Uniqueness of Invariant Lagrangian Graphs in a Homology or a Cohomology Class.

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Given a smooth compact Riemannian manifold M and a Hamiltonian H on the cotangent space T^*M, strictly convex and superlinear in the momentum variables, we prove uniqueness of certain "ergodic" invariant Lagrangian graphs within a given homology or cohomology class. In particular, in the context of quasi-integrable Hamiltonian systems, our result implies global uniqueness of Lagrangian KAM tori with rotation vector ρ. This result extends generically to the C^0-closure of KAM tori.

1 Introduction

A particularly interesting and fruitful approach to the study of local and global properties of dynamical systems is concerned with the study of invariant submanifolds, rather than single orbits, paying particular attention to their existence, their "fate" and their geometric properties. In the context of quasi-integrable Hamiltonian systems, one of the most celebrated breakthroughs in this kind of approach was KAM theory, which provided a method to construct invariant submanifolds diffeomorphic to tori, on which the dynamics is conjugated to a quasi-periodic motion with rotation vector ρ, sometimes referred to as *KAM tori*. KAM theory finally settled the old question about *existence* of such invariant submanifolds in "generic" quasi-integrable Hamiltonian systems, dating back at least to Poincaré, and opened the way to a new understanding of the nature of Hamiltonian systems, of their stability and of the onset of chaos in classical mechanics. However, the natural question about the *uniqueness* of these invariant submanifolds for a fixed rotation vector ρ remained open for many more years and, quite surprisingly, even nowadays, for many respects it is still unanswered. A possible reason for this is that the analytic methods, which the KAM algorithm is based on, are not well suited for studying global questions, while, on the other hand, the natural variational methods to approach this problem have been developed only much more recently and they are still not so widely well-known.

In this paper we address the above problem and prove global uniqueness of certain "ergodic" invariant Lagrangian graphs within a given homology or cohomology class, for a large class of convex Hamiltonians, known as Tonelli Hamiltonians. Our work will be based on the variational approach provided by the so-called Aubry-Mather theory, as well as its functional side called weak KAM theory.

The paper is organized as follows. In Section 2 we define the geometric objects we shall look at (the invariant Lagrangian graphs), we introduce some concepts (homology class of an invariant measure and Schwartzman ergodicity), which will turn out to be useful for illustrating their dynamical properties, and we state our main uniqueness results. In Section 3 we review the main definitions and results of Aubry-Mather and weak KAM theories, to the extent of what we need for our proofs. In Section 4 we prove our main results. In Section 5 we discuss in detail the implications of our results for KAM theory and compare them with some previous local uniqueness theorems for KAM tori. In Appendix A we discuss some details concerning the definition of Schwartzman ergodicity, give some examples and describe some properties of Schwartzman ergodic flows. In Appendices B and C we prove some properties of Lagrangian graphs and the Mañé set.

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2 Setting and Main results

Let M be a compact and connected smooth manifold without boundary of dimension n. Denote by $\mathrm{T}M$ its tangent bundle and T^*M the cotangent one. A point of $\mathrm{T}M$ will be denoted by (x, v), where $x \in M$ and $v \in \mathrm{T}_x M$, and a point of T^*M by (x, p), where $p \in \mathrm{T}_x^*M$ is a linear form on the vector space $\mathrm{T}_x M$. Let us fix a Riemannian metric g on M and let $\|\cdot\|_x$ be the norm induced by g on $\mathrm{T}_x M$; we shall use the same notation for the norm induced on the cotangent space T_x^*M. This cotangent space T^*M can be canonically endowed with a symplectic structure, given by the exact 2-form $\omega = dx \wedge dp = -d(pdx)$, where $(\mathcal{U}, x)$ is a local coordinate chart for M and $(\mathrm{T}^*\mathcal{U}, x, p)$ the associated cotangent coordinates. The 1-form $\lambda = pdx$ is also called *tautological form* (or *Liouville form*) and is intrisically defined, *i.e.,* independently of the choice of local coordinates (see Appendix B). A distinguished role in the study of the geometry of a symplectic space is played by the so-called *Lagrangian submanifolds*.

Definition 2.1. Let Λ be an n-dimensional C^1 submanifold of (T^*M, ω). We say that Λ is *Lagrangian* if for any $(x, p) \in \Lambda$, $\mathrm{T}_{(x,p)}\Lambda$ is a Lagrangian subspace, *i.e.,* $\omega\big|_{\mathrm{T}_{(x,p)}\Lambda} = 0$.

We shall mainly be concerned with Lagrangian graphs, that is Lagrangian manifolds $\Lambda \subset T^*M$ such that $\Lambda = \{(x, \eta(x)),\ x \in M\}$. It is straightforward to check that the graph Λ is Lagrangian if and only if η is a closed 1-form (see Appendix B). The element $c = [\eta] \in \mathrm{H}^1(M; \mathbb{R})$ is called the *cohomology class*, or *Liouville class*, of Λ. This motivates the following extension of the notion of Lagrangian graph to the continuous case.

Definition 2.2. A continuous section Λ of T^*M is a C^0-Lagrangian graph if it locally coincides with the graph of an exact differential.

We will consider the dynamics on T^*M generated by a Tonelli Hamiltonian.

Definition 2.3. A function $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ is called a *Tonelli (or optical) Hamiltonian* if:

i) the Hamiltonian H is of class C^k, with $k \geq 2$;

ii) the Hamiltonian H is strictly convex in the fiber in the C^2 sense, *i.e.,* the second partial vertical derivative $\partial^2 H/\partial p^2(x,p)$ is positive definite, as a quadratic form, for any $(x,p) \in \mathrm{T}^*M$;

iii) the Hamiltonian H is superlinear in each fiber, *i.e.,*

$$\lim_{\|p\|_x \to +\infty} \frac{H(x,p)}{\|p\|_x} = +\infty$$

(by the compactness of M, this condition is independent of the choice of the Riemannian metric).

Given H, we can define the associated *Lagrangian*, as a function on the tangent bundle:

$$\begin{aligned} L:\ \mathrm{T}M &\longrightarrow \mathbb{R} \\ (x,v) &\longmapsto \sup_{p\in \mathrm{T}_x^*M} \{\langle p,\, v\rangle_x - H(x,p)\} \end{aligned}$$

where $\langle \cdot, \cdot \rangle_x$ represents the canonical pairing between the tangent and cotangent space.

If H is a Tonelli Hamiltonian, one can easily prove that L is finite everywhere, of class C^k, superlinear and strictly convex in the fiber in the C^2 sense (*i.e., L is a Tonelli Lagrangian*) and the associated Euler-Lagrange flow Φ_t^L of L is conjugated to the Hamiltonian flow Φ_t^H of H via the *Legendre transform*:

$$\mathcal{L}:\ \mathrm{T}M \longrightarrow \mathrm{T}^*M \tag{1}$$

$$(x,v) \longmapsto \left(x,\, \frac{\partial L}{\partial v}(x,v)\right). \tag{2}$$

From now on we shall fix H and denote by L its conjugated Lagrangian and, when referring to an "invariant" measure or set, we will understand "invariant with respect to the Hamilton flow generated by H" or "with respect to the Euler-Lagrange flow generated by L".

Given an invariant probability measure μ on $\mathrm{T}M$ one can associate to it an element $\rho(\mu)$ of the homology group $\mathrm{H}_1(M;\mathbb{R})$, known as *rotation vector* or *Schwartzman asymptotic cycle*, which generalizes the notion of rotation vector given by Poincaré and describes how, asymptotically, a μ-average orbit winds around $\mathrm{T}M$. See Section 3 and Appendix A for a precise definition. This allows us to define the homology class of certain invariant Lagrangian graphs.

Definition 2.4. A Lagrangian graph Λ is called *Schwartzman uniquely ergodic* if all invariant measures supported on Λ have the same rotation vector ρ, which will be called *homology class of* Λ. Moreover, if there exists an invariant measure with full support, Λ will be called *Schwarztman strictly ergodic.*

We are now ready to state our main results.

Main Result. *For any given $\rho \in \mathrm{H}_1(M;\mathbb{R})$, there exists at most one Schwarzman strictly ergodic invariant Lagrangian graph with homology class ρ* [Theorem 4.4, Section 4]

For sake of completeness, we also recall the following well-known result, which is a corollary of the results in [25] (see also Section 4 for a proof).

Well-known Result. *For any given $c \in \mathrm{H}^1(M;\mathbb{R})$, there exists at most one invariant Lagrangian graph Λ with cohomology class c, carrying an invariant measure whose support is the whole of* Λ. [Theorem 4.2, Section 4]

If $M = \mathbb{T}^n$, it is natural to ask for the implications of our result for KAM theory. In this case, the homology group $\mathrm{H}_1(\mathbb{T}^n;\mathbb{R})$ is canonically identified with $\mathbb{R}^n$, and the invariant manifolds of interest are the so-called KAM tori, defined as follows.

Definition 2.5. $\mathcal{T} \subset \mathbb{T}^n \times \mathbb{R}^n$ is a (maximal) KAM torus with rotation vector ρ if:
i) $\mathcal{T} \subset \mathbb{T}^n \times \mathbb{R}^n$ is a continuous graph over $\mathbb{T}^n$;
ii) $\mathcal{T}$ is invariant under the Hamiltonian flow Φ_t^H generated by H;
iii) the Hamiltonian flow on $\mathcal{T}$ is conjugated to a uniform rotation on $\mathbb{T}^n$; *i.e.,* there exists a diffeomorphism $\varphi : \mathbb{T}^n \to \mathcal{T}$ such that $\varphi^{-1} \circ \Phi_t^H \circ \varphi = R_\rho^t$, $\forall t \in \mathbb{R}$, where $R_\rho^t : x \to x + \rho t$.

The celebrated KAM Theorem, whose statement will be recalled in Section 5, gives sufficient conditions on H and on the rotation vector ρ, allowing one to construct a KAM torus with rotation vector ρ and prescribed regularity (depending on the regularity class of H). Its proof is constructive and the invariant torus one manages to construct is locally unique (in a sense that will be clarified in Section 5). In spite of the long history and the huge literature dedicated to the KAM theorem, the issue of global uniqueness of such tori is still object of some debate and study, see for instance [5]. Our main result settles such question in the case of Tonelli Hamiltonians.

Corollary 2.6 (Global uniqueness of KAM tori)**.** *Every Tonelli Hamiltonian H on $\mathrm{T}^*\mathbb{T}^n$ possesses at most one Lagrangian KAM torus for any given rotation vector ρ. In particular, if H and ρ satisfy the assumptions of the KAM Theorem, then there exists one and only one KAM torus with rotation vector ρ.*

The property of being Lagrangian plays a crucial role. As was observed by Herman [18] (see also Appendix B), when ρ is *rationally independent*, *i.e.,* $\langle \rho, \nu \rangle \neq 0$, $\forall \nu \in \mathbb{Z}^n \setminus \{0\}$, as assumed in the KAM theorem, every KAM torus with frequency ρ is automatically Lagrangian. On the other hand, the existence of Lagrangian KAM tori with rationally dependent frequency is not typical. In some cases, a variant of the classical KAM algorithm allows one to construct *resonant* invariant tori with a given rational rotation vector ρ, also known as low dimensional tori [15, 16]. However, typically they do not foliate any Lagrangian submanifold. Therefore, the question of uniqueness of resonant tori is more subtle and, to our knowledge, apart from a few partial results [8], it is still open.

Note that the fact that the orbits on a KAM torus are action-minimizing is a key remark used in the proof of the Corollary. We deduce this property from the fact that such a Lagrangian torus gives a solution to the Hamilton-Jacobi equation. It can be also deduced from a classical result by Weierstrass, as already pointed out by Jürgen Moser. We thank John Mather for having drawn our attention to the remark in [25]. [In fact, Weierstrass method or the use of the Hamilton-Jacobi Equation are essentially two sides of the same coin].

In Section 5 we will extend Corollary 2.6 to generic invariant tori contained in the C^0-closure of the set of KAM tori. As remarked by Herman [19], generically this set is much larger than the set of KAM tori, and typically the flow on such invariant manifolds is not conjugated to a rotation. See Section 5 for a more detailed discussion of these issues.

3 An introduction to Aubry-Mather theory

This section is meant to provide a brief (but comprehensive) introduction to Mather's theory for Lagrangian systems and weak KAM theory. We shall recall most of the results that we are going to use, trying to give general ideas rather than rigourous proofs (for which we refer the reader to [12, 25]).

The setting is the same introduced in Section 2. Let M be a compact and connected smooth manifold without boundary of dimension n. We denote by H a fixed Tonelli Hamiltonian and L the associated Tonelli Lagrangian. Before entering into the details of Mather's theory, let us make a crucial remark. Observe that if η is a 1-form on M, we can easily define a function on the tangent space (linear on each fiber)

$$\begin{aligned} \hat{\eta} : \mathrm{T}M &\longrightarrow \mathbb{R} \\ (x,v) &\longmapsto \langle \eta(x),\, v \rangle_x \end{aligned}$$

and consider a new Tonelli Lagrangian $L_\eta := L - \hat{\eta}$. The associated Hamiltonian will be $H_\eta(x,p) = H(x, \eta(x) + p)$. Moreover, if η is closed, then $\int L\, dt$ and $\int L_\eta dt$ will have the same extremals and therefore the Euler-Lagrange flows on $\mathrm{T}M$ associated to L and L_η will be the same. This last point may be seen by observing that the variational equations $\delta \int L\, dt = 0$ and $\delta \int (L - \hat{\eta})\, dt = 0$ for the fixed end-point problem clearly have the same solutions, since η is closed. Although the extremals are the same, this is not generally true for the orbits

"minimizing the action" (we shall give a precise definition of "minimizers" later in this section). What one can say is that they stay the same when we change the Lagrangian by an exact 1-form. Thus, for a fixed L, the minimizers will depend only on the de Rham cohomology class $c = [\eta] \in \mathrm{H}^1(M; \mathbb{R})$. Considering modified Lagrangians corresponding to different cohomology classes represents a keystone of Mather's theory of Lagrangians systems.

In order to generalize to more degrees of freedom Aubry and Mather's variational approach to twist maps, a first important notion is that of *minimal measures*, which replaces that of action minimizing orbits: Aubry-Mather theory in higher dimension cannot deal with such orbits, due to a lack of them; there is, in fact, a classical example due to Hedlund (see [17]) of a Riemannian metric on a three-dimensional torus, for which minimal geodesics exist only in three directions. Instead, Mather proposed to look at the closely related notion of action minimizing invariant probability measures. Let $\mathfrak{M}(L)$ be the space of probability measures on $\mathrm{T}M$ invariant under the Euler-Lagrange flow of L. To every $\mu \in \mathfrak{M}(L)$, we may associate its *average action*

$$A_L(\mu) = \int_{\mathrm{T}M} L\, d\mu\,.$$

Since L is bounded below (because of the superlinear growth condition), this integral exists although it might be $+\infty$. In [25], Mather showed the existence of $\mu \in \mathfrak{M}(L)$ such that $A_L(\mu) < +\infty$. The argument is mainly the same as Krylov-Bogoliubov's theorem concerning existence of invariant measures for flows on compact spaces. This argument is applied to a one-point compactification of $\mathrm{T}M$, and the main step consists in showing that the measure provided by this construction has no atomic part supported at ∞ (which is a fixed point for the extended system). Note that Mather's approach works for periodic time-dependent Lagrangians. For time independent Lagrangians, finding such a μ is much easier. By conservation of energy, the levels of the energy function are compact and invariant under the Euler-Lagrange flow Φ_t^L, and therefore carry such measures. In case $A_L(\mu) < \infty$, thanks to the superlinearity of L, for any closed 1-form η on M, the integral $\int_{\mathrm{T}M} \hat{\eta} d\mu$ is well defined and finite, see [25]. Moreover, it is quite easy to show (again see [25]) that since μ is invariant by the Euler-Lagrangian flow Φ_t^L, if $\eta = df$ is an exact form, then $\int \widehat{df}\, d\mu = 0$. Therefore, we can define a linear functional

$$\begin{array}{rcl} \mathrm{H}^1(M;\mathbb{R}) & \longrightarrow & \mathbb{R} \\ c & \longmapsto & \displaystyle\int_{\mathrm{T}M} \hat{\eta} d\mu\,, \end{array}$$

where η is any closed 1-form on M, whose cohomology class is c. By duality, there exists $\rho(\mu) \in \mathrm{H}_1(M;\mathbb{R})$ such that

$$\int_{\mathrm{T}M} \hat{\eta} d\mu = \langle c, \rho(\mu)\rangle \qquad \forall\, c \in \mathrm{H}^1(M;\mathbb{R})$$

(the bracket on the right–hand side denotes the canonical pairing between cohomology and homology). We call $\rho(\mu)$ the *rotation vector* of μ. It is the same as the Schwartzman's asymptotic cycle of μ (see Appendix A and [31]). One can show that the action functional $A_L : \mathfrak{M}(L) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and that for every $h \in \mathrm{H}_1(M;\mathbb{R})$ there exists $\mu \in \mathfrak{M}(L)$ with $\rho(\mu) = h$. In other words the map $\rho : \mathfrak{M}(L) \longrightarrow \mathrm{H}_1(M;\mathbb{R})$ is surjective.

Following Mather, let us consider the minimal value of the average action A_L over the probability measures with rotation vector h:

$$\begin{array}{rcl} \beta : \mathrm{H}_1(M;\mathbb{R}) & \longrightarrow & \mathbb{R} \\ h & \longmapsto & \displaystyle\min_{\mu \in \mathfrak{M}(L):\, \rho(\mu)=h} A_L(\mu)\,. \end{array} \tag{3}$$

In fact, the minimum above is actually achieved (see [25]). This function β is what is generally known as *Mather's β-function*. A measure $\mu \in \mathfrak{M}(L)$ realizing such a minimum, *i.e.,* $A_L(\mu) = \beta(\rho(\mu))$, is called a *minimal* (or a *Mather*) *measure*. The β-function is convex, and therefore one can consider its *conjugate* function (given by Fenchel's duality) $\alpha : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$\begin{array}{rcl} \alpha(c) & := & \displaystyle\max_{h \in \mathrm{H}_1(M;\mathbb{R})} \left(\langle c, h\rangle - \beta(h)\right) = \\ & = & -\displaystyle\min_{h \in \mathrm{H}_1(M;\mathbb{R})} \left(\beta(h) - \langle c, h\rangle\right) = \\ & = & -\displaystyle\min_{\mu \in \mathfrak{M}(L)} \left(A_L(\mu) - \langle c, \rho(\mu)\rangle\right) = \\ & = & -\displaystyle\min_{\mu \in \mathfrak{M}(L)} A_{L-c}(\mu). \end{array}$$

Note that for a given μ invariant under the Euler-Lagrange flow Φ_t^L, we have $\langle c, \rho(\mu)\rangle = \int_{\mathrm{T}M} \hat{\eta}\, d\mu$, for any closed 1-form η on M in the cohomology class c. Therefore, we have $A_{L-c}(\mu) = \int_{\mathrm{T}M} (L - \hat{\eta})\, d\mu$, for η a closed 1-form on M such that $[\eta] = c$.

It is important to notice that the value of this α-function coincides with what is called *Mañé's critical value*, which will be introduced later in this section.

An important fact is the next Lemma. To state it, recall that, like any convex function on a finite-dimensional space, the Mather function β admits a subderivative at any point $h \in \mathrm{H}_1(M;\mathbb{R})$, *i.e.,* we can find $c \in \mathrm{H}^1(M;\mathbb{R})$ such that

$$\forall h' \in \mathrm{H}_1(M;\mathbb{R}), \quad \beta(h') - \beta(h) \geq \langle c, h' - h \rangle.$$

As it is usually done, we will denote by $\partial\beta(h)$ the set of $c \in \mathrm{H}^1(M;\mathbb{R})$ which are subderivatives of β at h, *i.e.,* the set of c which satisfy the inequality above. By Fenchel's duality, we have

$$c \in \partial\beta(h) \iff \langle c, h \rangle = \alpha(c) + \beta(h).$$

Lemma 3.1. *If $\mu \in \mathfrak{M}(L)$, then $A_L(\mu) = \beta(\rho(\mu))$ if and only if there exists $c \in \mathrm{H}^1(M;\mathbb{R})$ such that μ minimizes A_{L-c} (i.e., $A_{L-c}(\mu) = -\alpha(c)$). Moreover, if $\mu \in \mathfrak{M}(L)$ satisfies $A_L(\mu) = \beta(\rho(\mu))$, and $c \in \mathrm{H}^1(M;\mathbb{R})$, then μ minimizes A_{L-c} if and only if $c \in \partial\beta(\rho(\mu))$ (or $\langle c, h \rangle = \alpha(c) + \beta(\rho(\mu))$.*

Proof. We will prove both statement at the same time. Assume $A_L(\mu_0) = \beta(\rho(\mu_0))$. Let $c \in \partial\beta(\rho(\mu_0))$, by Fenchel's duality this is equivalent to

$$\begin{aligned}\alpha(c) &= \langle c, \rho(\mu_0) \rangle - \beta(\rho(\mu_0)) \\ &= \langle c, \rho(\mu_0) \rangle - A_L(\mu_0) \\ &= -A_{L-c}(\mu_0).\end{aligned}$$

Therefore by what was obtained above $A_{L-c}(\mu_0) = \min_{\mu \in \mathfrak{M}} A_{L-c}(\mu)$.

Assume conversely that $A_{L-c}(\mu_0) = \min_{\mu \in \mathfrak{M}} A_{L-c}(\mu)$, for some given cohomology class c. Therefore by what was obtained above

$$\alpha(c) = -A_{L-c}(\mu_0),$$

which can be written as

$$\langle c, \rho(\mu_0) \rangle = \alpha(c) + A_L(\mu_0).$$

It now suffices to use the Fenchel inequality $\langle c, \rho(\mu_0) \rangle \leq \alpha(c) + \beta(\rho(\mu_0)$, and the inequality $\beta(\rho(\mu_0) \leq A_L(\mu_0)$, given by the definition of β, to obtain the equalities

$$\langle c, \rho(\mu_0) \rangle = \alpha(c) + A_L(\mu_0).$$

In particular, we have $A_L(\mu_0) = \beta(\rho(\mu_0))$. □

If $\mu \in \mathfrak{M}(L)$ and μ minimizes A_{L-c}, we shall say that μ is a *c-action minimizing measure.* For such μ, c is a subderivative of β at $\rho(\mu)$, *i.e.,* the slope of a supporting hyperplane of the epigraph of β at $\rho(\mu)$.
The above discussion leads to two equivalent formulations for the minimality of a measure μ:

- there exists a homology class $h \in \mathrm{H}_1(M;\mathbb{R})$, namely its rotation vector $\rho(\mu)$, such that μ minimizes A_L amongst all measures in $\mathfrak{M}(L)$, with rotation vector h; *i.e.,* $A_L(\mu) = \beta(\rho(\mu))$;
- there exists a cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, namely any subderivative of β at $\rho(\mu)$, such that μ minimizes A_{L-c} amongst all measures in $\mathfrak{M}(L)$; *i.e.,* $A_{L-c}(\mu) = -\alpha(c)$.

For $h \in \mathrm{H}_1(M;\mathbb{R})$ and $c \in \mathrm{H}^1(M;\mathbb{R})$, let us define

$$\begin{aligned} \mathfrak{M}^h &:= \{\mu \in \mathfrak{M}(L):\ A_L(\mu) < +\infty,\ \rho(\mu) = h \text{ and } A_L(\mu) = \beta(h)\} \\ \mathfrak{M}_c &:= \{\mu \in \mathfrak{M}(L):\ A_L(\mu) < +\infty \text{ and } A_{L-c}(\mu) = -\alpha(c)\}. \end{aligned}$$

Observe that because of the superlinear growth condition in the fiber, $A_L(\mu) < +\infty$ implies $A_{L-c}(\mu) < +\infty$.

We have

$$\bigcup_{h \in \mathrm{H}_1(M;\mathbb{R})} \mathfrak{M}^h = \bigcup_{c \in \mathrm{H}^1(M;\mathbb{R})} \mathfrak{M}_c\,.$$

This leads to the definition of a first important family of invariant sets: *Mather sets*. For a cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we call *Mather set of cohomology class* c the set:

$$\widetilde{\mathcal{M}}_c := \overline{\bigcup_{\mu \in \mathfrak{M}_c} \operatorname{supp}\mu} \subset \mathrm{T}M\,; \tag{4}$$

the projection on the base manifold $\mathcal{M}_c = \pi\left(\widetilde{\mathcal{M}}_c\right) \subseteq M$ is called *projected Mather set* (with cohomology class c). In [25], Mather proved the celebrated *graph theorem*:

Theorem 3.2. *Let $\widetilde{\mathcal{M}}_c$ be defined as in (4). The set $\widetilde{\mathcal{M}}_c$ is compact, invariant under the Euler-Lagrange flow and $\pi|\widetilde{\mathcal{M}}_c$ is an injective mapping of $\widetilde{\mathcal{M}}_c$ into M, and its inverse $\pi^{-1} : \mathcal{M}_c \longrightarrow \widetilde{\mathcal{M}}_c$ is Lipschitz. Moreover this set is contained in the energy level corrisponding to the value $\alpha(c)$, i.e.,*

$$H \circ \mathcal{L}(x,v) = \alpha(c) \qquad \forall\,(x,v) \in \widetilde{\mathcal{M}}_c\,. \tag{5}$$

Remark 3.3. The last statement, corresponding to (5), is due to Dias-Carneiro [7].

Analogously, one can consider the *Mather set corresponding to a rotation vector* $h \in \mathrm{H}_1(M;\mathbb{R})$ as

$$\widetilde{\mathcal{M}}^h := \overline{\bigcup_{\mu \in \mathfrak{M}^h} \operatorname{supp}\mu} \subset \mathrm{T}M\,, \tag{6}$$

and the projected one $\mathcal{M}^h = \pi\left(\widetilde{\mathcal{M}}^h\right) \subseteq M$.

Notice that by Lemma 3.1, if $c \in \partial\beta(h)$, we have

$$\widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c.$$

Therefore, although this was not shown in [25], the set $\widetilde{\mathcal{M}}^h$ also has a Lipschitz graph over the basis.

Theorem 3.4. *Let $\widetilde{\mathcal{M}}^h$ be defined as in (6). $\widetilde{\mathcal{M}}^h$ is compact, invariant under the Euler-Lagrange flow and $\pi|\widetilde{\mathcal{M}}^h$ is an injective mapping of $\widetilde{\mathcal{M}}^h$ into M and its inverse $\pi^{-1} : \mathcal{M}^h \longrightarrow \widetilde{\mathcal{M}}^h$ is Lipschitz.*

Remark 3.5. Though the graph property for $\widetilde{\mathcal{M}}^h$ is not proved in [25], it was shown there that the support of an h-minimizing measure has the graph property. The graph property for $\widetilde{\mathcal{M}}^h$ can be easily deduced from this last property. In fact, since the space of probability measures on $\mathrm{T}M$ is a separable metric space, one can take a countable dense set $\{\mu_n\}_{n=1}^{\infty}$ of Mather's measures with rotation vector h and consider the new measure $\tilde{\mu} = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n$. This is still an invariant probability measure with rotation vector h and $\operatorname{supp}\tilde{\mu} = \widetilde{\mathcal{M}}_h$. Therefore, as the support of a single h-minimizing measure, $\widetilde{\mathcal{M}}^h$ has the graph property.

Moreover, this remark points out that there always exist Mather's measures μ^h and μ_c of full support, *i.e.,* $\operatorname{supp}\mu^h = \widetilde{\mathcal{M}}^h$ and $\operatorname{supp}\mu_c = \widetilde{\mathcal{M}}_c$.

We will say that an h-minimizing (resp. c-minimizing) measure μ has maximal support if $\operatorname{supp}\mu = \widetilde{\mathcal{M}}^h$ (resp. $\operatorname{supp}\mu = \widetilde{\mathcal{M}}_c$).

In addition to the Mather sets, one can construct other compact invariant sets, that play an interesting role both from a dynamical system and a geometric point of view: the *Aubry sets* and the *Mañé sets*. Instead of considering action minimizing invariant probability measures, we now shift our attention to *c-minimizing curves*.

Let us fix η a closed 1-form on M, with cohomology class c. As done by Mather in [26], it is convenient to introduce, for $t > 0$ and $x,\, y \in M$, the following quantity:

$$h_{\eta,t}(x,y) = \inf \int_0^t L_\eta(\gamma(s),\dot{\gamma}(s))\, ds\,,$$

where the infimum is taken over all piecewise C^1 paths $\gamma : [0,t] \longrightarrow M$, such that $\gamma(0) = x$ and $\gamma(t) = y$. We define the *Peierls' barrier* as:

$$h_\eta(x,y) = \liminf_{t\to+\infty} (h_{\eta,t}(x,y) + \alpha(c)t)\,.$$

It can be shown that this function is Lipschitz and that, only in the autonomous case, this $\liminf$ can be replaced with a $\lim$ (see [12]). Observe that h_η does not depend only on the cohomology class c, but also on the choice of the representant; namely, if $\eta' = \eta + df$, then $h_{\eta'}(x,y) = h_\eta(x,y) + f(y) - f(x)$.

Proposition 3.6. *The values of the map h_η are finite. Moreover, the following properties hold:*

i) *h_η is Lipschitz;*

ii) *for each $x \in M$, $h_\eta(x,x) \geq 0$;*

iii) *for each $x,\, y,\, z \in M$, $h_\eta(x,y) \leq h_\eta(x,z) + h_\eta(z,y)$;*

iv) *for each $x,\, y \in M$, $h_\eta(x,y) + h_\eta(y,x) \geq 0$.*

For a proof of the above claims and more, see [26, 12, 10]. Inspired by these properties, one can consider its symmetrization:

$$\begin{array}{rccl} \delta_c : & M \times M & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & h_\eta(x,y) + h_\eta(y,x) \end{array}$$

(observe that this function does depend only on the cohomology class). This function is positive, symmetric and satisfies the triangle inequality; therefore, it is a pseudometric on

$$\mathcal{A}_c = \{x \in M :\ \delta_c(x,x) = 0\}\,.$$

$\mathcal{A}_c$ is called the *projected Aubry set* associated to L and c, and δ_c is *Mather's pseudometric*. In [26], Mather has showed that this is a non-empty compact subset of M, that can be Lipschitzly lifted to a compact invariant subset of $\mathrm{T}M$.

Remark 3.7. One can easily construct a metric space out of it. We call *quotient Aubry set*, or *Mather quotient*, the metric space $(\bar{\mathcal{A}}_c,\, \bar{\delta}_c)$ obtained by identifying two points in $\mathcal{A}_c$, if their δ_c-pseudodistance is zero. We shall denote an element of this quotient by $\bar{x} = \{y \in \mathcal{A}_c :\ \delta_c(x,y) = 0\}$. These elements (that are also called *c-static classes*, see [10]) provide a partition of $\mathcal{A}_c$ into compact subsets, that can be lifted to invariant subsets of $\mathrm{T}M$.

We say that an absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a *c-minimizer*, if for any interval $[a,b]$ and any other absolutely continuous curve $\gamma_1 : [a,b] \longrightarrow M$ such that $\gamma(a) = \gamma_1(a)$ and $\gamma(b) = \gamma_1(b)$, we have

$$\int_a^b L_\eta(\gamma(t),\dot\gamma(t))dt \leq \int_a^b L_\eta(\gamma_1(t),\dot\gamma_1(t))dt\,.$$

In other words, $\gamma : \mathbb{R} \longrightarrow M$ is a c-minimizer, if for any $a < b$

$$\int_a^b L_\eta(\gamma(t),\dot\gamma(t))dt = h_{\eta,b-a}(\gamma(a),\gamma(b))\,.$$

This curves are also called *semi-static curves*. A peculiar property of these curves is that their α-limit set is contained in a static class and their ω-limit set too (possibly in a different one). See [26, 12, 10] for more details.

We define the *Mañé set* (with cohomology class c) as:

$$\widetilde{\mathcal{N}}_c = \bigcup \{(\gamma(t),\dot\gamma(t)) :\ \gamma \text{ is a } c\text{-minimizer and } t \in \mathbb{R}\}\,. \tag{7}$$

In particular, amongst these minimizers, one can consider particular ones that satisfy some extra conditions on the α and ω-limit sets. Namely, let $\gamma : \mathbb{R} \longrightarrow M$ be a c-minimizer and α, α' be in the α-limit set of γ and let ω, ω' be in the ω-limit set of γ. From the discussion in [26], it follows that $\delta_c(\alpha,\alpha') = \delta_c(\omega,\omega') = 0$ and $\delta_c(\alpha,\omega) = \delta_c(\alpha',\omega')$. We say that γ is a *c-regular minimizer* if a further condition is satisfied: $\delta_c(\alpha,\omega) = 0$. An equivalent condition is: γ is semi-static and for any interval $a < b$, we have that $h_{\eta,b-a}(\gamma(a),\gamma(b)) = -h_{\eta,b-a}(\gamma(b),\gamma(a))\,.$ These curves are also called *static curves*. Observe that the above condition can be interpreted saying that the α and ω-limit sets of γ are both contained in the same c-static class.

We define the *Aubry set* (with cohomology class c) as:

$$\widetilde{\mathcal{A}}_c = \bigcup \{(\gamma(t),\dot\gamma(t)) :\ \gamma \text{ is a } c\text{-regular minimizer and } t \in \mathbb{R}\}\,. \tag{8}$$

Similarly to the Mather set, the Aubry set is also a graph over the basis, see [26, 12].

Theorem 3.8. *Let $\widetilde{\mathcal{A}}_c$ be defined as in (8). By definition $\widetilde{\mathcal{A}}_c \subseteq \widetilde{\mathcal{N}}_c$. Both $\widetilde{\mathcal{A}}_c$ and $\widetilde{\mathcal{N}}_c$ are compact subsets of TM invariant under the Euler-Lagrange flow Φ_t^L. Moreover, $\pi\big(\widetilde{\mathcal{A}}_c\big) = \mathcal{A}_c$ and $\pi|\widetilde{\mathcal{A}}_c$ is a bi-Lipschitz homeomorphism.*

Note that the graph property does not hold for the Mañé set. Moreover, there is a clear relation between the support of Mather's measures and these sets.

Theorem 3.9. *For any $c \in \mathrm{H}^1(M;\mathbb{R})$, $\widetilde{\mathcal{M}}_c \subseteq \widetilde{\mathcal{A}}_c$. Moreover:*

$$\mu \in \mathfrak{M}_c \iff \mu \in \mathfrak{M}(L) \text{ and } \operatorname{supp}\mu \subseteq \widetilde{\mathcal{A}}_c \iff \mu \in \mathfrak{M}(L) \text{ and } \operatorname{supp}\mu \subseteq \widetilde{\mathcal{N}}_c\,. \tag{9}$$

The fact that an invariant measure whose support is contained in $\widetilde{\mathcal{N}}_c$ (or $\widetilde{\mathcal{A}}_c$) is c-minimizing might be seen, for example, as a consequence of the finiteness of the Peierls' barrier. The proof of the first equivalence in (9) goes back to Mañé [22]. As far as the second equivalence is concerned, it follows from the fact that the *non-wandering set* of $\Phi_t^L\big|\widetilde{\mathcal{N}}_c$ is contained in $\widetilde{\mathcal{A}}_c$, see Appendix C.

We can summarize what discussed so far in the following diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_c \arrow[d] & \subseteq & \widetilde{\mathcal{A}}_c \arrow[d] & \subseteq & \widetilde{\mathcal{N}}_c & \subseteq & \widetilde{\mathcal{E}}_c & \subseteq & \mathrm{T}M \arrow[d, "\pi"] \\
\mathcal{M}_c & \subseteq & \mathcal{A}_c & \subseteq & & & & \subseteq & M
\end{tikzcd}$$

where $\widetilde{\mathcal{E}}_c$ is the energy level corresponding to $\alpha(c)$.

Another interesting approach to the study of these invariant sets is provided by *weak KAM theory*. This is mainly based on the concept of "critical" subsolutions and "weak" solutions of Hamilton-Jacobi equation and can be interpreted, from a symplectic geometric point of view, as the study of particular Lagrangian graphs and their non-removable intersections (see [28]). This latter approach is particularly interesting, since it relates the dynamics of the system to the geometry of the space and might potentially open the way to a "symplectic" definition of Aubry-Mather theory.

Definition 3.10. A locally Lipschitz function $u : M \longrightarrow \mathbb{R}$ is a *subsolution* of $H_\eta(x, d_x u) = k$, with $k \in \mathbb{R}$, if $H_\eta(x, d_x u) \leq k$ for almost every $x \in M$.

This definition makes sense because, by Rademacher's theorem, $d_x u$ exists almost everywhere. It is possible to show that there exists $c[\eta] \in \mathbb{R}$, such that $H_\eta(x, d_x u) = k$ admits no subsolutions for $k < c[\eta]$ and has subsolutions for $k \geq c[\eta]$, see [21, 12]. The constant $c[\eta]$ is called *Mañé's critical value* and coincides with $\alpha(c)$, where $c = [\eta]$ (see [10, 12]).

Definition 3.11. A function $u : M \longrightarrow \mathbb{R}$ is an *η-critical subsolution*, if $H_\eta(x, d_x u) \leq \alpha(c)$ for almost every $x \in M$.

In particular, Mañé's critical energy level $\mathcal{E}_c^* := \{(x,p) \in \mathrm{T}^*M : \; H(x,p) = \alpha(c)\}$ is the only energy level in which there exist classical or weak solutions of Hamilton-Jacobi equation $H_\eta(x, du) = \alpha(c)$. Denote by $\mathcal{S}_\eta$ the set of critical subsolutions. This set $\mathcal{S}_\eta$ is non-empty. In fact, it is showed in [12] that:

Proposition 3.12. *If $u : M \longrightarrow \mathbb{R}$ is an η-critical subsolution, then for every $x,\, y \in M$*

$$u(y) - u(x) \le h_\eta(x, y)\,.$$

Moreover, for any $x \in M$, the function $h_{\eta,x}(\cdot) := h_\eta(x,\,\cdot)$ is a η-critical subsolution.

There is a more geometric way to define the Aubry set in the cotangent space, using directly these critical subsolutions; it is possible to show that this set is given by the intersection of the graph of the differentials of this critical subsolutions (where they are defined) plus the 1-form η. In particular, it turns out that each critical subsolution is differentiable on the (projected) Aubry set and their differentials coincide on this set. The following can be proven, see [12, 14].

Theorem 3.13. *Let*

$$\mathcal{A}_c^* := \bigcap_{u \in \mathcal{S}_\eta} \{(x, \eta_x + d_x u) : \ u \text{ is differentiable at } x\} \subset \mathrm{T}^* M\,. \tag{10}$$

Then $\mathcal{A}_c^* = \mathcal{L}(\widetilde{\mathcal{A}}_c)$, where $\mathcal{L}$ is the Legendre transform defined in (2).

Using critical subsolutions, one can also obtain a nice representation of h_η; for $x \in \mathcal{A}_c$ and $y \in M$:

$$h_\eta(x, y) = \sup_{u \in \mathcal{S}_\eta} \{u(y) - u(x)\}\,,$$

where this supremum is actually attained, for fixed $x, y \in \mathcal{A}_c$. Moreover, for $x, y \in \mathcal{A}_c$:

$$\delta_c(x, y) = \sup_{u, v \in \mathcal{S}_\eta} \{(u - v)(y) - (u - v)(x)\}\ .$$

These results can be improved (see [14]):

Theorem 3.14. *For any η-critical subsolution $u : M \longrightarrow \mathbb{R}$ and for each $\varepsilon > 0$, there exists a C^1 function $\tilde{u} : M \longrightarrow \mathbb{R}$ such that:*

i) *$\tilde{u}(x) = u(x)$ and $H_\eta(x, d_x\tilde{u}) = \alpha(c)$ on $\mathcal{A}_c$;*

ii) *$|\tilde{u}(x) - u(x)| < \varepsilon$ and $H_\eta(x, d_x\tilde{u}) < \alpha(c)$ on $M \setminus \mathcal{A}_c$.*

This implies that C^1 η-critical subsolutions are dense in $\mathcal{S}_\eta$ with the C^0-topology. This result has been recently extended by Patrick Bernard (see [3]), showing that every η-critical subsolution coincides, on the Aubry set, with a $C^{1,1}$ η-critical subsolution (that is the best regularity that one can generally expect). Denoting the set of C^1 η-critical subsolutions by $\mathcal{S}_\eta^1$, one can rewrite:

$$\mathcal{A}_c^* = \bigcap_{u \in \mathcal{S}_\eta^1} \mathrm{Graph}(\eta + du); \tag{11}$$

in particular, there exists a C^1 η-critical subsolution $\tilde{u}$ such that:

$$\mathcal{A}_c^* = \mathrm{Graph}(\eta + d\tilde{u}) \cap \mathcal{E}_c^*. \tag{12}$$

In the same way, one obtains:

$$\begin{aligned} h_\eta(x, y) &= \max_{u \in \mathcal{S}_\eta^1} \{u(y) - u(x)\} \\ \delta_c(x, y) &= \max_{u, v \in \mathcal{S}_\eta^1} \{(u - v)(y) - (u - v)(x)\}\,, \end{aligned}$$

for any $x,\, y \in \mathcal{A}_c$.

4 Minimizing properties of Lagrangian graphs and uniqueness results.

In this section we shall prove the main results announced in Section 2. Let us start by proving some action minimizing properties of probability measures supported on Lagrangian graphs.

Given a Lagrangian graph Λ with Liouville class c, we shall say that Λ is *c-subcritical*, or simply *subcritical*, if $\Lambda \subset \{(x,p) \in \mathrm{T}^*M : \ H(x,p) \leq \alpha(c)\}$, where $\alpha : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is Mather's α-function. Given a subcritical Lagrangian graph Λ with Liouville class c, we shall call $\Lambda_{crit} = \{(x,p) \in \Lambda : \ H(x,p) = \alpha(c)\}$ its *critical* part. The key result we need to prove is the following characterization of minimizing measures.

Lemma 4.1. *Let μ be an invariant probability measure on $\mathrm{T}M$ and $\mu^* = \mathcal{L}_*\mu$ its push-forward to T^*M, via the Legendre transform $\mathcal{L}$. Then, μ is a Mather's measure if and only if* $\operatorname{supp}\mu^*$ *is contained in the critical part of a subcritical Lagrangian graph. In particular, any invariant probability measure μ^* on T^*M, whose support is contained in an invariant Lagrangian graph with Liouville class c, is the image, via the Legendre transform, of a c-action minimizing measure.*

Proof. (i) If μ is a Mather's measure with cohomology class c, then the support of μ^* is contained in $\mathcal{L}(\widetilde{\mathcal{M}}_c) \subseteq \mathcal{A}^*_c$. By (11), $\mathcal{A}^*_c$ is an intersection of subcritical Lagrangian graphs, so $\operatorname{supp}\mu^*$ is contained in at least one subcritical Lagrangian graph Λ. In particular $\operatorname{supp}\mu^*$ is contained in the critical part of Λ, simply because $\mathcal{L}(\widetilde{\mathcal{M}}_c)$ is contained in the energy level $\mathcal{E}^*_c = \{(x,p) \in \mathrm{T}^*M : \ H(x,p) = \alpha(c)\}$, see Theorem 3.2.

(ii) Let us fix η to be a closed 1-form with $[\eta] = c$. Since we are assuming that Λ is a c-subcritical Lagrangian graph, we can write $\Lambda = \{(x, \eta(x) + du(x))\,,\, x \in M\}$, where $u : M \to \mathbb{R}$ is C^1. By Theorem 3.9, in order to show that μ is a c-action minimizing measure, it is enough to show that $\operatorname{supp}\mu \subseteq \widetilde{\mathcal{N}}_c$, *i.e.,* that any orbit in $\operatorname{supp}\mu$ is a c-minimizer, see (7). To this purpose, let us consider $(x,v) \in \operatorname{supp}\mu$ and let $\gamma(t) \equiv \pi(\Phi_t(x,v))$, where Φ_t is the Euler-Lagrange flow and π the canonical projection on M. Given any interval $[a,b] \subset \mathbb{R}$, let us consider the difference $u(\gamma(b)) - u(\gamma(a))$ and rewrite it as:

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b d_{\gamma(s)}u(\gamma(s))\dot{\gamma}(s)\,ds = \int_a^b \big[L_\eta(\gamma(s),\dot{\gamma}(s)) + H_\eta(\gamma(s), d_{\gamma(s)}u)\big]ds\,, \quad (13)$$

where the second equality follows from the definition of the Hamiltonian as the Legendre-Fenchel transform of the Lagrangian and the fact that $\gamma(s)$ is an orbit of the Euler-Lagrange flow. Note that along the orbit $H_\eta(\gamma(s), d_{\gamma(s)}u) = \alpha(c)$, because $\operatorname{supp}\mu$ is invariant and $\operatorname{supp}\mu^*$ is in the critical part of Λ. Then

$$\int_a^b L_\eta(\gamma(s),\dot{\gamma}(s))ds = u(\gamma(b)) - u(\gamma(a)) - \alpha(c)(b-a)\,. \quad (14)$$

On the other hand, any other curve $\gamma_1 : [a,b] \to M$ such that $\gamma_1(a) = \gamma(a)$ and $\gamma_1(b) = \gamma(b)$ satisfies:

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b d_{\gamma_1(s)}u(\gamma_1(s))\dot{\gamma}(s)\,ds \leq \int_a^b \big[L_\eta(\gamma_1(s),\dot{\gamma}_1(s)) + H_\eta(\gamma_1(s), d_{\gamma_1(s)}u)\big]ds \quad (15)$$

where the second inequality follows again by the duality between Hamiltonian and Lagrangian. Note that now $H_\eta(\gamma_1(s), d_{\gamma_1(s)}u) \le \alpha(c)$, because $\Lambda = \{(x, \eta(x) + du(x))\}$ is subcritical. Then

$$\int_a^b L_\eta(\gamma_1(s), \dot\gamma_1(s))ds \ge u(\gamma(b)) - u(\gamma(a)) - \alpha(c)(b-a) \tag{16}$$

and this proves that γ is a c-minimizer.

Let us finally observe that the Hamilton function on any invariant Lagrangian graph $\Lambda = \{(x, \eta + du)\}$ is a constant: $H(x, \eta + du) = k$ (see Appendix B). Then u is a classical solution of the Hamilton-Jacobi equation with cohomology class c. As discussed in Section II, there is only one possible value of k for which such solutions can exist, namely $k = \alpha(c)$, and this shows that Λ coincides with its critical part. By the result proved in item (*ii*), if μ^* is supported on Λ, then μ is a c-action minimizing measure and this proves the last claim in the statement of the Lemma. □

We can now prove the well-known uniqueness result for Lagrangian graphs supporting invariant measures of full support, in a fixed cohomology class, stated in the Introduction.

Theorem 4.2. *If $\Lambda \subset \mathrm{T}^*M$ is a Lagrangian graph on which the Hamiltonian dynamics admits an invariant measure μ^* with full support, then $\Lambda = \mathcal{L}\big(\widetilde{\mathcal{M}}_c\big) = \mathcal{A}_c^*$, where c is the cohomology class of Λ. Therefore, if Λ_1 and Λ_2 are two Lagrangian graphs as above, with the same cohomology class, then $\Lambda_1 = \Lambda_2$.*

Proof. By Lemma 4.1, the measure $\mu = \mathcal{L}_*^{-1}\mu^*$ is c-minimizing. This means that $\mathcal{L}^{-1}(\Lambda) = \operatorname{supp}\mu \subseteq \widetilde{\mathcal{M}}_c \subseteq \widetilde{\mathcal{A}}_c$, where the last inclusion follows from Theorem 3.9. Note however that, by Theorems 3.2 and 3.8, $\widetilde{\mathcal{M}}_c$ and $\widetilde{\mathcal{A}}_c$ are graphs over their bases and, since $\operatorname{supp}\mu$ is a graph over the whole M, it follows that

$$\mathcal{L}^{-1}(\Lambda) = \operatorname{supp}\mu = \widetilde{\mathcal{M}}_c = \widetilde{\mathcal{A}}_c\,. \tag{17}$$

□

One can deduce something more from the above proof.

Theorem 4.3. *If Λ and μ are as in Theorem 4.2 and ρ is the rotation vector of $\mu = \mathcal{L}^{-1}\mu^*$, then $\Lambda = \mathcal{L}\big(\widetilde{\mathcal{M}}^\rho\big)$. Therefore, if Λ_1 and Λ_2 are two Lagrangian graphs supporting measures of full support and the same rotation vector ρ, then $\Lambda_1 = \Lambda_2$. Moreover, Mather's β-function is differentiable at ρ with $\partial\beta(\rho) = c$, where c is the cohomology class of Λ.*

Proof. The first claim follows from the fact that $\widetilde{\mathcal{M}}^\rho$ is a graph over M and that by definition $\widetilde{\mathcal{M}}^\rho \supseteq \operatorname{supp}\mu = \mathcal{L}^{-1}(\Lambda)$. As far as the differentiability of β at ρ is concerned, suppose that $c' \in \mathrm{H}^1(M;\mathbb{R})$ is a subderivative of β at ρ. Therefore, $\beta(\rho) = \langle c', \rho\rangle - \alpha(c')$ and this implies that $\mathfrak{M}^\rho \subseteq \mathfrak{M}_{c'}$; in fact, for any $\mu \in \mathfrak{M}^\rho$:

$$\int_{\mathrm{T}M} (L - \hat\eta')\, d\mu = \int_{\mathrm{T}M} L\, d\mu - \int_{\mathrm{T}M} \hat\eta'\, d\mu = \beta(\rho) - \langle c', \rho\rangle = -\alpha(c')\,,$$

where η' is a closed 1-form of cohomology c'. As a result, $\widetilde{\mathcal{M}}^\rho = \mathcal{L}^{-1}(\Lambda) \subseteq \widetilde{\mathcal{M}}_{c'}$. The graph property of $\widetilde{\mathcal{M}}_{c'}$ and of $\widetilde{\mathcal{A}}_{c'}$ implies that $\widetilde{\mathcal{A}}_{c'} = \widetilde{\mathcal{M}}_{c'} = \mathcal{L}^{-1}(\Lambda)$ and $\mathcal{A}^*_{c'} = \Lambda$. As a consequence,

$c' = c$. In fact, by Theorem 3.14 and Eq.(11), there exists a C^1 function $v : M \longrightarrow \mathbb{R}$, such that $\Lambda = \{(x, \eta' + dv) : \, x \in M\}$, where η' is a closed 1-form with $[\eta'] = c'$. This, by definition, means that c' is the cohomology class of Λ and therefore $c' = c$. □

We are now in the position of proving the main uniqueness result in a homology class, stated in the Introduction.

Theorem 4.4. *Let Λ be a Schwartzman strictly ergodic invariant Lagrangian graph with homology class ρ. The following properties are satisfied:*

(i) *if $\Lambda \cap \mathcal{A}^*_c \neq \emptyset$, then $\Lambda = \mathcal{A}^*_c$ and $c = c_\Lambda$, where c_Λ is the cohomology class of Λ.*

(ii) *the Mather function α is differentiable at c_Λ and $\partial\alpha(c_\Lambda) = \rho$.*

Therefore,

(iii) *any invariant Lagrangian graph that carries a measure with rotation vector ρ is equal to the graph Λ;*

(iv) *any invariant Lagrangian graph is either disjoint from Λ or equal to Λ.*

We shall need the following Lemma.

Lemma 4.5. *Let ρ, c be respectively an arbitrary homology class in $\mathrm{H}_1(M;\mathbb{R})$ and an arbitrary cohomology class $\mathrm{H}^1(M;\mathbb{R})$. We have*

$$(1)\ \widetilde{\mathcal{M}}^\rho \cap \widetilde{\mathcal{A}}_c \neq \emptyset \quad \Longleftrightarrow \quad (2)\ \widetilde{\mathcal{M}}^\rho \subseteq \widetilde{\mathcal{A}}_c \quad \Longleftrightarrow \quad (3)\ \rho \in \partial\alpha(c)\,.$$

Proof of Lemma 4.5. The implication (2) $\Longrightarrow$ (1) is trivial. Let us prove that (1) $\Longrightarrow$ (3). If $\widetilde{\mathcal{M}}^\rho \cap \widetilde{\mathcal{A}}_c \neq 0$, then there exists a c-minimizing invariant measure μ with rotation vector ρ. Let η be a closed 1-form with $[\eta] = c$; from the definition of α and β:

$$-\alpha(c) = \int_{\mathrm{T}M} (L - \hat{\eta})\, d\mu = \int_{\mathrm{T}M} L\, d\mu - \langle c, \rho\rangle = \beta(\rho) - \langle c, \rho\rangle\,;$$

since β and α are convex conjugated, then ρ is a subderivative of α at c.
Finally, in order to show (3) $\Longrightarrow$ (2), let us prove that any action minimizing measure with rotation vector ρ is c-minimizing. In fact, if $\rho \in \partial\alpha(c)$ then $\alpha(c) = \langle c, \rho\rangle - \beta(\rho)$; therefore for any $\mu \in \mathfrak{M}^\rho$ and η as above:

$$-\alpha(c) = \beta(\rho) - \langle c, \rho\rangle = \int_{\mathrm{T}M} (L - \hat{\eta})\, d\mu.$$

This proves that $\mu \in \mathfrak{M}_c$ and concludes the proof. □

Proof of Theorem 4.4 (i) From Theorem 4.2, it follows that $\Lambda = \mathcal{A}^*_{c_\Lambda}$. Let us show that it does not intersect any other Aubry set. Suppose by contradiction that Λ intersects another Aubry set $\mathcal{A}^*_c$. By Theorem 4.3, $\Lambda = \mathcal{L}^{-1}\left(\widetilde{\mathcal{M}}^\rho\right)$, then $\widetilde{\mathcal{M}}^\rho \cap \widetilde{\mathcal{A}}_c \neq \emptyset$ and, because of the previous lemma and the graph property of $\widetilde{\mathcal{A}}_c$, we can conclude that $\mathcal{A}^*_c = \Lambda$. The same argument used in the proof of Theorem 4.3 allows us to conclude that $c = c_\Lambda$.

(*ii*) Suppose that $h \in \partial\alpha(c_\Lambda)$. The previous lemma implies that $\widetilde{\mathcal{M}}^h \subseteq \Lambda$; the Schwartzman unique ergodicity property of Λ implies $h = \rho$. Therefore α is differentiable at c_Λ and $\partial\alpha(c_\Lambda) = \rho$.

To prove (*iv*), let Λ_1 be an invariant Lagrangian graph, and call c_1 its cohomology class. If the compact invariant set $\Lambda \cap \Lambda_1$ is not empty, then we can find a probability measure μ^* invariant under the flow and whose support is contained in this intersection. Since μ^* is contained in the Lagrangian graph Λ_1, by Lemma 4.1, it is c_1-minimizing. Hence, the support of μ^* is contained in $\mathcal{A}^*_{c_1}$. This shows that the intersection $\Lambda \cap \mathcal{A}^*_{c_1}$ contains the support of μ^* and is therefore not empty. By (*i*), $\Lambda = \mathcal{A}^*_{c_1}$. Moreover, note that $\mathcal{A}^*_{c_1} \subseteq \Lambda_1$, because, on the one hand, $\Lambda_1 = \mathrm{Graph}(\eta_1 + du_1)$, with $[\eta_1] = c_1$ and u_1 a classical solution to the Hamilton-Jacobi equation (see proof of Lemma 4.1), and, on the other hand, $\mathcal{A}^*_{c_1} = \cap_{u \in \mathcal{S}^1_\eta} \mathrm{Graph}(\eta_1 + du_1)$, see (11). Therefore, $\Lambda = \Lambda_1$, since they are both graphs over the base.

To prove (*iii*), consider an invariant Lagrangian graph Λ_1, with cohomology class c_1, which carries an invariant measure μ^* whose rotation vector is ρ. By Lemma 4.1, the measure μ^* is c_1-minimizing. Therefore, we have $\widetilde{\mathcal{M}}^\rho \cap \widetilde{\mathcal{A}}_{c_1} \neq \emptyset$. By Lemma 4.5, it follows that $\mathcal{L}^{-1}(\Lambda) = \widetilde{\mathcal{M}}^\rho \subseteq \widetilde{\mathcal{A}}_{c_1} \subseteq \mathcal{L}^{-1}(\Lambda_1)$. Again, this forces the equality $\Lambda = \Lambda_1$ by the graph property. $\square$

Finally, observe that Lemma 3.1, Theorem 3.9 and Lemma 4.5 imply the following property.

Corollary 4.6. *The Mather function α is differentiable at c if and only if the restriction of the Euler-Lagrange flow to $\widetilde{\mathcal{A}}_c$ is Schwartzman uniquely ergodic, i.e., if and only if all invariant measures supported on $\widetilde{\mathcal{A}}_c$ have the same rotation vector (see Appendix A for the definition and a discussion of Schwartzman ergodic flows).*

5 Global uniqueness of KAM tori

In this section we motivate more precisely the problem of uniqueness of KAM tori and we prove Corollary 2.6. We also show how to generalize Corollary 2.6 to cover the case of invariant tori belonging to the closure of the set of KAM tori.

KAM theory concerns the study of existence of KAM tori (see Definition 2.5) in quasi-integrable Hamiltonian systems of the form $H(x,p) = H_0(p) + \varepsilon f(x,p)$, where: (x,p) are local coordinates on $\mathbb{T}^n \times \mathbb{R}^n$, ε is a "small" parameter and $f(x,p)$ a smooth function. If $\varepsilon = 0$ the system is integrable, in the sense that the dynamics can be explictly solved: in particular each torus $\mathbb{T}^n \times \{p_0\}$ is invariant and the motion on it corresponds to a rotation with frequency $\rho(p_0) = \frac{\partial H_0}{\partial p}(p_0)$. The question addressed by KAM theory is whether this foliation of phase space into invariant tori, on which the motion is (quasi-)periodic, persists even if $\varepsilon \neq 0$. In 1954 Kolmogorov stated (and Arnol'd and Moser proved it later in different contexts) that, in spite of the generic disappearence of the invariant submanifolds filled by periodic orbits, pointed out by Poincaré, for small ε it is always possible to find KAM tori corresponding to "*strongly non-resonant*", *i.e., Diophantine*, rotation vectors. Let us recall here the definition and some properties of Diophantine vectors: given $C, \tau > 0$, we say that $\rho \in \mathbb{R}^n$ is a (C,τ)-Diophantine vector if and only if $C|\langle \rho, \nu\rangle| \geq |\nu|^{-\tau}$, $\forall \nu \in \mathbb{Z}^n \setminus \{0\}$. The set of (C,τ)-Diophantine vectors will be denoted by $\mathcal{D}(C,\tau)$. Note that, if $\tau < n-1$, $\mathcal{D}(C,\tau) = \emptyset$, while for $\tau > n-1$, the Diophantine vectors have full measure in $\mathbb{R}^n$, that is $\lim_{R\to\infty} \mu_0(\cup_{C>0}\mathcal{D}(C,\tau) \cap B_R)/\mu_0(B_R) = 1$, where μ_0 is the Lesbegue measure and B_R is the ball of radius R centered at 0; for $\tau = n-1$, $\cup_{C>0}\mathcal{D}(C,\tau)$ has measure zero but Haussdorf dimension n. The celebrated KAM Theorem (in one of its several versions) not only shows the existence of such tori, but also provides an explicit method to construct them.

Theorem (Kolmogorov–Arnol'd–Moser) [30]. *Let $n \geq 2$, $\tau > n-1$, $C > 0$, $\ell > 2\tau + 2$, $M > 0$ and $r > 0$ be given. Let $B_r \in \mathbb{R}^n$ be the open ball of radius r centered at the origin. Let $H \in C^\ell(\mathbb{T}^n \times B_r)$ be of the form*

$$H(x,p) = H_0(p) + \varepsilon f(x,p) \tag{18}$$

with $|H_0|_{C^\ell} \leq M$, $|f|_{C^\ell} \leq M$, $\left|\frac{\partial^2 H_0}{\partial p^2}\right| \geq M^{-1}$ and $\rho = \frac{\partial H_0}{\partial p}(0) \in \mathcal{D}(C,\tau)$. Then, for any $s < \ell - 2\tau - 1$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ the Hamiltonian (18) *admits a $C^{s,s+\tau}$ KAM torus with rotation vector ρ, i.e., a $C^{s+\tau}$ invariant torus such that the Hamiltonian flow on it is C^s-conjugated with a rotation with frequency ρ.*

Remarks.
1) If $H \in C^\infty$ then the KAM torus mentioned in the theorem above is C^∞. If H is real analytic then the KAM torus is real analytic.
2) As already mentioned, the proof of this theorem is constructive and it actually contains much more informations than those summarized in the above statement. For instance, in the analytic case, the proof consists of an iterative method allowing one to construct order by order the series defining the conjugation function (and to prove convergence of the formal series). In the differentiable case the proof is based on the idea of iteratively approximating differentiable functions by analytic ones and then using the inductive approximation scheme of the analytic case. In the differentiable case the proof provides an explicit construction of a KAM torus (however the construction *a priori* depends on a number of arbitrary choices one has to make along the proof – *e.g.,* the choice of cutoffs one needs to introduce in the iterative approximation scheme).
3) The invariant torus constructed in the proof of the KAM Theorem is locally unique, in the sense that for any prescribed (and admissible) s there is at most one $C^{s,s+\tau}$ KAM torus with rotation vector ρ within a C^s-distance $\delta(n,s,C,\tau)$ to the one constructed in the proof of the KAM Theorem, see [5, 29, 30]. Note that the C^s-distance δ within which one can prove uniqueness of the KAM torus in a prescribed regularity class depends both on the irrationality properties of ρ and on the regularity class s itself. It is then *a priori* possible that even for small ε there exist different KAM tori *within a prescribed C^1-distance* from the one constructed in the proof of the Theorem, possibly less regular than that torus. Quite surprisingly, even in the analytic case, we are not aware of any proof of "global" uniqueness of the invariant analytic KAM torus with rotation vector ρ (of course in the analytic case the analytic torus one manages to construct is unique within the class of analytic tori – however nothing *a priori* guarantees that less regular invariant tori with the same rotation vector exist).

The questions arisen in remark (3) is our main motivation for the study of the problem of global uniqueness of KAM tori. Our result, in the form stated in the Corollary in Section 2, settles the question and shows that, at least in the case of Tonelli Hamiltonians, it is not possible to have two different KAM tori with the same rotation vector. Note that the assumption of strict convexity of the Hamiltonian is necessary to exclude trivial sources of non-uniqueness: for instance, in the context of quasi-integrable Hamiltonians, global uniqueness could be lost simply because the unperturbed Hamiltonian induces a map $p \to \partial_p H_0(p)$ from actions to frequencies that is not one to one. Let us also remark that, apparently, the Hamiltonian considered in KAM Theorem is not a Tonelli Hamiltonian, since the latter, by definition, is defined globally on the whole $\mathbb{T}^n \times \mathbb{R}^n$. However any C^ℓ strictly convex Hamiltonian defined on $\mathbb{T}^n \times B_r$ for some $r > 0$ can be extended to a global C^ℓ Tonelli Hamiltonian. Then in the statement of the KAM Theorem above it is actually enough to assume H to be a C^ℓ Tonelli Hamiltonian, locally satisfying the (in)equalities listed after (18).

Given the proof of our main results in Section 4, the proof of Corollary 2.6 is very simple.

Proof of Corollary 2.6. Since the Lagrangian KAM torus $\mathcal{T}$ admits an invariant measure μ^* of full support, which is the image via the conjugation φ of the uniform measure on $\mathbb{T}^n$, then the claims follow from Theorem 4.3. Note that for rationally independent rotation vectors, a classical remark, generally attributed to Herman [18] (see Appendix B for a proof), implies that $\mathcal{T}$ is automatically Lagrangian. $\square$

An interesting generalization of the result of Corollary 2.6 concerns the invariant tori belonging to the C^0-closure $\overline{\Upsilon}$ of the set Υ of all Lagrangian KAM tori. Note that, for quasi-integrable systems, Υ is not empty. The set Υ can be seen as a subset of $\mathrm{Lip}(\mathbb{T}^n, \mathbb{R}^n)$. This follows from Theorem 4.3, and from Mather's graph theorem, see Theorems 3.2, 3.4, and the results in [25]. Moreover, any family of invariant Lagrangians graphs on which the function α (or H) is bounded gives rise to a family of functions in $\mathrm{Lip}(\mathbb{T}^n, \mathbb{R}^n)$ with uniformly bounded Lipschitz constant. This is because, given Λ in such a family and denoting by $(\eta + du)$ its graph, for any pair of points $x, y \in \mathbb{T}^n$ and any smooth curve $\gamma(t)$ on Λ connecting x to y with unit speed, we have that $u(x) - u(y) \leq \int_0^{|x-y|} L_\eta(\gamma(t), \dot\gamma(t)) + \alpha(c)|x - y|$, where $c = [\eta]$, see (15). By Ascoli-Arzelà theorem, it follows that $\overline{\Upsilon}$ is also a subset of $\mathrm{Lip}(\mathbb{T}^n, \mathbb{R}^n)$, consisting of functions whose graphs are invariant C^0-Lagrangian tori. Herman [19] showed that, for a generic Hamiltonian H close enough to an integrable Hamiltonian H_0, the dynamics on the generic tori in $\overline{\Upsilon}$ is not conjugated to a rotation. These "new" tori therefore represent the majority, in the sense of topology, and hence most invariant tori cannot be obtained by the KAM algorithm. More precisely, Herman showed that in $\overline{\Upsilon}$ there exists a dense G_δ set (*i.e.,* a dense countable intersection of open sets) of invariant Lagrangian graphs on which the dynamics is strictly ergodic and weakly mixing, and for which the rotation vector, in the sense of Section 3, is not Diophantine. These invariant graphs are therefore not obtained by the KAM theorem, however our uniqueness result do still apply to these graphs since strict ergodicity implies Schwartzman strict ergodicity.

More generally, given any Tonelli Lagrangian on $\mathbb{T}^n$, we consider the set $\tilde{\Upsilon}$ of invariant Lagrangian graphs on which the dynamics of the flow is topologically conjugated to an *ergodic* linear flow on $\mathbb{T}^n$ (of course, far from the canonical integrable Lagrangian the set $\tilde{\Upsilon}$ may be empty). The dynamics on anyone of the invariant graph in $\tilde{\Upsilon}$ is strictly ergodic. Since the set of strictly ergodic flows on a compact set is a G_δ in the C^0 topology, see for example [13, Corollaire 4.5], it follows that there exists a dense G_δ subset $\mathcal{G}$ of the C^0 closure of $\tilde{\Upsilon}$ in $\mathrm{Lip}(\mathbb{T}^n, \mathbb{R}^n)$, such that the dynamics on any $\Lambda \in \mathcal{G}$ is strictly ergodic. Therefore we have the following proposition.

Proposition 5.1. *There exists a dense G_δ set $\mathcal{G}$ in the C^0 closure of $\tilde{\Upsilon}$ consisting of strictly ergodic invariant Lagrangian graphs. Any $\Lambda \in \mathcal{G}$ satisfies the following properties:*

(i) *the invariant graph Λ has a well-defined rotation vector $\rho(\Lambda)$.*

(ii) *Any invariant Lagrangian graph that intersects Λ coincides with Λ.*

(iii) *Any Lagrangian invariant graph that carries an invariant measure whose rotation is $\rho(\Lambda)$ coincides with Λ.*

Appendices

A Schwartzman unique and strict ergodicity

In Section 3 we have introduced the concept of rotation vector of a measure. This is closely related to the notion of Schwartzman asymptotic cycle of a flow, introduced by Sol Schwartzman in [31], as a first attempt to develop an algebraic topological approach to the study of dynamics. In particular, we would like to provide some examples and investigate some properties of what we call *Schwartzman uniquely ergodic flows.*

One can give a different description of the Schwartzman asymptotic cycle of a flow. This is also known as the flux homomorphism in volume preserving and symplectic geometry, see [2][Chapter 3]. We will use the description given in [11][pages 67-70]. This definition has the technical advantadge of not relying on the Krylov-Bogolioubov theory of generic orbits in a dynamical system, although a more geometrical definition showing that "averaged" pieces of long orbits converge almost everywhere in the first homology group for any invariant measure is certainly more heuristic and intuitive.

Let us start with some standard facts. As usual we set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The space $\mathbb{T}$ is a topological group for the addition. An important feature of $\mathbb{T}$ is that the canonical projection $\pi : \mathbb{R} \to \mathbb{T}$ is a covering map. Therefore given any continuous path $\gamma : [a, b] \to \mathbb{T}$, with $a \leq b$, we can find a continuous lift $\bar{\gamma} : [a, b] \to \mathbb{T}$ such that $\gamma = \pi\bar{\gamma}$. Any two such lifts differ by an integer. It follows that the quantity $\bar{\gamma}(b) - \bar{\gamma}(a)$ does not depend on the lift. We will set

$$\mathcal{V}(\gamma) = \bar{\gamma}(b) - \bar{\gamma}(a) \in \mathbb{R}.$$

This quantity remains constant on the homotopy class, with fixed end points, of the path γ. Morever, if γ is a closed path, *i.e.,* we have $\gamma(a) = \gamma(b)$ then $\mathcal{V}(\gamma) \in \mathbb{Z}$, and, if such a closed path is homotopic to 0 (with fixed endpoint) then $\mathcal{V}(\gamma) = 0$. It is also clear that for $c \in [a, b]$, we have

$$\mathcal{V}(\gamma|[a, b]) = \mathcal{V}(\gamma|[a, c]) + \mathcal{V}(\gamma|[c, b]).$$

Note also that two continuous paths $\gamma_1, \gamma_2 : [a, b] \to \mathbb{T}$ can be added by the formula

$$(\gamma_1 + \gamma_2)(t) = \gamma_1(t) + \gamma_2(t).$$

For this addition we have

$$\mathcal{V}(\gamma_1 + \gamma_2) = \mathcal{V}(\gamma_1) + \mathcal{V}(\gamma_2).$$

Another important property of the map $\mathcal{V}$ is its continuity on the functional space $C^0([a,b],\mathbb{T})$, endowed with the topology of uniform convergence. If we call θ the closed 1-form on $\mathbb{T}$ whose lift to $\mathbb{R}$ is the usual differential form dt on $\mathbb{R}$, where dt is the differential of the identity map $\mathbb{R}\to\mathbb{R}, t\mapsto t$. It is well-known that when $\gamma:[a,b]\to\mathbb{T}$ is C^1, we have

$$\mathcal{V}(\gamma)=\int_\gamma \theta=\int_a^b \theta_{\gamma(t)}(\dot\gamma(t))\,dt.$$

If X is a topological space, and $F: X\times[a,b]\to\mathbb{T}$ is a given map, we will define $\mathcal{V}(F): X\to\mathbb{R}$ by

$$\forall x\in X,\ \mathcal{V}(F)(x)=\mathcal{V}(F_x),$$

where $F_x:[a,b]\to\mathbb{T}$ is defined by $F_x(t)=F(x,t)$. The continuity of $\mathcal{V}$ on $C^0([a,b],\mathbb{T})$ implies that $\mathcal{V}(F)$ is continuous. Furthermore, the continuity of $\mathcal{V}$ on $C^0([a,b],\mathbb{T})$ also implies that the map $C^0(X\times[a,b],\mathbb{T})\to C^0(X,\mathbb{T}), F\mapsto\mathcal{V}(F)$ is continuous , when we provide spaces of continuous maps with the compact open topology.

If F can be lifted to a continuous map $\bar F: X\times[a,b]\to\mathbb{R}$ with $F=\pi\bar F$, then

$$\mathcal{V}(F)(x)=\bar F(x,b)-\bar F(x,a).$$

Suppose now that X is a topological space, and that $(\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on X. We will define $\Phi: X\times[0,1]\to X$ by $\Phi(x,t)=\phi_t(x)$. If $f: X\to\mathbb{T}$ is continuous, we set

$$\mathcal{V}(f,\phi_t)=\mathcal{V}(f\circ\Phi): X\to\mathbb{R}.$$

There is another another way to define $\mathcal{V}(f,\phi_t)$ which is used in [11]. The function $F(f,\Phi):X\times[0,1]\to\mathbb{T}$, defined by

$$F(f,\Phi)(x,t)=f(\phi_t(x))-f(x)$$

is continuous and identically 0 on $X\times\{0\}$, it is therefore homotopic to a constant and can be lifted to a continuous map $F(f,\Phi): X\times[0,1]\to\mathbb{R}$, with $F(f,\Phi)|X\times\{0\}$ identically 0. We have

$$\mathcal{V}(f,\phi_t)(x)=F(f,\Phi)(x,1).$$

Note that is f is homotopic to 0 then it can be lifted continuously to $\bar f: X\to\mathbb{R}$. In that case $\bar F(f,\Phi)=\bar f\Phi-\bar f$, and

$$\mathcal{V}(f,\phi_t)(x)=\bar f(\phi_1(x))-\bar f.$$

If μ is a measure with compact support invariant under the flow ϕ_t, for a continuous $f: X\to\mathbb{T}$, we define $\mathcal{S}(\mu,\phi_t)(f)$, or simply $\mathcal{S}(\mu)(f)$ when ϕ_t is fixed, by

$$\mathcal{S}(\mu)(f)=\int_X \mathcal{V}(f,\phi_t)(x)\,d\mu(x).$$

It is not difficult to verify that for $f_1,f_2: X\to\mathbb{T}$, then

$$\mathcal{S}(\mu)(f_1+f_2)=\mathcal{S}(\mu)(f_1)+\mathcal{S}(\mu)(f_2).$$

Moreover, if $f: X\to\mathbb{T}$ is homotopic to 0 it can be lifted to $\bar f: X\to\mathbb{R}$ and

$$\mathcal{S}(\mu)(f)=\int_X[\bar f(\phi_1(x))-\bar f(x)]\,d\mu(x)=\int_X \bar f(\phi_1(x))\,d\mu(x)-\int_X \bar f(x)\,d\mu(x)=0$$

since μ is invariant by ϕ_1. Therefore, if we denote by $[X, \mathbb{T}]$ the set of homotopy classes of continuous maps from X to $\mathbb{T}$, which is an additive group, the map $\mathcal{S}(\mu)$ is a well-defined additive homomorphism from the additive group $[X, \mathbb{T}]$ to $\mathbb{R}$.

When X is a good space (like a manifold or a locally finite polyhedron), it is well-known that $[X, \mathbb{T}]$ is canonically identified with the first cohomology group $\mathrm{H}^1(X; \mathbb{Z})$. In that case $\mathcal{S}(\mu)$ is in $\mathrm{Hom}(\mathrm{H}^1(X; \mathbb{Z}), \mathbb{R})$. Since the first cohomology group with real coefficients $\mathrm{H}^1(X; \mathbb{R})$ is $\mathrm{H}^1(X; \mathbb{Z}) \otimes \mathbb{R}$, we can view $\mathcal{S}(\mu)$ as an element of the dual $\mathrm{H}^1(X; \mathbb{R})^*$ of the $\mathbb{R}$-vector space $\mathrm{H}^1(X; \mathbb{R})$. When $\mathrm{H}^1(X; \mathbb{R})$ is finite-dimensional then $\mathrm{H}^1(X; \mathbb{R})^*$ is in fact equal to the first homology group $\mathrm{H}_1(X; \mathbb{R})$, and therefore $\mathcal{S}(\mu)$ defines an element of $\mathrm{H}_1(X; \mathbb{R})$, *i.e.,* a 1-cycle. This 1-cycle $\mathcal{S}(\mu)$ is called the *Schwartzman asymptotic cycle* of μ. Note that $\mathrm{H}^1(X; \mathbb{R})$ is finite dimensional when X is a finite poilyhedron or a compact manifold. It should be also noted that for a manifold M the projection $TM \to M$ is a homotopy equivalence. Therefore $\mathrm{H}^1(TM; \mathbb{R}) = \mathrm{H}^1(M; \mathbb{R})$ is finite dimensional when M is a compact manifold.

We now study the behavior of Schwartzman asymptotic cycles under semi-conjugacy.

Proposition A.1. *Suppose $\phi_t^i : X_i \to X_i, i = 1, 2$ are two continuous flows. Suppose also that $\psi : X_1 \to X_2$ is a continuous semi-conjugation between the flows, i.e., $\psi \circ \phi_t^1 = \phi_t^2 \circ \psi$, for every $t \in \mathbb{R}$. Given a probability measure μ with compact support on N_1 invariant under ϕ_t^1, then, for every continuous map $f : X_2 \to \mathbb{T}$, we have*

$$\mathcal{S}(\psi_* \mu, \phi_t^2)([f]) = \mathcal{S}(\mu, \phi_t^1)([f \circ \psi]),$$

where $\psi_* \mu$ is the image of μ under ψ. In particular, if we are in the situation where $\mathrm{Hom}([X_i, \mathbb{T}]) \equiv \mathrm{H}_1(X_i; \mathbb{R}), i = 1, 2,$ *we obtain*

$$\mathcal{S}(\psi_* \mu, \phi_t^2) = H_1(\psi)(\mathcal{S}(\mu, \phi_t^1)).$$

Proof. Notice that $f \psi \phi_t^1(x) - f \psi(x) = f \phi_t^2(\psi(x)) - f(\psi(x))$. Therefore by uniqueness of liftings $\mathcal{V}(f\psi, \phi_t^1)(x) = \mathcal{V}(f, \phi_t^2)(\psi(x))$. An integration with respect to μ finishes the proof. □

We would like now to relate the Schwartzman asymptotic cycles to the rotation vectors $\rho(\mu)$ defined for Lagrangian flows. We first consider the case of a C^1 flow ϕ_t on the manifold N. We call X the continuous vector field on N generating ϕ_t, *i.e.,*

$$\forall x \in N, \quad X(x) = \frac{d\phi_t(x)}{dt}\Big|_{t=0}.$$

By the flow property $\phi_{t+t'} = \phi_t \circ \phi_{t'}$, this implies

$$\forall x \in N, \ \forall t \in \mathbb{R}, \quad \frac{d\phi_t(x)}{dt} = X(\phi_t(x)).$$

In the case of a manifold N, the identification of $[N, \mathbb{T}]$ with $\mathrm{H}^1(N; \mathbb{Z})$ is best described with the de Rham cohomology. We consider the natural map $I_N : [X, \mathbb{T}] \to \mathrm{H}^1(X; \mathbb{R})$ defined by

$$I_N([f]) = [f^* \theta],$$

where $[f]$ on the left hand side denotes the homotopy class of the C^∞ map $f : N \to \mathbb{T}$, and $[f^* \theta]$ on the right hand side is the cohomology class of the pullback by f of the closed 1-form on

$\mathbb{T}$ whose lift to $\mathbb{R}$ is dt. Note that any homotopy class in $[N, \mathbb{T}]$ contains smooth maps because C^∞ maps are dense in C^0 maps (for the Whitney topology). Therefore the map I_N is indeed defined on the whole of $[N, \mathbb{T}]$. As it is well-known, this map I_N induces an isomorphism of $[N, \mathbb{T}]$ on $\mathrm{H}^1(N; \mathbb{Z}) \subset \mathrm{H}^1(N; \mathbb{R}) = \mathrm{H}^1(N; \mathbb{Z}) \otimes \mathbb{R}$.

Given a C^∞ map $f : N \to \mathbb{T}$, the C^1 flow ϕ_t on N, and $x \in N$, we compute $\mathcal{V}(f, \phi_t)(x)$. If $\gamma_x : [0, 1] \to N$ is the path $t \mapsto \phi_t(x)$, by definition, we have $\mathcal{V}(f, \phi_t)(x) = \mathcal{V}(f \circ \gamma_x)$. Since γ_x is C^1, we get

$$\mathcal{V}(f, \phi_t)(x) = \int_{f \circ \gamma_x} \theta = \int_{\gamma_x} f^*\theta.$$

Since $\gamma_x(t) = \phi_t(x)$, we have $\dot{\gamma}_x(t) = X(\phi_t(x))$. It follows that

$$\mathcal{V}(f, \phi_t)(x) = \int_0^1 (f^*\theta)_{\phi_t(x)}(X[\phi_t(x)])\, dt.$$

Recall that the interior product $i_X\omega$ of a differential form ω with X is given by

$$(i_X\omega)_x(\cdots) = \omega_x(X(x), \cdots).$$

When ω is a differential 1-form then $i_X\omega$ is a function. With this notation, we get

$$\mathcal{V}(f, \phi_t)(x) = \int_0^1 (i_X f^*\theta)(\phi_t(x))\, dt.$$

Therefore if μ is an invariant measure for ϕ_t, which we will assume to have a compact support, we obtain

$$\mathcal{S}(\mu) = \int_N \int_0^1 (i_X f^*\theta)(\phi_t(x))\, dt d\mu(x).$$

Since $i_X f^*\theta$ is continuous, and we are assuming that μ has a compact support, we have

$$\mathcal{S}(\mu) = \int_0^1 \int_N (i_X f^*\theta)(\phi_t(x))\, d\mu(x)dt.$$

By invariance of μ under ϕ_t, we get $\int_N (i_X f^*\theta)(\phi_t(x))\, d\mu(x) = \int_N (i_X f^*\theta)(x)\, d\mu(x)$, and therefore

$$\mathcal{S}(\mu) = \int_0^1 \int_N (i_X f^*\theta)(x)\, d\mu(x)dt = \int_N (i_X f^*\theta)\, d\mu.$$

This shows that as an element of $\mathrm{H}^1(M; \mathbb{R})^*$, the Schwarztman asymptotic cycle $\mathcal{S}(\mu)$ is given by

$$\mathcal{S}(\mu)([\omega]) = \int_N i_X\omega\, d\mu.$$

We can now easily compute Schwartzman asypmtotic cycles for linear flows on $\mathbb{T}^n$. Such a flow is determined by a constant vector field $\alpha \in \mathbb{R}^n$ on $\mathbb{T}^n$ (here we use the canonical trivialisation of the tangent bundle of $\mathbb{T}^n$), the associated flow $R_t^\alpha : \mathbb{T}^n \to \mathbb{T}^n$ is defined by $R_t^\alpha(x) = x + [t\alpha]$, where $[t\alpha]$ is the class in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ of the vector $t\alpha \in \mathbb{R}^n$. If ω is a 1-form with constant coefficients, *i.e.*, $\omega = \sum_{i=1}^n a_i dx_i$, with $a_i \in \mathbb{R}$, the interior product $i_\alpha\omega$ is the constant function $\sum_{i=1}^n \alpha_i a_i$. Therefore, it follows that $\mathcal{S}(\mu) = \alpha \in \mathbb{R}^n \equiv \mathrm{H}_1(\mathbb{T}^n; \mathbb{R})$.

We now compute Schwartzman asymptotic cycles for Euler-Lagrange flows. In this case $N = TM$ and ϕ_t is an Euler-Lagrange flow ϕ_t^L of some Lagrangian L. If we call X_L the vector field generating ϕ_t^L, since this flow is obtained from a second order ODE on M, we get

$$\forall x \in M, \forall v \in T_xM, \quad T\pi(X_L(x,v)) = v,$$

where $T\pi : T(TM) \to TM$ denotes the canonical projection. Since this projection π is a homotopy equivalence, to compute $\mathcal{S}(\mu)$ we only need to consider forms of the type $\pi^*\omega$ where ω is a closed 1-form on the base M. In this case $(i_{X_L}\pi^*\omega)(x,v) = \omega_x(T\pi(X_L(x,v)) = \omega_x(v)$. Therefore, for any measure probability measure μ on TM with compact support and invariant under ϕ_t^L, we obtain

$$\mathcal{S}(\mu)[\pi^*\omega] = \int_{TM} \omega_x(v)\, d\mu(x,v) = \int_{TM} \hat{\omega}\, d\mu.$$

This is precisely $\rho(\mu)$ as it was defined above in section 3. Note that the only property we have used is the fact that ϕ_t is the flow of a second order ODE on the base M.

To simplify things, in the remainder of this appendix, we will assume that X is a compact space, for which we have $[X, \mathbb{T}] = \mathrm{H}^1(X;\mathbb{Z})$, and $\mathrm{H}_1(X;\mathbb{Z})$ is finitely generated. In that case, the dual space $\mathrm{H}^1(X;\mathbb{R})^*$ is $\mathrm{H}_1(X;\mathbb{R})$, and for every flow ϕ_t on X and every probability measure μ on X invariant under ϕ_t, the Schwartzman asymptotic cycle is an element of the finite dimensional-vector space $\mathrm{H}_1(X;\mathbb{R})$.

Suppose that x is a periodic point of ϕ_t or period $T > 0$. One can define an invariant probability measure μ_{x,t_0} for ϕ_t by

$$\int_X g(x)\, d\mu_{x,t_0} = \frac{1}{t_0}\int_0^{t_0} g((\phi_t(x))\, dt,$$

where $g : X \to \mathbb{R}$ is a measurable function. We let the reader verify that $\mathcal{S}(\mu_{x,t_0})$ is equal in $\mathrm{H}_1(X;\mathbb{R})$ to the homology class $[\gamma_{x,t_0}]/t_0$, where γ_{x,t_0} is the loop $t \mapsto \phi_t(x), t \in [0, t_0]$. When x is a fixed point of ϕ_t, then the Dirac mass δ_x at x is invariant under ϕ_t, and in that case $\mathcal{S}(\delta_x) = 0$.

Definition A.2. For a flow ϕ_t on X, we denote by $\mathcal{S}(\phi_t)$ the set of all Scwartzman asymptotic cycles $\mathcal{S}(\mu)$, where μ is an arbitrary probability measure on X invariant under ϕ_t.

Since X is compact, note that for the weak topology the set $\mathfrak{M}(X)$ of probability Borel measures on X is compact and convex. It is even metrizable, since we are assuming X metrizable. Furthermore the subset $\mathfrak{M}(X,\phi_t) \subseteq \mathfrak{M}(X)$ of probability measures invariant under ϕ_t is, as it is well-known, compact convex and non empty. Therefore $\mathcal{S}(\phi_t)$ is a compact convex non-empty subset of $\mathrm{H}_1(X;\mathbb{R})$.

For the case of a linear flow R^α on $\mathbb{T}^n$, we have shown above that $\mathcal{S}(R_t^\alpha) = \{\alpha\} \subset \mathbb{R}^n \equiv \mathrm{H}_1(\mathbb{T}^n;\mathbb{R})$.

The following corollary is an easy consequence of Proposition A.1.

Corollary A.3. *For $i = 1, 2$, suppose that ϕ_t^i is a continous flow on the compact space X_i, which satisfies $\mathrm{Hom}([X_i,\mathbb{T}],\mathbb{R}) \equiv \mathrm{H}_1(X_i;\mathbb{R})$. If $\psi : X_1 \to X_2$ is a topological conjugacy between ϕ_t^1 and ϕ_t^2 (i.e., the map ψ is a homeomorphism that satisfies $\psi\phi_t^1 = \phi_t^2\psi$, for all $t \in \mathbb{R}$), then we have*

$$\mathcal{S}(\phi_t^2) = H_1(\psi)[\mathcal{S}(\phi_t^1)].$$

We denote by $\mathfrak{F}(X)$ the set of continuous flows on X. We can embed $\mathfrak{F}(X)$ in $C^0(X\times[0,1],X)$ by the map $\phi_t \mapsto F^{\phi_t} \in C^0(X\times[0,1],X)$, where

$$F^{\phi_t}(x,t) = \phi_t(x).$$

The topology on $C^0(X\times[0,1],X)$ is the compact open (or uniform) topology, and we endow $\mathfrak{F}(X)$ with the topology inherited from the embedding given above.

Lemma A.4. *The map $\phi_t \mapsto \mathcal{S}(\phi_t)$ is upper semi-continuous on $\mathfrak{F}(X)$. This means that for each open subset $U \subseteq \mathrm{H}_1(X;\mathbb{R})$, the set $\{\phi_t \in \mathfrak{F}(X) \mid \mathcal{S}(\phi_t) \subset U\}$ is open in $\mathfrak{F}(X)$.*

Proof. Since the topology on $C^0(X\times[0,1],X)$ is metrizable, if this where not true we could find an open set $U \subset \mathrm{H}_1(X;\mathbb{R})$ and a sequence ϕ_t^n of continuous flows on X converging uniformly to a flow ϕ_t, with $\mathcal{S}(\phi_t) \subset U$, and $\mathcal{S}(\phi_t^n)$ is not contained in U. this means that for each n we can find a probability measure μ_n on X invariant under ϕ_t^n and such that its Schwartzman asymptotic cycle $\mathcal{S}(\mu_n,\phi_t^n)$ for ϕ_n^t is not in the open set U. Since $\mathfrak{M}(X)$ is compact for the weak topology, extracting a subsequence if necessary, we can assume that $\mu_n \to \mu$. It is not difficult to show that μ is invariant under the flow ϕ_t. We now show that $\mathcal{S}(\mu_n,\phi_t^n) \to \mathcal{S}(\mu,\phi_t)$. This will yield a contradiction and finish the proof because $\mathcal{S}(\mu_n,\phi_t^n)$ is in the closed set $\mathrm{H}_1(X;\mathbb{R})\backslash U$, for every n, and $\mathcal{S}(\mu,\phi_t) \in U$.

To show that the linear maps $\mathcal{S}(\mu_n,\phi_t^n) \in \mathrm{H}_1(X;\mathbb{R}) = \mathrm{H}^1(X;\mathbb{R})^*$ converge to the linear map $\mathcal{S}(\mu,\phi_t)$, it suffices to show that $\mathcal{S}(\mu_n,\phi_t^n)([f]) \to \mathcal{S}(\mu,\phi_t)([f])$, for every $[f] \in [X,\mathbb{T}] = \mathrm{H}^1(X;\mathbb{Z}) \subset \mathrm{H}^1(X;\mathbb{R}) = \mathrm{H}^1(X;\mathbb{Z})\otimes\mathbb{R}$. Fix now a continuous map $f: X \to \mathbb{T}$. Denote by $F_n, F : X\times[0,1] \to \mathbb{T}$ the maps defined by

$$F_n(x,t) = f(\phi_t^n(x)) - f(x) \text{ and } F(x,t) = f(\phi_t(x)) - f(x).$$

By the uniform continuity of f on the compact metric space X, the sequence F_n converges uniformly to F. Since $F_n|X\times\{0\} \equiv 0$, if we call $\tilde{F}_n : X\times[0,1] \to \mathbb{R}$ the lift of F_n such that $\tilde{F}_n|X\times\{0\} \equiv 0$, then the sequence $\tilde{F}_n$ also converges uniformly to $\tilde{F}$, that is the lift of F such that $\tilde{F}|X\times\{0\} \equiv 0$. Since the μ_n are probability measures, we have

$$\left|\int_X \tilde{F}_n(x,1)\mu_n(x) - \int_X \tilde{F}(x,1)\mu_n(x)\right| \leq \|\tilde{F}_n - \tilde{F}\|_\infty \longrightarrow 0.$$

Since $\mu_n \to \mu$ weakly, we also have

$$\left|\int_X \tilde{F}(x,1)\mu_n(x) - \int_X \tilde{F}(x,1)\mu(x)\right| \longrightarrow 0.$$

Therefore $\mathcal{S}(\mu_n,\phi_t^n)([f]) = \int_X \tilde{F}_n(x,1)\mu_n(x) \to \mathcal{S}(\mu,\phi_t)([f]) = \int_X \tilde{F}(x,1)\mu_0(x)$. □

Definition A.5. [Schwartzman unique ergodicity] We say that a flow ϕ_t is Schwartzman uniquely ergodic if $\mathcal{S}(\phi_t)$ is reduced to one point.

By the computation done above linear flows on the torus $\mathbb{T}^n$ are Schwartzman uniquely ergodic. Of course, all uniquely ergodic flows (*i.e.,* flows having exactly one invariant probability measure) are also Schwartzman uniquely ergodic. Moreover, by Corollary A.3, any flow topologically conjugate to a Schwartzman uniquely ergodic flow is itself Schwartzman uniquely ergodic.

Theorem A.6. *The set $\mathfrak{S}(N)$ of Schwartzman uniquely ergodic flows is a G_δ in $\mathfrak{F}(X)$.*

Proof. . Fix some norm on $\mathrm{H}^1(X;\mathbb{R})$. We will measure diameters of subsets of $\mathrm{H}^1(X;\mathbb{R})$. with respect to that norm. Fix $\epsilon > 0$. Call $\mathcal{U}_\epsilon$ the set of flows ϕ_t such that the diameter of $\mathcal{S}(\phi_t) \subset \mathrm{H}_1(X;\mathbb{R})$ is $< \epsilon$. If $\phi_t^0 \in \mathcal{U}_\epsilon$, we can find U an open subset of $\mathrm{H}_1(X;\mathbb{R})$ of diameter $< \epsilon$ and containing $\mathcal{S}(\phi_t^0)$. By the lemma above the set $\{\phi_t \in \mathfrak{F}(X) \mid \mathcal{S}(\phi_t) \subset U\}$ is open in $\mathfrak{F}(X)$ contains ϕ_t^0 and is contained in $\mathcal{U}_\epsilon$. The set of Schwartzman uniquely ergodic flows is $\cap_{n\geq 1}\mathcal{U}_{1/n}$. ☐

Proposition A.7. *Let $\varphi_t : X \longrightarrow X$ be a continuous flow on the compact path connected space X. Suppose that there exist $t_i \uparrow +\infty$ such that $\varphi_{t_i} \longrightarrow \varphi$ in $C(N,N)$ (with the C^0-topology). Then, φ_t is Schwartzman uniquely ergodic. In particular, periodic flows and (uniformly) recurrent flows are Schwartzman uniquely ergodic (in both cases $\varphi = \mathrm{Id}$).*

Proof. Fix a continuous map $f : X \to \mathbb{T}$. Consider the function $F : X \times [0, +\infty) \to \mathbb{T}, (x,t) \mapsto f(\phi_t(x)) - f(x)$. We have $F(x,0) = 0$, for every $x \in X$. Call $\bar{F} : X \times [0, +\infty) \to \mathbb{R}$ the (unique) continuous lift of F such that $\bar{F}(x,0) = 0$, for every $x \in X$. The definition of the Schwartzman asymptotic cycle gives

$$\mathcal{S}(\mu)([f]) = \int_X \bar{F}(x,1)\, d\mu(x),$$

for every probability measure invariant under ϕ_t. We claim that we have

$$\forall t, t' \geq 0, \forall x \in X, \quad \bar{F}(x, t+t') = \bar{F}(\phi_t(x), t') + F(x,t).$$

In fact, if we fix t and we consider each side of the equality above as a (continuous) function of (x, t') with values in $\mathbb{R}$, we see that the two sides are equal for $t' = 0$, and that they both lift the function

$$(x,t') \mapsto f(\phi_{t+t'}(x)) - f(x) = f(\phi_t'(\phi_t(x)) - f(\phi_t(x)) + f(\phi_t(x)) - f(x)$$

with values in $\mathbb{T}$. By induction, it follows easily that

$$\forall k \in \mathbb{N}, \quad \bar{F}(x,k) = \sum_{j=0}^{k-1} \bar{F}(\phi_j(x), 1).$$

Therefore, if $t \geq 0$ and $[t]$ is its integer part, we also obtain

$$\bar{F}(x,t) = \bar{F}(\phi_{[t]}(x), t - [t]) + \sum_{j=0}^{[t]-1} \bar{F}(\phi_j(x), 1). \tag{$*$}$$

It follows that

$$\forall t \geq 0, \forall x \in X, \quad |\bar{F}(x,t)| \leq ([t]+1)\|\bar{F}|X \times [0,1]\|_\infty. \tag{$**$}$$

By compactness $\|\bar{F}|X \times [0,1]\|_\infty$ is finite. If we integrate equality $(*)$ with respect to a probability measure μ on X invariant under the flow ϕ_t, we obtain

$$\int_X \bar{F}(x,t)\, d\mu(x) = \int_X \bar{F}(x, t-[t])\, d\mu(x) + [t] \int_X \bar{F}(x,1)\, d\mu(x).$$

Therefore we have

$$\mathcal{S}(\mu)([f]) = \lim_{t\to+\infty}\int_X \frac{\bar{F}(x,t)}{t}\,d\mu(x). \qquad (***)$$

Suppose now that we set $\gamma_x(s) = \phi_s(x)$; we have $\bar{F}(x,t) = \mathcal{V}(f\gamma_x|[0,t])$. Fix now some point $x_0 \in X$, and consider $t_i \to +\infty$ such that $\phi_{t_i} \to \phi$ in the C^0 topology. Since $\bar{F}(x_0,t)/t$ is bounded in absolute value by $2\|\bar{F}|X\times[0,1]\|_\infty$, for $t \geq 1$, extracting a subsequence if necessary, we can assume that $\bar{F}(x_0,t_i)/t_i \to c \in \mathbb{R}$. If $x \in X$, we can find a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. The map $\Gamma : [0,1]\times[0,t] \to \mathbb{T}, (s,s') \to \phi_{s'}(\gamma(s))$ is continuous, therefore we can lift it to a continuous function with values in $\mathbb{R}$, and this implies the equality

$$\mathcal{V}(\Gamma|[0,1]\times\{0\}) + \mathcal{V}(\Gamma|\{1\}\times[0,t]) - \mathcal{V}(\Gamma|[0,1]\times\{1\}) - \mathcal{V}(\Gamma|\{0\}\times[0,t]) = 0.$$

This can be rewritten as

$$\mathcal{V}(f\gamma_x|[0,t]) - \mathcal{V}(f\gamma_{x_0}|[0,t]) = \mathcal{V}(f\phi_t\gamma) - \mathcal{V}(f\gamma),$$

which translates to

$$\bar{F}(x,t) - \bar{F}(x_0,t) = \mathcal{V}(f\phi_t\gamma) - \mathcal{V}(f\gamma).$$

Since $\phi_{t_i} \to \phi$ uniformly, by continuity of $\mathcal{V}$, the left hand-side remains bounded as $t = t_i \to +\infty$. It follows that $(\bar{F}(x,t_i) - \bar{F}(x_0,t_i))/t_i \to 0$. Hence for every $x \in X$, we also have that $\bar{F}(x,t_i)/t_i$ tends to the same limit c as $\bar{F}(x_0,t_i)/t_i$. Since $\bar{F}(x,t)/t$ is uniformly bounded for $t \geq 1$, by $(**)$, by Lebesgue's dominated convergence we obtain from $(***)$ that $\mathcal{S}(\mu)([f]) = c$, where c is independent of the invariant measure μ. This is of course true for any $f : X \to \mathbb{T}$. Therefore $\mathcal{S}(\mu)$ does not depend on the invariant measure μ. □

An interesting property of Schwartzman uniquely ergodic flows (which also shows that they have some kind of rigidity) is the following proposition, that follows immediately from the definition of Schwartzman unique ergodicity and what we remarked above about the asymptotic cycles of fixed and periodic points (see also [31]).

Proposition A.8. *Suppose that ϕ_t is a Schwartzman uniquely ergodic flow on X. If there exists either a fixed point or a closed orbit homologous to zero, then all closed orbits are homologous to zero. In the remaining case, if C_1 and C_2 are closed orbits with periods τ_1 and τ_2, then $\frac{C_1}{\tau_1}$ and $\frac{C_2}{\tau_2}$ are homologous. Since $[C_1]$ and $[C_2]$ are in $\mathrm{H}_1(X;\mathbb{Z})$, it follows in this case that the ratio of the periods of any two closed orbits must be rational. Consequently, for any continuous family of periodic orbits of φ_t, all orbits have the same period.*

Definition A.9. [Schwartzman strict ergodicity] We say that a flow ϕ_t is Schwartzman strictly ergodic if it is Schwartzman uniquely ergodic and it has an invariant measure μ of full support (*i.e.,* $\mu(U) > 0$ for every non-empty open subset U of X).

Linear flows on the torus $\mathbb{T}^n$ are Schwartzman strictly ergodic (they preserve Lebesgue measure). Of course, all strictly ergodic flows (*i.e.,* flows having exactly one invariant probability measure, and the support of this measure is full) are also Schwartzman strictly ergodic. A minimal flow which is Schwartzman uniquely ergodic is in fact Schwartzman strictly ergodic (because all invariant measures have full support). Moreover, any flow topologically conjugate to a Schwartzman strictly ergodic flow is also Schwartzman strictly ergodic.

B Lagrangian graphs and KAM tori

In this Appendix we recall some very well-known properties of Lagrangian graphs and KAM tori used in the previous sections. We provide the standard proofs for the convenience of the reader. First of all, let us observe that, as remarked in Section 2, the canonical symplectic structure is intrinsically defined, *i.e.,* it does not depend on the choice of the local coordinates. In fact, it is easy to check that the Liouville form λ can be equivalently defined as

$$\lambda(x,p) = (d\pi(x,p))^* p \ \in \mathrm{T}^*_{(x,p)}\mathrm{T}^*M,$$

where $\pi : \mathrm{T}^*M \longrightarrow M$ is the canonical projection on M. Now, let us prove that smooth Lagrangian graphs correspond to closed 1-forms.

Proposition B.1. *Let $\Lambda = \{(x,\eta(x)),\ x \in M\}$ a smooth section of T^*M. Λ is Lagrangian if and only if η is a closed 1-form.*

Proof. Let us consider:

$$\begin{array}{rcl} s_\eta : M & \longrightarrow & \mathrm{T}^*M \\ x & \longmapsto & (x,\eta(x))\,. \end{array}$$

We want to prove first that $s_\eta^*\lambda = \eta$, where λ is the tautological form introduced above and $s_\eta^*\lambda$ denotes its pull-back. Recalling that $\lambda(x,p) = (d\pi(x,p))^*p$, we get:

$$\begin{array}{rcl} (s_\eta^*\lambda)(x) & = & (ds_\eta(x))^*\lambda(x,p) = (ds_\eta(x))^*(d\pi(x,p))^*\eta(x) = \\ & = & d(\pi \circ s_\eta(x,p))^*\eta(x) = \eta(x)\,, \end{array}$$

where in the last equality we used that $\pi \circ s_\eta$ is the identity map. Using this property, the claim follows immediately. In fact:

$$\begin{array}{rcccccc} \Lambda \text{ is Lagrangian} & \Longleftrightarrow & \omega\big|_{\mathrm{T}\Lambda} = 0 & \Longleftrightarrow & s_\eta^*\omega = 0 & \Longleftrightarrow & s_\eta^* d\lambda = 0 \\ & \Longleftrightarrow & ds_\eta^*\lambda = 0 & \Longleftrightarrow & d\eta = 0 & \Longleftrightarrow & \eta \text{ is closed.} \end{array}$$

□

Let us now consider a Hamiltonian $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$.

Proposition B.2. *Let Λ be a Lagrangian submanifold. Then Λ is invariant if and only if* $H\big|_\Lambda \equiv \mathrm{const.}$

Proof. (*i*) The Hamiltonian vector field X_H is defined by $\omega(X_H,\cdot) = dH$. Since Λ is invariant, $X_H\big|_\Lambda$ is tangent to Λ. But Λ is Lagrangian, therefore $0 = \omega(X_H,V) = dH \cdot V$ for any $V \in T\Lambda$, and this implies that H is constant on Λ. (*ii*) Since H is constant on Λ, we have that, for every $V \in \mathrm{T}\Lambda$, $0 = dH \cdot V = \omega(X_H,V)$. Since Λ is Lagrangian, X_H belongs to $\mathrm{T}\Lambda$ itself and therefore Λ is invariant. □

Remark B.3. As observed in the proof of Lemma 4.1, if Λ is an invariant Lagrangian graph, then the value of the constant is given by $\alpha(c_\Lambda)$, where α is the α-function associated to H and c_Λ the cohomology class of Λ.

To conclude, let us show that in the case of KAM tori with rationally independent rotation vectors, the condition of being Lagrangian is automatically satisfied. Actually, the same result holds in a slightly more general setting, as was observed in [18][page 52, Proposition 3.2], from where we borrowed the proof.

Proposition B.4. *Given a Hamiltonian H, let $\mathcal{T} \subset \mathbb{T}^n \times \mathbb{R}^n$ be an invariant graph over $\mathbb{T}^n$ such that the Hamiltonian flow on $\mathcal{T}$ is conjugated to a flow R_t on $\mathbb{T}^n$, which is transitive, i.e., with a dense orbit. Then $\mathcal{T}$ is Lagrangian.*

Proof. Let $\varphi : \mathbb{T}^n \to \mathcal{T}$ be the conjugation: $\varphi^{-1} \circ \Phi_t \circ \varphi = R_t$, $\forall t \in \mathbb{R}$. Consider the inclusion $i_{\mathcal{T}}$ of $\mathcal{T}$ into $\mathbb{T}^n \times \mathbb{R}^n$. We want to prove that $\omega\big|_{\mathcal{T}} = i_{\mathcal{T}}^*\omega \equiv 0$. Let us start by proving that the restriction of the symplectic form $i_{\mathcal{T}}^*\omega$ is invariant under the Hamiltonian flow Φ_t. In fact,

$$\Phi_t^*\left(i_{\mathcal{T}}^*\omega\right) = \left(i_{\mathcal{T}} \circ \Phi_t\right)^*\omega = \left(\Phi_t \circ i_{\mathcal{T}}\right)^*\omega = i_{\mathcal{T}}^*\left(\Phi_t^*\omega\right) = i_{\mathcal{T}}^*\omega\,,$$

where we used that $\mathcal{T}$ is invariant ($\Phi_t \circ i_{\mathcal{T}} = i_{\mathcal{T}} \circ \Phi_t$) and Φ_t is a symplectomorphism for any $t \in \mathbb{R}$ (*i.e.*, $\Phi_t^*\omega = \omega$). Consider now the 1-form on $\mathbb{T}^n$ given by $\omega_1 = \varphi^*\left(i_{\mathcal{T}}^*\omega\right)$. Let us show that ω_1 is invariant under R_t; in fact,

$$\begin{aligned}(R_t)^*\omega_1 &= (R_t)^*\left(\varphi^*\left(i_{\mathcal{T}}^*\omega\right)\right) = \left(\varphi \circ R_t\right)^* i_{\mathcal{T}}^*\omega = \left(\Phi_t \circ \varphi\right)^* i_{\mathcal{T}}^*\omega = \\ &= \varphi^*\left(\Phi_t^*\left(i_{\mathcal{T}}^*\omega\right)\right) = \varphi^*\left(i_{\mathcal{T}}^*\omega\right) = \omega_1\,,\end{aligned}$$

where we used that $\varphi^{-1} \circ \Phi_t \circ \varphi = R_t$. Since R_t is transitive, then ω_1 invariant implies ω_1 constant: $\omega_1 = \sum_{i<j} a_{ij}\, dx_i \wedge dx_j$. But ω_1 is exact (since $\omega = -d\lambda$ is exact), therefore $\omega_1 = \varphi^*\left(i_{\mathcal{T}}^*\omega\right) \equiv 0$ and (using that φ is invertible) the result is proved. ☐

C Proof of Theorem 3.9

In this Appendix we shall give the proof of the second equivalence in (9) and this will conclude the proof of Theorem 3.9, as already discussed after its statement. We want to show that

$$\mu \in \mathfrak{M}(L) \text{ and } \operatorname{supp}\mu \subseteq \widetilde{\mathcal{N}}_c \implies \operatorname{supp}\mu \subseteq \widetilde{\mathcal{A}}_c\,. \tag{19}$$

Let us first recall the definition of *non-wandering point* for a flow $\Phi_t : X \longrightarrow X$.

Definition C.1. *A point $x \in X$ is called non-wandering if for each neighborhood $\mathcal{U}$ and each positive integer n, there exists $t > n$ such that $f^t(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$.*

We will denote the set of *non-wandering* points for Φ_t by $\Omega(\Phi_t)$. Note that, if μ is an invariant measure, then $\operatorname{supp}\mu \subseteq \Omega(\Phi_t)$. In fact, by the ergodic decomposition theorem, every point $x \in \operatorname{supp}\mu$ is in the support of an ergodic invariant measure μ_1: therefore, x is non-wandering, by the ergodicity of μ_1. Given this remark, (19) is a simple consequence of the following Proposition.

Proposition C.2. *If M is a compact manifold and L a Tonelli Lagrangian on $\mathrm{T}M$, then $\Omega\left(\Phi_t^L\big|\widetilde{\mathcal{N}}_c\right) \subseteq \widetilde{\mathcal{A}}_c$ for each $c \in \mathrm{H}^1(M;\mathbb{R})$.*

Remarks.
1) Proposition C.2 also shows that the Aubry set is non-empty. In fact, any continuous flow on a compact space possesses non-wandering points.
2) As remarked above, every point in the support of an invariant measure μ is non-wandering. Therefore, if $x \in \text{supp } \mu \subseteq \widetilde{\mathcal{N}}_c$, then $x \in \Omega\left(\Phi_t^L\big|\widetilde{\mathcal{N}}_c\right)$ and, by the above proposition, $x \in \widetilde{\mathcal{A}}_c$. This proves (19).

Before proving the above statement, we need some preliminary results. Let $u : M \longrightarrow \mathbb{R}$ be an η-critical subsolution, with $[\eta] = c$ (see Definition 3.11), and $\gamma : [a,b] \longrightarrow M$ a curve. It is easy to check (see for instance [12]), that for all $a \leq t \leq t' \leq b$:

$$u(\gamma(t')) - u(\gamma(t)) \leq \int_t^{t'} L(\gamma(s), \dot\gamma(s))\, ds + \alpha(c)(t'-t)\,, \tag{20}$$

see also (13) and (15). We will say that $\gamma : [a,b] \longrightarrow M$ is $(u, L, \alpha(c))$-*calibrated* on $[a,b]$ if for all $a \leq t \leq t' \leq b$ the above inequality is an equality:

$$u(\gamma(t')) - u(\gamma(t)) = \int_t^{t'} L(\gamma(s), \dot\gamma(s))\, ds + \alpha(c)(t'-t)\,.$$

In [12] many properties of such curves have been studied. In particular, they provide a useful characterization of the Aubry and Mañé sets in terms of critical subsolutions. Let us denote by $\widetilde{\mathcal{I}}_c(u)$ the set of points $(x,v) \in \mathrm{T}M$ such that the curve $\gamma(t) = \pi\Phi_t^L(x,v)$ is $(u, L, \alpha(c))$-calibrated on $\mathbb{R}$. In [12] it is proven that:

$$\widetilde{\mathcal{A}}_c = \bigcap_{u \in \mathcal{S}^1_\eta} \widetilde{\mathcal{I}}_c(u) \quad \text{and} \quad \widetilde{\mathcal{N}}_c = \bigcup_{u \in \mathcal{S}^1_\eta} \widetilde{\mathcal{I}}_c(u). \tag{21}$$

Proof. [**Proposition C.2**] Without any loss of generality we can assume that $c = 0$ and $\alpha(0) = 0$. Let $(x,v) \in \Omega\left(\Phi_t^L\big|\widetilde{\mathcal{N}}_0\right)$. By the definition of non-wandering point, there exist a sequence $(x_k, v_k) \in \widetilde{\mathcal{N}_0}$ and $t_k \to +\infty$, such that $(x_k, v_k) \to (x,v)$ and $\Phi^L_{t_k}(x_k, v_k) \to (x,v)$ as $k \to +\infty$. From (21), for each (x_k, v_k) there exists a critical subsolution u_k, such that the curve $\gamma_k(t) = \pi\Phi_t^L(x_k, v_k)$ is $(u_k, L, 0)$-calibrated. Moreover, up to extracting a subsequence, we can assume that, on any compact interval, γ_k converge in the C^1-topology to $\gamma(t) = \Phi_t^L(x,v)$.

Pick now any critical subsolution u. If we show that γ is $(u, L, 0)$-calibrated, using (21) we can conclude that $(x,v) \in \widetilde{\mathcal{A}}_0$. First of all, observe that, by the continuity of u,

$$u(\gamma_k(t_k)) - u(x_k) \stackrel{k\to\infty}{\longrightarrow} 0\,.$$

Using that η-critical subsolutions are equi-Lipschitz [12], we can also conclude that

$$u_k(\gamma_k(t_k)) - u_k(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$$

and, therefore,

$$\int_0^{t_k} L(\gamma_k(s), \dot\gamma_k(s))\, ds = u_k(\gamma_k(t_k)) - u_k(x_k) \stackrel{k\to\infty}{\longrightarrow} 0\,. \tag{22}$$

Let $0 \le a \le b$ and choose $t_k \ge b$. Observe now that

$$\begin{aligned} u(\gamma_k(b)) - u(\gamma_k(a)) &= u(\gamma_k(t_k)) - u(x_k) - [u(\gamma_k(t_k)) - u(\gamma_k(b))] - [u(\gamma_k(a)) - u(x_k)] \ge \\ &\ge u(\gamma_k(t_k)) - u(x_k) - \int_b^{t_k} L(\gamma_k(s), \dot\gamma_k(s))\, ds - \int_0^a L(\gamma_k(s), \dot\gamma_k(s))\, ds = \\ &= u(\gamma_k(t_k)) - u(x_k) + \int_a^b L(\gamma_k(s), \dot\gamma_k(s))\, ds - \int_0^{t_k} L(\gamma_k(s), \dot\gamma_k(s))\, ds\,; \end{aligned}$$

taking the limit as $k \to \infty$ on both sides, one can conclude:

$$u(\gamma(b)) - u(\gamma(a)) \ge \int_a^b L(\gamma(s), \dot\gamma(s))\, ds$$

and therefore, from (20), it follows the equality. This shows that γ is $(u, L, 0)$-calibrated on $[0, \infty)$. To show that it is indeed calibrated on all $\mathbb{R}$, one can make a symmetric argument, letting $(y_k, w_k) = \Phi^L_{t_k}(x_k, v_k)$ play the role of (x_k, v_k) in the previous argument. In fact, one has $(y_k, w_k) \to (x, v)$ and $\Phi^L_{-t_k}(y_k, w_k) \to (x, v)$ as $k \to +\infty$ and the very same argument works.

□

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