LONG-TIME ASYMPTOTICS FOR THE KORTEWEG–DE VRIES EQUATION VIA NONLINEAR STEEPEST DESCENT

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Abstract. We apply the method of nonlinear steepest descent to compute the long-time asymptotics of the Korteweg–de Vries equation for decaying initial data in the soliton and similarity region.

1. Introduction

One of the most famous examples of completely integrable wave equations is the Korteweg–de Vries (KdV) equation

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where, as usual, the subscripts denote the differentiation with respect to the corresponding variables.

Following the seminal work of Gardner, Green, Kruskal, and Miura [13], one can use the inverse scattering transform to establish existence and uniqueness of (real-valued) classical solutions for the corresponding initial value problem with rapidly decaying initial conditions. We refer to, for instance, the monographs by Marchenko [18] or Eckhaus and Van Harten [12]. Our concern here are the long-time asymptotics of such solutions. The classical result is that an arbitrary short-range solution of the above type will eventually split into a number of solitons travelling to the right plus a decaying radiation part travelling to the left, as illustrated in Figure 1. This was first established numerically by Zabusky and Kruskal [27] and later rigorously proved by Eckhaus and Schuur [11]. Precise asymptotics for the radiation part were first formally derived by Zakharov and Manakov [26] in case of the nonlinear Schrödinger equation and in the KdV case by Ablowitz and Segur [1], [22] (assuming absence of solitons) with further extensions by Buslaev and Sukhanov [4].

To describe the asymptotics in more detail, we recall the well-known fact (see e.g. [7], [18]) that $q(x,t)$ is uniquely determined by the (right) scattering data of the associated Schrödinger operator

$$H(t) = -\frac{d^2}{dx^2} + q(x,t). \tag{1.2}$$

The scattering data consist of the reflection coefficient $R(k,t)$, a finite number of (t independent) eigenvalues $-\kappa_j^2$ with $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_N$, and norming constants $\gamma_j(t)$. We will write $R(k) = R(k,0)$ and $\gamma_j = \gamma_j(0)$ for the scattering

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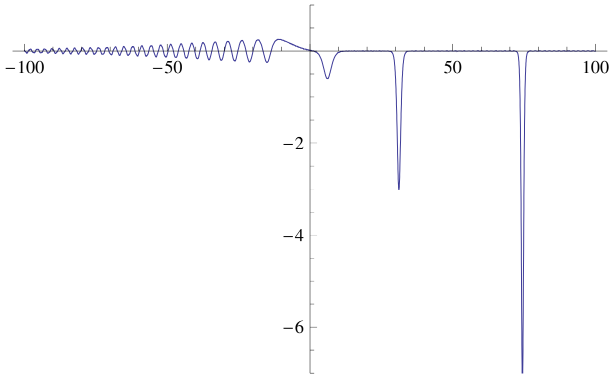

FIGURE 1. Numerically computed solution $q(x,t)$ of the KdV equation at time $t=5$, with initial condition $q(x,0)=\operatorname{sech}(x+3)-5\operatorname{sech}(x-1)$.

data of the initial condition. Then the long-time asymptotics can be described by distinguishing the following main regions:

(i). The soliton region, $x/t>C$ for some $C>0$, in which the solution is asymptotically given by a sum of one-soliton solutions

$$q(x,t)\sim -2\sum_{j=1}^{N}\frac{\kappa_j^2}{\cosh^2(\kappa_j x-4\kappa_j^3 t-p_j)}, \tag{1.3}$$

where the phase shifts are given by

$$p_j=\frac{1}{2}\log\left(\frac{\gamma_j^2}{2\kappa_j}\prod_{l=j+1}^{N}\left(\frac{\kappa_l-\kappa_j}{\kappa_l+\kappa_j}\right)^2\right). \tag{1.4}$$

In the case of a pure N-soliton solution (i.e., $R(k,t)=0$) this was first established independently by Hirota [14], Tanaka [23], and Wadati and Toda [24]. In the general case it was first established rigorously by Eckhaus and Schuur [11] (see also [21]).

(ii). The self-similar region, $|x/(3t)^{1/3}|\le C$ for some $C>0$, in which the solution is connected with the Painléve II transcendent. This was first established by Segur and Ablowitz [22].

(iii). The collisionless shock region, $x<0$ and $C^{-1}<\frac{-x}{(3t)^{1/3}(\log(t))^{2/3}}<C$, for some $C>1$, which only occurs in the generic case (i.e., when $R(0)=-1$). Here the asymptotics can be given in terms of elliptic functions as was pointed out by Segur and Ablowitz [22] with further extensions in Deift, Venakides, and Zhou [8].

(vi). The similarity region, $x/t<-C$ for some $C>0$, where

$$q(x,t)\sim\left(\frac{4\nu(k_0)k_0}{3t}\right)^{1/2}\sin(16tk_0^3-\nu(k_0)\log(192tk_0^3)+\delta(k_0)), \tag{1.5}$$

with

$$\begin{aligned}\nu(k_0) =& -\frac{1}{2\pi}\log(1-|R(k_0)|^2),\\ \delta(k_0) =& \frac{\pi}{4} - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu(k_0))) + 4\sum_{j=1}^{N}\arctan\big(\frac{\kappa_j}{k_0}\big)\\ &+ \frac{1}{\pi}\int_{-k_0}^{k_0}\log(|\zeta-k_0|)d\log(1-|R(\zeta)|^2).\end{aligned}$$

Here $k_0 = \sqrt{-\frac{x}{12t}}$ denotes the stationary phase point, $R(k) = R(k,t=0)$ the reflection coefficient, and Γ the Gamma function.

Again this was found by Ablowitz and Segur [1] with further extensions by Buslaev and Sukhanov [4].

However, to the best of our knowledge a full rigorous treatment of these results is still missing, except for the paper by Eckhaus and Schuur [11], which covers the soliton region together with the fact that the solution decays in the other regions.

Our aim here is to use the nonlinear steepest descent method for oscillatory Riemann–Hilbert problems from Deift and Zhou [9] and apply it to rigorously establish the long-time asymptotics in the soliton and similarity regions (Theorem 4.4 respectively 5.4 below). Solitons will be added using the techniques from Deift, Kamvissis, Kriecherbauer, and Zhou [6] respectively Krüger and Teschl [16]. Overall we closely follow the recent review article [17], where Krüger and Teschl applied these methods to compute the long-time asymptotics for the Toda lattice.

Finally, note that if $q(x,t)$ solves the KdV equation, then so does $q(-x,-t)$. Therefore it suffices to investigate the case $t\to\infty$.

2. The Inverse scattering transform and the Riemann–Hilbert problem

In this section we want to derive the Riemann–Hilbert problem for the KdV equation from scattering theory. This is essentially classical (compare, e.g., [3]) except for two points. The eigenvalues will be added by appropriate pole conditions which are then turned into jumps following Deift, Kamvissis, Kriecherbauer, and Zhou [6]. We will impose an additional symmetry conditions to ensure uniqueness later on following Krüger and Teschl [16].

For the necessary results from scattering theory respectively the inverse scattering transform for the KdV equation we refer to [18] (see also [3] and [7]). We consider real-valued classical solutions $q(x,t)$ of the KdV equation (1.1), which decay rapidly, that is

$$\max_{|t|\le T}\int_{\mathbb{R}}(1+|x|)^{1+l}|q(x,t)|dx<\infty, \qquad \text{for all } T>0, \tag{2.1}$$

for some $l\in\mathbb{N}_0$. Existence of such solutions can for example be established via the inverse scattering transform if one assumes (cf. [18, Sect. 4.2]) that the initial condition satisfies

$$\int_{\mathbb{R}}(1+|x|)\left(|q(x,0)|+|q_x(x,0)|+|q_{xx}(x,0)|+|q_{xxx}(x,0)|\right)dx<\infty. \tag{2.2}$$

Associated with $q(x,t)$ is a self-adjoint Schrödinger operator

$$H(t) = -\frac{d^2}{dx^2} + q(.,t), \qquad \mathfrak{D}(H) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}). \tag{2.3}$$

Here $L^2(\mathbb{R})$ denotes the Hilbert space of square integrable (complex-valued) functions over $\mathbb{R}$ and $H^k(\mathbb{R})$ the corresponding Sobolev spaces.

By our assumption (2.1) the spectrum of H consists of an absolutely continuous part $[0,\infty)$ plus a finite number of eigenvalues $-\kappa_j^2 \in \mathbb{R}\backslash(-\infty,0]$, $1 \le j \le N$. In addition, there exist two Jost solutions $\psi_\pm(k,x,t)$ which solve the differential equation

$$H(t)\psi_\pm(k,x,t) = k^2\psi_\pm(k,x,t), \qquad \mathrm{Im}(k) > 0, \tag{2.4}$$

and asymptotically look like the free solutions

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mp\mathrm{i}kx}\psi_\pm(k,x,t) = 1. \tag{2.5}$$

Both $\psi_\pm(k,x,t)$ are analytic for $\mathrm{Im}(k) > 0$ and continuous for $\mathrm{Im}(k) \ge 0$.

The asymptotics of the two Jost solutions are

$$\psi_\pm(k,x,t) = \mathrm{e}^{\pm\mathrm{i}kx}\Big(1 + Q_\pm(x,t)\frac{1}{2\mathrm{i}k} + O\big(\frac{1}{k^2}\big)\Big), \tag{2.6}$$

as $k \to \infty$ with $\mathrm{Im}(k) > 0$, where

$$Q_+(x,t) = -\int_x^\infty q(y,t)dy, \quad Q_-(x,t) = -\int_{-\infty}^x q(y,t)dy. \tag{2.7}$$

Furthermore, one has the scattering relations

$$T(k)\psi_\mp(k,x,t) = \overline{\psi_\pm(k,x,t)} + R_\pm(k,t)\psi_\pm(k,x,t), \qquad k \in \mathbb{R}, \tag{2.8}$$

where $T(k)$, $R_\pm(k,t)$ are the transmission respectively reflection coefficients. They have the following well-known properties:

Lemma 2.1. *The transmission coefficient* $T(k)$ *is meromorphic for* $\mathrm{Im}(k) > 0$ *with simple poles at* $\mathrm{i}\kappa_1,\dots,\mathrm{i}\kappa_N$ *and is continuous up to the real line. The residues of* $T(k)$ *are given by*

$$\mathrm{Res}_{\mathrm{i}\kappa_j} T(k) = \mathrm{i}\mu_j(t)\gamma_{+,j}(t)^2 = \mathrm{i}\mu_j\gamma_{+,j}^2, \tag{2.9}$$

where

$$\gamma_{+,j}(t)^{-1} = \|\psi_+(\mathrm{i}\kappa_j,.,t)\|_2 \tag{2.10}$$

and $\psi_+(\mathrm{i}\kappa_j,x,t) = \mu_j(t)\psi_-(\mathrm{i}\kappa_j,x,t)$.

Moreover,

$$T(k)\overline{R_+(k,t)} + \overline{T(k)}R_-(k,t) = 0, \qquad |T(k)|^2 + |R_\pm(k,t)|^2 = 1. \tag{2.11}$$

In particular one reflection coefficient, say $R(k,t) = R_+(k,t)$, and one set of norming constants, say $\gamma_j(t) = \gamma_{+,j}(t)$, suffices. Moreover, the time dependence is given by:

Lemma 2.2. *The time evolutions of the quantities* $R(k,t)$ *and* $\gamma_j(t)$ *are given by*

$$R(k,t) = R(k)\mathrm{e}^{8\mathrm{i}k^3t}, \tag{2.12}$$

$$\gamma_j(t) = \gamma_j\mathrm{e}^{4\kappa_j^3t}, \tag{2.13}$$

where $R(k) = R(k,0)$ *and* $\gamma_j = \gamma_j(0)$.

We will set up a Riemann–Hilbert problem as follows:

$$m(k,x,t) = \begin{cases} \begin{pmatrix} T(k)\psi_-(k,x,t)\mathrm{e}^{\mathrm{i}kx} & \psi_+(k,x,t)\mathrm{e}^{-\mathrm{i}kx}\end{pmatrix}, & \mathrm{Im}(k)>0,\\ \begin{pmatrix} \psi_+(-k,x,t)\mathrm{e}^{\mathrm{i}kx} & T(-k)\psi_-(-k,x,t)\mathrm{e}^{-\mathrm{i}kx}\end{pmatrix}, & \mathrm{Im}(k)<0.\end{cases} \tag{2.14}$$

We are interested in the jump condition of $m(k,x,t)$ on the real axis $\mathbb{R}$ (oriented from negative to positive). To formulate our jump condition we use the following convention: When representing functions on $\mathbb{R}$, the lower subscript denotes the non-tangential limit from different sides. By $m_+(k)$ we denote the limit from above and by $m_-(k)$ the one from below. Using the notation above implicitly assumes that these limits exist in the sense that $m(k)$ extends to a continuous function on the real axis.

Theorem 2.3 (Vector Riemann–Hilbert problem)**.** *Let $\mathcal{S}_+(H(0)) = \{R(k),\ k\geq 0;\ (\kappa_j,\gamma_j),\ 1\leq j\leq N\}$ be the right scattering data of the operator $H(0)$. Then $m(k)=m(k,x,t)$ defined in (2.14) is meromorphic away from the real axis with simple poles at $\mathrm{i}\kappa_j$, $-\mathrm{i}\kappa_j$ and satisfies:*

(i) *The jump condition*

$$m_+(k) = m_-(k)v(k), \qquad v(k) = \begin{pmatrix} 1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-t\Phi(k)}\\ R(k)\mathrm{e}^{t\Phi(k)} & 1\end{pmatrix}, \tag{2.15}$$

for $k\in\mathbb{R}$,

(ii) *the pole conditions*

$$\begin{aligned} \mathrm{Res}_{\mathrm{i}\kappa_j}\, m(k) &= \lim_{k\to\mathrm{i}\kappa_j} m(k)\begin{pmatrix} 0 & 0\\ \mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} & 0\end{pmatrix},\\ \mathrm{Res}_{-\mathrm{i}\kappa_j}\, m(k) &= \lim_{k\to -\mathrm{i}\kappa_j} m(k)\begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\\ 0 & 0\end{pmatrix}, \end{aligned} \tag{2.16}$$

(iii) *the symmetry condition*

$$m(-k) = m(k)\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \tag{2.17}$$

(iv) *and the normalization*

$$\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = (1 \quad 1). \tag{2.18}$$

Here the phase is given by

$$\Phi(k) = 8\mathrm{i}k^3 + 2\mathrm{i}k\frac{x}{t}. \tag{2.19}$$

Proof. The jump condition (2.15) is a simple calculation using the scattering relations (2.8) plus (2.11). The pole conditions follow since $T(k)$ is meromorphic for $\mathrm{Im}(k)>0$ with simple poles at $\mathrm{i}\kappa_j$ and residues given by (2.9). The symmetry condition holds by construction and the normalization (2.18) is immediate from the following lemma below. □

Observe that the pole condition at $\mathrm{i}\kappa_j$ is sufficient since the one at $-\mathrm{i}\kappa_j$ follows by symmetry.

Moreover, using

$$T(k)\psi_-(k,x,t)\psi_+(k,x,t) = 1 + \frac{q(x,t)}{2k^2} + O(\frac{1}{k^4}) \tag{2.20}$$

as $k \to \infty$ with $\mathrm{Im}(k) > 0$ (observe that the right-hand side is just the diagonal Green's functions of $H(t)$ divided by the free one) we obtain from (2.6)

Lemma 2.4. *The function $m(k,x,t)$ defined in (2.14) satisfies*

$$m(k,x,t) = \begin{pmatrix} 1 & 1 \end{pmatrix} + Q(x,t)\frac{1}{2\mathrm{i}k}\begin{pmatrix} -1 & 1 \end{pmatrix} + O\left(\frac{1}{k^2}\right). \tag{2.21}$$

Here $Q(x,t) = Q_+(x,t)$ is defined in (2.7).

For our further analysis it will be convenient to rewrite the pole condition as a jump condition and hence turn our meromorphic Riemann–Hilbert problem into a holomorphic Riemann–Hilbert problem following [6]. Choose ε so small that the discs $|k - \mathrm{i}\kappa_j| < \varepsilon$ lie inside the upper half plane and do not intersect. Then redefine $m(k)$ in a neighborhood of $\mathrm{i}\kappa_j$ respectively $-\mathrm{i}\kappa_j$ according to

$$m(k) = \begin{cases} m(k)\begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, & |k-\mathrm{i}\kappa_j| < \varepsilon, \\ m(k)\begin{pmatrix} 1 & \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, & |k+\mathrm{i}\kappa_j| < \varepsilon, \\ m(k), & \text{else}. \end{cases} \tag{2.22}$$

Note that for $\mathrm{Im}(k) < 0$ we redefined $m(k)$ such that it respects our symmetry (2.17). Then a straightforward calculation using $\mathrm{Res}_{\mathrm{i}\kappa}\, m(k) = \lim_{k\to\mathrm{i}\kappa}(k-\mathrm{i}\kappa)m(k)$ shows:

Lemma 2.5. *Suppose $m(k)$ is redefined as in (2.22). Then $m(k)$ is holomorphic away from the real axis and the small circles around $\mathrm{i}\kappa_j$ and $-\mathrm{i}\kappa_j$. Furthermore it satisfies (2.15), (2.17), (2.18) and the pole condition is replaced by the jump condition*

$$\begin{aligned} m_+(k) &= m_-(k)\begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, \quad |k-\mathrm{i}\kappa_j| = \varepsilon, \\ m_+(k) &= m_-(k)\begin{pmatrix} 1 & \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, \quad |k+\mathrm{i}\kappa_j| = \varepsilon, \end{aligned} \tag{2.23}$$

where the small circle around $\mathrm{i}\kappa_j$ is oriented counterclockwise.

Next we turn to uniqueness of the solution of this vector Riemann–Hilbert problem. This will also explain the reason for our symmetry condition. We begin by observing that if there is a point $k_1 \in \mathbb{C}$, such that $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, then $n(k) = \frac{1}{k-k_1}m(k)$ satisfies the same jump and pole conditions as $m(k)$. However, it will clearly violate the symmetry condition! Hence, without the symmetry condition, the solution of our vector Riemann–Hilbert problem will not be unique in such a situation. Moreover, a look at the one-soliton solution verifies that this case indeed can happen.

Lemma 2.6 (One-soliton solution)**.** *Suppose there is only one eigenvalue and that the reflection coefficient vanishes, that is, $\mathcal{S}_+(H(t)) = \{R(k,t) \equiv 0,\ k \in$*

$\mathbb{R}$; $(\kappa,\gamma(t)), \kappa>0, \gamma>0\}$. *Then the unique solution of the Riemann–Hilbert problem (2.15)–(2.18) is given by*

$$m_0(k) = \begin{pmatrix} f(k) & f(-k) \end{pmatrix} \tag{2.24}$$
$$f(k) = \frac{1}{1+(2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}} \left(1 + \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}(2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}\right).$$

In particular,

$$Q(x,t) = \frac{2\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}{1+(2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}. \tag{2.25}$$

Proof. By assumption the reflection coefficient vanishes and so the jump along the real axis disappears. Therefore and by the symmetry condition, we know that the solution is of the form $m_0(k) = \begin{pmatrix} f(k) & f(-k) \end{pmatrix}$ where $f(z)$ is meromorphic. Furthermore the function $f(k)$ has only a simple pole at $\mathrm{i}\kappa$, so that we can make the ansatz $f(k) = C + D\frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}$. Then the constants C and D are uniquely determined by the pole conditions and the normalization. □

In fact, observe $f(k_1) = f(-k_1) = 0$ if and only if $k_1 = 0$ and $2\kappa = \gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}$. Furthermore, even in the general case $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ can only occur at $k_1 = 0$ as the following lemma shows.

Lemma 2.7. *If $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ for m defined as in (2.14), then $k_1 = 0$. Moreover, the zero of at least one component is simple in this case.*

Proof. By (2.14) the condition $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ implies that the Jost solutions $\psi_-(k,x)$ and $\psi_+(k,x)$ are linearly dependent or that the transmission coefficient $T(k_1) = 0$. This can only happen, at the band edge, $k_1 = 0$ or at an eigenvalue $k_1 = \mathrm{i}\kappa_j$.

We begin with the case $k_1 = \mathrm{i}\kappa_j$. In this case the derivative of the Wronskian $W(k) = (\psi_+(k,x)\psi_-'(k,x) - \psi_+'(k,x)\psi_-(k,x))$ does not vanish by the well-known formula $\frac{d}{dk}W(k)|_{k=k_1} = -2k_1\int_{\mathbb{R}}\psi_+(k_1,x)\psi_-(k_1,x)dx \neq 0$. Moreover, the diagonal Green's function $g(z,x) = W(k)^{-1}\psi_+(k,x)\psi_-(k,x)$ is Herglotz as a function of $z = -k^2$ and hence can have at most a simple zero at $z = -k_1^2$. Since $z \to -k^2$ is conformal away from $z = 0$ the same is true as a function of k. Hence, if $\psi_+(\mathrm{i}\kappa_j,x) = \psi_-(\mathrm{i}\kappa_j,x) = 0$, both can have at most a simple zero at $k = \mathrm{i}\kappa_j$. But $T(k)$ has a simple pole at $\mathrm{i}\kappa_j$ and hence $T(k)\psi_-(k,x)$ cannot vanish at $k = \mathrm{i}\kappa_j$, a contradiction.

It remains to show that one zero is simple in the case $k_1 = 0$. In fact, one can show that $\frac{d}{dk}W(k)|_{k=k_1} \neq 0$ in this case as follows: First of all note that $\dot\psi_\pm(k)$ (where the dot denotes the derivative with respect to k) again solves $H\dot\psi_\pm(k_1) = -k_1^2\dot\psi_\pm(k_1)$ if $k_1 = 0$. Moreover, by $W(k_1) = 0$ we have $\psi_+(k_1) = c\,\psi_-(k_1)$ for some constant c (independent of x). Thus we can compute

$$\begin{aligned} \dot W(k_1) &= W(\dot\psi_+(k_1),\psi_-(k_1)) + W(\psi_+(k_1),\dot\psi_-(k_1)) \\ &= c^{-1}W(\dot\psi_+(k_1),\psi_+(k_1)) + cW(\psi_-(k_1),\dot\psi_-(k_1)) \end{aligned}$$

by letting $x \to +\infty$ for the first and $x \to -\infty$ for the second Wronskian (in which case we can replace $\psi_\pm(k)$ by $\mathrm{e}^{\pm\mathrm{i}kx}$), which gives

$$\dot W(k_1) = -\mathrm{i}(c + c^{-1}).$$

Hence the Wronskian has a simple zero. But if both functions had more than simple zeros, so would the Wronskian, a contradiction. □

3. A uniqueness result for symmetric vector Riemann–Hilbert problems

In this chapter we want to investigate uniqueness for the holomorphic vector Riemann–Hilbert problem

$$\begin{aligned} & m_+(k) = m_-(k)v(k), \qquad k \in \Sigma, \\ & m(-k) = m(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ & \lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \end{aligned} \tag{3.1}$$

Hypothesis H.3.1. *Let Σ consist of a finite number of smooth oriented curves in $\mathbb{C}$ such that the distance between Σ and $\{\mathrm{i}y | y \geq y_0\}$ is positive for some $y_0 > 0$. Assume that the contour Σ is invariant under $k \mapsto -k$. Moreover, suppose the jump matrix v can be factorized according to $v = b_-^{-1} b_+ = (\mathbb{I} - w_-)^{-1}(\mathbb{I} + w_+)$, where $w_\pm$ are continuous, bounded, square integrable and satisfy*

$$w_\pm(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_\mp(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in \Sigma. \tag{3.2}$$

Now we are ready to show that the symmetry condition in fact guarantees uniqueness.

Theorem 3.2. *Suppose there exists a solution $m(k)$ of the Riemann–Hilbert problem (3.1) for which $m(k) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ can happen at most for $k = 0$ in which case $\limsup_{k\to 0} \frac{k}{m_j(k)}$ is bounded from any direction for $j = 1$ or $j = 2$.*

Then the Riemann–Hilbert problem (3.1) with norming condition replaced by

$$\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} \alpha & \alpha \end{pmatrix} \tag{3.3}$$

for given $\alpha \in \mathbb{C}$, has a unique solution $m_\alpha(k) = \alpha\, m(k)$.

Proof. Let $m_\alpha(k)$ be a solution of (3.1) normalized according to (3.3). Then we can construct a matrix valued solution via $M = (m, m_\alpha)$ and there are two possible cases: Either $\det M(k)$ is nonzero for some k or it vanishes identically.

We start with the first case. By lemma 2.5, we can rewrite all poles as jumps with determinant one. Hence, the determinant of this modified Riemann–Hilbert problem has no jump. Since, it is bounded at infinity, the determinant is constant. But taking determinants in

$$M(-k) = M(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

gives a contradiction.

It remains to investigate the case where $\det(M) \equiv 0$. In this case we have $m_\alpha(k) = \delta(k) m(k)$ with a scalar function δ. Moreover, $\delta(k)$ must be holomorphic for $k \in \mathbb{C}\backslash\Sigma$ and continuous for $k \in \Sigma$ except possibly at the points where $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. Since it has no jump across Σ,

$$\delta_+(k) m_+(k) = m_{\alpha,+}(k) = m_{\alpha,-}(k) v(k) = \delta_-(k) m_-(k) v(k) = \delta_-(k) m_+(k),$$

it is even holomorphic in $\mathbb{C}\backslash\{0\}$ with at most a simple pole at $k=0$. Hence it must be of the form

$$\delta(k) = A + \frac{B}{k}.$$

Since δ has to be symmetric, $\delta(k) = \delta(-k)$, we obtain $B = 0$. Now, by the normalization we obtain $\delta(k) = A = \alpha$. This finishes the proof. □

Furthermore, the requirements cannot be relaxed to allow (e.g.) second order zeros in stead of simple zeros. In fact, if $m(k)$ is a solution for which both components vanish of second order at, say, $k=0$, then $\tilde{m}(k) = \frac{1}{k^2} m(k)$ is a nontrivial symmetric solution of the vanishing problem (i.e. for $\alpha = 0$).

By Lemma 2.7 we have

Corollary 3.3. *The function $m(k,x,t)$ defined in (2.14) is the only solution of the vector Riemann–Hilbert problem (2.15)–(2.18).*

Observe that there is nothing special about $k\to\infty$ where we normalize, any other point would do as well. However, observe that for the one-soliton solution (2.24), $f(k)$ vanishes at

$$k = \mathrm{i}\kappa \frac{1-(2\kappa)^2\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}{1+(2\kappa)^2\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}$$

and hence the Riemann–Hilbert problem normalized at this point has a nontrivial solution for $\alpha = 0$ and hence, by our uniqueness result, no solution for $\alpha = 1$. This shows that uniqueness and existence are connected, a fact which is not surprising since our Riemann–Hilbert problem is equivalent to a singular integral equation which is Fredholm of index zero (see Appendix A).

4. Conjugation and Deformation

This section demonstrates how to conjugate our Riemann-Hilbert problem and how to deform our jump contour, such that the jumps will be exponentially close to the identity away from the stationary phase points. Throughout this and the following section, we will assume that the $R(k)$ has an analytic extension to a small neighborhood of the real axis. This is for example the case if we assume that our solution is exponentially decaying. In Section 6 we will show how to remove this assumption.

For easy reference we note the following result:

Lemma 4.1 (Conjugation). *Assume that $\widetilde{\Sigma} \subseteq \Sigma$. Let D be a matrix of the form*

$$D(k) = \begin{pmatrix} d(k)^{-1} & 0 \\ 0 & d(k) \end{pmatrix}, \tag{4.1}$$

where $d: \mathbb{C}\backslash\widetilde{\Sigma} \to \mathbb{C}$ is a sectionally analytic function. Set

$$\tilde{m}(k) = m(k) D(k), \tag{4.2}$$

then the jump matrix transforms according to

$$\tilde{v}(k) = D_-(k)^{-1} v(k) D_+(k). \tag{4.3}$$

If d satisfies $d(-k) = d(k)^{-1}$ and $d(k) = 1 + O(\frac{1}{|k|})$ as $k\to\infty$. Then the transformation $\tilde{m}(k) = m(k)D(k)$ respects our symmetry, that is, $\tilde{m}(k)$ satisfies (2.17) if and only if $m(k)$ does, and our normalization condition.

In particular, we obtain

$$\tilde{v} = \begin{pmatrix} v_{11} & v_{12} d^2 \\ v_{21} d^{-2} & v_{22} \end{pmatrix}, \qquad k \in \Sigma \backslash \widetilde{\Sigma}, \tag{4.4}$$

respectively

$$\tilde{v} = \begin{pmatrix} \frac{d_-}{d_+} v_{11} & v_{12} d_+ d_- \\ v_{21} d_+^{-1} d_-^{-1} & \frac{d_+}{d_-} v_{22} \end{pmatrix}, \qquad k \in \Sigma \cap \widetilde{\Sigma}. \tag{4.5}$$

In order to remove the poles there are two cases to distinguish. If $k_0 \in \mathrm{i}\mathbb{R}$ and $\kappa_j < \kappa_0$ then the corresponding jump is exponentially close to the identity and there is nothing to do. Otherwise we use conjugation to turn the jumps into one with exponentially decaying off-diagonal entries, again following Deift, Kamvissis, Kriecherbauer, and Zhou [6]. It turns out that we will have to handle the poles at $\mathrm{i}\kappa_j$ and $-\mathrm{i}\kappa_j$ in one step in order to preserve symmetry and in order to not add additional poles elsewhere.

Lemma 4.2. *Assume that the Riemann–Hilbert problem for m has jump conditions near $\mathrm{i}\kappa$ and $-\mathrm{i}\kappa$ given by*

$$\begin{aligned} m_+(k) &= m_-(k) \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma^2}{k - \mathrm{i}\kappa} & 1 \end{pmatrix}, \quad |k - \mathrm{i}\kappa| = \varepsilon, \\ m_+(k) &= m_-(k) \begin{pmatrix} 1 & \frac{\mathrm{i}\gamma^2}{k + \mathrm{i}\kappa} \\ 0 & 1 \end{pmatrix}, \qquad |k + \mathrm{i}\kappa| = \varepsilon. \end{aligned} \tag{4.6}$$

Then this Riemann–Hilbert problem is equivalent to a Riemann–Hilbert problem for $\tilde{m}$ which has jump conditions near $\mathrm{i}\kappa$ and $-\mathrm{i}\kappa$ given by

$$\begin{aligned} \tilde{m}_+(k) &= \tilde{m}_-(k) \begin{pmatrix} 1 & -\frac{(k + \mathrm{i}\kappa)^2}{\mathrm{i}\gamma^2 (k - \mathrm{i}\kappa)} \\ 0 & 1 \end{pmatrix}, \qquad |k - \mathrm{i}\kappa| = \varepsilon, \\ \tilde{m}_+(k) &= \tilde{m}_-(k) \begin{pmatrix} 1 & 0 \\ \frac{(k - \mathrm{i}\kappa)^2}{\mathrm{i}\gamma^2 (k + \mathrm{i}\kappa)} & 1 \end{pmatrix}, \qquad |k + \mathrm{i}\kappa| = \varepsilon, \end{aligned}$$

and all remaining data conjugated (as in Lemma 4.1) by

$$D(k) = \begin{pmatrix} \frac{k - \mathrm{i}\kappa}{k + \mathrm{i}\kappa} & 0 \\ 0 & \frac{k + \mathrm{i}\kappa}{k - \mathrm{i}\kappa} \end{pmatrix}. \tag{4.7}$$

Proof. To turn γ^2 into γ^{-2}, introduce D by

$$D(k) = \begin{cases} \begin{pmatrix} 1 & -\frac{k - \mathrm{i}\kappa}{\mathrm{i}\gamma^2} \\ \frac{\mathrm{i}\gamma^2}{k - \mathrm{i}\kappa} & 0 \end{pmatrix} \begin{pmatrix} \frac{k - \mathrm{i}\kappa}{k + \mathrm{i}\kappa} & 0 \\ 0 & \frac{k + \mathrm{i}\kappa}{k - \mathrm{i}\kappa} \end{pmatrix}, & |k - \mathrm{i}\kappa| < \varepsilon, \\ \begin{pmatrix} 0 & -\frac{\mathrm{i}\gamma^2}{k + \mathrm{i}\kappa} \\ \frac{k + \mathrm{i}\kappa}{\mathrm{i}\gamma^2} & 1 \end{pmatrix} \begin{pmatrix} \frac{k - \mathrm{i}\kappa}{k + \mathrm{i}\kappa} & 0 \\ 0 & \frac{k + \mathrm{i}\kappa}{k - \mathrm{i}\kappa} \end{pmatrix}, & |k + \mathrm{i}\kappa| < \varepsilon, \\ \begin{pmatrix} \frac{k - \mathrm{i}\kappa}{k + \mathrm{i}\kappa} & 0 \\ 0 & \frac{k + \mathrm{i}\kappa}{k - \mathrm{i}\kappa} \end{pmatrix}, & \text{else}, \end{cases}$$

and note that $D(k)$ is analytic away from the two circles. Now set $\tilde{m}(k) = m(k) D(k)$and note that $D(k)$ is also symmetric. Therefore the jump conditions can be verified by straightforward calculations and Lemma 4.1. □

The jump along the real axis is of oscillatory type and our aim is to apply a contour deformation following [9] such that all jumps will be moved into regions where the oscillatory terms will decay exponentially. Since the jump matrix v contains both $\exp(t\Phi)$ and $\exp(-t\Phi)$ we need to separate them in order to be able to move them to different regions of the complex plane.

We recall that the phase of the associated Riemann–Hilbert problem is given by

$$\Phi(k) = 8\mathrm{i}k^3 + 2\mathrm{i}k\frac{x}{t} \tag{4.8}$$

and the stationary phase points, $\Phi'(k) = 0$, are denoted by

$$k_0 = \sqrt{-\frac{x}{12t}}, \quad -k_0 = -\sqrt{-\frac{x}{12t}}, \qquad \lambda_0 = \frac{x}{12t}. \tag{4.9}$$

For $\frac{x}{t} > 0$ we have $k_0 \in \mathrm{i}\mathbb{R}$, and for $\frac{x}{t} < 0$ we have $k_0 \in \mathbb{R}$. For $\frac{x}{t} > 0$ we will also need the value $\mathrm{i}\kappa_0 \in \mathrm{i}\mathbb{R}$ defined via $\mathrm{Re}(\Phi(\mathrm{i}\kappa_0)) = 0$, that is,

$$\frac{x}{t} = 4\kappa_0^2. \tag{4.10}$$

We will set $\kappa_0 = 0$ if $\frac{x}{t} < 0$ for notational convenience. A simple analysis shows that for $\frac{x}{t} > 0$ we have $0 < k_0/\mathrm{i} < \kappa_0$.

As mentioned above we will need the following factorizations of the jump condition (2.15):

$$v(k) = b_-(k)^{-1} b_+(k), \tag{4.11}$$

where

$$b_-(k) = \begin{pmatrix} 1 & \overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}, \qquad b_+(k) = \begin{pmatrix} 1 & 0 \\ R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}.$$

for $\mathrm{Re}(k_0) < |k|$ and

$$v(k) = B_-(k)^{-1} \begin{pmatrix} 1 - |R(k)|^2 & 0 \\ 0 & \frac{1}{1-|R(k)|^2} \end{pmatrix} B_+(k), \tag{4.12}$$

where

$$B_-(k) = \begin{pmatrix} 1 & 0 \\ -\frac{R(k)\mathrm{e}^{t\Phi(k)}}{1-|R(k)|^2} & 1 \end{pmatrix}, \qquad B_+(k) = \begin{pmatrix} 1 & -\frac{\overline{R(k)}\mathrm{e}^{-t\Phi(k)}}{1-|R(k)|^2} \\ 0 & 1 \end{pmatrix}.$$

for $|k| < \mathrm{Re}(k_0)$.

To get rid of the diagonal part in the factorization corresponding to $|k| < \mathrm{Re}(k_0)$ and to conjugate the jumps near the eigenvalues we need the partial transmission coefficient $T(k, k_0)$.

We define the partial transmission coefficient with respect to k_0 by

$$T(k,k_0) = \begin{cases} \displaystyle\prod_{\kappa_j \in (\kappa_0,\infty)} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j}, & k_0 \in \mathrm{i}\mathbb{R}^+, \\ \displaystyle\prod_{j=1}^{N} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j} \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \frac{\log(|T(\zeta)|^2)}{\zeta - k} d\zeta \right), & k_0 \in \mathbb{R}^+, \end{cases} \tag{4.13}$$

for $k \in \mathbb{C}\backslash\Sigma(k_0)$, where $\Sigma(k_0) = [-\mathrm{Re}(k_0), \mathrm{Re}(k_0)]$. Thus $T(k, k_0)$ is meromorphic for $k \in \mathbb{C}\backslash\Sigma(k_0)$. Note that $T(k, k_0)$ can be computed in terms of the scattering data since $|T(k)|^2 = 1 - |R_+(k,t)|^2$. Moreover, we set

$$T_1(k_0) = \begin{cases} \sum\limits_{\kappa_j \in (\kappa_0,\infty)} 2\kappa_j, & k_0 \in \mathrm{i}\mathbb{R}^+, \\ \sum\limits_{j=1}^N 2\kappa_j + \frac{1}{2\pi}\int_{-k_0}^{k_0} \log(|T(\zeta)|^2) d\zeta, & k_0 \in \mathbb{R}^+. \end{cases} \tag{4.14}$$

Thus

$$T(k,k_0) = 1 + T_1(k_0)\frac{\mathrm{i}}{k} + O\left(\frac{1}{k^2}\right), \qquad \text{as } k \to \infty. \tag{4.15}$$

Theorem 4.3. *The partial transmission coefficient $T(k,k_0)$ is meromorphic in $\mathbb{C}\backslash\Sigma(k_0)$, where $\Sigma(k_0) = [-\operatorname{Re}(k_0), \operatorname{Re}(k_0)]$, with simple poles at $\mathrm{i}\kappa_j$ and simple zeros at $-\mathrm{i}\kappa_j$ for all j with $\kappa_0 < \kappa_j$, and satisfies the jump condition*

$$T_+(k,k_0) = T_-(k,k_0)(1 - |R(k)|^2), \qquad \text{for } k \in \Sigma(k_0). \tag{4.16}$$

Moreover,

(i) $T(-k,k_0) = T(k,k_0)^{-1}$, $k \in \mathbb{C}\backslash\Sigma(k_0)$ and $\lim_{k\to\infty} T(k,k_0) > 0$

(ii) $\overline{T(\overline{k},k_0)} = T(k,k_0)^{-1}$.

Proof. That $\mathrm{i}\kappa_j$ are simple poles and $-\mathrm{i}\kappa_j$ are simple zeros is obvious from the Blaschke factors and that $T(k,k_0)$ has the given jump follows from Plemelj's formulas. (i) and (ii) are straightforward to check. □

Now we are ready to perform our conjugation step. Introduce

$$D(k) = \begin{cases} \begin{pmatrix} 1 & -\frac{k-\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}} \\ \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 0 \end{pmatrix} D_0(k), & |k - \mathrm{i}\kappa_j| < \varepsilon,\ \kappa_0 < \kappa_j, \\ \begin{pmatrix} 0 & -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ \frac{k+\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}} & 1 \end{pmatrix} D_0(k), & |k + \mathrm{i}\kappa_j| < \varepsilon,\ \kappa_0 < \kappa_j, \\ D_0(k), & \text{else}, \end{cases}$$

where

$$D_0(k) = \begin{pmatrix} T(k,k_0)^{-1} & 0 \\ 0 & T(k,k_0) \end{pmatrix}.$$

Note that we have

$$D(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we conjugate our problem using $D(k)$ and set

$$\tilde{m}(k) = m(k)D(k). \tag{4.17}$$

Then using Lemma 4.1 and Lemma 4.2 the jump corresponding to $\kappa_0 < \kappa_j$ (if any) is given by

$$\begin{aligned} \tilde{v}(k) &= \begin{pmatrix} 1 & -\frac{k-\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{-2}} \\ 0 & 1 \end{pmatrix}, \qquad |k - \mathrm{i}\kappa_j| = \varepsilon, \\ \tilde{v}(k) &= \begin{pmatrix} 1 & 0 \\ \frac{k+\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^2} & 1 \end{pmatrix}, \qquad |k + \mathrm{i}\kappa_j| = \varepsilon, \end{aligned} \tag{4.18}$$

and corresponding to $\kappa_0 > \kappa_j$ (if any) by

$$
\begin{aligned}
\tilde{v}(k) &= \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{-2}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, \qquad |k-\mathrm{i}\kappa_j| = \varepsilon, \\
\tilde{v}(k) &= \begin{pmatrix} 1 & \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{2}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, \qquad |k+\mathrm{i}\kappa_j| = \varepsilon.
\end{aligned} \tag{4.19}
$$

In particular, all jumps corresponding to poles, except for possibly one if $\kappa_j = \kappa_0$, are exponentially close to the identity. In this case we will keep the pole condition which now reads

$$
\begin{aligned}
\operatorname{Res}_{\mathrm{i}\kappa_j} \tilde{m}(k) &= \lim_{k\to \mathrm{i}\kappa_j} \tilde{m}(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j,k_0)^{-2} & 0 \end{pmatrix}, \\
\operatorname{Res}_{-\mathrm{i}\kappa_j} \tilde{m}(k) &= \lim_{k\to -\mathrm{i}\kappa_j} \tilde{m}(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j,k_0)^{-2} \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{4.20}
$$

Furthermore, the jump along $\mathbb{R}$ is given by

$$
\tilde{v}(k) = \begin{cases} \tilde{b}_-(k)^{-1} \tilde{b}_+(k), & \operatorname{Re}(k_0) < |k|, \\ \tilde{B}_-(k)^{-1} \tilde{B}_+(k), & \operatorname{Re}(k_0) > |k|, \end{cases} \tag{4.21}
$$

where

$$
\tilde{b}_-(k) = \begin{pmatrix} 1 & \frac{R(-k)\mathrm{e}^{-t\Phi(k)}}{T(-k,k_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(k) = \begin{pmatrix} 1 & 0 \\ \frac{R(k)\mathrm{e}^{t\Phi(k)}}{T(k,k_0)^2} & 1 \end{pmatrix}, \tag{4.22}
$$

and

$$
\begin{aligned}
\tilde{B}_-(k) &= \begin{pmatrix} 1 & 0 \\ -\frac{T_-(k,k_0)^{-2}}{1-|R(k)|^2} R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{T_-(-k,k_0)}{T_-(k,k_0)} R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, \\
\tilde{B}_+(k) &= \begin{pmatrix} 1 & -\frac{T_+(k,k_0)^2}{1-|R(k)|^2} R(-k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{T_+(k,k_0)}{T_+(-k,k_0)} R(-k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Here we have used

$$
T_\pm(-k,k_0) = T_\mp(k,k_0)^{-1}, \qquad k \in \Sigma(k_0)
$$

and the jump condition for the partial transmission coefficient $T(k,k_0)$ along $\Sigma(k_0)$ in the last step, which shows that the matrix entries are bounded for $k\in\mathbb{R}$.

Note also that we have used $T(k,k_0)^{-1} = \overline{T(k,k_0)}$ and $R(-k) = \overline{R(k)}$ for $k\in\mathbb{R}$ to show that there exists an analytic continuation into the neighborhood of the real axis.

Now we deform the jump along $\mathbb{R}$ to move the oscillatory terms into regions where they are decaying. There are two cases to distinguish:

Case 1: $k_0 \in \mathrm{i}\mathbb{R}$, $k_0 \neq 0$:

We set $\Sigma_\pm = \{k \in \mathbb{C} | \operatorname{Im}(k) = \pm\varepsilon\}$ for some small ε such that $\Sigma_\pm$ lies in the region with $\pm\operatorname{Re}(k) < 0$ and such that we don't intersect any other contours. Then we can split our jump by redefining $\tilde{m}(k)$ according to

$$
\widehat{m}(k) = \begin{cases} \tilde{m}(k)\tilde{b}_+(k)^{-1}, & 0 < \operatorname{Im}(k) < \varepsilon, \\ \tilde{m}(k)\tilde{b}_-(k)^{-1}, & -\varepsilon < \operatorname{Im}(k) < 0, \\ \tilde{m}(k), & \text{else}. \end{cases} \tag{4.23}
$$

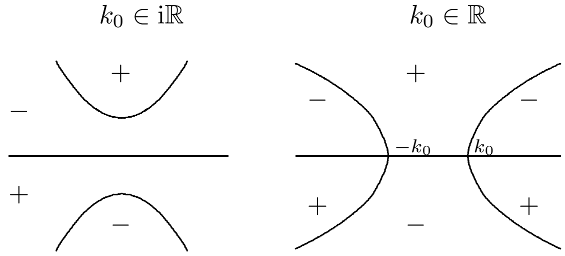

FIGURE 2. Sign of $\mathrm{Re}(\Phi(k))$ for different values of k_0

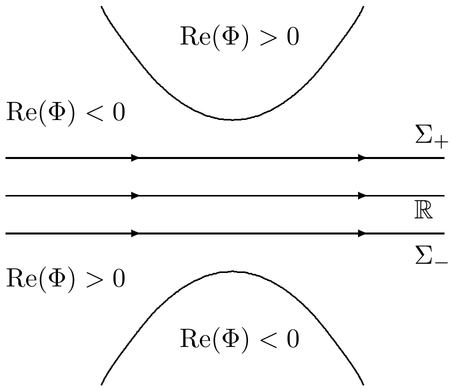

FIGURE 3. Deformed contour for $k_0 \in \mathrm{i}\mathbb{R}^+$

Thus the jump along the real axis disappears and the jump along $\Sigma_\pm$ is given by

$$\widehat{v}(k) = \begin{cases} \tilde{b}_+(k), & k \in \Sigma_+ \\ \tilde{b}_-(k)^{-1}, & k \in \Sigma_-. \end{cases} \tag{4.24}$$

All other jumps are unchanged. Note that the resulting Riemann-Hilbert problem still satisfies our symmetry condition (2.17), since we have

$$\tilde{b}_\pm(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{b}_\mp(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.25}$$

By construction the jump along $\Sigma_\pm$ is exponentially close to the identity as $t \to \infty$.

Case 2: $k_0 \in \mathbb{R}$, $k_0 \neq 0$:

We set $\Sigma_\pm = \Sigma_\pm^1 \cup \Sigma_\pm^2$. Again note that $\Sigma_\pm^1$ respectively $\Sigma_\pm^2$ lie in the region with $\pm\mathrm{Re}(\Phi(k)) < 0$. Then we can split our jump by redefining $\tilde{m}(k)$ according to

$$\widehat{m}(k) = \begin{cases} \tilde{m}(k)\tilde{b}_+(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_+^1, \\ \tilde{m}(k)\tilde{b}_-(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_-^1, \\ \tilde{m}(k)\tilde{B}_+(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_+^2, \\ \tilde{m}(k)\tilde{B}_-(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_-^2, \\ \tilde{m}(k), & \text{else}. \end{cases} \tag{4.26}$$

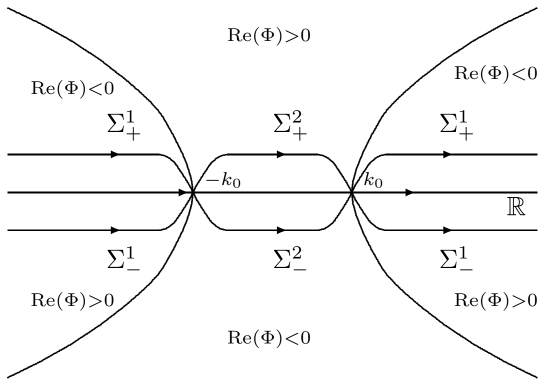

FIGURE 4. Deformed contour for $k_0 \in \mathbb{R}^+$

One checks that the jump along $\mathbb{R}$ disappears and the jump along $\Sigma_\pm$ is given by

$$\widehat{v}(k) = \begin{cases} \tilde{b}_+(k), & k \in \Sigma_+^1, \\ \tilde{b}_-(k)^{-1}, & k \in \Sigma_-^1, \\ \tilde{B}_+(k), & k \in \Sigma_+^2, \\ \tilde{B}_-(k)^{-1}, & k \in \Sigma_-^2. \end{cases} \tag{4.27}$$

All other jumps are unchanged. Again the resulting Riemann-Hilbert problem still satisfies our symmetry condition (2.17) and the jump along $\Sigma_\pm \backslash \{k_0, -k_0\}$ is exponentially decreasing as $t \to \infty$

Theorem 4.4. *Assume (2.1) and abbreviate by $c_j = 4\kappa_j^2$ the velocity of the j'th soliton determined by $\mathrm{Re}(\Phi(\mathrm{i}\kappa_j)) = 0$. Then the asymptotics in the soliton region, $x/t \geq C$ for some $C > 0$, are as follows:*

Let $\varepsilon > 0$ sufficiently small such that the intervals $[c_j - \varepsilon, c_j + \varepsilon]$, $1 \leq j \leq N$, are disjoint and lie inside $\mathbb{R}^+$.

If $|\frac{x}{t} - c_j| < \varepsilon$ for some j, one has

$$\int_x^\infty q(y,t)dy = -4 \sum_{i=j+1}^N \kappa_i - \frac{2\gamma_j^2(x,t)}{1 + (2\kappa_j)^{-1}\gamma_j^2(x,t)} + O(t^{-l}), \tag{4.28}$$

respectively

$$q(x,t) = \frac{-4\kappa_j \gamma_j^2(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} + O(t^{-l}), \tag{4.29}$$

where

$$\gamma_j^2(x,t) = \gamma_j^2 \mathrm{e}^{-2\kappa_j x + 8\kappa_j^3 t} \prod_{i=j+1}^N \left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right)^2. \tag{4.30}$$

If $|\frac{x}{t} - c_j| \geq \varepsilon$, for all j, one has

$$\int_x^\infty q(y,t)dy = -4 \sum_{\kappa_i \in (\kappa_0, \infty)} \kappa_i + O(t^{-l}), \qquad \kappa_0 = \sqrt{\frac{x}{4t}},$$

respectively

$$q(x,t) = O(t^{-l}). \tag{4.31}$$

Proof. Since $\widehat{m}(k) = \widehat{m}(k)$ for k away from $\mathbb{R}$ equations (4.15) and (2.21) imply the following asymptotics

$$\widehat{m}(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} + (-2T_1(k_0) + Q(x,t)) \frac{1}{2\mathrm{i}k} \begin{pmatrix} -1 & 1 \end{pmatrix} + O\left(\frac{1}{k^2}\right). \tag{4.32}$$

By construction, the jump along $\Sigma_\pm$ is exponentially decreasing as $t \to \infty$. Hence we can apply Theorem A.6 as follows:

If $\left|\frac{x}{t} - c_j\right| > \varepsilon$ for all j we can choose $\gamma_0 = 0$ and $\widehat{w}_0^t \equiv 0$ in Theorem A.6. Since the error between $\widehat{w}_0^t$ and $\widehat{w}^t$ is exponentially small as $t \to \infty$, the solutions of the associated Riemann-Hilbert problems only differ by $O(t^{-l})$ for any $l \geq 1$. Comparing $\widehat{m}_0 = \begin{pmatrix} 1 & 1 \end{pmatrix}$ with the above asymptotics shows $Q_+(x,t) = 2T_1(k_0) + O(t^{-l})$.

If $\left|\frac{x}{t} - c_j\right| < \varepsilon$ for some j, we choose $\gamma_0^t = \gamma_k(x,t)$ and $\widehat{w}_0^t \equiv 0$. As before we conclude that the error between the solutions of $\widehat{w}^t$ and $\widehat{w}_0^t$ is exponentially small and so the associated solutions of the Riemann-Hilbert problems only differ by $O(t^{-l})$. From Lemma 2.6, we have the one-soliton solution $\widehat{m}_0(k) = \begin{pmatrix} \widehat{f}(k) & \widehat{f}(-k) \end{pmatrix}$ with $\widehat{f}(k) = \frac{1}{1+(2\kappa_j)^{-1}\gamma_j^2(x,t)}\big(1 + \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j}(2\kappa_j)^{-1}\gamma_j^2(x,t)\big)$, where

$$\gamma_j^2(x,t) = \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j, \mathrm{i}\frac{\kappa_j}{\sqrt{3}})^{-2} = \gamma_j^2 \mathrm{e}^{-2\kappa_j x + 8\kappa_j^3 t} \prod_{i=j+1}^{N} \left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}\right)^2.$$

As before, comparing with the above asymptotics shows

$$Q(x,t) = 2T_1(k_0) + \frac{2\gamma_j^2(x,t)}{1 + (2\kappa_j)^{-1}\gamma_j^2(x,t)} + O(t^{-l}).$$

To see the second part just use (2.20) in place of (2.21). This finishes the proof in the case where $R(z)$ has an analytic extensions. We will remove this assumption in Section 6 thereby completing the proof. □

Since the one-soliton solution is exponentially decaying away from its minimum, we also obtain the form stated in the introduction:

Corollary 4.5. *Assume (2.1), then the asymptotic in the soliton region, $x/t \geq C$ for some $C > 0$, is given by*

$$q(x,t) = -2 \sum_{j=1}^{N} \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 - p_j)} + O(t^{-l}), \tag{4.33}$$

where

$$p_j = \frac{1}{2} \log\left(\frac{\gamma_j^2}{2\kappa_j} \prod_{i=j+1}^{N} \left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}\right)^2 \right). \tag{4.34}$$

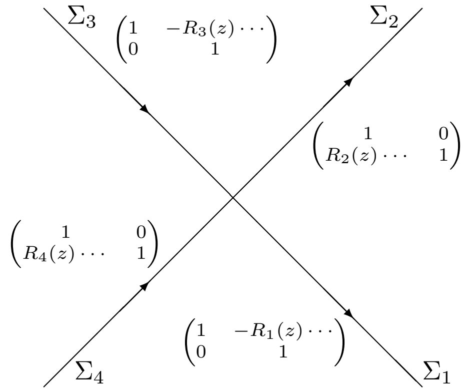

FIGURE 5. Contours of a cross

5. Reduction to a Riemann–Hilbert problem on a small cross

In the previous section we have seen that for $k_0 \in \mathbb{R}$ we can reduce everything to a Riemann-Hilbert problem for $\widehat{m}(k)$ such that the jumps are exponentially close to the identity except in small neighborhoods of the stationary phase points k_0 and $-k_0$. Hence we need to continue our investigation of this case in this chapter.

Denote by $\Sigma^c(\pm k_0)$ the parts of $\Sigma_+ \cup \Sigma_-$ inside a small neighborhood of $\pm k_0$. We will now show that solving the two problems on the small crosses $\Sigma^c(k_0)$ respectively $\Sigma^c(-k_0)$ will lead us to the solution of our original problem.

Introduce the cross $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_4$ (see Figure 5) by

$$\begin{aligned} &\Sigma_1 = \{u\mathrm{e}^{-\mathrm{i}\pi/4},\, u \in [0,\infty)\} && \Sigma_2 = \{u\mathrm{e}^{\mathrm{i}\pi/4},\, u \in [0,\infty)\} \\ &\Sigma_3 = \{u\mathrm{e}^{3\mathrm{i}\pi/4},\, u \in [0,\infty)\} && \Sigma_4 = \{u\mathrm{e}^{-3\mathrm{i}\pi/4},\, u \in [0,\infty)\}. \end{aligned} \tag{5.1}$$

Orient Σ such that the real part of z increases in the positive direction. Denote by $\mathbb{D} = \{z,\, |z| < 1\}$ the open unit disc. Throughout this section $z^{\mathrm{i}\nu}$ will denote the function $\mathrm{e}^{\mathrm{i}\nu\log(z)}$, where the branch cut of the logarithm is chosen along the negative real axis $(-\infty, 0)$.

Introduce the following jump matrices (v_j for $z \in \Sigma_j$)

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 & -R_1(z) z^{2\mathrm{i}\nu}\mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, & v_2 &= \begin{pmatrix} 1 & 0 \\ R_2(z) z^{-2\mathrm{i}\nu}\mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}, \\ v_3 &= \begin{pmatrix} 1 & -R_3(z) z^{2\mathrm{i}\nu}\mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, & v_4 &= \begin{pmatrix} 1 & 0 \\ R_4(z) z^{-2\mathrm{i}\nu}\mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix} \end{aligned} \tag{5.2}$$

and consider the RHP given by

$$\begin{aligned} m_+(z) &= m_-(z) v_j(z), && z \in \Sigma_j, \quad j = 1,2,3,4, \\ m(z) &\to \mathbb{I}, && z \to \infty. \end{aligned} \tag{5.3}$$

The solution is given in the following theorem of Deift and Zhou [9] (for a proof of the version stated below see Krüger and Teschl [17]).

Theorem 5.1 ([9])**.** *Assume there is some $\rho_0 > 0$ such that $v_j(z) = \mathbb{I}$ for $|z| > \rho_0$. Moreover, suppose that within $|z| \le \rho_0$ the following estimates hold:*

(i) *The phase satisfies* $\Phi(0) = \mathrm{i}\Phi_0 \in \mathrm{i}\mathbb{R}$, $\Phi'(0) = 0$, $\Phi''(0) = \mathrm{i}$ *and*

$$\pm \operatorname{Re}\big(\Phi(z) - \Phi(0)\big) \geq \frac{1}{4}|z|^2, \qquad \begin{cases} + & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\ - & \text{else}, \end{cases} \tag{5.4}$$

$$|\Phi(z) - \Phi(0) - \frac{\mathrm{i}z^2}{2}| \leq C|z|^3. \tag{5.5}$$

(ii) *There is some* $r \in \mathbb{D}$ *and constants* $(\alpha, L) \in (0,1] \times (0,\infty)$ *such that* R_j, $j = 1,\dots,4$, *satisfy Hölder conditions of the form*

$$|R_1(z) - \overline{r}| \leq L|z|^\alpha, \qquad |R_2(z) - r| \leq L|z|^\alpha,$$

$$|R_3(z) - \frac{\overline{r}}{1-|r|^2}| \leq L|z|^\alpha, \qquad |R_4(z) - \frac{r}{1-|r|^2}| \leq L|z|^\alpha. \tag{5.6}$$

Then the solution of the RHP (5.3) satisfies

$$m(z) = \mathbb{I} + \frac{1}{z}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + O(t^{-\frac{1+\alpha}{2}}), \tag{5.7}$$

for $|z| > \rho_0$, *where*

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-\mathrm{i}t\Phi_0}t^{-\mathrm{i}\nu}, \qquad \nu = -\frac{1}{2\pi}\log(1-|r|^2). \tag{5.8}$$

Furthermore, if $R_j(z)$ *and* $\Phi(z)$ *depend on some parameter, the error term is uniform with respect to this parameter as long as* r *remains within a compact subset of* $\mathbb{D}$ *and the constants in the above estimates can be chosen independent of the parameters.*

Theorem 5.2 (Decoupling). *Consider the Riemann–Hilbert problem*

$$\begin{aligned} m_+(k) &= m_-(k)v(k), \qquad k \in \Sigma, \\ m(\infty) &= \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{aligned} \tag{5.9}$$

and let $0 < \alpha < \beta \leq 2\alpha$, $\rho(t) \to \infty$ *be given.*

Suppose that for every sufficiently small $\varepsilon > 0$ *both the* L^2 *and the* L^∞ *norms of* v *are* $O(t^{-\beta})$ *away from some* ε *neighborhoods of some points* $k_j, 1 \leq j \leq n$. *Moreover, suppose that the solution of the matrix problem with jump* $v(k)$ *restricted to the* $\frac{\varepsilon}{2}$ *neighborhood has a solution given by*

$$M_j(k) = \mathbb{I} + \frac{1}{\rho(t)^\alpha}\frac{M_j}{k-k_j} + O(\rho(t)^{-\beta}), \qquad |k-k_j| > \frac{\varepsilon}{2}. \tag{5.10}$$

Then the solution $m(k)$ *is given by*

$$m(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{\rho(t)^\alpha}\begin{pmatrix} 1 & 1 \end{pmatrix}\sum_{j=1}^n \frac{M_j}{k-k_j} + O(\rho(t)^{-\beta}), \tag{5.11}$$

where the error term depends on the distance of k *to* Σ.

Proof. In this proof we will use the theory developed in Appendix A with $m_0(k) = \mathbb{I}$ and the usual Cauchy kernel $\Omega_\infty(s,k) = \mathbb{I}\frac{ds}{s-k}$. Assume that $m(k)$ exists. Introduce $\tilde{m}(k)$ by

$$\tilde{m}(k) = \begin{cases} m(k)M_j(k)^{-1}, & |k-k_j| \leq \varepsilon, \\ m(k), & \text{else}. \end{cases} \tag{5.12}$$

The Riemann–Hilbert problem for $\tilde{m}(k)$ has jumps given by

$$\tilde{v}(k) = \begin{cases} M_j(k)^{-1}, & |k-k_j| = \varepsilon, \\ M_j(k)v(k)M_j(k)^{-1}, & k \in \Sigma, \frac{\varepsilon}{2} < |k-k_j| < \varepsilon, \\ \mathbb{I}, & k \in \Sigma, |k-k_j| < \frac{\varepsilon}{2}, \\ v(k), & \text{else}. \end{cases} \tag{5.13}$$

By assumption the jumps are $\mathbb{I} + O(\rho(t)^{-\alpha})$ on the circles $|k-k_j| = \varepsilon$ and even $\mathbb{I} + O(\rho(t)^{-\beta})$ on the rest (both in the L^2 and L^∞ norms). In particular, we infer

$$\big\|\tilde{\mu} - \begin{pmatrix}1 & 1\end{pmatrix}\big\|_2 = O(\rho(t)^{-\alpha}), \tag{5.14}$$

because μ satisfies $(\mu - \begin{pmatrix}1 & 1\end{pmatrix}) = (\mathbb{I} - C_w)^{-1} C_w \begin{pmatrix}1 & 1\end{pmatrix}$ and therefore we can estimate $\mu - \begin{pmatrix}1 & 1\end{pmatrix}$ by using the Neumann series to obtain $\|\mu - \begin{pmatrix}1 & 1\end{pmatrix}\|_2 \le \frac{c\|w\|_2}{1-c\|w\|_\infty}$.

Thus we conclude

$$\begin{aligned} m(k) &= \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{\tilde{\Sigma}} \tilde{\mu}(s)\tilde{w}(s)\frac{ds}{s-k} \\ &= \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^n \int_{|k-k_j|=\varepsilon} \tilde{\mu}(s)(M_j(s)^{-1} - \mathbb{I})\frac{ds}{s-k} + O(\rho(t)^{-\beta}) \\ &= \begin{pmatrix}1 & 1\end{pmatrix} - \rho(t)^{-\alpha} \begin{pmatrix}1 & 1\end{pmatrix} \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^n M_j \int_{|k-k_j|=\varepsilon} \frac{1}{s-k_j}\frac{ds}{s-k} + O(\rho(t)^{-\beta}) \\ &= \begin{pmatrix}1 & 1\end{pmatrix} + \rho(t)^{-\alpha} \begin{pmatrix}1 & 1\end{pmatrix} \sum_{j=1}^n \frac{M_j}{k-k_j} + O(\rho(t)^{-\beta}), \end{aligned} \tag{5.15}$$

and hence the claim is proven. □

Now let us turn to the solution of the problem on $\Sigma^c(k_0) = (\Sigma_+ \cup \Sigma_-) \cap \{k \,|\, |k-k_0| < \varepsilon\}$ for some small $\varepsilon > 0$. Without loss we can also deform our contour slightly such that $\Sigma^c(k_0)$ consists of two straight lines. Next, note

$$\Phi(k_0) = -16\mathrm{i}k_0^3, \qquad \Phi''(k_0) = 48\mathrm{i}k_0.$$

As a first step we make a change of coordinates

$$\zeta = \sqrt{48k_0}(k-k_0), \qquad k = k_0 + \frac{\zeta}{\sqrt{48k_0}} \tag{5.16}$$

such that the phase reads $\Phi(k) = \Phi(k_0) + \frac{\mathrm{i}}{2}\zeta^2 + O(\zeta^3)$.

Next we need the behavior of our jump matrix near k_0, that is, the behavior of $T(k,k_0)$ near k_0.

Lemma 5.3. *Let $k_0 \in \mathbb{R}$, then*

$$T(k,k_0) = \left(\frac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu} \tilde{T}(k,k_0), \tag{5.17}$$

where $\nu = -\frac{1}{\pi}\log(|T(k_0)|)$ and the branch cut of the logarithm is chosen along the negative real axis. Here

$$\tilde{T}(k,k_0) = \prod_{j=1}^N \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j} \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{1}{\zeta-k} d\zeta\right) \tag{5.18}$$

is Hölder continuous of any exponent less then 1 *at the stationary phase point* $k = k_0$ *and satisfies* $\tilde{T}(k_0, k_0) \in \mathbb{T}$.

Proof. First of all observe that

$$\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{k_0}\log(|T(k_0)|^2)\frac{1}{\zeta - k}d\zeta\right) = \left(\frac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu}. \tag{5.19}$$

Hölder continuity of any exponent less than 1 is well-known (cf. [19]). □

If $k(\zeta)$ is defined as in (5.16) and $0 < \alpha < 1$, then there is an $L > 0$ such that

$$\left|T(k(\zeta), k_0) - \zeta^{\mathrm{i}\nu}\tilde{T}(k_0, k_0)\mathrm{e}^{-\mathrm{i}\nu\log(2k_0\sqrt{48k_0})}\right| \le L\,|\zeta|^{\alpha}, \tag{5.20}$$

where the branch cut of $\zeta^{\mathrm{i}\nu}$ is chosen along the negative real axis.

We also have

$$|R(k(\zeta)) - R(k_0)| \le L\,|\zeta|^{\alpha} \tag{5.21}$$

and thus the assumptions of Theorem 5.1 are satisfied with

$$r = R(k_0)\tilde{T}(k_0, k_0)^{-2}\mathrm{e}^{2\mathrm{i}\nu\log(2k_0\sqrt{48k_0})}. \tag{5.22}$$

Therefore we can conclude that the solution on $\Sigma^c(k_0)$ is given by

$$\begin{aligned} M_1^c(k) &= \mathbb{I} + \frac{1}{\zeta}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & -\overline{\beta} \\ \beta & 0 \end{pmatrix} + O(t^{-\alpha}) \\ &= \mathbb{I} + \frac{1}{\sqrt{48k_0}(k-k_0)}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & -\overline{\beta} \\ \beta & 0 \end{pmatrix} + O(t^{-\alpha}), \end{aligned} \tag{5.23}$$

where β is given by

$$\begin{aligned} \beta &= \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-t\Phi(k_0)}t^{-\mathrm{i}\nu} \\ &= \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu)))}\tilde{T}(k_0, k_0)^2(192k_0^3)^{-\mathrm{i}\nu}\mathrm{e}^{-t\Phi(k_0)}t^{-\mathrm{i}\nu}. \end{aligned} \tag{5.24}$$

and $1/2 < \alpha < 1$.

We also need the solution $M_2^c(k)$ on $\Sigma^c(-k_0)$. We make the following ansatz, which is inspired by the symmetry condition for the vector Riemann–Hilbert problem, outside the two small crosses:

$$\begin{aligned} M_2^c(k) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_1^c(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \mathbb{I} - \frac{1}{\sqrt{48k_0}(k+k_0)}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & \overline{\beta} \\ -\beta & 0 \end{pmatrix} + O(t^{-\alpha}). \end{aligned} \tag{5.25}$$

Applying Theorem 5.2 yields the following result:

Theorem 5.4. *Assume (2.1) with* $l = 5$, *then the asymptotics in the similarity region,* $x/t \le -C$ *for some* $C > 0$, *are given by*

$$\begin{aligned} \int_x^{\infty} q(y,t)dy = &-4\sum_{\kappa_j \in (\kappa_0, \infty)}\kappa_j - \frac{1}{\pi}\int_{-k_0}^{k_0}\log(|T(\zeta)|^2)d\zeta \\ &- \sqrt{\frac{\nu(k_0)}{3k_0 t}}\cos(16tk_0^3 - \nu(k_0)\log(192tk_0^3) + \delta(k_0)) + O(t^{-\alpha}) \end{aligned} \tag{5.26}$$

respectively

$$q(x,t) = \sqrt{\frac{4\nu(k_0)k_0}{3t}} \sin(16tk_0^3 - \nu(k_0)\log(192tk_0^3) + \delta(k_0)) + O(t^{-\alpha}) \tag{5.27}$$

for any $1/2 < \alpha < 1$. *Here* $k_0 = \sqrt{-\frac{x}{12t}}$ *and*

$$\nu(k_0) = -\frac{1}{\pi}\log(|T(k_0)|), \tag{5.28}$$

$$\begin{aligned}\delta(k_0) =& \frac{\pi}{4} - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu(k_0))) + 4\sum_{j=1}^{N} \arctan\big(\frac{\kappa_j}{k_0}\big)\\ & - \frac{1}{\pi}\int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right)\frac{1}{\zeta - k_0}d\zeta.\end{aligned} \tag{5.29}$$

Proof. By Theorem 5.2 we have

$$\begin{aligned}\widehat{m}(k) &= \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{\sqrt{48k_0}}\frac{\mathrm{i}}{t^{1/2}}\left(\frac{1}{k-k_0}\begin{pmatrix}\overline{\beta} & -\beta\end{pmatrix} - \frac{1}{k+k_0}\begin{pmatrix}-\beta & \overline{\beta}\end{pmatrix}\right) + O(t^{-\alpha})\\ &= \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{\sqrt{48k_0}}\frac{\mathrm{i}}{t^{1/2}}\frac{1}{k}\left(\sum_{l=0}^{\infty}\left(\frac{k_0}{k}\right)^l\begin{pmatrix}\overline{\beta} & -\beta\end{pmatrix} - \sum_{l=0}^{\infty}\left(-\frac{k_0}{k}\right)^l\begin{pmatrix}-\beta & \overline{\beta}\end{pmatrix}\right)\\ &\quad + O(t^{-\alpha}),\end{aligned}$$

which leads to

$$Q(x,t) = 2T_1(k_0) + \frac{4}{\sqrt{48k_0}}\frac{1}{t^{1/2}}(\mathrm{Re}(\beta)) + O(t^{-\alpha})$$

upon comparison with (4.32). Using the fact that $|\beta/\sqrt{\nu}| = 1$ proves the first claim.

To see the second part, as in the proof of Theorem 4.4, just use (2.20) in place of (2.21), which shows

$$q(x,t) = \sqrt{\frac{4k_0}{3t}}\,\mathrm{Im}(\beta) + O(t^{-\alpha}).$$

This finishes the proof in the case where $R(z)$ has an analytic extensions. We will remove this assumption in Section 6 thereby completing the proof. □

Equivalence of the formula for $\delta(k_0)$ given in the previous theorem with the one given in the introduction follows after a simple integration by parts.

Remark 5.5. *Formally the equation (5.27) for q can be obtained by differentiating the equation (5.26) for Q with respect to x. That this step is admissible could be shown as in Deift and Zhou* [10], *however our approach avoids this step.*

Remark 5.6. *Note that Theorem 5.2 does not require uniform boundedness of the associated integral operator in contradistinction to Theorem A.6. We only need the knowledge of the solution in some small neighborhoods. However it cannot be used in the soliton region, because our solution is not of the form $\mathbb{I} + o(1)$.*

6. Analytic Approximation

In this chapter we want to present the necessary changes in the case where the reflection coefficient does not have an analytic extension. The idea is to use an analytic approximation and to split the reflection in an analytic part plus a small rest. The analytic part will be moved to the complex plane while the rest remains on the real axis. This needs to be done in such a way that the rest is of $O(t^{-1})$ and the growth of the analytic part can be controlled by the decay of the phase.

In the soliton region a straightforward splitting based on the Fourier transform

$$R(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} \hat{R}(x) dx \tag{6.1}$$

will be sufficient. It is well-known that our decay assumption (2.1) implies $\hat{R} \in L^1(\mathbb{R})$ and the estimate (cf. [18, Sect. 3.2])

$$|\hat{R}(-2x)| \le const \int_x^\infty q(r)dr, \qquad x \ge 0, \tag{6.2}$$

implies $x^l \hat{R}(-x) \in L^1(0,\infty)$.

Lemma 6.1. *Suppose $\hat{R} \in L^1(\mathbb{R})$, $x^l\hat{R}(-x) \in L^1(0,\infty)$ and let $\varepsilon, \beta > 0$ be given. Then we can split the reflection coefficient according to $R(k) = R_{a,t}(k) + R_{r,t}(k)$ such that $R_{a,t}(k)$ is analytic in $0 < \mathrm{Im}(k) < \varepsilon$ and*

$$|R_{a,t}(k)\mathrm{e}^{-\beta t}| = O(t^{-l}), \quad 0 < \mathrm{Im}(k) < \varepsilon, \qquad |R_{r,t}(k)| = O(t^{-l}), \quad k \in \mathbb{R}. \tag{6.3}$$

Proof. We choose $R_{a,t}(k) = \int_{-K(t)}^\infty \mathrm{e}^{\mathrm{i}kx}\hat{R}(x)dx$ with $K(t) = \frac{\beta_0}{\varepsilon} t$ for some positive $\beta_0 < \beta$. Then, for $0 < \mathrm{Im}(k) < \varepsilon$,

$$\left|R_{a,t}(k)\mathrm{e}^{-\beta t}\right| \le \mathrm{e}^{-\beta t} \int_{-K(t)}^\infty |\hat{R}(x)| \mathrm{e}^{-\mathrm{Im}(k)x} dx \le \mathrm{e}^{-\beta t}\mathrm{e}^{K(t)\varepsilon}\|\hat{R}\|_1 = \|\hat{R}\|_1 \mathrm{e}^{-(\beta-\beta_0)t},$$

which proves the first claim. Similarly, for $\mathrm{Im}(k) = 0$,

$$|R_{r,t}(k)| = \int_{K(t)}^\infty \frac{x^l|\hat{R}(-x)|}{x^l} dx \le \frac{\|x^l\hat{R}(-x)\|_{L^1(0,\infty)}}{K(t)^l} \le \frac{const}{t^l}$$

□

To apply this lemma in the soliton region $k_0 \in \mathrm{i}\mathbb{R}^+$ we choose

$$\beta = \min_{\mathrm{Im}(k) = -\varepsilon} -\mathrm{Re}(\Phi(k)) > 0. \tag{6.4}$$

and split $R(k) = R_{a,t}(k) + R_{r,t}(k)$ according to Lemma 6.1 to obtain

$$\tilde{b}_\pm(k) = \tilde{b}_{a,t,\pm}(k)\tilde{b}_{r,t,\pm}(k) = \tilde{b}_{r,t,\pm}(k)\tilde{b}_{a,t,\pm}(k). \tag{6.5}$$

Here $\tilde{b}_{a,t,\pm}(k)$, $\tilde{b}_{r,t,\pm}(k)$ denote the matrices obtained from $\tilde{b}_\pm(k)$ as defined in (4.22) by replacing $R(k)$ with $R_{a,t}(k)$, $R_{r,t}(k)$, respectively. Now we can move the analytic parts into the complex plane as in Chapter 4 while leaving the rest on the real axis. Hence, rather then (4.24), the jump now reads

$$\hat{v}(k) = \begin{cases} \tilde{b}_{a,t,+}(k), & k \in \Sigma_+, \\ \tilde{b}_{a,t,-}(k)^{-1}, & k \in \Sigma_-, \\ \tilde{b}_{r,t,-}(k)^{-1}\tilde{b}_{r,t,+}(k), & k \in \mathbb{R}. \end{cases} \tag{6.6}$$

By construction we have $\hat{v}(k) = \mathbb{I} + O(t^{-l})$ on the whole contour and the rest follows as in Section 4.

In the similarity region not only $\tilde{b}_\pm$ occur as jump matrices but also $\tilde{B}_\pm$. These matrices $\tilde{B}_\pm$ have at first sight more complicated off diagonal entries, but a closer look shows that they have indeed the same form. To remedy this we will rewrite them in terms of left rather then right scattering data. For this purpose let us use the notation $R_r(z) \equiv R_+(z)$ for the right and $R_l(z) \equiv R_-(z)$ for the left reflection coefficient. Moreover, let $T_r(z, z_0) \equiv T(z, z_0)$ be the right and $T_l(z, z_0) \equiv T(z)/T(z, z_0)$ be the left partial transmission coefficient.

With this notation we have

$$\tilde{v}(k) = \begin{cases} \tilde{b}_-(k)^{-1}\tilde{b}_+(k), & \mathrm{Re}(k_0) < |k|, \\ \tilde{B}_-(k)^{-1}\tilde{B}_+(k), & \mathrm{Re}(k_0) > |k|, \end{cases} \tag{6.7}$$

where

$$\tilde{b}_-(k) = \begin{pmatrix} 1 & \frac{R_r(-k)\mathrm{e}^{-t\Phi(k)}}{T_r(-k,k_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(k) = \begin{pmatrix} 1 & 0 \\ \frac{R_r(k)\mathrm{e}^{t\Phi(k)}}{T_r(k,k_0)^2} & 1 \end{pmatrix},$$

and

$$\tilde{B}_-(k) = \begin{pmatrix} 1 & 0 \\ -\frac{T_{r,-}(k,k_0)^{-2}}{|T(k)|^2} R_r(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix},$$
$$\tilde{B}_+(k) = \begin{pmatrix} 1 & -\frac{T_{r,+}(k,k_0)^2}{|T(k)|^2} R_r(-k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}.$$

Using (2.11) we can further write

$$\tilde{B}_-(k) = \begin{pmatrix} 1 & 0 \\ \frac{R_l(-k)\mathrm{e}^{t\Phi(k)}}{T_l(-k,k_0)^2} & 1 \end{pmatrix}, \quad \tilde{B}_+(k) = \begin{pmatrix} 1 & \frac{R_l(k)\mathrm{e}^{-t\Phi(k)}}{T_l(k,k_0)^2} \\ 0 & 1 \end{pmatrix}. \tag{6.8}$$

Now we can proceed as before with $\tilde{B}_\pm(z)$ as with $\tilde{b}_\pm(z)$ by splitting $R_l(z)$ rather than $R_r(z)$.

In the similarity region we need to take the small vicinities of the stationary phase points into account. Since the phase is cubic near these points, we cannot use it to dominate the exponential growth of the analytic part away from the unit circle. Hence we will take the phase as a new variable and use the Fourier transform with respect to this new variable. Since this change of coordinates is singular near the stationary phase points, there is a price we have to pay, namely, requiring additional smoothness for $R(k)$. In this respect note that (2.1) implies $R(k) \in C^l(\mathbb{R})$ (cf. [15]). We begin with

Lemma 6.2. *Suppose $R(k) \in C^5(\mathbb{R})$. Then we can split $R(k)$ according to*

$$R(k) = R_0(k) + (k - k_0)(k + k_0)H(k), \qquad k \in \Sigma(k_0), \tag{6.9}$$

where $R_0(k)$ is a real rational function in k such that $H(k)$ vanishes at k_0, $-k_0$ of order three and has a Fourier transform

$$H(k) = \int_{\mathbb{R}} \hat{H}(x)\mathrm{e}^{x\Phi(k)} dx, \tag{6.10}$$

with $x\hat{H}(x)$ integrable.

Proof. We can construct a rational function, which satisfies $f_n(-k) = \overline{f_n(k)}$ for $k \in \mathbb{R}$, by making the ansatz $f_n(k) = \frac{k_0^{2n+4}+1}{k^{2n+1}} \sum_{j=0}^n \frac{1}{j!(2k_0)^j}(\alpha_j + \mathrm{i}\beta_j \frac{k}{k_0})(k-k_0)^j(k+k_0)^j$. Furthermore we can choose α_j, $\beta_j \in \mathbb{R}$ for $j = 1, \dots, n$, such that we can match the values of R and its first four derivatives at k_0, $-k_0$ at $f_n(k)$. Thus we will set $R_0(k) = f_4(k)$ with $\alpha_0 = \mathrm{Re}(R(k_0))$, $\beta_0 = \mathrm{Im}(R(k_0))$, and so on. Since $R_0(k)$ is integrable we infer that $H(k) \in C^4(\mathbb{R})$ and it vanishes together with its first three derivatives at k_0, $-k_0$.

Note that $\Phi(k)/\mathrm{i} = 8(k^3 - 3k_0^2 k)$ is a polynomial of order three which has a maximum at $-k_0$ and a minimum at k_0. Thus the phase $\Phi(k)/\mathrm{i}$ restricted to $\Sigma(k_0)$ gives a one to one coordinate transform $\Sigma(k_0) \to [\Phi(k_0)/\mathrm{i}, \Phi(-k_0)/\mathrm{i}] = [-16k_0^3, 16k_0^3]$ and we can hence express $H(k)$ in this new coordinate (setting it equal to zero outside this interval). The coordinate transform locally looks like a cube root near k_0 and $-k_0$, however, due to our assumption that H vanishes there, H is still C^2 in this new coordinate and the Fourier transform with respect to this new coordinates exists and has the required properties. □

Moreover, as in Lemma 6.1 we obtain:

Lemma 6.3. *Let $H(k)$ be as in the previous lemma. Then we can split $H(k)$ according to $H(k) = H_{a,t}(k) + H_{r,t}(k)$ such that $H_{a,t}(k)$ is analytic in the region* $\mathrm{Re}(\Phi(k)) < 0$ *and*

(6.11)
$$|H_{a,t}(k)\mathrm{e}^{\Phi(k)t/2}| = O(1),\ \mathrm{Re}(\Phi(k)) < 0, \mathrm{Im}(k) \le 0, \quad |H_{r,t}(k)| = O(t^{-1}),\ k \in \mathbb{R}.$$

Proof. We choose $H_{a,t}(k) = \int_{-K(t)}^{\infty} \hat{H}(x)\mathrm{e}^{x\Phi(k)}dx$ with $K(t) = t/2$. Then we can conclude as in Lemma 6.1:

$$|H_{a,t}(k)\mathrm{e}^{\Phi(k)t/2}| \le \|\hat{H}(x)\|_1 |\mathrm{e}^{-K(t)\Phi(k)+\Phi(k)t/2}| \le \|\hat{H}(x)\|_1 \le const$$

and

$$|H_{r,t}(k)| \le \int_{-\infty}^{-K(t)} |\hat{H}(x)|dx \le const \sqrt{\int_{-\infty}^{-K(t)} \frac{1}{x^4}dx} \le const \frac{1}{K(t)^{3/2}} \le const \frac{1}{t}.$$

□

By construction $R_{a,t}(k) = R_0(k) + (k-k_0)(k+k_0)H_{a,t}(k)$ will satisfy the required Lipschitz estimate in a vicinity of the stationary phase points (uniformly in t) and all jumps will be $\mathbb{I} + O(t^{-1})$. Hence we can proceed as in Chapter 5.

Appendix A. Singular integral equations

In this section we show how to transform a meromorphic vector Riemann–Hilbert problem with simple poles at $\mathrm{i}\kappa$, $-\mathrm{i}\kappa$,

(A.1)
$$\begin{aligned}
&m_+(k) = m_-(k)v(k), \qquad k \in \Sigma,\\
&\mathrm{Res}_{\mathrm{i}\kappa}\, m(k) = \lim_{k\to \mathrm{i}\kappa} m(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma^2 & 0 \end{pmatrix}, \quad \mathrm{Res}_{-\mathrm{i}\kappa}\, m(k) = \lim_{k\to -\mathrm{i}\kappa} m(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma^2 \\ 0 & 0 \end{pmatrix},\\
&m(-k) = m(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\
&\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}
\end{aligned}$$

into a singular integral equation. Since we require the symmetry condition (2.17) for our Riemann–Hilbert problem we need to adapt the usual Cauchy kernel to preserve this symmetry. Moreover, we keep the single soliton as an inhomogeneous term which will play the role of the leading asymptotics in our applications.

The classical Cauchy-transform of a function $f : \Sigma \to \mathbb{C}$ which is square integrable is the analytic function $Cf : \mathbb{C}\backslash\Sigma \to \mathbb{C}$ given by

$$Cf(k) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{f(s)}{s-k} ds, \qquad k \in \mathbb{C}\backslash\Sigma. \tag{A.2}$$

Denote the tangential boundary values from both sides (taken possibly in the L^2-sense — see e.g. [5, eq. (7.2)]) by $C_+ f$ respectively $C_- f$. Then it is well-known that C_+ and C_- are bounded operators $L^2(\Sigma) \to L^2(\Sigma)$, which satisfy $C_+ - C_- = \mathbb{I}$ (see e.g. [5]). Moreover, one has the Plemelj–Sokhotsky formula ([19])

$$C_\pm = \frac{1}{2}(\mathrm{i}H \pm \mathbb{I}),$$

where

$$Hf(k) = \frac{1}{\pi} \fint_\Sigma \frac{f(s)}{k-s} ds, \qquad k \in \Sigma, \tag{A.3}$$

is the Hilbert transform and $\fint$ denotes the principal value integral.

In order to respect the symmetry condition we will restrict our attention to the set $L^2_s(\Sigma)$ of square integrable functions $f : \Sigma \to \mathbb{C}^2$ such that

$$f(-k) = f(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.4}$$

Clearly this will only be possible if we require our jump data to be symmetric as well (i.e., Hypothesis 3.1 holds).

Next we introduce the Cauchy operator

$$(Cf)(k) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma f(s) \Omega_\kappa(s,k) \tag{A.5}$$

acting on vector-valued functions $f : \Sigma \to \mathbb{C}^2$. Here the Cauchy kernel is given by

$$\Omega_\kappa(s,k) = \begin{pmatrix} \frac{k+\mathrm{i}\kappa}{s+\mathrm{i}\kappa}\frac{1}{s-k} & 0 \\ 0 & \frac{k-\mathrm{i}\kappa}{s-\mathrm{i}\kappa}\frac{1}{s-k} \end{pmatrix} ds = \begin{pmatrix} \frac{1}{s-k} - \frac{1}{s+\mathrm{i}\kappa} & 0 \\ 0 & \frac{1}{s-k} - \frac{1}{s-\mathrm{i}\kappa} \end{pmatrix} ds, \tag{A.6}$$

for some fixed $\mathrm{i}\kappa \notin \Sigma$. In the case $\kappa = \infty$ we set

$$\Omega_\infty(s,k) = \begin{pmatrix} \frac{1}{s-k} & 0 \\ 0 & \frac{1}{s-k} \end{pmatrix} ds. \tag{A.7}$$

and one easily checks the symmetry property:

$$\Omega_\kappa(-s,-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Omega_\kappa(s,k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.8}$$

The properties of C are summarized in the next lemma.

Lemma A.1. *Assume Hypothesis 3.1. The Cauchy operator C has the properties, that the boundary values $C_\pm$ are bounded operators $L^2_s(\Sigma) \to L^2_s(\Sigma)$ which satisfy*

$$C_+ - C_- = \mathbb{I} \tag{A.9}$$

and

$$(Cf)(-\mathrm{i}\kappa) = (0 \quad *), \qquad (Cf)(\mathrm{i}\kappa) = (* \quad 0). \tag{A.10}$$

Furthermore, C restricts to $L^2_s(\Sigma)$, that is

$$(Cf)(-k) = (Cf)(k)\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad k\in\mathbb{C}\backslash\Sigma \tag{A.11}$$

for $f\in L^2_s(\Sigma)$ or $L^\infty_s(\Sigma)$ and if $w_\pm$ satisfy (H.3.1) we also have

$$C_\pm(fw_\mp)(-k) = C_\mp(fw_\pm)(k)\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad k\in\Sigma. \tag{A.12}$$

Proof. Everything follows from (A.8) and the fact that C inherits all properties from the classical Cauchy operator. □

We have thus obtained a Cauchy transform with the required properties. Following Section 7 and 8 of [2], we can solve our Riemann–Hilbert problem using this Cauchy operator.

Introduce the operator $C_w: L^2_s(\Sigma)\to L^2_s(\Sigma)$ by

$$C_w f = C_+(fw_-) + C_-(fw_+), \quad f\in L^2_s(\Sigma) \tag{A.13}$$

and this operator is also well-defined for $f\in L^\infty_s(\Sigma)$ and $C_w f\in L^2_s(\Sigma)$. Furthermore recall from Lemma 2.6 that the unique solution corresponding to $v\equiv\mathbb{I}$ is given by

$$m_0(k) = \begin{pmatrix} f(k) & f(-k)\end{pmatrix},$$
$$f(k) = \frac{1}{1+(2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}\left(1+\frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}(2\kappa)^{-1}\gamma^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa)}\right).$$

Observe that for $\gamma=0$ we have $f(k)=1$ and for $\gamma=\infty$ we have $f(k)=\frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}$. In particular, $m_0(k)$ is uniformly bounded away from $\mathrm{i}\kappa$ for all $\gamma\in[0,\infty]$.

Then we have the next result.

Theorem A.2. *Assume Hypothesis 3.1.*

Suppose m solves the Riemann–Hilbert problem (A.1). Then

$$m(k) = (1-c_0)m_0(k) + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mu(s)(w_+(s)+w_-(s))\Omega_\kappa(s,k), \tag{A.14}$$

where

$$\mu = m_+ b_+^{-1} = m_- b_-^{-1} \quad \text{and} \quad c_0 = \left(\frac{1}{2\pi\mathrm{i}}\int_\Sigma \mu(s)(w_+(s)+w_-(s))\Omega_\kappa(s,\infty)\right)_1.$$

Here $(m)_j$ denotes the j'th component of a vector. Furthermore, μ solves

$$(\mathbb{I}-C_w)(\mu(k)-(1-c_0)m_0(k)) = C_w(1-c_0)m_0(k) \tag{A.15}$$

Conversely, suppose $\tilde\mu$ solves

$$(\mathbb{I}-C_w)(\tilde\mu(k)-m_0(k)) = C_w m_0(k), \tag{A.16}$$

and

$$\tilde c_0 = \left(\frac{1}{2\pi\mathrm{i}}\int_\Sigma \tilde\mu(s)(w_+(s)+w_-(s))\Omega_\kappa(s,\infty)\right)_1 \neq 1,$$

then m defined via (A.14), with $(1-c_0)=(1-\tilde c_0)^{-1}$ and $\mu=(1-\tilde c_0)^{-1}\tilde\mu$, solves the Riemann–Hilbert problem (A.1) and $\mu = m_\pm b_\pm^{-1}$.

Proof. First of all note that by (A.12) $(\mathbb{I} - C_w)$ satisfies the symmetry condition and hence so do $m_0 + (\mathbb{I} - C_w)^{-1} C_w m_0$ and m.

So if m solves (A.1) and we set $\mu = m_\pm b_\pm^{-1}$, then m satisfies an additive jump given by

$$m_+ - m_- = \mu(w_+ + w_-).$$

Hence, if we denote the left hand side of (A.14) by $\tilde{m}$, both functions satisfy the same additive jump. So $m - \tilde{m}$ has no jump and must thus solve (A.1) with $v \equiv \mathbb{I}$. By uniqueness (Corollary 3.3) $m - \tilde{m} = \alpha m_0$ for some $\alpha \in \mathbb{C}$ and by looking at the first component at $k \to \infty$ we see $\alpha = 0$, that is $m = \tilde{m}$.

Moreover, if m is given by (A.14), then (A.9) implies

$$\begin{aligned} m_\pm &= (1 - c_0) m_0 + C_\pm(\mu w_-) + C_\pm(\mu w_+) \\ &= (1 - c_0) m_0 + C_w(\mu) \pm \mu w_\pm \\ &= (1 - c_0) m_0 - (\mathbb{I} - C_w)\mu + \mu b_\pm. \end{aligned} \tag{A.17}$$

From this we conclude that $\mu = m_\pm b_\pm^{-1}$ solves (A.15).

Conversely, if $\tilde{\mu}$ solves (A.16) then (A.17) implies $m_\pm = \mu b_\pm$ which shows that m defined via (A.14) solves the Riemann–Hilbert problem (A.1). □

Remark A.3. *In our case $m_0(k) \in L^\infty(\Sigma)$, but $m_0(k)$ is not square integrable and so $\mu \in L^2(\Sigma) + L^\infty(\Sigma)$ in general.*

In the case where the contour Σ is bounded $m_0(k) \in L^\infty(\Sigma)$ implies that $m_0(k)$ square integrable and we can directly apply $(\mathbb{I} - C_w)^{-1}$ to $m_0(k)$.

Note also that in the special case $\gamma = 0$ we have $m_0(k) = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and we can choose κ as we please, say $\kappa = \infty$ such that $c_0 = \tilde{c}_0 = 0$ in the above theorem.

Hence we have a formula for the solution of our Riemann–Hilbert problem $m(k)$ in terms of $m_0 + (\mathbb{I} - C_w)^{-1} C_w m_0$ and this clearly raises the question of bounded invertibility of $\mathbb{I} - C_w$ as a map from $L^2_s(\Sigma) \to L^2_s(\Sigma)$. This follows from Fredholm theory (cf. e.g. [25]):

Lemma A.4. *Assume Hypothesis 3.1. The operator $\mathbb{I} - C_w$ is Fredholm of index zero,*

$$\operatorname{ind}(\mathbb{I} - C_w) = 0. \tag{A.18}$$

By the Fredholm alternative, it follows that to show the bounded invertibility of $\mathbb{I} - C_w$ we only need to show that $\ker(\mathbb{I} - C_w) = 0$. The latter being equivalent to unique solvability of the corresponding vanishing Riemann–Hilbert problem.

Corollary A.5. *Assume Hypothesis 3.1. A unique solution of the Riemann–Hilbert problem (A.1) exists if and only if the corresponding vanishing Riemann–Hilbert problem, where the normalization condition is replaced by $m(k) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ as $k \to \infty$, has at most one solution.*

We are interested in comparing two Riemann–Hilbert problems associated with respective jumps w_0 and w with $\|w - w_0\|_\infty$ and $\|w - w_0\|_2$ small, where

$$\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)} \tag{A.19}$$

and

$$\|w\|_2 = \|w_+\|_{L^2(\Sigma)} + \|w_-\|_{L^2(\Sigma)}. \tag{A.20}$$

For such a situation we have the following result:

Theorem A.6. *Assume that for some data w_0^t the operator*

$$\mathbb{I} - C_{w_0^t} : L_s^2(\Sigma) \to L_s^2(\Sigma) \tag{A.21}$$

has a bounded inverse, where the bound is independent of t, and let $\kappa = \kappa_0$, $\gamma^t = \gamma_0^t$.

Furthermore, assume w^t satisfies

$$\|w^t - w_0^t\|_\infty \le \alpha(t) \text{ and } \left\|w^t - w_0^t\right\|_2 \le \alpha(t) \tag{A.22}$$

for some function $\alpha(t) \to 0$ as $t \to \infty$. Then $(\mathbb{I} - C_{w^t})^{-1} : L_s^2(\Sigma) \to L_s^2(\Sigma)$ also exists for sufficiently large t and the associated solutions of the Riemann–Hilbert problems (A.1) only differ by $O(\alpha(t))$.

Proof. By the boundedness of the Cauchy transform we conclude that

$$\left\|C_{w^t} - C_{w_0^t}\right\|_{L^2(\Sigma)\to L^2(\Sigma)} = O(\alpha(t)).$$

Thus by the second resolvent identity, we infer that $(\mathbb{I} - C_{w^t})^{-1}$ exists for large t and

$$\left\|(\mathbb{I} - C_{w_0^t})^{-1} - (\mathbb{I} - C_{w^t})^{-1}\right\|_{L^2(\Sigma)\to L^2(\Sigma)} = O(\alpha(t)). \tag{A.23}$$

Next we observe that

$$\mu_0^t - \mu^t = (\mathbb{I} - C_{w_0^t})^{-1} C_{w_0^t}(1 - c_0)m_0(k) - (\mathbb{I} - C_{w^t})^{-1} C_{w^t}(1 - c_0)m_0(k) \in L_s^2(\Sigma)$$

and we can therefore conclude

$$\left\|\mu_0^t - \mu^t\right\|_{L^2(\Sigma)} = O(\alpha(t)). \tag{A.24}$$

This now implies that for k away from Σ

$$\left|m^t(k) - m_0^t(k)\right| = O(\alpha(t)). \tag{A.25}$$

For $k \in \Sigma$ we can conclude

$$\left\|m_\pm^t(k) - m_{0,\pm}^t(k)\right\|_{L^2(\Sigma)} = O(\alpha(t)), \tag{A.26}$$

Note that in all conclusions we have used that $\mu \in L^2(\Sigma) + L^\infty(\Sigma)$. □

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