the integral we have $|t-u| \ge \inf_n |\sigma_n| > 0$, we conclude there is a fixed C such that

$$|q'(u)| < C\left(\frac{a_n^+ - u}{u - a_n^-}\right)^{\frac{1}{2}} + C\left(\frac{u - a_n^-}{a_n^+ - u}\right)^{\frac{1}{2}}.$$

From this we conclude that the pointwise $q'(u)$ coincides with the distributional derivative and that $\int_{g_n}\langle u\rangle^N|q'(u)|^r du \le \frac{C\langle n\rangle^N}{2-r}|g_n|$ for some fixed C. By Lemma 2.1 we conclude $\|\langle u\rangle^N q'\|_r \le \frac{C_N}{2-r}$ for some C_N.

From the second claim in Lemma 5.1 we obtain:

Lemma 5.2. *For any integer N there is a constant $C_N > 0$ such that, for any $w = u + iv$ with $v \ge 0$ and $|u| > 1$, we have*

$$w - k(w) = \sum_{\ell=0}^{N}\frac{Q_\ell}{w^{\ell+1}} + R_N(w)\,, \quad \left|R_N(w)\right| \le \frac{C_N}{\langle w\rangle^{N+1}}.$$

Proof. We have for $v > 0$ by (4.1) [K1] $k(w) - w = \frac{1}{\pi}\int_{\cup g_n}\frac{q(t)}{t-w}dt$ and so

$$w - k(w) = \frac{1}{\pi}\sum_{\ell=0}^{N}\frac{1}{w^{\ell+1}}\int_{\cup g_n} t^\ell q(t)dt + \frac{1}{w^{N+1}}\left(Q_v - iP_v\right) * \left(t^{N+1}q(t)\right)(u)$$

with in the last term the convolution of $t^{N+1}q(t)$ with the Poisson kernels

$$P_v(x) = \frac{1}{\pi}\frac{v}{x^2+v^2}\,, \quad Q_v(x) = \frac{1}{\pi}\frac{x}{x^2+v^2}.$$

Since for $1 < r < 2$ there is a C_r such that for any $v > 0$, see p. 121 [Ste],

$$\|\left(Q_v - iP_v\right) * \left(t^{N+1}q(t)\right)\|_{W^{1,r}} \le C_r\|t^{N+1}q(t)\|_{W^{1,r}}\,,$$

by Lemma 5.1 and by the Sobolev embedding theorem there is a fixed constant $C_N > 0$ such that $|\left(Q_v - iP_v\right) * \left(t^{N+1}q(t)\right)(u)| \le C_N$.

As an immediate corollary of Lemma 5.2 we obtain:

Lemma 5.3. *There are two constants C_1 and C_2 such that, for $w = u + iv$ with $1 \geq v \geq 0$ and $|u| > C_1$, we have*

$$\Big|u - p - \frac{Q_0}{u} + \frac{Q_0 v^2}{u^3} - \frac{Q_2}{u^3}\Big| \leq \frac{C_2}{|u|^4}.$$

By $w-k = Q_0\frac{\overline{w}}{|w|^2}+Q_2\frac{\overline{w}^3}{|w|^6}+O(w^{-4})$, we get $u-p = Q_0\frac{u}{u^2+v^2}+Q_2\frac{u^3}{(u^2+v^2)^3}+O(u^{-4})$. Then by Taylor series $\frac{1}{u^2+v^2} = u^{-2} - v^2u^{-4} + \dots$, we get the desired result.

§6 Relation between v and $q(u + iv)$

In the proof of Lemma 4.4 we will need to use the relative size of the coordinates of $w = u + iv$ and $k(w) = p(w) + iq(w)$. Lemma 5.3 gives some information on $u - p(u + iv)$. We now consider the relation between v and $q(u + iv)$. Recall that

$$q(u + iv) = v + \frac{v}{\pi}\int_{\mathbb{R}} \frac{q(t)\,dt}{(t-u)^2 + v^2}. \tag{6.1}$$

The unnatural restriction on u in the following lemma is not sharp and is only justified by estimates needed later.

Lemma 6.1. *Let $a_n^+ + |g_n|^{\frac{1}{4}} \leq u \leq a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}$ for $n \gg 1$ large, and let $1 \geq v \geq 0$ Then there is a constant C such that*

$$|q(u + iv) - v - \frac{v}{u^2}Q_0| \leq Cvu^{-4}.$$

Proof. By (6.1) write

$$q = v + \frac{v}{\pi u^2}\int_{\mathbb{R}} q(t)\left(1 + \frac{2t}{u} - \frac{t^2}{u^2} - \frac{v^2}{u^2}\right)dt + \frac{v}{\pi}\int_{\mathbb{R}} \frac{q(t)\left(\frac{2t}{u} - \frac{t^2}{u^2} - \frac{v^2}{u^2}\right)^2}{(t-u)^2 + v^2}dt$$

and use the formulas for Q_ℓ above (5.1) to express the second term on the rhs as $\frac{v}{u^2}((1-\frac{v^2}{u^2})Q_0 - \frac{Q_2}{u^2})$. In the second integral we expand the square on the numerator, treating the resulting terms separately. For example we write

$$\frac{4v}{\pi u^2}\int_{\mathbb{R}} \frac{t^2 q(t)}{(t-u)^2 + v^2}dt = \frac{4v}{\pi u^4}\int_{\mathbb{R}} t^2 q(t)\left(1 + \frac{2t}{u} - \frac{t^2}{u^2} - \frac{v^2}{u^2}\right)dt +$$
$$+ \frac{4v}{\pi u^4}\int_{\mathbb{R}} \frac{t^2 q(t)\left(2t - \frac{t^2}{u} - \frac{v^2}{u}\right)^2}{(t-u)^2 + v^2}dt.$$

The first term on the rhs is $O(vu^{-4})$. To show that the second term is $O(vu^{-4})$, we need to show bounds of the form $\int_{\mathbb{R}} \frac{|t|^N q(t)}{(t-u)^2+v^2} dt \le C_N$. Say that $u \le \frac{a_n^+ + a_{n+1}^-}{2}$, with the case $u \ge \frac{a_n^+ + a_{n+1}^-}{2}$ treated similarly. Then

$$\int_{\mathbb{R}} \frac{|t|^N q(t)}{(t-u)^2+v^2} dt = \left(\sum_{\ell \neq n} \int_{g_\ell} \frac{|t|^N q(t)}{(t-u)^2+v^2} dt \right) + \int_{a_n^-}^{a_n^+} \frac{|t|^N q(t)}{(t-u)^2+v^2} dt$$

where the first term in the rhs is bounded by a $C_N \langle u \rangle^{-2}$ thanks to $|t-u| \approx |\ell - n|$, Lemma 5.1 and $\langle u \rangle^{-2} * \langle u \rangle^{-N} \lesssim \langle u \rangle^{-2}$. Next we write

$$\int_{a_n^-}^{a_n^+} \frac{|t|^N q(t)}{(t-u)^2+v^2} dt \le \frac{C|n|^N |g_n|^2}{(a_n^+ - u)^2} \le C_N$$

by $q(t) \le (1 + Cn^{-2})|g_n|$, see [KK], by $a_n^+ - a_n^- = |g_n|$, by our restriction on u and by Lemma 2.1.

We have the following corollary:

Lemma 6.2. *In $\{1 \ge v \ge 0\}$ for any preassigned $C_1 > 0$ in $a_n^+ + |g_n|^{\frac{1}{4}} \le u \le a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}$ there is a C such that*

$$\left| \Im(E - k^2) \right| \le C \frac{v}{u^3}.$$

Proof. Write $\Im(E-k^2) = (v-q)(u+p) + (u-p)(v+q)$ with $0 \le v \le q$. By Lemma 5.3 we have $u - p = \frac{Q_0}{u} - \frac{Q_0 v^2}{u^3} + \frac{Q_2}{u^3} + O(u^{-4})$ and $u + p = 2u - \frac{Q_0}{u} + \frac{Q_0 v^2}{u^3} - \frac{Q_2}{u^3} + O(u^{-4})$. By Lemma 6.1 $v - q = -vu^{-2}Q_0 + O(vu^{-4})$ and $q + v = 2v + vu^{-2}Q_0 + O(vu^{-4})$. The $O(vu^{-1})$ term in $\Im(E - k^2)$ cancels and the following one is $O(vu^{-3})$.

Now we extend Lemma 6.2 without the restriction on u. From $q(t) \ge 0$ and $q(t) \not\equiv 0$ we get $q(u+iv) > v$ by (6.1). The following is an elementary consequence of Lemmas 2.1 and 5.1:

Lemma 6.3. *For any $u \in \mathbb{R}$ there is at most one g_n such that dist$(u, g_n) \ll 1$. For such an n we have*

$$0 \le \sum_{\ell \neq n} \frac{v}{\pi} \int_{g_\ell} \frac{q(t)\, dt}{(t-u)^2+v^2} < C \frac{v}{\langle u \rangle^2}.$$

If such an n does not exist the above formula holds summing over all $\ell \in \mathbb{Z}$.

Suppose now that u is close to the gap $g_n = (a_n^-, a_n^+)$ and set

$$I_n(u,v) = \frac{v}{\pi} \int_{a_n^-}^{a_n^+} \frac{q(t)\, dt}{(t-u)^2+v^2}.$$

Lemma 6.4. *There is a fixed C independent from n such that:*

(1) *Let $u \in \sigma_n$. Set $m = n, n+1$, a_m either $a_m = a_n^+$ for $m = n$ or $a_m = a_{n+1}^-$ for $m = n+1$. Suppose w close to a_m. Then if $|u - a_m| > \frac{|g_m|}{4}$ or if $v \geq |g_m|$ we have*

$$\frac{1}{C} \frac{|g_m|^2 v}{(u - a_m)^2 + v^2} < I_n(u, v) < C \frac{|g_m|^2 v}{(u - a_m)^2 + v^2};$$

(2) *if $u \in g_n = [a_n^-, a_n^+]$ for $v \geq \frac{|g_n|}{2}$ we have*

$$\frac{1}{C} \frac{|g_n|^2}{v} < I_n(u, v) < C \frac{|g_n|^2}{v};$$

(3) *for $u \in g_n$ and $v < 2 \min\{|u - a_n^-|, |u - a_n^+|\}$ we have*

$$\frac{1}{C} |g_n|^{\frac{1}{2}} \sqrt{\min\{|u - a_n^-|, |u - a_n^+|\}} < I_n(u, v) < C |g_n|^{\frac{1}{2}} \sqrt{\min\{|u - a_n^-|, |u - a_n^+|\}}.$$

Proof.

(1) For definiteness let $u \leq (a_n^+ + a_{n+1}^-)/2$. Set $\tilde{u} = u - a_n^+$. We have

$$I_n(u, v) = \frac{v}{\pi} \left[\int_0^{\frac{|g_n|}{2}} + \int_{\frac{|g_n|}{2}}^{|g_n|} \right] \frac{\sqrt{t}\sqrt{|g_n| - t}}{(t + \tilde{u})^2 + v^2} dt.$$

We have

$$v \int_{\frac{|g_n|}{2}}^{|g_n|} \frac{\sqrt{t}\sqrt{|g_n| - t}}{(t + \tilde{u})^2 + v^2} dt \approx \frac{|g_n|^2 v}{(|g_n| + \tilde{u})^2 + v^2}$$

Set $\tilde{I}_n(u, v) = v \int_0^{\frac{|g_n|}{2}} \frac{\sqrt{t}\sqrt{|g_n| - t}}{(t + \tilde{u})^2 + v^2} dt$. For $\tilde{u} \gtrsim |g_n|$ or for $v \gtrsim |g_n|$ then

$$\tilde{I}_n(u, v) = \frac{v}{\tilde{u}^2 + v^2} \int_0^{\frac{|g_n|}{2}} \sqrt{t}\sqrt{|g_n| - t} dt \approx \frac{|g_n|^2 v}{\tilde{u}^2 + v^2}.$$

(2) For definiteness here and below $u \leq \frac{a_n^+ + a_n^-}{2}$. We set $\tilde{u} = u - a_n^-$. We have by (5.2)

$$I_n(u, v) \approx \frac{v}{\pi} \int_{a_n^-}^{a_n^+} \frac{\sqrt{a_n^+ - t}\sqrt{t - a_n^-}}{(t - u)^2 + v^2} dt \approx v\sqrt{|g_n|} \int_0^{\frac{|g_n|}{2}} \frac{\sqrt{t} dt}{(t - \tilde{u})^2 + v^2}$$

Then for $v \gtrsim |g_n|$ we get $I_n(u, v) \approx \frac{|g_n|^2 v}{\tilde{u}^2 + v^2}$ and hence claim 1 in Lemma 6.4.

(3) We write for $\tilde u = u - a_n^-$

$$I_n(u,v) \approx v\sqrt{|g_n|}\left[\int_0^{\frac{\tilde u}{2}} + \int_{\frac{\tilde u}{2}}^{2\tilde u} + \int_{2\tilde u}^{\frac{|g_n|}{2}}\right] \frac{\sqrt{t}dt}{(t-\tilde u)^2+v^2} \approx \frac{v\sqrt{|g_n|}\tilde u^{\frac32}}{\tilde u^2+v^2} +$$
$$+ \sqrt{|g_n|\tilde u}\arctan(\frac{\tilde u}{2v}) + \sqrt{|g_n|v}\int_{\frac{2\tilde u}{v}}^{\frac{|g_n|}{2v}} \frac{\sqrt{t}dt}{t^2+1}.$$

For $2\tilde u > v$ we get $I_n(u,v) \approx \sqrt{|g_n|\tilde u}$ and hence claim 3 in Lemma 6.4.

§7 Estimates on the band function

We will need to bound $\frac{dE}{dk}$, $\frac{d^2E}{dk^2}$ and $\frac{d^3E}{dk^3}$.

Lemma 7.1. *There are constants $C_1 > C_2 > 0$ such $\forall m$ and $\forall u \in \sigma_m = [a_m^+, a_{m+1}^-]$ and if $v=0$, we have for $A(u) = \frac{|g_m|^2}{(u-a_m^+)^{\frac12}(u-a_m^++|g_m|)^{\frac32}} + \frac{|g_{m+1}|^2}{(a_{m+1}^- - u)^{\frac12}(a_{m+1}^- - u + |g_{m+1}|)^{\frac32}}$*

$$1 + C_2\left(A(u) + \frac{1}{\langle u\rangle^2}\right) \ge p'(u) \ge 1 + C_1 A(u). \tag{1}$$

Correspondingly for $p \in [m\pi, (m+1)\pi]$ we have

$$\frac{1}{1 + C_2\left(A(u) + \frac{1}{\langle u\rangle^2}\right)} \le \frac{du}{dp} \le \frac{1}{1 + C_1 A(u)} \tag{2}$$

$$\frac{2|u|}{1 + C_2\left(A(u) + \frac{1}{\langle u\rangle^2}\right)} \le |\frac{dE}{dp}| \le \frac{2|u|}{1 + C_1 A(u)}. \tag{3}$$

If $a_n^+ + |n|^3|g_n| \le u \le a_{n+1}^- - |n+1|^3|g_{n+1}|$ with $n \gg 1$ large and $v=0$, then there is a fixed C such that

$$|\dot w - 1| + |\dot E(k) - 2k| \le C\langle k\rangle^{-2}. \tag{4}$$

There is n_0 such that if for $n \ge n_0$, $|a_n^+ - u| + |v| \ge |k||g_n|$ with $a_n^+ \le u \le \frac{a_n^+ + a_{n+1}^-}{2}$ and $1 \ge v \ge 0$ then there is a C such that for the corresponding $k = p + iq$ we have

$$\left|\frac{\dot E}{2k} - 1\right| \le C|k|^{-1}. \tag{5}$$

(5) holds also in $|a_{n+1}^- - u| + |v| \ge |k||g_{n+1}|$ and in particular for $a_n^+ + |g_n|^{\frac14} \le u \le a_{n+1}^- - |g_{n+1}|^{\frac35}$ and $1 \ge v \ge 0$.

Estimate (5) is used in the proof of Lemma 4.4.

Proof of Lemma 7.1. By [K1], $p'(u) = 1 + \frac{1}{\pi}\sum_n \int_{g_n} \frac{q(t)}{(t-u)^2} dt \geq 1 + I(u)$ with

$$I(u) = \frac{1}{2}\sum_n \frac{|g_n|^2}{\sqrt{|(u-a_n^-)(a_n^+-u)|}\left(\sqrt{|u-a_n^-|}+\sqrt{|a_n^+-u|}\right)^2}.$$

For $u \in \sigma_m$, ignoring all the terms in the sum defining $I(u)$ except for $n = m, m+1$, we get the lower bound for $p'(u)$ in (1) by $\sum_{n=m,m+1} \cdots \approx A(u)$. Turning to the upper bound, by (5.2) we have $p'(u) \leq 1 + C_0 I(u)$. We split now $2I(u) =$

$$\left(\sum_{n\neq m,m+1} + \sum_{n=m,m+1}\right) \frac{|g_n|^2}{\sqrt{|(u-a_n^-)(a_n^+-u)|}\left(\sqrt{|u-a_n^-|}+\sqrt{|a_n^+-u|}\right)^2}.$$

Since there is a fixed $c > 0$ such that for $n \neq m, m+1$ and for $u \in \sigma_m$ we have $|a_n^\pm - u| \geq c\langle n-m\rangle$ then

$$\sum_{n\neq m,m+1} \cdots \leq C\sum_n \frac{|g_n|^2}{\langle n-m\rangle^2} \leq C_2\frac{1}{\langle u\rangle^2}.$$

This gives the upper bound for $p'(u)$ in (1). (2) is obtained taking the inverses in (1) and (3) follows from $\frac{dE}{dk} = 2w\frac{dw}{dk}$.

To prove (4) we claim

Claim. *For $a_n^+ + C_1|n|^3|g_n| \leq u \leq a_{n+1}^- - C_1|n+1|^3|g_{n+1}|$ for any fixed C_1, $n \gg 1$ large and $v = 0$, we have $p'(u) = 1 + \frac{Q_0}{u^2} + O(u^{-3})$.*

Assume for a moment the Claim. Then $\dot{w} = 1 - \frac{Q_0}{u^2} + O(u^{-3})$ and by Lemma 5.3

$$\dot{E} = 2u\dot{w} = 2u - 2\frac{Q_0}{u} + O(u^{-2}) = 2p + 2\frac{Q_0}{u} - 2\frac{Q_0}{u} + O(u^{-2}).$$

Proof of the Claim. Is suggested by formal differentiation of $p(u) = u - \frac{Q_0}{u} - \frac{Q_2}{u^3} + \ldots$, but for a proof we return to $p'(u) = 1 + \frac{1}{\pi}\sum_n \int_{g_n} \frac{q(t)}{(t-u)^2} dt$. By the argument in the proof of Lemma 6.1, simply setting $v = 0$ in the appropriate integral,

$$p'(u) = 1 + \frac{Q_0}{u^2} - \frac{Q_2}{u^4} + \frac{1}{\pi u^2}\int_{\mathbb{R}} \frac{q(t)\left(2t - \frac{t^2}{u}\right)^2}{(t-u)^2} dt.$$

Then

$$\int_{\mathbb{R}} \frac{q(t)\left(2t-\frac{t^2}{u}\right)^2}{(t-u)^2} = \left[\left(\sum_{\ell\neq n}\int_{g_\ell}\right)+\int_{a_n^-}^{a_n^+}\right]\frac{q(t)\left(2t-\frac{t^2}{u}\right)^2}{(t-u)^2}dt$$

where the first term in the rhs is bounded by a $C\langle u\rangle^{-2}$ thanks to $|t-u|\approx|\ell-n|$ and where

$$\int_{a_n^-}^{a_n^+}\frac{q(t)\left(2t-\frac{t^2}{u}\right)^2}{(t-u)^2}dt \lesssim \int_{a_n^-}^{a_n^+}\frac{q(t)t^2}{(t-u)^2}dt \le \frac{C|n|^2|g_n|^2}{(a_n^+-u)^2}\lesssim n^{-4}$$

by $t\approx u\approx n$ i.e. Lemma 2.1, $q(t)\le(1+Cn^{-2})|g_n|$, see [KK], $a_n^+-a_n^-=|g_n|$, see definition of $|g_n|$, and by our restriction on u, that is $u-a_n^+\gtrsim n^3|g_n|$.

To prove (5) we write

$$k'(w)=1+\frac{1}{\pi}\sum_{\ell\neq n}\int_{g_\ell}\frac{q(t)}{(t-w)^2}dt+\frac{1}{\pi}\int_{a_n^-}^{a_n^+}\frac{q(t)}{(t-w)^2}dt.$$

The second term in the rhs is $O(n^{-2})$. For $|a_n^+-u|+|a_n^--u|+|v|\ge|k||g_n|$, so in particular for $a_n^++cn^{\frac13}|g_n|^{\frac23}\le u\le\frac{a_n^++a_{n+1}^-}{2}$ and $1\ge v\ge 0$, the third term has absolute value less than

$$\frac{1}{\pi}\int_{a_n^-}^{a_n^+}\frac{q(t)}{(t-u)^2+v^2}dt\le C\frac{|g_n|^2}{|k|^2|g_n|^2}=O(k^{-2}).$$

So $\dot w=1+O(k^{-2})$ and $\dot E=2w\,\dot w=2w+O(k^{-1})=2k+O(k^{-1})$ by Lemma 5.2.

We will need the following formulas, see (6.1) [K1]:

Lemma 7.2. *For any $u\notin[a,b]$ we have*

(1) $$\int_a^b\frac{\sqrt{(t-a)(b-t)}}{(t-u)^3}dt=\frac{\pi}{8}sign(u-a)\frac{(b-a)^2}{|u-a|^{\frac32}|u-b|^{\frac32}}$$

(2) $$\int_a^b\frac{\sqrt{(t-a)(b-t)}}{(t-u)^4}dt=\frac{\pi}{16}\frac{(b-a)^2}{|u-a|^{\frac32}|u-b|^{\frac32}}\left(\frac{1}{|u-a|}+\frac{1}{|u-b|}\right).$$

The proof is as Lemma 6.1 [K1]. Let us suppose $u>b$. Then setting $\ell=b-a$, $\ell s=t-a$, $\ell h=a-u$, $q=\sqrt{1+\frac1h}$ and introducing a new variable x defined by $x\,s=\sqrt{s(1-s)}$ and so $s=1/(1+x^2)$, $ds=-2xs^2dx$, we obtain

$$\frac1\ell\int_0^1\frac{\sqrt{s(1-s)}}{(s+h)^3}ds=\frac{1}{\ell h^3}\int_0^1\frac{\sqrt{s(1-s)}}{s^3(\frac1s+\frac1h)^3}ds=\frac{1}{\ell h^3}\int_{\mathbb{R}}\frac{x^2}{(x^2+q^2)^3}dx$$

which by the Residue Theorem is equal to $\frac{\pi i}{\ell h^3}\left[\frac{z^2}{(z+iq)^3}\right]''_{z=iq}=\frac{\pi}{8\ell h^3 q^3}$. Proceeding similarly we get (1) also for $u<a$. (2) follows by differentiation.

Lemma 7.3. *There are positive constants C_0, C_1, C_2, C_3, α and $m_0\geq 0$ such that for any $m\geq m_0$ and for any $u\in]a_m^+,a_{m+1}^-[$ we have*

$$(1)\qquad |p''(u)|\leq C_1\langle u\rangle^{-3}\quad \forall u\in[a_m^++\alpha,a_{m+1}^--\alpha]$$

$$(2)\qquad p''(u)\leq -C_1\langle u\rangle^{-3}-\frac{1}{4}\frac{(a_m^+-a_m^-)^2}{|u-a_m^-|^{\frac{3}{2}}|u-a_m^+|^{\frac{3}{2}}}\quad \forall u\leq a_m^++\alpha$$

$$(3)\qquad p''(u)\geq -C_2\langle u\rangle^{-3}-C_0\frac{(a_m^+-a_m^-)^2}{|u-a_m^-|^{\frac{3}{2}}|u-a_m^+|^{\frac{3}{2}}}\quad \forall u\leq a_m^++\alpha$$

$$(4)\qquad p''(u)\geq -\frac{C_2}{\langle u\rangle^3}+\frac{1}{4}\frac{(a_{m+1}^+-a_{m+1}^-)^2}{|u-a_{m+1}^-|^{\frac{3}{2}}|u-a_{m+1}^+|^{\frac{3}{2}}}\quad \forall u\geq a_{m+1}^--\alpha$$

$$(5)\qquad p''(u)\leq -\frac{C_3}{\langle u\rangle^3}+\frac{C_0}{4}\frac{(a_{m+1}^+-a_{m+1}^-)^2}{|u-a_{m+1}^-|^{\frac{3}{2}}|u-a_{m+1}^+|^{\frac{3}{2}}}\quad \forall u\geq a_{m+1}^--\alpha.$$

Since $p(u)$ is odd, for $m\leq -m_0$ there is an analogous statement.

We start with
$$p''(u)=\frac{2}{\pi}\sum_n\int_{g_n}\frac{q(t)}{(t-u)^3}dt.$$

We are assuming $u\in\sigma_m=[a_m^+,a_{m+1}^-]$. The terms with $n\leq m$ (resp. $n>m$) are negative (resp. positive). We have

(6)
$$p''(u)\leq \frac{2}{\pi}\sum_{n\leq m}\int_{g_n}\frac{\sqrt{(t-a_n^-)(a_n^+-t)}}{(t-u)^3}dt+\frac{2C_0}{\pi}\sum_{n>m}\int_{g_n}\frac{\sqrt{(t-a_n^-)(a_n^+-t)}}{(t-u)^3}dt$$
$$=-\frac{1}{4}\sum_{n\leq m}\frac{(a_n^+-a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}}+\frac{C_0}{4}\sum_{n>m}\frac{(a_n^+-a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}}.$$

Similarly we have

$$(7)\qquad p''(u)\geq -\frac{C_0}{4}\sum_{n\leq m}\frac{(a_n^+-a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}}+\frac{1}{4}\sum_{n>m}\frac{(a_n^+-a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}}.$$

We are considering $m\gg 1$. Observe that by Lemma 2.1 for $u\in\sigma_m$ there are fixed constants such that for arbitrary N

$$
(8) \qquad \begin{aligned}
&\sum_{n\geq m+1} \frac{(a_n^+ - a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}} \approx \sum_{n>m+1} \frac{|g_n|^2}{\langle n-m\rangle^3} \leq \frac{C_N}{\langle u\rangle^N} \\
&\sum_{n\leq m-1} \frac{(a_n^+ - a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}} \approx \sum_{n\leq m-1} \frac{|g_n|^2}{\langle n-m\rangle^3} \approx \langle u\rangle^{-3}.
\end{aligned}
$$

Hence for $u \in [a_m^+ + \alpha, a_{m+1}^- - \alpha]$ with $\alpha > 0$, (6)-(8) and $|g_n| \lesssim \langle n\rangle^{-N}$ imply (1). Assume now $u \in (a_m^+, a_m^+ + \alpha]$. From (6)-(8), $|g_{m+1}| \lesssim \langle m+1\rangle^{-N}$ and the signs, we get (2) and (3)

$$
p''(u) \approx -\langle u\rangle^{-3} - \frac{(a_m^+ - a_m^-)^2}{|u-a_m^-|^{\frac{3}{2}}|u-a_m^+|^{\frac{3}{2}}}.
$$

Now we consider u close to a_{n+1}^-. We now prove (4). From (7) we get

$$
\begin{aligned}
p''(u) \geq &-\frac{C_0}{4}\sum_{n\leq m} \frac{(a_n^+ - a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}} + \frac{1}{4}\sum_{n>m+1} \frac{(a_n^+ - a_n^-)^2}{|u-a_n^-|^{\frac{3}{2}}|u-a_n^+|^{\frac{3}{2}}} + \\
&+ \frac{1}{4}\frac{(a_{m+1}^+ - a_{m+1}^-)^2}{|u-a_{m+1}^-|^{\frac{3}{2}}|u-a_{m+1}^+|^{\frac{3}{2}}}.
\end{aligned}
$$

We absorb the first two terms in the right hand side inside the term $\frac{-C_2}{\langle u\rangle^3}$ of (4) and we get (4). The proof of (5) proceeds similarly starting from (6).

In the following two lemmas the symbols $\approx$, $\lesssim$ and $\ll$ involve fixed constants. We remark that $E(k)$ is even in k so for this reason we will assume now only $k \gg 1$.

Lemma 7.4. *There are fixed $C_1, C_2, c > 0$, with $C_1 > C_2$, and $m_0 > 0$ such that for any $m \geq m_0 > 0$ and any $u \in \sigma_m$ we have* $\left|\frac{d^2E}{du^2}\right| \approx$

$$
\begin{aligned}
&\frac{m}{|g_m|} && \textit{for} \quad u - a_m^+ \leq c|g_m|; \\
1 + &\frac{m|g_m|^2}{|u-a_m^+|^3} && \textit{for} \quad a_m^+ + c|g_m| \leq u \leq a_{m+1}^- - C_1|m+1|^{\frac{1}{3}}|g_{m+1}|^{\frac{2}{3}}; \\
&\frac{-m|g_{m+1}|^2}{|u-a_{m+1}^-|^{\frac{3}{2}}|u-a_{m+1}^+|^{\frac{3}{2}}} && \textit{for} \quad c|g_{m+1}| \leq a_{m+1}^- - u \leq C_2|m+1|^{\frac{1}{3}}|g_{m+1}|^{\frac{2}{3}}; \\
&\frac{-m}{|g_{m+1}|} && \textit{for} \quad a_{m+1}^- - u \leq c|g_{m+1}|.
\end{aligned}
$$

For $a_m^+ + |g_m|^{\frac14} \le w \le a_{m+1}^- - |g_{m+1}|^{\frac35}$ *we have* $\ddot w = O(k^{-3})$.

We start by the last statement. We have $\ddot w = -(\dot w)^3 k''$ with $\dot w \approx 1$ for $a_m^+ + |g_m|^{\frac14} \le w \le a_{m+1}^- - |g_{m+1}|^{\frac35}$. In this interval we have $|g_m|^2|u-a_m^+|^{-\frac32}|u-a_m^-|^{-\frac32} \le |g_m|^{\frac54} \ll k^{-3}$ and $|g_{m+1}|^2|u-a_{m+1}^+|^{-\frac32}|u-a_{m+1}^-|^{-\frac32} \le |g_{m+1}|^{\frac15} \ll k^{-3}$. We have

$$\ddot E = 2\,(\dot u)^2 - 2u\,(\dot u)^3\,p''. \tag{1}$$

Let us first assume $u \le a_{m+1}^- - \alpha$. Then $A(u) \approx \sqrt{\frac{|g_m|}{u-a_m^+}} + |g_{m+1}|^2$. By (1-3) Lemma 7.3 and by Lemma 7.1 we have

$$\frac{d^2E}{dp^2} \approx \frac{1}{(1+\sqrt{\frac{|g_m|}{u-a_m^+}})^2} + \frac{m}{(1+\sqrt{\frac{|g_m|}{u-a_m^+}})^3}\left(\frac{(a_m^+-a_m^-)^2}{|u-a_m^-|^{\frac32}|u-a_m^+|^{\frac32}} + \frac{1}{\langle m\rangle^3}\right).$$

For $|u-a_m^+| \lesssim |g_m|$ we have $\frac{d^2E}{dp^2} \approx \frac{m}{|g_m|}$. For $a_m^+ + |g_m| \lesssim u \le a_{m+1}^- - \alpha$ we have

$$\frac{d^2E}{dp^2} \approx 1 + \frac{m|g_m|^2}{|u-a_m^+|^3}.$$

Now we consider $u \ge a_{m+1}^- - \alpha$. Then $A(u) \approx \sqrt{\frac{|g_{m+1}|}{u-a_{m+1}^-}} + |g_m|^2$. The two terms in the right hand side of (1) can be equal for $|u-a_{m+1}^-| \approx |m+1|^{\frac13}|g_{m+1}|^{\frac23}$. For $|u-a_{m+1}^-| \gg |m+1|^{\frac13}|g_{m+1}|^{\frac23}$ we claim that $\frac{d^2E}{dp^2} \approx 1$. Indeed $A(u) \ll 1$, $\frac{du}{dp} \approx 1$, and

$$|\frac{d^2p}{du^2}| \lesssim \langle u\rangle^{-3} + \frac{m|g_{m+1}|^2}{|u-a_{m+1}^-|^{\frac32}|u-a_{m+1}^+|^{\frac32}} \ll 1.$$

For $|u-a_{m+1}^-| \ll |m+1|^{\frac13}|g_{m+1}|^{\frac23}$ we distinguish between $|u-a_{m+1}^-| \gtrsim |g_{m+1}|$ and $|u-a_{m+1}^-| \lesssim |g_{m+1}|$. If $|g_{m+1}| \lesssim |u-a_{m+1}^-| \ll |m+1|^{\frac13}|g_{m+1}|^{\frac23}$ then $A(u) \lesssim 1$, $\frac{du}{dp} \approx 1$ and by (4-5) Lemma 7.3

$$\frac{d^2E}{dp^2} \approx \frac{-m|g_{m+1}|^2}{|u-a_{m+1}^-|^{\frac32}|u-a_{m+1}^+|^{\frac32}}.$$

For $|g_{m+1}| \gtrsim |u-a_{m+1}^-|$, as $u \nearrow a_{m+1}^-$ then $A(u)$ starts getting larger and $\frac{du}{dp}$ starts getting smaller without however matching p'' which is very large, and we have

$$u\left(\frac{du}{dp}\right)^3\frac{d^2p}{du^2} \approx (m+1)\frac{|u-a_{m+1}^-|^{\frac32}}{|g_{m+1}|^{\frac32}}\frac{(a_m^+-a_m^-)^2}{|u-a_{m+1}^-|^{\frac32}|u-a_{m+1}^+|^{\frac32}} \approx \frac{m}{|g_{m+1}|}$$

and hence $\frac{d^2E}{dp^2} \approx -\frac{m+1}{|g_{m+1}|}$.

Lemma 7.5. *Let here $w = u + i0$. There are fixed $c > 0$, $c_1 > 0$ and n_0 such that for $m \geq n_0$ and for $\alpha \in (1/2, 1)$ then $a^-_{m+1} - |g_{m+1}|^\alpha < u < a^-_{m+1} - c|g_{m+1}|$ implies $|\dddot{E}| \geq c_1|m+1|\,|g_{m+1}|^{4(\frac{1}{2}-\alpha)}$. Similarly $a^+_m + |g_m|^\alpha > u > a^+_m + c|g_m|$ implies $|\dddot{E}| \geq c_1|m|\,|g_m|^{4(\frac{1}{2}-\alpha)}$.*

We prove the $m+1$ case, the other being similar. We have

$$\frac{d^3E}{dp^3} = -6\left(\frac{du}{dp}\right)^4 \frac{d^2p}{du^2} + 6u\left(\frac{du}{dp}\right)^5 \left(\frac{d^2p}{du^2}\right)^2 - 2u\left(\frac{du}{dp}\right)^4 \frac{d^3p}{du^3}. \tag{1}$$

If c is large, in our domain $\frac{du}{dp} \approx 1$ and $\frac{d^2p}{du^2} \approx \frac{|g_{m+1}|^2}{|u-a^+_{m+1}|^{\frac{3}{2}}\,|u-a^-_{m+1}|^{\frac{3}{2}}}$. We claim that the dominating term in the rhs of (1) is the third. We write

$$p'''(u) = \frac{2}{\pi}\sum_n \int_{g_n} \frac{q(t)}{(t-u)^4}dt.$$

For u as in the statement, by (2) Lemma 7.2

$$\left|\frac{d^3p}{du^3}\right| \approx \frac{|g_{m+1}|^2}{|u-a^+_{m+1}|^{\frac{3}{2}}|u-a^-_{m+1}|^{\frac{5}{2}}} \gtrsim |g_{m+1}|^{4(\frac{1}{2}-\alpha)}.$$

Since $\frac{du}{dp} \approx 1$ for $|g_{m+1}| \ll |u - a^-_{m+1}|$, we have $(p'')^2 \ll |p'''|$ and the last term in the rhs of (1) is the dominating one.

In Lemma 4.6 we have defined $\chi_{int}(k) = \sum_{n\geq n_0} \chi_1(\frac{w-a^+_n}{|g_n|^{\frac{1}{3}}})\chi_1(\frac{a^-_{n+1}-w}{|g_{n+1}|^{\frac{3}{5}}})$ for $n_0 \gg 1$. In the support of χ_{int} we have $w \approx k$, $k' = 1 + O(\langle w\rangle^{-2})$, $k'' = O(\langle w\rangle^{-3})$. Now, to complete the proof of Lemma 4.7 it is enough to prove:

Lemma 7.6. *We can extend the restriction of $p(u)$ on the support of $\chi_{int}(p(u))$ to the whole of $\mathbb{R}$ so that the extension $\widetilde{p}(u)$ satisfies the same relations $w \approx \widetilde{p}$, $\widetilde{p}' = 1 + O(\langle u\rangle^{-2})$, $\widetilde{p}'' = O(\langle u\rangle^{-3})$.*

Recall $p(u) = u + \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\int_{g_n}\frac{q(t)}{t-u}dt$ for $u \in \sigma = \cup\sigma_n$. For $u \in supp\chi_{int}(p(u))$ we have $p(u) = \widetilde{p}(u)$ with

$$\widetilde{p}(u) := u + \frac{1}{\pi}\sum_{n<n_0}\int_{g_n}\frac{q(t)}{t-u}dt + \frac{1}{\pi}\sum_{n\geq n_0}\chi_1(\frac{u-a^+_n}{|g_n|^{\frac{1}{3}}})\chi_1(\frac{a^-_n-u}{|g_n|^{\frac{3}{5}}})\int_{g_n}\frac{q(t)}{t-u}dt$$

for a fixed even smooth cutoff χ_1 with $\chi_1(t) = 0$ for t near 0 and $\chi_1(t) = 1$ for $t \gtrsim 1$. If we consider $\widetilde{p}(u)$ for any $u > a^+_{n_0}$ we obtain that $\widetilde{p}(u)$ satisfies the relations stated in in the statement. This follows from the fact that near a gap g_m we have

$$\begin{aligned}
\widetilde{p}' &= 1 + O(\langle u\rangle^{-2}) + \frac{1}{\pi}\left(\chi_1(\frac{u-a_m^+}{|g_m|^{\frac{1}{3}}})\chi_1(\frac{a_m^- - u}{|g_m|^{\frac{3}{5}}})\int_{g_m}\frac{q(t)}{t-u}dt\right)' \\
&= 1 + O(\langle u\rangle^{-2}) + O(|g_m|^{\frac{4}{5}}) \\
\widetilde{p}'' &= O(\langle u\rangle^{-3}) + \frac{1}{\pi}\left(\chi_1(\frac{u-a_m^+}{|g_m|^{\frac{1}{3}}})\chi_1(\frac{a_m^- - u}{|g_m|^{\frac{3}{5}}})\int_{g_m}\frac{q(t)}{t-u}dt\right)'' \\
&= O(\langle u\rangle^{-3}) + O(|g_m|^{\frac{1}{5}}).
\end{aligned}$$

§8 Estimates on fundamental solutions

In what follows $\dot{f} = \frac{d}{dk}f$ and $f' = \frac{d}{dx}f$. Referring to formulas (2.1) and (2.2), we write $\theta(x,k)$ and $\varphi(x,k)$ for $\theta(x,w(k))$ and $\varphi(x,w(k))$. Then we have:

Lemma 8.1. *For $x \in [0,1]$ we have*

$$\big|\theta(x,k) - \cos(kx)\big| \le \frac{1}{|k|}e^{\frac{x}{|k|}(\|P\|_\infty + |k^2 - E(k)|)} \tag{1}$$

$$\big|\varphi(x,k) - \frac{\sin(kx)}{k}\big| \le \frac{1}{|k|^2}\,e^{\frac{x}{|k|}(\|P\|_\infty + |k^2 - E(k)|)}. \tag{2}$$

Furthermore there is a fixed constant C such that for $x \in [0,1]$

$$\Big|\dot{\theta}(x,k) + x\sin(kx) - \int_0^x \theta(s,k)\frac{\partial}{\partial k}\frac{\sin(k(x-s))\left[P(s) + k^2 - E(k)\right]}{k}ds\Big| \le \frac{C}{\langle k\rangle}$$

$$\Big|\dot{\varphi}(x,k) - \frac{\partial}{\partial k}\frac{\sin(kx)}{k} - \int_0^x \varphi(s,k)\frac{\partial}{\partial k}\frac{\sin(k(x-s))\left[P(s) + k^2 - E(k)\right]}{k}ds\Big| \le \frac{C}{\langle k\rangle^2}.$$

Proof. The argument is routine. $\theta(x,k)$ and $\varphi(x,k)$ satisfy the following integral equations:

$$\theta(x,k) = \cos(kx) + \frac{1}{k}\int_0^x \sin(k(x-s))\left[P(s) + k^2 - E(k)\right]\theta(s,k)ds \tag{3}$$

$$\varphi(x,k) = \frac{\sin(kx)}{k} + \frac{1}{k}\int_0^x \sin(k(x-s))\left[P(s) + k^2 - E(k)\right]\varphi(s,k)ds. \tag{4}$$

Now we write

$$\theta(x,k) = \sum_{n=0}^{\infty} \theta_n(x,k)\,, \quad \varphi(x,k) = \sum_{n=0}^{\infty} \varphi_n(x,k)$$

$$\theta_0(x,k) = \cos(kx)\,, \quad \varphi_0(x,k) = \frac{\sin(kx)}{k}$$

$$\theta_{n+1}(x,k) = \frac{1}{k}\int_0^x \sin\left(k(x-s)\right)\left[P(s)+k^2-E(k)\right]\theta_n(s,k)ds$$

$$\varphi_{n+1}(x,k) = \frac{1}{k}\int_0^x \sin\left(k(x-s)\right)\left[P(s)+k^2-E(k)\right]\varphi_n(s,k)ds.$$

Singling out $\theta(x,k)$, we have for $x_{n+1}=x$: $\theta_{n+1}(x,k) =$

$$= \frac{1}{k^{n+1}}\int_{0\le x_1\le\cdots\le x_n\le x}\prod_{j=1}^{n}\left\{\sin\left(k(x_{j+1}-x_j)\right)\left[P(x_j)+k^2-E(k)\right]dx_j\right\}\cos(kx_1).$$

This implies the following estimate which implies (1):

$$\left|\theta_{n+1}(x,k)\right| \le \frac{\left(\int_0^x(|P(s)|+|k^2-E(k)|)ds\right)^n}{k^{n+1}\,n!} \le \frac{x^n}{k^{n+1}\,n!}(\|P\|_\infty+|k^2-E(k)|)^n.$$

Proceeding similarly we obtain the following inequality, which gives us (2):

$$\left|\varphi_{n+1}(x,k)\right| \le \frac{\left(\int_0^x(|P(s)|+|k^2-E(k)|)ds\right)^n}{k^{n+2}\,n!} \le \frac{x^n}{k^{n+2}\,n!}(\|P\|_\infty+|k^2-E(k)|)^n.$$

Next we consider

$$\dot\theta(x,k) = -x\sin(kx) + \int_0^x \theta(s,k)\frac{\partial}{\partial k}\frac{\sin\left(k(x-s)\right)\left[P(s)+k^2-E(k)\right]}{k}ds +$$
$$+\frac{1}{k}\int_0^x \sin\left(k(x-s)\right)\left[P(s)+k^2-E(k)\right]\dot\theta(s,k)ds$$

and

$$\dot\varphi(x,k) = \frac{\partial}{\partial k}\frac{\sin(kx)}{k} + \int_0^x \varphi(s,k)\frac{\partial}{\partial k}\frac{\sin\left(k(x-s)\right)\left[P(s)+k^2-E(k)\right]}{k}ds +$$
$$+\frac{1}{k}\int_0^x \sin\left(k(x-s)\right)\left[P(s)+k^2-E(k)\right]\dot\varphi(s,k)ds.$$

We have for a fixed C_1

$$\begin{aligned}&\left|\partial_k\left(k^{-1}\sin\left(k(x-s)\right)P(s)\right)+\left(k^2-E(k)\right)\partial_k\left(k^{-1}\sin\left(k(x-s)\right)\right)+\right.\\ &\left.+\left(2k-\dot E(k)\right)k^{-1}\sin\left(k(x-s)\right)\right| \le C\langle k\rangle^{-1}+C\langle k\rangle^{-1}\left|2k-\dot E(k)\right| \le C_1.\end{aligned} \tag{8.1}$$

By this estimate, by (1) and (2) and by the above arguments we obtain the last two inequalities of lemma 8.1.

Lemma 8.2. *There is a fixed C such that*

$$|\int_0^1 \theta^2(x,k)dx - \frac{1}{2} - \frac{\sin(2k)}{4k}| \le Ck^{-2}; \tag{1}$$

$$\big|k^2 \int_0^1 \varphi^2(x,k)dx - \frac{1}{2} + \frac{\sin(2k)}{4k}| \le Ck^{-2}; \tag{2}$$

$$\big|2k \int_0^1 \theta(x,k)\varphi(x,k)dx - \frac{1-\cos(2k)}{2k}| \le Ck^{-2}. \tag{3}$$

Proof. We use the notation in the proof of Lemma 8.1. To prove (1) is enough to show that $\int_0^1 \cos(xk)\theta_1(x,k)dx = O(k^{-2})$. By its definition and elementary computation

$$\theta_1(x,k) = \frac{\sin(xk)}{2k}\left[\int_0^x P(s)ds + (k^2 - E)x\right] + O(k^{-2}). \tag{4}$$

By $k^2 - E = -2Q_0 + o(1)$ and elementary integration, (1) follows.

To prove (2) is enough to show that $\int_0^1 \sin(xk)k\varphi_1(x,k)dx = O(k^{-2})$. By its definition and elementary computation

$$k\,\varphi_1(x,k) = -\frac{\cos(xk)}{2k}\left[\int_0^x P(s)ds + (k^2 - E)x\right] + O(k^{-2}). \tag{5}$$

Elementary integration gives (2).

To prove (3) is enough $\int_0^1 (\cos(xk)k\varphi_1(x,k) + \sin(xk)\theta_1(x,k))\,dx = O(k^{-2})$. So (3) follows from $\int_0^1 (\cos(xk)k\varphi_1(x,k) + \sin(xk)\theta_1(x,k))\,dx =$

$$= -\frac{1}{2k}\int_0^1 \cos(2kx)\left[\int_0^x P(s)ds + (k^2 - E)x\right]dx + O(k^{-2}).$$

§9 Proof of Lemma 4.4

Formulas (2.2), (2.3), (2.7) and Lemma 8.1 imply with Lemma 9.1 below gives the first claim of Lemma 4.4.

Lemma 9.1. *There is a fixed constant $C > 0$ such that for $w = u + iv$ with $v = 0$ and for $a_n^+ + Cn^5|g_n| \le u \le a_{n+1}^- - Cn^5|g_{n+1}|$, we have*

$$\big|N^2(k) - 1\big| + \big|\frac{\sin k}{k\varphi(k)} - 1\big| + \big|\frac{\varphi'(k) - \theta(k)}{2k\varphi(k)}\big| \le \frac{C}{\langle k\rangle}.$$

Proof. Expanding in the definition of $N^2(k)$ we write

$$N^2(k) = \int_0^1 \left(\theta(x,k) + \frac{\varphi'(k)-\theta(k)}{2\varphi(k)}\varphi(x,k)\right)^2 dx + \int_0^1 \frac{\sin^2 k}{\varphi^2(k)}\varphi^2(x,k)dx =$$
$$= \int_0^1 \left[\theta^2(x,k) + 2\theta(x,k)\varphi(x,k)\frac{\varphi'(k)-\theta(k)}{2\varphi(k)} + \left(\frac{\varphi'(k)-\theta(k)}{2\varphi(k)}\varphi(x,k)\right)^2\right] +$$
$$+ \int_0^1 \frac{\sin^2 k}{\varphi^2(k)}\varphi^2(x,k)dx.$$

We recall now from formulas (1.4) and (3.1) in [F2]:

$$\dot{E}(k) = \frac{2\sin k}{\varphi(k)N^2(k)}. \tag{9.1}$$

For $a_n^+ + Cn^5|g_n| \le u \le a_{n+1}^- - Cn^5|g_{n+1}|$ we have $\left|\dot{E}(k) - 2k\right| \lesssim \langle n\rangle^{-2}$ by claim 4 in the statement of Lemma 7.1. So $N^2(k) = \frac{\sin k}{k\varphi(k)}\left(1 + O_2(\frac{1}{k^2})\right)$ with O_2 a big O. Hence by Lemma 8.2 we obtain, for a certain number of big O's,

$$\left(\frac{\sin k}{k\varphi(k)}\right)^2 - 2\frac{\sin k}{k\varphi(k)}\frac{1+O_2(k^{-2})}{1-\frac{\sin(2k)}{2k}+O_1(k^{-2})} + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2 +$$
$$+ \frac{1+\frac{\sin(2k)}{2k}+O_3(k^{-2})}{1-\frac{\sin(2k)}{2k}+O_1(k^{-2})} + O_4(k^{-1})\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)} = 0$$

which implies $\frac{\sin k}{k\varphi(k)} = \frac{1+O_2(k^{-2})}{1-\frac{\sin(2k)}{2k}+O_1(k^{-2})} \pm \sqrt{\Delta}$, for

$$\Delta = \frac{\frac{\sin^2(2k)}{4k^2}}{\left(1-\frac{\sin(2k)}{2k}\right)^2} + O(k^{-2}) - \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2 - O_4(k^{-1})\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}.$$

So $\frac{\sin k}{k\varphi(k)} = 1 + O(\frac{1}{k})$ and $\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)} = O(\frac{1}{k})$ when $k \in \mathbb{R}$ and under the restriction $a_n^+ + Cn^5|g_n| \le u \le a_{n+1}^- - Cn^5|g_{n+1}|$.

The proof of the second claim of Lemma 4.4 is trickier and proceeds in several steps. We set $\|m_\pm^0(k)\|_2 = (\int_0^1 |m_\pm^0(x,k)|^2 dx)^{\frac{1}{2}}$. First of all we consider the Fourier series expansion $m_\pm^0(x,k) = \sum \widehat{m}_\pm(\ell,k)e^{2\pi i\ell x}$ and show that the L^2 is concentrated in two harmonics. One harmonic is $|\widehat{m_\pm}(0,k)| \approx \|m_\pm(k)\|_2$. If there is an n such that $|n\pi + k| \ll 1$ then also $\widehat{m_\pm}(\mp n,k)$ can be significant. We then bound $N^{-1}(k)$, $m^\pm(k)k^{-1}N^{-1}(k)$ in terms $\|m_\pm(k)\|_2$. Next, we express $m_\pm^0(x,k)$ in terms of $\varphi(x,k)$ and $\theta(x,k)$, we expand the latter in terms of $\sin(kx)$ and $\cos(kx)$ and a reminder, and we conclude that in L^∞ sense $m^0_\pm(x,k)$ can be approximated by the two terms of the Fourier expansion discussed above. Next we look at the normalization of the Bloch functions. Slightly off the slits we have $|\widehat{m_\pm}(0,k)| \gg |\widehat{m_\pm}(\mp n,k)|$ and so $1 \approx \widehat{m_+}(0,k)\widehat{m_-}(0,k)$. From this we get the desired bound on $|m_+(x,k)m_-(y,k)| \lesssim 1$ off the slits. Near the slits we have $|\widehat{m_\pm}(0,k)| \approx |\widehat{m_\pm}(\mp n,k)|$ so to exploit the normalization we have to exclude a significant cancelation in a certain formula. Let us start with the first step, and show that there are at most two significant harmonics.

Lemma 9.2. *For all n except possibly for one n_0, we have $|n\pi + k| \gtrsim 1$. Then for $n \neq 0, n_0$ we have $\sum_{n\neq 0,n_0} |\widehat{m_+}(n)|^2 \le C|k|^{-2}\|m^0_+(k)\|_2^2$ for a fixed C. Furthermore for a fixed C we have $\sum_{n\neq 0,n_0} n^2|\widehat{m_+}(n)|^2 \le C\|m^0_+(k)\|_2^2$. The same statement holds for m^0_- with n_0 replaced by $-n_0$. If for all $n \neq 0$ we have $|n\pi+k| \gtrsim 1$ we can extend the above inequalities to the sum on all $n \neq 0$.*

Proof. The first sentence is straightforward. We will assume there is n_0 with $|n_0\pi + k| \ll 1$. If such n_0 does not exist, the proof is almost the same. Set for $n \neq 0, n_0$

$$\begin{aligned} &\widehat{m_+}(n) + \sum_{\ell\neq 0,n_0} T(n,\ell)\widehat{m_+}(\ell) = -\frac{\widehat{P}(n)\widehat{m_+}(0) + \widehat{P}(n-n_0)\widehat{m_+}(n_0)}{4\pi n(n\pi+k)} \\ &T(n,\ell) = \frac{\widehat{P}(0) + k^2 - E}{4\pi n(n\pi+k)}\delta(n-\ell) + \frac{\widehat{P}(n-\ell)}{4\pi n(n\pi+k)}. \end{aligned} \tag{1}$$

We have $\widehat{P}(0)+k^2-E = O(k^{-2})$ by Lemma 5.2 and by $\widehat{P}(0) = 2Q_0$. Equation (1) is of the form $(I+T)u = f$ where $\|f\|_{l^2} \le C|k|^{-1}\|m^0_\pm(k)\|_2$ and $\|f\|_{l^2_1} \le C\|m^0_\pm(k)\|_2$ where $\|f\|^2_{l^2_1} = \sum_{n\neq 0,n_0} n^2|f(n)|^2$. By

$$\sup_\ell \sum_n \left(|kT(n,\ell)| + |nT(n,\ell)|\right) + \sup_n \sum_\ell \left(|kT(n,\ell)| + |nT(n,\ell)|\right) \le C$$

for $u \in l^2(\mathbb{N})$ we have $\sum_n |(Tu)(n)|^2 \le C|k|^{-2}\|u\|^2_{l^2}$ and $\sum_n n^2|(Tu)(n)|^2 \le C\|u\|^2_{l^2}$. So inverting and after a Neumann expansion, we see that for $n \neq 0, n_0$ we have

$$\widehat{m_+}(n) = -\frac{\widehat{P}(n)\widehat{m_+}(0) + \widehat{P}(n-n_0)\widehat{m_+}(n_0)}{4\pi n(n\pi+k)} + \widehat{e}(n) \tag{2}$$

where $\widehat{e} = \sum_{m=1}^\infty (-)^m T^m f$ satisfies $\widehat{e}(0) = \widehat{e}(n_0) = 0$ and

$$\|\widehat{e}\|_{l^2} \le \|\widehat{e}\|_{l^2_1} \le C\|\sum_{m=0}^\infty (-)^m T^m f\|_{l^2} \le C|k|^{-1}\sum_{m=0}^\infty (C|k|^{-1})^m \|m^0_+(k)\|_2.$$

Hence by (2), $\sum_{n\neq 0,n_0} |\widehat{m_+}(n)|^2 \leq C|k|^{-2}\|m_+^0(k)\|_2^2$ and $\sum_{n\neq 0,n_0} n^2|\widehat{m_+}(n)|^2 \leq C\|m_+^0(k)\|_2^2$. The proof for m_- is similar.

We express now the Bloch functions in terms of the fundamental solutions as in §2. Using the notation in the proof of Lemma 8.1 and for $m^\pm(k) = \frac{\varphi'(k)-\theta(k)}{2\varphi(k)} \pm i\frac{\sin k}{\varphi(k)}$,

$$\begin{aligned} m_\pm^0(x,k) &= e^{\mp ikx}\left(\frac{\cos(xk)}{N} + \frac{m^\pm(k)}{kN}\sin(xk)\right) + \\ &+ \frac{1}{N}e^{\mp ikx}\sum_{j=1}^{\infty}\theta_j(x,k) + \frac{m^\pm(k)}{kN}e^{\mp ikx}\sum_{j=1}^{\infty}k\varphi_j(x,k). \end{aligned} \tag{9.2}$$

By the proof of Lemma 8.1 $\left|e^{\mp ikx}\sum_{j=1}^\infty \theta_j(x,k)\right| + \left|e^{\mp ikx}\sum_{j=1}^\infty k\,\varphi_j(x,k)\right| = O(k^{-1})$ for $x\in[0,1]$. Then we have

$$\begin{aligned} &m_\pm^0(x,k) = A_\pm(k) + B_\pm(k)e^{\mp i2kx} + O(N^{-1}k^{-1}) + O(m^\pm(k)N^{-1}k^{-2}),\\ &A_\pm(k) = \frac{1 - i\,k^{-1}m^\pm(k)}{2N(k)} = \frac{1 + \frac{\sin(k)}{k\varphi(k)} \mp i\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}}{2N(k)}\\ &B_\pm(k) = \frac{1 + i\,k^{-1}m^\pm(k)}{2N(k)} = \frac{1 - \frac{\sin(k)}{k\varphi(k)} \pm i\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}}{2N(k)}. \end{aligned}$$

We have

$$\widehat{m_\pm^0(\cdot,k)}(0) = A_\pm(k) + O\left(k^{-1}B_\pm(k)\right) + O(N^{-1}k^{-1}) + O(m^\pm(k)N^{-1}k^{-2}). \tag{9.3}$$

For the n_0 of Lemma 9.2 we have

$$\widehat{m_\pm^0(\cdot,k)}(\pm n_0) = B_\pm(k)\frac{e^{\pm 2i(\pi n_0 - k)} - 1}{\pm 2i(\pi n_0 - k)} + O(N^{-1}k^{-1}) + O(m^\pm(k)N^{-1}k^{-2}). \tag{9.4}$$

Lemma 9.3. *For a fixed $C>0$*

$$|1/N(k)| + |\frac{m^\pm(k)}{kN(k)}| + |A_\pm| + |B_\pm| \leq C\|m_\pm^0(k)\|_2.$$

Proof. Take

$$A_\pm(k) = \frac{1}{2N} + i\frac{m^\pm(k)}{2kN}\,, \quad B_\pm(k) = \frac{1}{2N} - i\frac{m^\pm(k)}{2kN}. \tag{1}$$

Then by (9,3-4)

(2) $\quad |A_\pm(k)| \lesssim \|m^0_\pm(k)\|_2 + O(N^{-1}k^{-1}) + O(m^\pm(k)N^{-1}k^{-2})$

(3) $\quad |B_\pm(k)| \lesssim \|m^0_\pm(k)\|_2 + O(N^{-1}k^{-1}) + O(m^\pm(k)N^{-1}k^{-2})$.

By the triangular inequality,

(4) $\quad |1/N(k)| \le |A_\pm(k)| + |B_\pm(k)| \lesssim \|m^0_\pm(k)\|_2 + O(m^\pm(k)N^{-1}k^{-2})$.

By (1-3) if one of $|1/N(k)|$ and $|m^\pm(k)N^{-1}k^{-1}|$ is $\gg \|m^0_\pm(k)\|_2$, then $|1/N(k)| \approx |m^+(k)N^{-1}k^{-1}|$. Then $|1/N(k)| \lesssim \|m^0_\pm(k)\|_2$ by (4) and $|m^\pm(k)N^{-1}k^{-1}| \lesssim \|m^0_\pm(k)\|_2$. So these last two formulas hold. So $|A_\pm(k)| + |B_\pm(k)| \lesssim \|m^0_\pm(k)\|_2$ by (2-3) .

We now use the normalization of Bloch functions $1 = \int_0^1 m^0_+(x,k)m^0_-(x,k)dx$. We will denote n_0 by n. By (9,3-4) and Lemma 9.3

(9.5) $\quad 1 = [A_+(k)A_-(k) + B_+(k)B_-(k)] + O(k^{-1}\|m^0_+(k)\|_2\|m^0_-(k)\|_2)$.

We can also write by Lemma 9.2

(9.6) $\quad 1 = \widehat{m}_+(0,k)\widehat{m}_-(0,k) + \widehat{m}_+(-n,k)\widehat{m}_-(n,k) + O(k^{-2}\|m^0_+(k)\|_2\|m^0_-(k)\|_2)$.

Lemma 9.4. *Suppose that k is in a region such that $|\widehat{m}_\pm(\mp n,k)| \le \frac{1}{2}|\widehat{m}_\pm(0,k)|$. Then for a fixed constant C we have $\|m^0_+(k)\|_2\|m^0_-(k)\|_2 \le C$. As a consequence also $|m^0_+(x,k)m^0_-(y,k)| < C$ for some fixed C.*

Proof. $\|m^0_+(k)\|_2\|m^0_-(k)\|_2 \le C_1$ for a fixed C_1 by $|\widehat{m}_\pm(0,k)| \approx \|m^0_\pm(k)\|_2$ and (9.6). By (9.2-4) and Lemma 9.3 we get $|m^0_+(x,k)m^0_-(y,k)| < C$ for a fixed C.

Lemma 9.4 applies to the case when k is not close to the slits, for example if there is no n with $|\pi n + k| \ll 1$. Let us suppose $n = n_0$ exists. We have:

Lemma 9.5. *Consider $k = p + iq$ and corresponding $w = u + iv$. For a fixed constant C and for $I_n(u,v) = \frac{v}{\pi}\int_{a_n^-}^{a_n^+} \frac{q(t)\,dt}{(t-u)^2+v^2}$ we have*

$$(9.7) \qquad \big||\widehat{m}_\pm(\mp n,k)|^2 - \frac{1}{2}\frac{|u|}{|n|\pi}\frac{|I_n(u,v)|}{|q|}\|m^0_\pm(k)\|_2^2\big| \le C|k|^{-2}\|m^0_\pm(k)\|_2^2.$$

Proof. It is enough to consider $q > 0$. Set $m = m^0_\pm$. Multiply by $\overline{m}$ the equation $m'' \pm 2ikm' - P(x)m + (E - k^2)m = 0$, integrate in $[0,1]$ and take imaginary part to obtain

$$(9.8) \qquad 2q\,\Im\int_0^1 \overline{m}\,m'dx = \Im(E - k^2)\|m\|_2^2.$$

Then by Lemma 9.2, $\int_0^1 \overline{m}\,m'dx = -2\pi n i|\widehat{m}(\mp n,k)|^2 + O(k^{-1}\|m\|_2^2)$ and $\Im(E - k^2) = (v-q)(u+p) + (u-p)(v+q) = -(2u + O(u^{-1}))I_n + O((|v| + |q|)u^{-1})$. Substitute in (9.8), divide by $4\pi q|n|$ and use $0 \le v \le q$ and $I_n \le q$.

Lemma 9.6. *There is a fixed $\Gamma > 0$ such that for $|p - n\pi| > \Gamma|g_n|$ we have $|\widehat{m}_\pm(\mp n, k)| \leq \frac{1}{2}|\widehat{m}_\pm(0,k)|$.*

Proof. For $\Gamma \gg 1$ we have $I_n/q \ll 1$ in (9.7) by Lemma 6.2. Furthermore $|u|/(|n|\pi) = 1 + O(1/|u|)$.

Lemma 9.7. *There is a fixed C such that for any fixed $\Gamma > 0$ there are $N, \delta > 0$ such that for $|n| \geq N$ in the region $|p - n\pi| < \Gamma|g_n|$, and $0 \leq q \leq \delta|g_n|$ we have $\|m^0_+(k)\|_2 \|m^0_-(k)\|_2 \leq C$.*

Proof. First of all, by Lemma 9.5 for k near the slit we have $|\widehat{m}^0_\pm(j,k)|^2 \approx \|m^0_+(k)\|_2^2/2$ for $j = 0, \mp n$. These harmonics could be large with a large cancelation in (9.6). We have

$$A_+(k)A_-(k) + B_+(k)B_-(k) = \frac{\left(1 + \frac{\sin(k)}{k\varphi(k)}\right)^2 + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2}{4N^2(k)} + \\ + \frac{\left(1 - \frac{\sin(k)}{k\varphi(k)}\right)^2 + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2}{4N^2(k)} = \frac{1 + \left(\frac{\sin(k)}{k\varphi(k)}\right)^2 + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2}{2N^2(k)}.$$

By (9.1) we have

$$A_+(k)A_-(k) - B_+(k)B_-(k) = \frac{\sin(k)}{k\varphi(k)N^2} = \frac{\dot{E}}{2k} = \frac{w}{k\frac{dk}{dw}} \approx \frac{1}{\frac{dk}{dw}}. \tag{9.9}$$

We have $\|m^0_+(k)\|_2\|m^0_-(k)\|_2 \approx$

$$\approx |A_+(k)A_-(k)| \leq 1 + 2/|k'(w)| + O(k^{-1}\|m^0_+(k)\|_2\|m^0_-(k)\|_2).$$

By Lemma 9.8 below we obtain $\|m^0_+(k)\|_2\|m^0_-(k)\|_2 \lesssim 1$.

Lemma 9.8. *There is a fixed $C > 0$ such that for any fixed $\Gamma > 0$ there are $N, \delta > 0$ such that for $|n| \geq N$ in the region $|p - n\pi| < \Gamma|g_n|$, and $0 \leq q \leq \delta|g_n|$ we have $|k'(w)| > C$.*

Proof. We recall by Schwartz Christoffel formula, for $c_m \in g_m$ with $k'(c_m) = 0$,

$$k'(w) = a\frac{1 - w/c_n}{\sqrt{(1 - w/a^-_n)(1 - w/a^+_n)}} \prod_{\ell \neq n} \frac{1 - w/c_\ell}{\sqrt{(1 - w/a^-_\ell)(1 - w/a^+_\ell)}}, \tag{1}$$

see (1.10) [MO]. By taking the derivative dq/du in (5.1) we see that $q'(u) > 0$ for $u \in (a^-_m, a^-_m + \varepsilon|g_m|]$ and $q'(u) < 0$ for $u \in [a^+_m - \varepsilon|g_m|, a^+_m)$ for a fixed $\varepsilon > 0$. So

$c_m \in [a_m^- + \varepsilon|g_m|, a_m^+ - \varepsilon|g_m|]$ for a fixed sufficiently small $\varepsilon > 0$. Since the infinite product in (1) has value approximately 1 for w near πn, because of the second factor there is a fixed $C > 0$ such that $|k'(w)| > C$ holds for $w = u + iv$ with $|v| \le \delta|g_n|$ and either $a_n^- - \Gamma|g_n| \le u \le a_n^- + \varepsilon|g_n|/2$ or $a_n^+ - \varepsilon|g_n|/2 \le u \le a_n^+ + \Gamma|g_n|$ with $\delta > 0$ fixed and sufficiently small. Now we need to show that the values of $k = p + iq$ in the statement are inside this region in the w plane. First of all $0 \le q \le \delta|g_n|$ implies $0 \le v \le q \le \delta|g_n|$ by (6.1). For $0 \le v \le \delta|g_n|$ and $a_n^- + \delta|g_n| \ll u \ll a_n^+ - \delta|g_n|$ by Lemma 6.3 we have $q \approx I_n(u, v)$ and by Lemma 6.4 we have $I_n(u, v) \approx |g_n|^{\frac{1}{2}}\sqrt{\min\{|u - a_n^-|, |u - a_n^+|\}} \gg \delta|g_n|$. Obviously the latter is incompatible with $q \le \delta|g_n|$. So for small $\delta > 0$ we have either $a_n^- - \Gamma|g_n| \lesssim u \le a_n^- + \varepsilon|g_n|/2$ or $a_n^+ - \varepsilon|g_n|/2 \le u \lesssim a_n^+ + \Gamma|g_n|$.

Notice that by (9.9) and (1) in Lemma 9.8 we can see that near the tip of the slit the product $\|m_+^0(k)\|_2\|m_-^0(k)\|_2$ is unbounded.

§10 Proof of Lemma 4.5: case $u \in [a_n^+ + |g_n|^{\frac{1}{4}}, a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}]$

Lemma 4.5 consists in 3 claims. The third one follows immediately from Lemma 4.4 by the Cauchy integral formula. The first two claims follow immediately from the Cauchy integral formula from Lemma 10.1 which is an improvement of the second claim of Lemma 4.4 in the case when $a_n^+ + |g_n|^{\frac{1}{4}} \le u \le a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}$. We will assume this restriction on u everywhere below in this section:

Lemma 10.1. *For $a_n^+ + |g_n|^{\frac{1}{4}} \le u \le a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}$ with $n \ge n_0$ and for $1 \ge v \ge 0$, there is a C such that $\left|m_-^0(x,k)m_+^0(y,k) - 1\right| \le \frac{C}{|k|}$.*

Set $m_\pm(x) = m_\pm^0(x,k)$ and consider the expansion $m_\pm(x) = \sum_n e^{2\pi i n x}\widehat{m_\pm}(n)$. For $a_n^+ + |g_n|^{\frac{1}{4}} \le u \le a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}$ we have the following strengthening of Lemma 9.3:

Lemma 10.2. *We have for a fixed C*

$$||\widehat{m_\pm}(0)|^2 - \|m_\pm^0(k)\|_2^2| + \sum_{n\neq 0} |\widehat{m_\pm}(n)|^2 \le C|k|^{-2}\|m_\pm^0(k)\|_2^2.$$

Proof . By (9.8) and Lemma 6.2 we get $\Im \int_0^1 \bar{m}_\pm m_\pm' dx = O(u^{-2})\|m_\pm^0(k)\|_2^2$. By Lemma 9.2 $\Im \int_0^1 \bar{m}_\pm m_\pm' dx = \mp 2\pi i n_0 |\widehat{m}_\pm(\mp n_0, k)|^2 + O(k^{-1}\|m_\pm^0(k)\|_2^2)$. Hence $|\widehat{m_\pm}(\pm n_0)| \le C|k|^{-1}\|m_\pm^0(k)\|_2$. The latter and Lemma 9.2 imply the inequality $||\widehat{m_\pm}(0)|^2 - \|m_\pm^0(k)\|_2^2| \le C|k|^{-2}\|m_\pm^0(k)\|_2^2$.

Now we have the following lemma:

Lemma 10.3. *For fixed constants we have* $|N^{-1}| \approx |m^{\pm}(k)N^{-1}k^{-1}| \approx |A_{\pm}| \approx \|m^0_{\pm}(k)\|_2$ *and* $|B_{\pm}| \lesssim |k|^{-1}\|m^0_{\pm}(k)\|_2$.

Proof of Lemma 10.3. We get $|B_{\pm}| \lesssim |k|^{-1}\|m^0_{\pm}(k)\|_2$ by Lemmas 10.2 & 9.3 and by (9.4) . By $|\widehat{m}_{\pm}(0,k)| \approx \|m^0_{\pm}(k)\|_2$ and by (9.3) we get $|A_{\pm}| \approx \|m^0_{\pm}(k)\|_2$. By definition of $A_{\pm}$ and of $B_{\pm}$, estimates $|B_{\pm}| \lesssim |k|^{-1}\|m^0_{\pm}(k)\|_2$ and $|A_{\pm}| \approx \|m^0_{\pm}(k)\|_2$ imply $|1/N(k)| \approx |m^{\pm}(k)N^{-1}k^{-1}| \approx \|m^0_{\pm}(k)\|_2$.

Lemma 10.4. *We have for fixed constants* $\|m^0_{\pm}(k)\|_2 \approx 1$.

Proof of Lemma 10.4. By Lemma 10.3 $|A_{\pm}(k)| \approx |1/N| \approx \|m^0_{\pm}(k)\|_2$. By Lemma 10.2 and the formulas immediately above (9.5) $m^0_{\pm}(x,k) \approx A_{\pm}(k)$. In particular by normalization of Bloch functions we have the following which gives us Lemma 10.4:

$$1 = \int_0^1 m^0_{+}(x,k)m^0_{-}(x,k)dx \approx A_{+}(k)A_{-}(k).$$

Proof of Lemma 10.1. We write

$$m^0_{-}(x,k)m^0_{+}(y,k) = \frac{\left(1+\frac{\sin(k)}{k\varphi(k)}\right)^2 + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2}{4N^2(k)} + O(k^{-1}).$$

By the expansion of $N^2(k)$ in Lemma 9.1 and by Lemmas 8.2, 10.3 and 10.4, we have

$$2N^2(k) = 1 + \left(\frac{\sin(k)}{k\varphi(k)}\right)^2 + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2 + O(k^{-1}).$$

This, formula (9.1) and $\frac{\dot{E}}{2k} = 1 + O(k^{-1})$, Lemma 7.1 (5), imply the following,

$$\frac{1+\left(\frac{\sin(k)}{k\varphi(k)}\right)^2 + \left(\frac{\varphi'(k)-\theta(k)}{2k\varphi(k)}\right)^2}{4N^2(k)} + \frac{\frac{\sin(k)}{k\varphi(k)}}{2N^2(k)} = \frac{1}{2} + \frac{\dot{E}}{4k} + O(k^{-1}) = 1 + O(k^{-1}),$$

which ends the proof of Lemma 10.1. An immediate consequence of Lemma 10.1, of the geometry of the comb $\mathcal{K}$ and of the Cauchy integral formula is:

Lemma 10.5. *There is* n_0 *such that if for* $n \geq n_0$ *we have* $a_n^+ + |g_n|^{\frac{1}{4}} \leq u \leq \frac{a_n^+ + a_{n+1}^-}{2}$ *and* $v = 0$, *then there is a* C *such that for the corresponding* $k = p + i0$ *we have* $\left|\partial_k(m^0_{-}(x,k)m^0_{+}(y,k))\right| \leq \frac{C}{k|k-\pi n|}$. *If* $\frac{a_n^+ + a_{n+1}^-}{2} \leq u \leq a_{n+1}^- - |g_{n+1}|^{\frac{3}{5}}$ *then* $\left|\partial_k(m^0_{-}(x,k)m^0_{+}(y,k))\right| \leq \frac{C}{k|k-\pi(n+1)|}$.

References

[Cai] K.Cai, *Dispersion for Schrödinger operators with one gap periodic potentials in* $\mathbb{R}$, Dynamics Part. Diff. Eq. **3** (2006), 71–92.

[Cu] S.Cuccagna, *Stability of standing waves for NLS with perturbed Lamé potential*, J.Diff. Eq. **223** (2006), 112–160.

[Ea] M.Eastham, *The spectral theory of periodic differential operators*, Scottish Academic Press, London, 1973.

[F1] N.Firsova, *On the time decay of a wave packet in a one-dimensional finite band periodic lattice*, J. Math. Phys. **37** (1996), 1171–1181.

[F2] ———, *A direct and inverse scattering problem for a one-dimensional perturbed Hill operator*, Math. USSR-Sb. **58** (1987), 351–388.

[K1] E.Korotyaev, *The propagation of the waves in periodic media at large time*, Math. Asymptot. Anal. **15** (1997), 1–24.

[K2] ———, *Some properties of the quasimomentum of the one-dimensional Hill operator*, J. Soviet Math. **6** (1992), 3081–3087.

[KK] P.Kargaev, E.Korotyaev, *Effective masses and conformal mapping*, Comm. Math. Phys. **169** (1995), 597–625.

[MO] V.Marcenko, I.Ostrovski, *A characterization of the spectrum of Hill's operator*, Math. URSS Sbornik **26** (1975), 402–554.

[RS] M.Reed, B.Simon, *Methods of mathematical physics*, Academic Press.

[Ste] E.Stein, *Harmonic analysis*, Princeton mathematical series 43, Princeton U. Press, 1993.

[Str] W.Strauss, *Nonlinear wave equations*, CBMS Regional Conf. Ser. Mat. 76, AMS, 1989, pp. 173–190.

DISMI University of Modena and Reggio Emilia, Via Amendola 2, Padiglione Morselli, Reggio Emilia 42100 Italy

E-mail address: `cuccagna.scipio@unimore.it`