A General Boundary Value Problem and its Weyl Function

Vladimir Ryzhov

April 14, 2007

Dedicated to Prof. Boris Pavlov on the occasion of his 70th birthday

Abstract

We study the abstract boundary value problem defined in terms of the Green identity and introduce the concept of Weyl operator function $M(\cdot)$ that agrees with other definitions found in the current literature. In typical cases of problems arising from the multidimensional partial equations of mathematical physics the function $M(\cdot)$ takes values in the set of unbounded densely defined operators acting on the auxiliary boundary space. Exact formulae are obtained and essential properties of $M(\cdot)$ are studied. In particular, we consider boundary problems defined by various boundary conditions and justify the well known procedure that reduces such problems to the "equation on the boundary" involving the Weyl function, prove an analogue of the Borg-Levinson theorem, and link our results to the classical theory of extensions of symmetric operators.

Keywords: *Abstract boundary value problem, symmetric operators, Green formula, Weyl function.*

AMS Subject Classification: Primary 47B25, 47F05; Secondary 35J25, 31B10

Introduction and notation

The abstract approach to various linear boundary value problems arising in mathematical physics is commonly based on the extension theory of symmetric operators that was developed in classic works of J. von Neumann, H. Weyl, D. Hilbert, K. Friedrichs, M. Krein and many others. In fact, the extension theory as we know it today can be seen as a cumulative result of intensive research on various problems of physics carried out in the last century. The basic assumption under which the fruitful theory was developed is the validity of the so-called Green formula that furnish a convenient setting for the study of various boundary conditions in the form of linear dependencies between traces of the solution being sought and its normal derivatives on the boundary. The abstract framework allowed one to treat classical boundary value problems such as problems of Dirichlet, Neumann, and Robin for partial differential equations of mathematical physics as direct applications of the extension theory.

An abstract approach based on the Green formula that only requires the main operator to be symmetric and densely defined and does not impose any constraints on its nature, be it differential or otherwise, was developed later and summarized in many works, see [8, 10, 11, 14, 17, 18, 22] and references therein. The main object of the theory is the so called boundary triple that consists of the main operator assumed to be the adjoint to a symmetric densely defined one, and two boundary mappings defined on its domain with values in an auxiliary Hilbert space. An abstract Green formula then can explicitly written in terms of these objects and subsequently studied. The authors of [10, 11] succeeded in advancing the theory to the point where an abstract version of the Weyl-Titchmarsh function, previously known only for the Sturm-Liouville type of problems, was successfully introduced and utilized for the following spectral analysis. Conditions imposed in [10, 11] are rather strict and don't allow one to consider the case of symmetric operators with non-surjective boundary mappings.

Unfortunately, this class includes many interesting partial differential operators of mathematical physics with infinite defect indices. A few successful attempts have been made recently in order to overcome these limitations and extend the theory of boundary triples and respective Weyl functions to situations where requirements of [10, 11] are not met, see [3, 18] for examples.

The paper is dedicated to the systematic treatment of general boundary value problems and associated Weyl functions by means of operator theory. The approach offered here is not based on the theory of symmetric operators and their extensions; we engage another line of reasoning. The starting point is the Green formula for the main operator and certain assumptions regarding its restriction to the null set of one of the boundary mappings. The illustrative example supplied at the end of paper deals with the Laplacian defined on its natural domain in the bounded region in $\mathbb{R}^3$ as the main operator and two boundary operators that map a smooth function defined inside this region to its trace and trace of its normal derivative on the boundary. The restriction mentioned above turns out to be the Dirichlet Laplacian which is a selfadjoint operator with the bounded inverse. These two properties of the Laplacian are in fact the main conditions imposed on the abstract problem under consideration. It is useful to keep this example in mind reading the paper. Further, the Weyl function defined in the manner consistent with the previous definitions is an analytic function whose values are unbounded operators. However, after substraction of its value at the origin it becomes an analytic bounded operator function of Herglotz class, i. e. the operator function with the positive imaginary part in the upper half plane. In the example cited above the Weyl function coincides with the Dirichlet-to-Neumann map for the Laplacian, see [25]. It has to be noted that all results of the paper are obtained under minimal assumptions about participating objects. In particular, the main operator is not assumed to be closed.

There exits a tight relationship of our results with the theory of open systems as developed by M. S. Livŝiç in his seminal book [21]. Broadly speaking, the main operator of the boundary problem can be understood as a main operator of a certain open system whose input and output transformations are described in terms of boundary mappings. Correspondingly, in light of this interpretation the Weyl function becomes a transfer function of this system and as such describes the system's properties. More importantly, the approach based on the open system theory offers an intuitive way to model various boundary value problems as closed systems perturbed by certain "channel vectors". The latter are naturally interpreted as external control and observation channels attached to the system. The author hopes to expound these ideas and address other relevant topics in a subsequent publication.

The paper is organized as follows.

In the Section 1 after definition of the Green formula two main assumptions are formulated. Then we study their immediate consequences and make a few observations that will be used later in the paper. The Section concludes with two equivalent descriptions of the boundary value problem subject to the main assumptions.

Section 2 links the setting introduced in the previous section with the extension theory of symmetric operators. It is possible to define the minimal operator and show that it is symmetric, and if its domain is dense, the standard extension theory can be applied. Then we formulate the criteria for complete nonselfadjointness of the minimal operator and distinguish two special cases which are equivalent to the Friedrichs and Krein extensions in more conventional setting of semibounded densely defined symmetric operators.

Third Section is dedicated to the Weyl function and its properties. Here we study the question of recovery of the boundary value problem by its Weyl function and prove an analogue of the Borg-Levinson theorem. The last paragraph deal with general boundary value conditions and offers a recipe to reduce the corresponding boundary value problem to the equation "on the boundary". Solvability criteria of such obtained boundary equation are formulated in terms of the Weyl function and boundary conditions.

Final Section 4 illustrates the theory by means of an example of the Laplacian on the smooth bounded domain. The Weyl function is expressed in terms of the surface potentials and some related results regarding its properties are formulated.

The author would like to thank Prof. S. N. Naboko for his interest to the work and continual encouragement and to the anonymous referee for the close attention and valuable remarks.

The symbol $\mathscr{B}(H_1, H_2)$ where H_1, H_2 are separable Hilbert spaces is used for the Banach algebra of bounded operators, defined everywhere in H_1 with values in H_2. The notation $A : H_1 \to H_2$ is equivalent to $A \in \mathscr{B}(H_1, H_2)$. Also, $\mathscr{B}(H) := \mathscr{B}(H, H)$. The real axis, complex plane are denoted as $\mathbb{R}$, $\mathbb{C}$, respectively. Further, $\mathbb{C}_\pm := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$, $\mathbb{R}_\pm := \{x \in \mathbb{R} : \pm x > 0\}$, where Im stands for the imaginary part of a complex number. The domain, range and kernel of a linear operator A are denoted as $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker(A)$; the symbol $\rho(A)$ is used for the resolvent set of A. For a Hilbert space the term *subspace* will denote a closed linear set. The orthogonal complement to a linear set in a Hilbert space is always closed, i. e. is a subspace.

1 Green formula

1. Let H be a separable Hilbert space and A be a linear operator on H with dense domain $\mathcal{D}(A)$.

Definition 1.1. We will call the **Green formula** for A the equality

$$(Au, v)_H - (u, Av)_H = (\Gamma_1 u, \Gamma_0 v)_E - (\Gamma_0 u, \Gamma_1 v)_E, \qquad u, v \in \mathcal{D}(A), \tag{1.1}$$

where linear operators Γ_0, Γ_1 map vectors from $\mathcal{D}(\Gamma_k) \subset H$, $k = 0, 1$ into an auxiliary Hilbert space E. It is assumed that $\mathcal{D}(\Gamma_k) \supset \mathcal{D}(A)$, $k = 0, 1$. In the sequel the space E is called **boundary space**.

Assumption 1. *Green formula (1.1) for operator A holds with some operators Γ_0, Γ_1 and space E.*

Remark 1.2. Note that we do not impose any conditions on boundedness or closability of the operators A, Γ_0, Γ_1. In particular, A, Γ_0, Γ_1 acting on the domain $\mathcal{D}(A)$ are not assumed to be closed or even closable. If the domains $\mathcal{D}(\Gamma_k)$, $k = 0, 1$ are wider than $\mathcal{D}(A)$, the restrictions of mappings Γ_0, Γ_1 to the sets $\mathcal{D}(\Gamma_0) \backslash \mathcal{D}(A)$ and $\mathcal{D}(\Gamma_1) \backslash \mathcal{D}(A)$ respectively are irrelevant to the following considerations. For that reason we will assume in the sequel that $\mathcal{D}(\Gamma_0) = \mathcal{D}(\Gamma_1) = \mathcal{D}(A)$ without any loss of generality.

Remark 1.3. The well known situation when Green formula (1.1) holds is the case when A is the adjoint of a symmetric operator with equal deficiency indices defined on the dense domain. The validity of (1.1) with some Γ_0, Γ_1, E is readily verified on the ground of the Neumann formula for $\mathcal{D}(A)$. (See, e. g. [5].) In a series of papers dedicated to this case with an additional condition $\Gamma_0 \mathcal{D}(A) = \Gamma_1 \mathcal{D}(A) = E$ the collection $\{\Gamma_0, \Gamma_1, E\}$ is called a *boundary triple* or a *boundary values space* of the operator A, see [8, 10, 11, 14, 17] for more details.

Let us introduce the null set $\ker(A)$ of operator A and denote it $\mathscr{H}$:

$$\mathscr{H} := \ker(A) = \{u \in \mathcal{D}(A) \mid Au = 0\} \tag{1.2}$$

The set $\mathscr{H}$ is closed in H if A is closed, as in the case described in Remark 1.3. For our purposes, however, the closedness of $\mathscr{H}$ is not relevant. Denote A_0 the restriction of A to the intersection of $\mathcal{D}(A)$ and $\ker(\Gamma_0)$:

$$A_0 := A|_{\mathcal{D}(A_0)}, \quad \text{where} \quad \mathcal{D}(A_0) := \{u \in \mathcal{D}(A) \mid \Gamma_0 u = 0\} \tag{1.3}$$

Following equalities are direct consequence of the definitions above:

$$\ker(A_0) = \ker(\Gamma_0) \cap \mathscr{H} \tag{1.4}$$

$$(A - zI)(A_0 - zI)^{-1} u = u, \quad u \in H, \ z \in \rho(A_0) \tag{1.5}$$

where the resolvent set in (1.5) is assumed to be non-empty. According to (1.1), (1.3), operator A_0 is symmetric on the possibly non-dense domain $\mathcal{D}(A_0)$.

Now we can formulate our second assumption imposed on the introduced objects:

Assumption 2. *In notation (1.1)–(1.3) following conditions are satisfied:*

1. *Operator A_0 is selfadjoint on the domain $\mathcal{D}(A_0)$ and there exists its bounded inverse A_0^{-1}:*

$$A_0 = A_0^*, \quad \exists \, A_0^{-1} : H \to H \tag{1.6}$$

2. *Set $\Gamma_0\mathscr{H}$ is dense in E:*

$$\overline{\Gamma_0\mathscr{H}} = E \tag{1.7}$$

In the sequel we will always assume that both Assumptions are valid and will use the following symbol

$$\{A, \Gamma_0, \Gamma_1, H, E\} \tag{1.8}$$

for the collection of three operators A, Γ_0, Γ_1, and two Hilbert spaces H, E subjected to these Assumptions.

Remark 1.4. Since A_0 is selfadjoint, it is closed and its resolvent set is not empty. It follows that (1.5) holds at least for $\operatorname{Im} z \neq 0$ and $z \in \mathbb{C}$ in some neighborhood of the origin. Furthermore, domains $\mathcal{D}(A_0)$ and $\mathcal{D}(A)$ are dense in H. The inverse operator A_0^{-1} is defined on the whole space H and its range coincides with $\mathcal{D}(A_0)$. Now formula (1.4) shows that

$$\mathcal{D}(A_0) \cap \mathscr{H} = \{0\}. \tag{1.9}$$

It means that the set $\mathscr{H}$ could not be "too large" in the space H. Particularly, (1.9) may be not valid if the set $\mathscr{H}$ is replaced with its closure $\overline{\mathscr{H}}$.

Remark 1.5. Operators Γ_0, Γ_1 are not determined uniquely by the Green formula. In particular, the equality (1.1) holds true if the operator Γ_1 is replaced by the sum $\Gamma_1 + N\Gamma_0$, where N is a symmetric operator on E defined on the domain $\Gamma_0\mathcal{D}(A)$. It is easily seen that Assumption 2 holds or not for the sets $\{A, \Gamma_0, \Gamma_1 + N\Gamma_0, H, E\}$ and (1.8) at the same time. Indeed, neither of conditions (1.6) and (1.7) affects the operator Γ_1.

Remark 1.6. For any $a \in \mathbb{R}$, the Green formula (1.1) is valid for both A and $A - aI$ simultaneously with the same E, Γ_0, Γ_1. It means that the requirement $0 \in \rho(A_0)$ in Assumption 2 always can be ensured provided that there exists a real number $a \in \mathbb{R}$ such that $a \in \rho(A_0)$. To that end, it is sufficient to pass to the boundedly invertible selfadjoint operator $A - aI$ in place of A. Note as well that the condition (1.7) will be satisfied automatically if the space E is *defined* as the closure of $\Gamma_0\mathscr{H}$.

2. Our next objective is to establish the relationships among vectors from linear sets $\ker(A - zI)$ for different values of $z \in \rho(A_0)$ and describe the domain of A in terms of operator A_0 and vectors from $\mathscr{H}$.

Proposition 1.7. *1. The mapping*

$$h \longmapsto (I - zA_0^{-1})^{-1}h, \quad h \in \mathscr{H},\ z \in \rho(A_0) \tag{1.10}$$

establishes a one-to-one correspondence between $\mathscr{H} = \ker(A)$ and $\ker(A - zI)$. For any $h \in \mathscr{H}$ and $h_z := (I - zA_0^{-1})^{-1}h \in \ker(A - zI)$, $z \in \rho(A_0)$ the equality $\Gamma_0 h = \Gamma_0 h_z$ holds.

2. *For any $z_1, z_2 \in \rho(A_0)$*

$$\ker(A - z_1 I) \cap \ker(A - z_2 I) = \{0\}, \qquad z_1 \neq z_2$$

3. *For any $z \in \rho(A_0)$ the domain $\mathcal{D}(A)$ allows the representation in the form of direct sum*

$$\mathcal{D}(A) = \mathcal{D}(A_0) \dot{+} \ker(A - zI) = \mathcal{D}(A_0) \dot{+} (I - zA_0^{-1})^{-1}\mathscr{H}, \qquad z \in \rho(A_0). \tag{1.11}$$

Proof. Let $z \in \rho(A_0)$ be a complex number.

(1) Due to (1.5) we have for any vector $h \in \mathscr{H}$

$$(A - zI)(I - zA_0^{-1})^{-1}h = (A - zI)[I + z(A_0 - zI)^{-1}]h = (A - zI)h + zh = 0.$$

Conversely, for $h_z \in \ker(A - zI)$ and $h := (I - zA_0^{-1})h_z$ we obtain

$$Ah = Ah_z - zAA_0^{-1}h_z = (A - zI)h_z = 0.$$

Finally, the equality $\Gamma_0 h = \Gamma_0 h_z$ follows directly from the relation $h = (I - zA_0^{-1})h_z$ and definition of the domain $\mathcal{D}(A_0)$.

(2) Let $z_1, z_2 \in \rho(A_0)$ be two complex numbers. If $(A - z_1)u = 0$ and $(A - z_2)u = 0$ for some vector $u \in \mathcal{D}(A)$, then $Au = z_1 u = z_2 u$. Therefore $(z_1 - z_2)u = 0$, which implies the required equality.

(3) For an arbitrary number $z \in \rho(A_0)$ the sum in (1.11) is direct because the equality $(A - zI)f = 0$ for a vector $f \in \mathcal{D}(A_0)$ implies $(A_0 - zI)f = 0$. Therefore $f = 0$ because $\ker(A_0 - zI) = \{0\}$. Further, any vector $f \in \mathcal{D}(A)$ can be represented in the form:

$$f = (A_0 - zI)^{-1}(A - zI)f + \left[f - (A_0 - zI)^{-1}(A - zI)f\right], \qquad z \in \rho(A_0) \tag{1.12}$$

The first summand here belongs to $\mathcal{D}(A_0)$, and the second one lies in $\ker(A - zI)$ according to the formula (1.5). The rest follows from the already proven statement (1). In the particular case of $z = 0$ we arrive at the decomposition

$$f = A_0^{-1}Af + (f - A_0^{-1}Af), \quad f \in \mathcal{D}(A). \tag{1.13}$$

Operator $A_0^{-1}A$ here can be interpreted as an oblique projection of the sum $\mathcal{D}(A) = \mathcal{D}(A_0) \dot{+} \mathscr{H}$ onto the first summand.

The proof is complete. □

3. Now we can introduce one of the main objects of the theory under development, operator Π. Its properties and relationships with operators A and Γ_0 play a significant role in the sequel.

Proposition 1.8. *1. Operator $\Gamma_1 A_0^{-1}$ is defined everywhere on the space H and bounded. Let Π be its adjoint.*

$$\Pi \equiv (\Gamma_1 A_0^{-1})^* : E \to H \tag{1.14}$$

2. The set $\mathscr{H}$ permits the characterization:

$$\mathscr{H} = \{u \in \mathcal{D}(A) \mid \Pi\Gamma_0 u = u\} \tag{1.15}$$

3. The range of operator Π is dense in the subspace $\overline{\mathscr{H}}$:

$$\overline{\mathcal{R}(\Pi)} = \overline{\mathscr{H}} \tag{1.16}$$

Proof. (1) Operator $\Gamma_1 A_0^{-1}$ is defined in the whole H because for any element $x \in H$ the vector $A_0^{-1}x$ belongs to the domain $\mathcal{D}(A_0) = \mathcal{D}(\Gamma_1)$. Let us write the Green formula (1.1) with $u \in \mathcal{D}(A)$ and $v = A_0^{-1}x$, where $x \in H$ and use the equality (1.5) with $z = 0$:

$$(Au, A_0^{-1}x) - (u, AA_0^{-1}x) = (\Gamma_1 u, \Gamma_0 A_0^{-1}x) - (\Gamma_0 u, \Gamma_1 A_0^{-1}x),$$

Since $AA_0^{-1}x = x$ and $\Gamma_0 A_0^{-1} = 0$, we obtain

$$(Au, A_0^{-1}x) = (u, x) - (\Gamma_0 u, \Gamma_1 A_0^{-1}x), \qquad u \in \mathcal{D}(A), \quad x \in H \tag{1.17}$$

Further, for $u = h \in \mathscr{H}$ the relation (1.17) implies

$$(h, x) = (\Gamma_0 h, \Gamma_1 A_0^{-1}x), \quad x \in H, \ h \in \mathscr{H}.$$

This identity means that the vector $\Gamma_0 h$ belongs to the domain of operator $(\Gamma_1 A_0^{-1})^*$, acting as

$$(\Gamma_1 A_0^{-1})^* : \ \Gamma_0 h \longmapsto h, \quad h \in \mathscr{H} \tag{1.18}$$

The set $\Gamma_0 \mathscr{H}$ is dense in E according to Assumption 2, so that the operator $\Gamma_1 A_0^{-1}$ defined on H has the densely defined adjoint $(\Gamma_1 A_0^{-1})^*$. Consequently, $\Gamma_1 A_0^{-1}$ is closable in H. (See [5], p. 70-71.) As the domain of $\Gamma_1 A_0^{-1}$ is the whole space H, it implies that $\Gamma_1 A_0^{-1}$ is already closed. By the virtue of closed graph theorem $\Gamma_1 A_0^{-1}$ is bounded, so is its adjoint $\Pi = (\Gamma_1 A_0^{-1})^*$.

(2) Taking into account the definition (1.14) of operator Π, the implication $u \in \mathscr{H} \Rightarrow \Pi\Gamma_0 u = u$ follows directly from (1.18). In order to prove the inverse assertion note that for a vector $u \in \mathcal{D}(A)$ for which $\Pi\Gamma_0 u = u$, the right hand side of (1.17) is zero for any $x \in H$. The set $\{A_0^{-1}x \mid x \in H\}$ is equal to $\mathcal{D}(A_0)$, hence is dense in H. Now the inclusion $u \in \ker(A)$ is a consequence of the equality to zero of the left hand side of (1.17).

(3) Equality (1.15) means that $\mathscr{H} \subset \mathcal{R}(\Pi)$. Let us show that any vector from $\mathcal{R}(\Pi)$ can be approximated by elements of $\mathscr{H}$. It is sufficient for the proof of (1.16). For a given $\Pi e \in \mathcal{R}(\Pi)$, $e \in E$ and $\varepsilon > 0$ we choose the vector $h \in \mathscr{H}$ such that $\|\Gamma_0 h - e\| \le \varepsilon \|\Pi\|^{-1}$. It is always possible because $\Gamma_0\mathscr{H}$ is dense in the space E according to Assumption 2. Since $h = \Pi\Gamma_0 h$ due to equality (1.15), we have $\|\Pi e - h\| = \|\Pi e - \Pi\Gamma_0 h\| \le \|\Pi\| \cdot \|e - \Gamma_0 h\| \le \varepsilon$.

The proof is complete. $\square$

From the description (1.15) we obtain following important equality

$$\Pi\Gamma_0 h = h, \quad h \in \mathscr{H}. \tag{1.19}$$

showing that the operator Π is the left inverse of the restriction of Γ_0 to the set $\mathscr{H}$. Let us take notice of a few consequences of (1.19).

Remark 1.9. Since $\mathscr{H} \subset \mathcal{D}(A) = \mathcal{D}(\Gamma_0)$, operator Γ_0 can be applied to the both sides of (1.19), which yields

$$(\Gamma_0\Pi - I)\Gamma_0 h = 0, \quad h \in \mathscr{H} \tag{1.20}$$

This equality shows that the restriction $\Pi|_{\Gamma_0\mathscr{H}}$ is the right inverse of Γ_0 restricted to the set $\mathscr{H}$.

Remark 1.10. Operator $\Pi\Gamma_0$ defined on the domain $\mathcal{D}(A) = \mathcal{D}(A_0)\dot{+}\mathscr{H}$ is an oblique projection to the second summand

$$\Pi\Gamma_0 \;:\; f + h \longmapsto h, \quad f \in \mathcal{D}(A_0),\, h \in \mathscr{H}.$$

In particular, $\Pi\Gamma_0\mathcal{D}(A)$ and $\mathscr{H}$ coincide. This assertion follows immediately from the definition of domain $\mathcal{D}(A_0) = \mathcal{D}(A) \cap \ker(\Gamma_0)$ and (1.19).

Remark 1.11. The equality (1.9) can be strengthened as follows

$$\mathcal{D}(A_0) \cap \mathcal{R}(\Pi) = \{0\} \tag{1.21}$$

Indeed, let $e \in E$ be a vector from E such that $\Pi e \in \mathcal{D}(A_0)$, in other words, $\Gamma_0\Pi e = 0$. Owing to the identity $\overline{\Gamma_0\Pi} = I_E$ for the closure of operator $\Gamma_0\Pi$ resulting from (1.20) and density of $\Gamma_0\mathscr{H}$ in E, we conclude that $\overline{\Gamma_0\Pi}e = e$. Now the equality $e = 0$ follows.

Remark 1.12. Since $f - A_0^{-1}Af \in \mathscr{H}$, the relation (1.19) and relation $\Gamma_0 A_0^{-1} = 0$ imply that the sum (1.13) can be rewritten in a more elaborated form

$$f = A_0^{-1}Af + \Pi\Gamma_0 f, \qquad f \in \mathcal{D}(A). \tag{1.22}$$

Let us insert this representation into the expression for the second summand in (1.12):

$$\begin{aligned}
&f - (A_0 - zI)^{-1}(A - zI)f = [I - (A_0 - zI)^{-1}(A - zI)]f \\
&= \left[I - (A_0 - zI)^{-1}(A - zI)\right] A_0^{-1}Af + \left[I - (A_0 - zI)^{-1}(A - zI)\right] \Pi\Gamma_0 f \\
&= \left[I + z(A_0 - zI)^{-1}\right] \Pi\Gamma_0 f = A_0(A_0 - zI)^{-1}\Pi\Gamma_0 f = (I - zA_0^{-1})^{-1}\Pi\Gamma_0 f.
\end{aligned}$$

Now (1.12) can be rewritten as

$$f = (A_0 - zI)^{-1}(A - zI)f + (I - zA_0^{-1})^{-1}\Pi\Gamma_0 f, \qquad f \in \mathcal{D}(A), \quad z \in \rho(A_0) \tag{1.23}$$

in complete accordance with the decomposition (1.11). Obviously, this formula is a generalization of (1.22).

Remark 1.13. If we represent $h \in \mathscr{H}$ in (1.19) as $h = (I - zA_0^{-1})h_z$, where $h_z \in \ker(A - zI)$, $z \in \rho(A_0)$, then (1.19) takes the form

$$(I - zA_0^{-1})^{-1}\Pi\Gamma_0 h_z = h_z, \quad h_z \in \ker(A - zI), \quad z \in \rho(A_0). \tag{1.24}$$

In particular, this equality shows that the intersection $\ker(\Pi) \cap \Gamma_0 \ker(A - zI)$ is trivial:

$$\ker(\Pi) \cap \Gamma_0 \ker(A - zI) = \{0\}, \qquad z \in \rho(A_0), \tag{1.25}$$

which is almost obvious if we take into consideration that $\Gamma_0 \ker(A - zI) = \Gamma_0\mathscr{H}$ due to the assertion (1) of Proposition 1.7 and recall that the operator Π is the left inverse of $\Gamma_0|_{\mathscr{H}}$.

4. The collection (1.8) as defined above naturally appears in the context of what could be called *the theory of null extensions of selfadjoint operators.*

Definition 1.14. Let T be an operator on the Hilbert space H with domain $\mathcal{D}(T)$. Linear operator S is called a *null extension* of T if $\mathcal{D}(T) \subset \mathcal{D}(S)$, $Sx = Tx$ for $x \in \mathcal{D}(T)$, and $Sx = 0$ for $x \in \mathcal{D}(S) \setminus \mathcal{D}(T)$.

Next two Propositions offer convenient ways to construct collections $\{A, \Gamma_0, \Gamma_1, H, E\}$ that automatically satisfy both Assumptions 1 and 2.

Proposition 1.15. *Let A_0 be a selfadjoint boundedly invertible operator on the space H with domain $\mathcal{D}(A_0)$ and $\mathscr{H}$ be a linear set in H with the property $\mathcal{D}(A_0) \cap \mathscr{H} = \{0\}$. Define operator A as a null extension of A_0 to the set $\mathcal{D}(A) := \mathcal{D}(A_0) \dot{+} \mathscr{H}$ according to the formula $A(A_0^{-1}f + h) := f$, $f \in H$, $h \in \mathscr{H}$. Assume that there exists a Hilbert space E and a linear mapping Γ_0 with domain $\mathcal{D}(A_0) \dot{+} \mathscr{H}$ and the range $\mathcal{R}(\Gamma_0)$ dense in E, such that* $\ker(\Gamma_0) = \mathcal{D}(A_0)$ *and there exists a bounded left inverse* Π *of the restriction $\Gamma_0|_{\mathscr{H}}$ defined on the whole space E. Let N be a symmetric operator on E with the dense domain $\Gamma_0\mathscr{H}$ and $\Gamma_1 := \Pi^* A + N\Gamma_0$, $\mathcal{D}(\Gamma_1) := \mathcal{D}(A_0) \dot{+} \mathscr{H}$ be a densely defined mapping from H to E. Then the collection $\{A, \Gamma_0, \Gamma_1, H, E\}$ of such defined objects satisfies both Assumptions 1 and 2.*

Proof. Assumption 2 follows directly from the premise of Proposition. With regard to Assumption 1, it is sufficient to show that the Green formula (1.1) is valid for $u = v \in \mathcal{D}(A)$. For any vector $u \in \mathcal{D}(A)$ of the form $u = A_0^{-1} + h$ with some $f \in H$ and $h \in \mathscr{H}$ we have $Au = f$. Therefore, since A_0^{-1} is selfadjoint and $\Gamma_0 A_0^{-1} = 0$, $\Pi\Gamma_0 h = h$,

$$\begin{aligned}
(Au, u) - (u, Au) &= (f, A_0^{-1}f + h) - (A_0^{-1}f + h, f) = (f, h) - (h, f) \\
&= (f, \Pi\Gamma_0 h) - (\Pi\Gamma_0 h, f) = (\Pi^* f, \Gamma_0 h) - (\Gamma_0 h, \Pi^* f) \\
&= (\Pi^* Au, \Gamma_0(A_0^{-1}f + h)) - (\Gamma_0(A_0^{-1}f + h), \Pi^* Au) \\
&= (\Pi^* Au + N\Gamma_0 u, \Gamma_0 u) - (\Gamma_0 h, \Pi^* Au + N\Gamma_0 u) \\
&= (\Gamma_1 u, \Gamma_0 u) - (\Gamma_0 u, \Gamma_1 u),
\end{aligned}$$

so that Assumption 1 is valid as well. ☐

As an alternative, collection (1.8) satisfying Assumptions 1 and 2 can be defined in terms of the pair $\{A, \Gamma_0\}$. Properties of $\{A, \Gamma_0\}$ that ensure existence of the corresponding collection (1.8) are easy to formulate. Arguments similar to the proof of Proposition 1.15 result in the following

Proposition 1.16. *Let A be a linear operator on Hilbert space H with domain $\mathcal{D}(A)$ and Γ_0 be an operator from H to an auxiliary Hilbert space E with the domain $\mathcal{D}(\Gamma_0) = \mathcal{D}(A)$. Assume that*

1. *Operator A_0 defined as a restriction of A to the linear set $\mathcal{D}(A_0) := \ker(\Gamma_0)$ is selfadjoint and boundedly invertible.*
2. *There exists a bounded mapping $\Pi : E \to H$ such that $\Pi\Gamma_0 h = h$ for any $h \in \ker(A)$.*
3. *Set $\Gamma_0 \ker(A)$ is dense in E.*

Then the collection $\{A, \Gamma_0, \Gamma_1, H, E\}$ where the operator Γ_1 is defined as $\Gamma_1 := \Pi^* A + N\Gamma_0$ with an arbitrary symmetric operator N on the domain $\mathcal{D}(N) := \Gamma_0 \ker(A)$, satisfies both Assumptions 1 and 2.

2 Minimal operator and Krein extension

1. Operator A_{00} defined as a restriction of A to the set $\mathcal{D}(A_{00}) := \{u \in \mathcal{D}(A_0) \mid \Gamma_1 u = 0\}$ is called **minimal**. Obviously, A_{00} is symmetric and A_0 is the selfadjoint extension of A_{00}. Furthermore, $A_{00} : A_0^{-1} f \mapsto f$ where $A_0^{-1} f \in \mathcal{D}(A_{00})$. It turns out that the domain $\mathcal{D}(A_{00})$ can be effectively described in other terms, not involving directly operators Γ_0 and Γ_1. Moreover, under some quite natural assumption A_{00} is the minimal operator indeed in the sense of extension theory of symmetric operators.

Proposition 2.1. *The equalities hold*

$$\mathcal{D}(A_{00}) := \{u \in \mathcal{D}(A) \mid \Gamma_0 u = \Gamma_1 u = 0\} = A_0^{-1} \mathscr{H}^\perp. \tag{2.1}$$

If the set $\mathcal{D}(A_{00})$ is dense in H then A is closable and A_{00}^* coincides with its closure $\bar{A}$. The domain of $\bar{A}$ allows decomposition $\mathcal{D}(\bar{A}) = \mathcal{D}(A_0) \dot{+} \overline{\mathscr{H}}$ where $\overline{\mathscr{H}}$ is the null subspace of $\bar{A}$.

Proof. For $f \in H$ and $u := A_0^{-1} f \in \mathcal{D}(A_0)$ we have $\Gamma_1 u = \Gamma_1 A_0^{-1} f = \Pi^* f$. Therefore, the inclusion $f \in \mathscr{H}^\perp = \ker(\Pi^*)$ is equivalent to the equality $\Gamma_1 u = 0$.

Further, in order for a densely defined operator to be closable it is necessary and sufficient that its adjoint be densely defined, see [5], p. 70. Let us consider the adjoint operator A^* and show that $A^* \supset A_{00}$. Then A is closable since A_{00} is densely defined. For $u := A_0^{-1} f + h \in \mathcal{D}(A)$ where $f \in H$, $h \in \mathscr{H}$ and $v := A_0^{-1} f_0 \in \mathcal{D}(A_{00})$ where $f_0 \in \mathscr{H}^\perp$ we have

$$(Au, v) = (A(A_0^{-1} f + h), A_0^{-1} f_0) = (f, A_0^{-1} f_0) = (A_0^{-1} f + h, f_0) = (u, A_{00} v)$$

so that $A^* \supset A_{00}$ as required.

Usual duality arguments yield $\bar{A} = A^{**} \subset A_{00}^*$, so that for the proof of equality $A_{00}^* = \bar{A}$ we only need to show that $\bar{A} \supset A_{00}^*$. We start with the observation that the subspace $\overline{\mathscr{H}}$ belongs to $\ker(\bar{A})$ because the null set of a closed operator is always closed. In particular, $\overline{\mathscr{H}} \subset \mathcal{D}(\bar{A})$. Further, assuming $x \in \mathcal{D}(A_{00}^*)$ by the definition of adjoint operator, $(A_{00} A_0^{-1} f_0, x) = (f_0, A_0^{-1} A_{00}^* x)$ for any $f_0 \in \mathscr{H}^\perp$. From the other side, obviously $(A_{00} A_0^{-1} f_0, x) = (f_0, x)$ since A_{00} is a restriction of A_0. Therefore, $x - A_0^{-1} A_{00}^* x \perp \mathscr{H}^\perp$, or the vector $\mathfrak{h}(x) := x - A_0^{-1} A_{00}^* x$ belongs to the subspace $\overline{\mathscr{H}}$. It follows that any vector $x \in \mathcal{D}(A_{00}^*)$ is represented in the form of sum $x = A_0^{-1} A_{00}^* x + \mathfrak{h}(x)$ where $A_0^{-1} A_{00}^* x \in \mathcal{D}(A_0)$ and $\mathfrak{h}(x) \in \overline{\mathscr{H}}$. In other words, $\mathcal{D}(A_{00}^*) \subset \mathcal{D}(A_0) + \overline{\mathscr{H}} \subset \mathcal{D}(\bar{A})$. The equality $\mathcal{D}(A_{00}^*) = \mathcal{D}(\bar{A}) = \mathcal{D}(A_0) \dot{+} \overline{\mathscr{H}}$ is therefore obtained. The sum here is direct because if the intersection $\mathcal{D}(A_0) \cap \overline{\mathscr{H}}$ was not trivial, then it would consist of vectors $x \in \mathcal{D}(A_0)$ such that $\bar{A} x = A_0 x = 0$, which contradict the assumption $\ker(A_0) = \{0\}$. Now we need to show that actions of A_{00}^* and $\bar{A}$ coincide on their domain. To that end note that $\overline{\mathscr{H}} = \mathcal{R}(A_{00})^\perp = \ker(A_{00}^*)$, so that parts of both operators A_{00}^* and $\bar{A}$ in the subspace $\overline{\mathscr{H}}$ is the null operator. Finally, action of $\bar{A}$ on the domain $\mathcal{D}(A_0)$ is $\bar{A} : A_0^{-1} f \mapsto f$ where $f \in H$. From the other side, for the operator A_{00}^* we obtain

$$(A_{00}^* A_0^{-1} f, v) = (A_0^{-1} f, A_{00} v) = (f, A_0^{-1} A_{00} v) = (f, v)$$

where $f \in H$, $v \in \mathcal{D}(A_{00})$. Because $\mathcal{D}(A_{00})$ is dense in H, it follows that $A_{00}^* : A_0^{-1} f \mapsto f$, $f \in H$. Therefore actions of operators A_{00}^* and $\bar{A}$ coincide on their domain $\mathcal{D}(A_0) \dot{+} \overline{\mathscr{H}}$.

The proof is complete. □

2. Operator A_{00} can have nontrivial selfadjoint parts, i. e. nontrivial reducing subspaces where it generates selfadjoint operators. If there are no such parts, operator A_{00} is called **completely non-selfadjoint** or **simple**. The criterion of complete non-selfadjointness of A_{00} is given in the following proposition.

Proposition 2.2. *Denote H_{nsa} the closed envelope of* $\ker(A - zI)$, $z \in \rho(A_0)$ *and let $H_0 := H \ominus H_{nsa}$ be the orthogonal complement. Then*

$$H_{nsa} = clos \bigvee_{z \in \rho(A_0)} \ker(A - zI) = clos \bigvee_{e \in E,\ z \in \rho(A_0)} (I - z A_0^{-1})^{-1} \Pi e = clos \bigvee_{e \in E,\ n \geq 0} A_0^{-n} \Pi e, \tag{2.2}$$

and H_0 is the maximal subspace where A_{00} induces a selfadjoint operator coinciding with $A_0|_{H_0}$.

Proof. Equalities in (2.2) follow from the description of $\ker(A - zI)$ given in Proposition 1.7, density of $\Gamma_0\mathscr{H}$ in E, and the power expansion of resolvent $(I - zA_0^{-1})^{-1}$, $z \in \rho(A_0)$.

From (2.2) we have that both H_{nsa} and H_0 reduce the selfadjoint operator A_0 and $H_0 \subset \mathscr{H}^\perp$. Taking into consideration the definition $\mathcal{D}(A_{00}) = A_0^{-1}\mathscr{H}^\perp$, we conclude that $A_0^{-1}H_0 \subset \mathcal{D}(A_{00}) \cap H_0$. Since A_{00} is a restriction of A_0, it follows that $A_{00}A_0^{-1}h_0 = h_0$ for $h_0 \in H_0$. Therefore $A_0|_{H_0} \subset A_{00}$, H_0 reduces A_{00}, and the part of A_{00} in H_0 is a selfadjoint operator $A_0|_{H_0}$. In order to show that H_0 is the maximal subspace where A_{00} induces a selfadjoint part, assume that H' is a reducing space of A_{00} and $A_{00}|_{H'}$ is selfadjoint. Since A_{00} is the restriction of A_0, the space H' also reduces the selfadjoint operator A_0. The set $A_0^{-1}H'$ is dense in H'; it is the domain of the selfadjoint part $A_0|_{H'} = A_{00}|_{H'}$ of A_{00}. It follows from the definition of $\mathcal{D}(A_{00})$ that $H' \subset \mathscr{H}^\perp$. Since H' is invariant for A_0^{-1}, we have $0 = (A_0^{-n}x, \Pi e) = (x, A_0^{-n}\Pi e)$ for $x \in H'$, $e \in E$, and $n \geq 0$. Therefore, H' is orthogonal to the subspace (2.2) and $H' \subset H_0$. $\square$

3. Consider the null extension A_K of operator A_{00} to the set $\mathcal{D}(A_K) := A_0^{-1}\mathscr{H}^\perp \dot{+} \mathscr{H}$. According to Definition 1.14,

$$A_K : A_0^{-1}f_0 + h \mapsto f_0, \qquad f_0 \in \mathscr{H}^\perp, \quad h \in \mathscr{H}$$

It is a simple exercise to see that A_K is symmetric, but not necessarily densely defined. The next Theorem summarizes other properties of A_K.

Theorem 2.3. *1. Symmetric operator A_K is a restriction of A to the domain*

$$\mathcal{D}(A_0) = \{u \in \mathcal{D}(A) \mid (\Gamma_1 - \Lambda\Gamma_0)u = 0\}$$

where $\Lambda := \Gamma_1\Pi$ is symmetric, densely defined and closable on $\mathcal{D}(\Gamma) := \Gamma_0\mathscr{H}$.

2. *The equation $A_Ku = f$ is solvable only when $\Pi^*f = 0$, that is, for $f \in \mathscr{H}^\perp$. The solutions are represented in the form $u = A_0^{-1}f + h$, where $h \in \mathscr{H}$ is arbitrary.*

3. *If the domain $\mathcal{D}(A_{00})$ is dense in H, then A_K is essentially selfadjoint. Its closure is the null extension of A_K to the set $A_0^{-1}\mathscr{H}^\perp \dot{+} \overline{\mathscr{H}}$.*

Proof. (1) Let $f \in H$ and $h \in \mathscr{H}$ be two arbitrary vectors and $u := A_0^{-1}f + h \in \mathcal{D}(A)$. Then

$$\begin{aligned}(\Gamma_1 - \Lambda\Gamma_0)u &= (\Gamma_1 - \Gamma_1\Pi\Gamma_0)\left(A_0^{-1}f + h\right)\\ &= \Gamma_1A_0^{-1}f + \Gamma_1h - \Gamma_1h = \Gamma_1A_0^{-1}f = \Pi^*f,\end{aligned}$$

where we used equalities $\Gamma_0A_0^{-1} = 0$ and $\Pi\Gamma_0h = h$. Since $\ker(\Pi^*) = \mathcal{R}(\Pi)^\perp = \mathscr{H}^\perp$, the inclusion $f \in \mathscr{H}^\perp$ is equivalent to $(\Gamma_1 - \Lambda\Gamma_0)u = 0$. If $h \in \mathscr{H}$, then

$$\begin{aligned}(\Lambda\Gamma_0h, \Gamma_0h) - (\Gamma_0h, \Lambda\Gamma_0h) &= (\Gamma_1\Pi\Gamma_0h, \Gamma_0h) - (\Gamma_0h, \Gamma_1\Pi\Gamma_0h)\\ &= (\Gamma_1h, \Gamma_0h) - (\Gamma_0h, \Gamma_1h) = 0\end{aligned}$$

Since Λ is symmetric and densely defined, it is closable.

(2) Obvious according to the definition $\mathcal{D}(A_K)$.

(3) Operator A_K is symmetric and densely defined. The representation $A_0^{-1}\mathscr{H}^\perp \dot{+} \overline{\mathscr{H}}$ for the domain of its closure is obtained in the way demonstrated in the proof of Proposition 2.1. Let us show that A_K is essentially selfadjoint. Since A_K is symmetric, $\mathcal{D}(\overline{A_K}) \subset \mathcal{D}(A_K^*)$ and we need to verify correctness of the inverse inclusion. Let $f_0 \in \mathscr{H}^\perp$, $h \in \mathscr{H}$, and $x \in \mathcal{D}(A_K^*)$, i. e.

$$(A_K(A_0^{-1}f_0 + h), x) = (A_0^{-1}f_0 + h, A_K^*x) = (f_0, A_0^{-1}A_K^*x) + (h, A_K^*x)$$

Since the left hand side is equal to (f_0, x), we have $(f_0, x - A_0^{-1}A_K^*x) = (h, A_K^*x)$. For $h = 0$ this equality yields the representation $x = A_0^{-1}A_K^*x + \mathfrak{h}$, where $\mathfrak{h} \in \overline{\mathscr{H}}$ is some vector. From the other side, if $f_0 = 0$, then $(h, A_K^*x) = 0$ for each $h \in \mathscr{H}$ and we see that $A_K^*x \in \mathscr{H}^\perp$. Thus any vector $x \in \mathcal{D}(A_K^*)$ is represented in the form $x = A_0^{-1}g + \mathfrak{h}$ with $g = A_K^*x \in \mathscr{H}^\perp$ and $\mathfrak{h} \in \overline{\mathscr{H}}$. Therefore, $x \in \mathcal{D}(\overline{A_K})$.

The proof is complete. $\square$

Remark 2.4. Operator A_K is an analogue of the Krein extension of operator A_{00}, see [2]. From the same point of view A_0 is equivalent to the Friedrichs extension of A_{00}. For any element $u \in \mathcal{D}(A)$ we have $(\Gamma_1 - \Lambda\Gamma_0)u = (\Gamma_1 - \Gamma_1\Pi\Gamma_0)u = \Gamma_1(I - \Pi\Gamma_0)u$, so that the condition $u \in \mathcal{D}(A_K)$ is equivalent to the identity $\Gamma_1 u = \Gamma_1\Pi\Gamma_0 u$. Operator Λ was studied by M. Vishik in the context of elliptic boundary value problems in [27]. Later $\Gamma_1 - \Lambda\Gamma_0$ was rewritten as $\Gamma_1(I - \Pi\Gamma_0)$ by G. Grubb [15] and independently by A. Alonso and B. Simon in [2]. See the note [16] for further references regarding boundary conditions for the Krein extension. In our case the extension A_K is not necessarily densely defined, nor semibounded. Extensions A_0 and A_K of the symmetric operator A_{00} are transversal in the sense of [10], i. e. $\mathcal{D}(A_0) \cap \mathcal{D}(A_K) = \mathcal{D}(A_{00})$ and $\mathcal{D}(A_0) \vee \mathcal{D}(A_K) = \mathcal{D}(A)$. Note at last that the operator A_K is uniquely determined by the pair $\{A_0, \mathscr{H}\}$. In particular, it does not depend on the choice of operator Γ_1 in the Green formula.

3 Spectral boundary value problem and Weyl function

Let $\{A, \Gamma_0, \Gamma_1, H, E\}$ be the collection (1.8) subject to Assumptions 1 and 2. In this section we consider the spectral boundary value problem defined by the set $\{A, \Gamma_0, \Gamma_1, H, E\}$, describe conditions of its solvability, and introduce the notion of Weyl function.

Definition 3.1. For given $f \in H$ and $\varphi \in E$ we call the **spectral boundary value problem** associated with collection (1.8) the problem of finding pairs (u, z) of vectors $u \in \mathcal{D}(A)$ and complex numbers $z \in \mathbb{C}$, satisfying equations:

$$\begin{cases} (A - zI)u = f \\ \Gamma_0 u = \varphi \end{cases} \tag{3.1}$$

Let us formulate a simple result regarding solvability of (3.1).

Theorem 3.2. *For any $z \in \rho(A_0)$, $\varphi \in \Gamma_0\mathscr{H}$, $f \in H$ the solution $u = u_z^{f,\varphi}$ to the problem (3.1) exists and is unique. It is represented in the form*

$$u_z^{f,\varphi} = (A_0 - zI)^{-1} f + [I + z(A_0 - zI)^{-1}]\Pi\varphi \tag{3.2}$$

Proof. Verification of the uniqueness of solution is more or less standard. If for some $z \in \rho(A_0)$, $\varphi \in \Gamma_0\mathscr{H}$, and $f \in H$ there exist two solutions $u_1, u_2 \in \mathcal{D}(A)$ to the system (3.1), then their difference $u_0 := u_1 - u_2$ satisfies both equations (3.1) with $f = 0$, $\varphi = 0$. Therefore the vector u_0 belongs to the domain of operator A_0 because $\Gamma_0 u_0 = 0$. Then it follows from the first equation (3.1) than $(A - zI)u_0 = (A_0 - zI)u_0 = 0$ and $u_0 = 0$ because $z \in \rho(A_0)$.

Further, from the equality (1.5) and the inclusion $\Pi\varphi \in \mathscr{H}$ where $\varphi \in \Gamma_0\mathscr{H}$, we obtain for the vector $u_z^{f,\varphi}$ defined by (3.2)

$$(A - zI)u_z^{f,\varphi} = f + (A - zI)[I + z(A_0 - zI)^{-1}]\Pi\varphi = f + (A - zI)\Pi\varphi + z\Pi\varphi = f$$

Hence, the first equation in (3.1) is satisfied. Next, using equalities $\Gamma_0(A_0 - zI)^{-1} = 0$ and (1.20) we obtain for the vector $u_z^{f,\varphi}$

$$\Gamma_0 u_z^{f,\varphi} = \Gamma_0\Pi\varphi = \varphi.$$

The proof is complete. □

Remark 3.3. From the representation (3.2) follows the norm estimate of the solution $u_z^{f,\varphi}$:

$$\|u_z^{f,\varphi}\| \le C(z)(\|f\| + \|\varphi\|), \quad f \in H,\ \varphi \in \Gamma_0\mathscr{H},\ z \in \rho(A_0)$$

with a constant $C(z) > 0$ depending on $z \in \rho(A_0)$. Owing to the density of $\Gamma_0\mathscr{H}$ in E (cf. (1.7)) this means that the mapping $(f, \varphi) \mapsto u_z^{f,\varphi}$ defined on the set of pairs $\{(f, \varphi) \mid f \in H, \varphi \in \Gamma_0\mathscr{H}\}$ can be uniquely extended to a bounded operator from $H \oplus E$ to H. Assuming $z \in \rho(A_0)$, it seems natural to interpret vectors $u_z^{f,\varphi}$ defined by the formula (3.2) with $f \in H$, $\varphi \in E$ as solutions to the problem (3.1) as well. Each solution $u_z^{f,\varphi}$ for $\varphi \in E\backslash\Gamma_0\mathscr{H}$ can be approximated by "regular" solutions u_z^{f,φ_n}, $n = 1, 2, \ldots$, where $\varphi_n \in \Gamma_0\mathscr{H}$. A short argument reveals that the passage from solutions $u_z^{f,\varphi}$ where $\varphi \in \Gamma_0\mathscr{H}$ to solutions for $\varphi \in E$ corresponds to the null extension of operator A from the set $\mathcal{D}(A_0)\dot{+}\mathscr{H}$ to the wider set $\mathcal{D}(A_0)\dot{+}\mathcal{R}(\Pi)$ as described in Proposition 1.15. The sum is direct due to (1.21). At last, since the vector $u_z^{f,\varphi}$ is unambiguously defined by the right hand side of (3.2) for any $\varphi \in E$, we can state that for $z \in \rho(A_0)$ the uniqueness property of solution (3.2) holds true for any $f \in H$, $\varphi \in E$.

Remark 3.4. In general, the solution (3.2) does not belong to the domain $\mathcal{D}(A)$ if $\varphi \in E \setminus \mathscr{H}$. Nevertheless, the vector $u_z^{f,\varphi}$ defined by (3.2) is a solution to a "weak variant" of the problem (3.1), thereby is a weak solution to (3.1). More precisely, the vector $u_z^{f,\varphi}$ solves the following variational problem for unknown $u \in H$

$$(u, (A_0 - \bar{z}I)v) = (f, v)_H + (\varphi, \Gamma_1 v)_E, \qquad v \in \mathcal{D}(A_0). \tag{3.3}$$

The proof is based on direct calculations.

$$\begin{aligned}(u_z^{f,\varphi}, (A_0 - \bar{z}I)v) &= ((A_0 - zI)^{-1}f + [I + z(A_0 - zI)^{-1}]\Pi\varphi, (A_0 - \bar{z}I)v) \\ &= (f, v) + ((I - zA_0^{-1})^{-1}\Pi\varphi, (A_0 - \bar{z}I)v) \\ &= (f, v) + (\varphi, \Gamma_1 A_0^{-1}(I - \bar{z}A_0^{-1})^{-1}(A_0 - \bar{z}I)v) \\ &= (f, v) + (\varphi, \Gamma_1 v),\end{aligned}$$

as required. In order to show that the equation (3.3) is the weak version of (3.1) assume that some $w \in H$ solves (3.1). In particular, $w \in \mathcal{D}(A)$ and $\Gamma_0 w = \varphi$. Then for any $v \in \mathcal{D}(A_0)$ by the Green formula

$$\begin{aligned}(w, (A_0 - \bar{z}I)v) &= (w, A_0 v) - (zw, v) = (w, Av) + (f - Aw, v) \\ &= (f, v) + (w, Av) - (Aw, v) = (f, v) + (\Gamma_0 w, \Gamma_1 v) - (\Gamma_1 w, \Gamma_0 v) = (f, v) + (\varphi, \Gamma_1 v),\end{aligned}$$

that is, the vector w solves the weak problem (3.3) as well. Note that if the set $\Gamma_1\mathcal{D}(A_0)$ is dense in E and $\mathcal{R}(A_0 - \bar{z}I)$ is dense in H, then the weak solution is determined by (3.3) uniquely.

3.1 Semi-homogeneous problems

The classical approach of boundary value problems theory (see [4, 28] for instance) considers two semi-homogeneous problems obtained from (3.1) by putting $f = 0$, $\varphi \neq 0$ and $f \neq 0$, $\varphi = 0$. The solution to (3.1) then is sought in the form of sum of solutions to these auxiliary problems. Below we give a brief account regarding their solvability in abstract setting of the paper.

Let us start our discission with the case $f = 0$, $\varphi \in E$.

$$\begin{cases} (A - zI)u = 0 \\ \Gamma_0 u = \varphi \end{cases} \tag{3.4}$$

According to Theorem 3.2 and Remark 3.3, for any $\varphi \in E$, $z \in \rho(A_0)$ there exists an unique solution u_z^φ to the problem (3.4). If $\varphi \in E \setminus \Gamma_0\mathscr{H}$, then the solution u_z^φ is the weak solution as explained in Remark 3.4. The representation (3.2) with $f = 0$ reduces to the expression:

$$u_z^\varphi = [I + z(A_0 - zI)^{-1}]\Pi\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi, \quad \varphi \in E \tag{3.5}$$

It is easily seen that vectors defined in (3.2) and (3.5) are connected by the relation:

$$u_z^{f,\varphi} = (A_0 - zI)^{-1}f + u_z^\varphi, \quad f \in H, \varphi \in E, z \in \rho(A_0), \tag{3.6}$$

Obviously, $(A_0 - zI)^{-1}f$ here is a solution to the problem (3.1) with $f \neq 0$, $\varphi = 0$. Thus, formula (3.6) represents the solution to (3.1) as a direct sum of the solution to (3.1) with $f \neq 0$, $\varphi = 0$ and the solution to (3.1) with $f = 0$, $\varphi \neq 0$. It is interesting to observe that these two problems can be seen as conjugate to each other in the following sense.

Remark 3.5. Let $z \in \rho(A_0)$ and $v_z^f = (A_0 - zI)^{-1} f$ be the solution to the semi-homogeneous problem

$$\begin{cases} (A - zI)v = f \\ \Gamma_0 v = 0 \end{cases} \qquad z \in \mathbb{C},\ f \in H.$$

Let u_z^φ be a solution to (3.4). Then

$$(u_z^\varphi, f)_H = (\varphi, \Gamma_1 v_{\bar z}^f)_E, \quad f \in H,\ \varphi \in E,\ z \in \rho(A_0).$$

Indeed, since

$$\big[(I - zA_0^{-1})^{-1}\Pi\big]^* = \Gamma_1 A_0^{-1}(I - \bar z A_0^{-1})^{-1} = \Gamma_1 (A_0 - \bar z I)^{-1},$$

the assertion follows directly from the representation (3.5).

Finally, the case of (3.1) with $f = 0$, $\varphi = 0$ is in fact the spectral problem for the eigenvalues and eigenvectors of selfadjoint operator A_0.

3.2 Weyl function

The vector u_z^φ defined in (3.5) for $\varphi \in \Gamma_0\mathscr{H}$ belongs to $\ker(A - zI)$, consequently lies in the domain of the operator Γ_1. Let us calculate $\Gamma_1 u_z^\varphi$ for a given pair of $\varphi \in \Gamma_0\mathscr{H}$ and $z \in \rho(A_0)$. We have

$$\begin{aligned}\Gamma_1 u_z^\varphi &= \Gamma_1[I + z(A_0 - zI)^{-1}]\Pi\varphi = \Gamma_1\Pi\varphi + z\Gamma_1(A_0 - zI)^{-1}\Pi\varphi \\ &= \Gamma_1\Pi\varphi + z\Gamma_1 A_0^{-1}(I - zA_0^{-1})^{-1}\Pi\varphi = \Gamma_1\Pi\varphi + z\Pi^*(I - zA_0^{-1})^{-1}\Pi\varphi \\ &= [\Gamma_1\Pi + z\Pi^*(I - zA_0^{-1})^{-1}\Pi]\varphi\end{aligned}$$

Introduce an operator-function $M(z)$, $z \in \rho(A_0)$ with values in the set of densely defined operators on E with the domain $\Gamma_0\mathscr{H}$:

$$M(z)\ :\ \varphi \longmapsto [\Gamma_1\Pi + z\Pi^*(I - zA_0^{-1})^{-1}\Pi]\varphi, \qquad \varphi \in \Gamma_0\mathscr{H}, \quad z \in \rho(A_0) \tag{3.7}$$

Since $\varphi = \Gamma_0 u_z^\varphi$ for the solution u_z^φ to the problem (3.4), the calculations conducted above show that

$$\Gamma_1 u_z^\varphi = M(z)\Gamma_0 u_z^\varphi, \quad z \in \rho(A_0),\ \varphi \in \Gamma_0\mathscr{H} \tag{3.8}$$

Definition 3.6. Function $M(z)$ is called **Weyl function** of the problem (3.1) (or (3.4)).

Remark 3.7. Note that the densely defined operator $M(0) = \Gamma_1\Pi$ is not necessarily bounded, whereas the component of $M(\cdot)$ depending on z, that is, the difference $M(z) - M(0)$, $z \in \rho(A_0)$, is an analytic operator function with values in $\mathscr{B}(E)$.

Remark 3.8. The Weyl function $M(z)$ can be rewritten in the form

$$M(z)\varphi = \Gamma_1(I - zA_0^{-1})^{-1}\Pi\varphi, \quad \varphi \in \Gamma_0\mathscr{H}, z \in \rho(A_0) \tag{3.9}$$

This representation directly follows from the definition above, since

$$I + z(A_0 - zI)^{-1} = A_0(A_0 - zI)^{-1} = (I - zA_0^{-1})^{-1}.$$

Assuming $\varphi = \Gamma_0 h$ with some $h \in \mathscr{H}$ and noting that for h and $h_z = (I - zA_0^{-1})^{-1}h \in \ker(A - zI)$ the equality $\varphi = \Gamma_0 h = \Gamma_0 h_z$ holds according to Proposition 1.7, we can rewrite (3.9) in the form:

$$M(z)\Gamma_0 h_z = \Gamma_1 h_z, \quad h_z \in \ker(A - zI),\ z \in \rho(A_0). \tag{3.10}$$

Let us formulate some properties of the constant term $M(0) = \Gamma_1\Pi$ of Weyl function $M(z)$ and its z-dependent part $M(z) - M(0)$, $z \in \rho(A_0)$.

Proposition 3.9. *1. Operator $M(0) = \Gamma_1\Pi$ defined on the domain $\Gamma_0\mathscr{H}$ is symmetric and closable.*

2. *The difference $M(z)-M(0)$ is an operator R-function, i. e. an analytic operator function taking values in $\mathscr{B}(E)$ with the positive imaginary part in the upper half plane.*

3. *Let $\{A,\Gamma_0,\Gamma_1+N\Gamma_0,H,E\}$ where N is a symmetric operator on E with domain $\Gamma_0\mathscr{H}$ be the collection (1.8) obtained from $\{A,\Gamma_0,\Gamma_1,H,E\}$ according to Remark 1.5. Then the Weyl function of boundary value problem associated with $\{A,\Gamma_0,\Gamma_1+N\Gamma_0,H,E\}$ is the sum $N+M(z)$, $z\in\rho(A_0)$.*

Proof. (1) Operator $M(0)=\Gamma_1\Pi$ coincides with the operator Λ defined in Theorem 2.3.

(2) Analyticity of $M(z)-M(0)$ for $z\in\rho(A_0)$ is obvious and a short calculation yields the equality

$$(M(z)\varphi,\psi)_E-(\varphi,M(\zeta)\psi)_E=(z-\overline{\zeta})\left((I-zA_0^{-1})^{-1}\Pi\varphi,(I-\zeta A_0^{-1})^{-1}\Pi\psi\right)_H, \quad \varphi,\psi\in\Gamma_0\mathscr{H},\quad z,\zeta\in\rho(A_0) \tag{3.11}$$

where the right hand side is defined correctly for all $\varphi,\psi\in E$. In the special case of $\zeta=z$, $z\notin\mathbb{R}$ and $\varphi=\psi$ we obtain

$$\operatorname{Im}(M(z)\varphi,\varphi)=(\operatorname{Im}z)\cdot\|(I-zA_0^{-1})\Pi\varphi\|^2,\qquad \varphi\in E,\quad z\notin\mathbb{R}$$

showing that the function $M(z)-M(0)$ is an operator R-function.

(3) Informally, the assertion is an consequence of the Weyl function definition (3.10). The formal proof reads as follows. Firstly, the operator Π does not depend on the particular choice of operator Γ_1 in the Green formula (1.1). Indeed, for $\widetilde{\Gamma}_1:=\Gamma_1+N\Gamma_0$ we have $\Pi^*=\Gamma_1A_0^{-1}=(\Gamma_1+N\Gamma_0)A_0^{-1}=\widetilde{\Gamma}_1A_0^{-1}$. Therefore, the z-dependent part of the Weyl function $M(z)-M(0)=z\Pi^*(I-zA_0^{-1})^{-1}\Pi$ is not affected by the choice of Γ_1. Moreover, for $x\in\Gamma_0\mathscr{H}$ we have $\widetilde{\Gamma}_1\Pi x=(\Gamma_1+N\Gamma_0)\Pi x=\Gamma_1\Pi x+N\Gamma_0\Pi x=(M(0)+N)x$, as required.

The proof is complete. ☐

Remark 3.10. Assuming $N=-M(0)$ in the statement (3) of the Proposition, we conclude that the Weyl function of the boundary value problem corresponding to collection $\{A,\Gamma_0,\Gamma_1-M(0)\Gamma_0,H,E\}$ is a bounded analytic operator function $M(z)-M(0)$ equal to the null operator at the origin. Operator $\widetilde{\Gamma}_1=\Gamma_1-M(0)\Gamma_0$ considered as an alternative to Γ_1 was introduced in the context of boundary value problems for elliptic partial differential equations by Vishik in [27].

Remark 3.11. Let $A_t:=A+tI$ where $t\in\mathbb{R}$. Then Assumption 1 is obviously satisfied for the operator A_t. Definition (3.8) makes it possible to introduce the concepts of Weyl function for the operator A_t. Since $\ker(A-zI)=\ker(A_t-\zeta I)$, where $\zeta=t+z$ and operators Γ_0, Γ_1 do not depend on $t\in\mathbb{R}$, the Weyl function $M_t(\cdot)$ for operator A_t can be defined in a quite natural way as $M_t(\zeta):=M(\zeta-t)$, $\zeta\in\rho(A_0+tI)$.

3.3 Uniqueness

This subsection is dedicated to the question of uniqueness as to what extent a boundary value problem can be recovered from its Weyl function. It turns out that the answer depends on the assumption of complete nonselfadjointness of the minimal operator A_{00}. If A_{00} is completely nonselfadjoint, then all components of the collection $\{A,\Gamma_0,\Gamma_1,H,E\}$ are determined by the Weyl function uniquely up to an unitary equivalence.

Theorem 3.12. *Let E, H be two Hilbert spaces and $M_1(\cdot)$, $M_2(\cdot)$ are the Weyl functions of two boundary value problems $\{A^j,\Gamma_0^j,\Gamma_1^j,H,E\}$, $j=1,2$ defined in H with the boundary space E whose minimal operators are completely nonselfadjoint. Other objects that corresponds to these problems, such as operators Π and A_0, will be distinguished similarly by employing the superscript based notation. Assume that symmetric operators $\Lambda^j=M_j(0)$, $j=1,2$ are defined on the common domain $\mathscr{D}:=\mathcal{D}(M_1(\cdot))=\mathcal{D}(M_2(\cdot))$ dense in E and the differences $m_j(z):=M_j(z)-M_j(0)$, $j=1,2$ are bounded operator functions analytic in some neighborhood of the origin $z=0$ where $m_1(z)=m_2(z)$. Then there exists an unitary $U:H\to H$ such that $UA_0^1=A_0^2U$, $U\Pi^1=\Pi^2$, $UA^1=A^2U$, $\Gamma_0^1=\Gamma_0^2U$, and $\Gamma_1^1=(\Gamma_1^2+N\Gamma_0^2)U$ where $N=\Lambda^1-\Lambda^2$ is a symmetric operator on $\mathscr{D}$.*

Proof. From the condition $m_1(z) = m_2(z)$ by using expansion $(I - zT)^{-1} = \sum_{n=0}^{\infty}(zT)^n$, $T \in \mathscr{B}(H)$ with $T = A_0^j$, we obtain

$$((A_0^1)^{-n}\Pi^1 e, (A_0^1)^{-m}\Pi^1 e) = ((A_0^2)^{-n}\Pi^2 e, (A_0^2)^{-m}\Pi^2 e), \qquad n, m = 0, 1, 2, \dots, \quad e \in E$$

Define the mapping $\widetilde{U}$ on the set $\{(A_0^1)^{-n}\Pi^1 e \mid n \geq 0,\, e \in E\}$ dense in H due to complete nonselfadjointness of A_{00}^1 as $\widetilde{U} : (A_0^1)^{-n}\Pi^1 e \mapsto (A_0^2)^{-n}\Pi^2 e$. The operator $\widetilde{U}$ is isometric and its range is dense in H. We extend $\widetilde{U}$ to the unitary U defined everywhere on H. Then $U(A_0^1)^{-n}\Pi^1 = (A_0^2)^{-n}\Pi^2$. For $n = 0$ we obtain $U\Pi^1 = \Pi^2$. Moreover, $UA_0^1(A_0^1)^{-n}\Pi^1 = A_0^2(A_0^2)^{-n}\Pi^2 = A_0^2 U(A_0^1)^{-n}\Pi^1$ for any $n \geq 0$; therefore $UA_0^1 = A_0^2 U$. A similar computation can be found in [6].

Now we can define the operator A^j on its domain $\mathcal{D}(A^j) := \mathcal{D}(A_0^j)\dot{+}\Pi^j\mathscr{D}$ as follows. Let the action of A^j on the set $\mathcal{D}(A_0^j)$ coincide with that of A_0^j and $A^j h = 0$ for $h \in \Pi^j\mathscr{D}$. Consequently, $UA^1 = A^2 U$. Operator Γ_0^j is defined as null on the set $\mathcal{D}(A_0^j)$ and as the left inverse to Π^j on $\Pi^j\mathscr{D}$. Therefore, $\mathcal{D}(\Gamma_0^j) = \mathcal{D}(A^j)$. For $h := \Pi^1\mathscr{D}$ the relation $U\Pi^1 = \Pi^2$ yields $\Gamma_0^1 h = \Gamma_0^2\Pi^2\Gamma_0^1 h = \Gamma_0^2 U\Pi^1\Gamma_0^1 h = \Gamma_0^2 Uh$. For $f \in \mathcal{D}(A_0^1)$ obviously $0 = \Gamma_0^1 f = \Gamma_0^2 Uf$ since $Uf \in \mathcal{D}(A_0^2)$. Therefore, $\Gamma_0^1 u = \Gamma_0^2 Uu$ for $u \in \mathcal{D}(A^1)$.

Having obtained pairs $\{A^j, \Gamma_0^j\}$ we can employ Proposition 1.16 and conclude that boundary value problems $\{A^j, \Gamma_0^j, \Gamma_1^j, H, E\}$ corresponding to the Weyl functions $M^j(\cdot)$ are uniquely determined with the exception of operators Γ_1^j. From the representation $\Gamma_1^j = (\Pi^j)^* A^j + \Lambda^j\Gamma_0^j$ we have $\Gamma_1^1 = ((\Pi^2)^* A^2 + \Lambda^1\Gamma_0^2)U = \Gamma_1^2 U + (\Lambda^1 - \Lambda^2)\Gamma_0^2 U = (\Gamma_1^2 + N\Gamma_0^2)U$.

The proof is complete. □

Note that the statements of Theorem regarding operators Γ_0, Γ_1 are only valid if $\mathcal{D}(\Lambda^1) \cap \mathcal{D}(\Lambda^2)$ is dense in E. With this assumption omitted, details of the proof allows us to formulate the following Corollary.

Corollary 3.13. *In the notation of Theorem 3.12 if $m_1(z) = m_2(z)$ for $z \in \mathbb{C}$ in some neighborhood of the origin, then there exists an unitary $U : H \to H$ such that $UA_0^1 = A_0^2 U$ and $U\Pi^1 = \Pi^2$.*

Theorem 3.12 sets forth questions of finding objects that would be sufficient to serve as the "inverse data" in the process of reconstructing the boundary value problem, as well as finding an explicit recipes for such reconstructions. One relevant result regarding the former is given below for the case when operator A_0 has no continuous spectrum. In a sense, it is a version of the Borg-Levinson theorem for the Schrödinger operator (see [24]) generalized to the case of boundary value problems under consideration.

Theorem 3.14. *Let the spectrum of operator A_0 be purely discrete. Denote $\{z_j\}_1^\infty$, $\{f_j\}_1^\infty$ the eigenvalues and the orthonormal basis of corresponding eigenvectors of A_0. If the symmetric operator A_{00} is completely nonselfadjoint, then operators A_0 and Π are determined uniquely up to an unitary equivalence by the collections $\{z_j\}_1^\infty$ and $\{\Gamma_1 f_j\}_1^\infty$.*

Proof. We only need to show that the function $m(z) = M(z) - M(0)$, $z \in \rho(A_0)$ can be recovered from the given collections $\{z_j\}$, $\{\Gamma_1 f_j\}$. The reconstruction of A_0 and Π knowing the function $m(z)$ is ensured by Corollary 3.13. Since $m(\cdot)$ is bounded, it is sufficient to consider the quadratic form $(m(z)e, e)$, $e \in E$ only. We have

$$\begin{aligned}
(m(z)e, e) &= (z\Pi^*(I - zA_0^{-1})^{-1}\Pi e, e) = (z(I - zA_0^{-1})^{-1}\Pi e, \Pi e) \\
&= z\sum_j \langle (I - zA_0^{-1})^{-1}\Pi e, f_j\rangle\langle f_j, \Pi e\rangle = z\sum_j \langle e, \Gamma_1 A_0^{-1}(I - \bar{z}A_0^{-1})^{-1} f_j\rangle\langle \Gamma_1 A_0^{-1} f_j, e\rangle \\
&= z\sum_j \frac{1}{z_j^2}\cdot\frac{\langle e, \Gamma_1 f_j\rangle\langle \Gamma_1 f_j, e\rangle}{1 - zz_j^{-1}} = \sum_j \frac{z}{z_j}\cdot\frac{|(e, \Gamma_1 f_j)|^2}{z_j - z}
\end{aligned}$$

Therefore, the function $m(z)$ is uniquely determined by the sets $\{z_j\}$ and $\{\Gamma_1 f_j\}$. □

3.4 General boundary conditions

Properties of the Weyl function play an important role in the study of spectral problems for restrictions of operator A to the domains described by various boundary conditions. Let B_0, B_1 be two linear operators on the space E such that $\mathcal{D}(B_0) \supset \Gamma_0\mathscr{H}$ and $B_1 \in \mathscr{B}(E)$. The problem of our interest is to find the vector $w \in \mathcal{D}(A)$ satisfying following equations:

$$\begin{cases} (A - zI)w = f \\ (B_0\Gamma_0 + B_1\Gamma_1)w = \varphi \end{cases} \tag{3.12}$$

for given $f \in H$, $\varphi \in E$, and $z \in \mathbb{C}$. Note that we can easily assume that the operators B_0 and B_1 in (3.12) depend on the complex number $z \in \mathbb{C}$. In this case the problem (3.12) is an example of the boundary value problem with the spectral parameter in the boundary condition.

The following theorem establishes a one-to-one correspondence between solutions to the system (3.12) and solutions to a certain "boundary equation" in the space E that involves the Weyl function. Such a reduction of the boundary value problem to the "equation on the boundary" is well known in the literature, see [1, 18, 19] for examples. The theorem can be seen as an abstract variant of this procedure. Note that its value is fully realized only if B_1 is not equal to the null operator.

Theorem 3.15. *Let $w_z^{f,\varphi} \in \mathcal{D}(A)$ be a solution to the problem (3.12) for given $z \in \rho(A_0)$, $f \in H$ and $\varphi \in E$. Then the vector*

$$\Psi_z^{f,\varphi} := \Gamma_0 w_z^{f,\varphi} \tag{3.13}$$

is a solution to the equation

$$(B_0 + B_1M(z))\Psi = \varphi - B_1\Pi^*(I - zA_0^{-1})^{-1}f, \quad \Psi \in \Gamma_0\mathscr{H} \tag{3.14}$$

Conversely, if $\Psi_z^{f,\varphi} \in \Gamma_0\mathscr{H}$ is a solution to (3.14) for given $z \in \rho(A_0)$, $f \in H$ and $\varphi \in E$, then the vector

$$w_z^{f,\varphi} := (A_0 - zI)^{-1}f + \left[I + z(A_0 - zI)^{-1}\right]\Pi\Psi_z^{f,\varphi} \tag{3.15}$$

is a solution to the problem (3.12).

Proof. Let $w_z^{f,\varphi} \in \mathcal{D}(A)$ solves the problem (3.12). Then

$$\begin{aligned}
&\varphi - B_1\Pi^*(I - zA_0^{-1})^{-1}f \\
&= (B_0\Gamma_0 + B_1\Gamma_1)w_z^{f,\varphi} - B_1\Gamma_1A_0^{-1}(I - zA_0^{-1})^{-1}f \\
&= B_0\Gamma_0w_z^{f,\varphi} + B_1\Gamma_1\left[w_z^{f,\varphi} - (A_0 - zI)^{-1}f\right] \\
&= B_0\Gamma_0w_z^{f,\varphi} + B_1\Gamma_1\left[I - (A_0 - zI)^{-1}(A - zI)\right]w_z^{f,\varphi} \\
&= B_0\Gamma_0w_z^{f,\varphi} + B_1M(z)\Gamma_0\left[I - (A_0 - zI)^{-1}(A - zI)\right]w_z^{f,\varphi} \\
&= \left[B_0 + B_1M(z)\right]\Gamma_0w_z^{f,\varphi}
\end{aligned}$$

where we made use of the inclusion $[I - (A_0 - zI)^{-1}(A - zI)]w_z^{f,\varphi} \in \ker(A - zI)$ and property (3.10) of Weyl function. We see that (3.13) is a solution to the problem (3.14). The required inclusion $\Gamma_0w_z^{f,\varphi} \in \Gamma_0\mathscr{H}$ follows from the decomposition

$$w_z^{f,\varphi} = (A_0 - zI)^{-1}f + h_z = (A_0 - zI)^{-1}f + (I - zA_0^{-1})^{-1}h,$$

where $h_z \in \ker(A - zI)$ and $h := (I - zA_0^{-1})h_z \in \mathscr{H}$, so that $\Gamma_0w_z^{f,\varphi} = \Gamma_0h_z = \Gamma_0h \in \Gamma_0\mathscr{H}$ according to Proposition 1.7.

To prove the inverse, let $\Psi_z^{f,\varphi} \in \Gamma_0\mathscr{H}$ be a solution to the equation (3.14). Then since the second summand in (3.15) belongs to $\ker(A - zI)$, the vector $w_z^{f,\varphi}$ defined by (3.15) lies in the domain of the operator A and obviously, $(A - zI)w_z^{f,\varphi} = f$. Let us verify that $w_z^{f,\varphi}$ is also a solution to the second equation (3.12). We have

$$
\begin{aligned}
&(B_0\Gamma_0 + B_1\Gamma_1) w_z^{f,\varphi} \\
&= (B_0\Gamma_0 + B_1\Gamma_1) \left[(A_0 - zI)^{-1} f + \left(I + z(A_0 - zI)^{-1} \right) \Pi \Psi_z^{f,\varphi} \right] \\
&= B_0\Gamma_0 \Pi \Psi_z^{f,\varphi} + B_1\Gamma_1 (A_0 - zI)^{-1} f + B_1\Gamma_1 (I - zA_0^{-1})^{-1} \Pi \Psi_z^{f,\varphi} \\
&= [B_0 + B_1 M(z)]\, \Psi_z^{f,\varphi} + B_1\Gamma_1 (A_0 - zI)^{-1} f \\
&= \varphi - B_1 \Pi^* (I - zA_0^{-1})^{-1} f + B_1\Gamma_1 (A_0 - zI)^{-1} f \\
&= \varphi - B_1\Gamma_1 \left[A_0^{-1} (I - zA_0^{-1})^{-1} - (A_0 - zI)^{-1} \right] f
\end{aligned}
$$

where we used representation (3.9) for the Weyl function, equality $\Gamma_0 \Pi \Psi_z^{f\varphi} = \Psi_z^{f\varphi}$ and assumption (3.14). Now the required result $(B_0\Gamma_0 + B_1\Gamma_1) w_z^{f,\varphi} = \varphi$ follows from the identity $A_0^{-1}(I - zA_0^{-1})^{-1} = (A_0 - zI)^{-1}$.

The proof is complete. $\square$

Remark 3.16. A simple condition that guarantees solvability of (3.14) for any $f \in H$, $\varphi \in E$ is the closedness of the operator $(B_0 + B_1 M(z))$ on its domain $\Gamma_0 \mathscr{H}$ and its invertibility, so that the range of its inverse $[B_0 + B_1 M(z)]^{-1}$ coincides with $\Gamma_0 \mathscr{H}$. As easily seen from (3.15), in this case the norm estimate $\|w_z^{f,\varphi}\| \le C(z)(\|f\| + \|\varphi\|)$ holds with some positive constant $C(z) < \infty$ depending on $z \in \rho(A_0)$.

Remark 3.17. Comparison of (3.15) and (3.2) shows that the solution $w_z^{f,\varphi}$ can be obtained by solving the problem (3.1) with $\varphi \in E$ replaced with the solution $\Psi_z^{f,\varphi}$ to (3.14) provided that the latter exists. In case if (3.14) is solvable and $\Psi_z^{f,\varphi} \in E$, but not necessarily $\Psi_z^{f,\varphi} \in \Gamma_0 \mathscr{H}$ as required in conditions of Theorem 3.15, solution $w_z^{f,\varphi}$ still can be defined as a weak solution $u_z^{f,\Psi}$ where $\Psi := \Psi_z^{f,\varphi}$ to the problem (3.1) described in Remark 3.4.

4 An example

This section illustrates the formalism developed above by means of an example. The setting can be extended to more general cases of operators and domains, most notably to the general strongly elliptic partial differential operators and domains of lesser smoothness. Such generalizations would be based upon the potential theory developed to the degree that embraces more generic domains and operators. Relevant results are partially available in the literature, see [12, 19]. However, we keep the exposition simple in hope to dedicate a separate publication to more advanced situations.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with the C^∞-boundary Γ, and $H := L_2(\Omega)$, $E := L_2(\Gamma)$. Operator A is the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ on domain $\mathcal{D}(A) = H^2(\Omega)$ where $H^2(\Omega)$ is the Sobolev space of functions $u \in L_2(\Omega)$ such that $\partial^2 u / \partial x_i^2 \in L_2(\Omega)$, $i = 1, 2, 3$. Denote $\partial_\nu := \frac{\partial}{\partial \nu}$ the derivative in the direction of the outer normal ν to the boundary Γ. For any function $u \in \mathcal{D}(A)$ there exist its trace $u|_\Gamma$ and the trace of its derivative $(\partial_\nu u)|_\Gamma$ on Γ. We define the mappings Γ_0 and Γ_1 on the domain $\mathcal{D}(A) = H^2(\Omega)$ as follows

$$
\Gamma_0 \,:\, u \mapsto u|_\Gamma\,, \qquad \Gamma_1 \,:\, u \mapsto (\partial_\nu u)|_\Gamma\,, \qquad u \in \mathcal{D}(A)
$$

It is known that Γ_0 and Γ_1 are bounded operators acting from $H^2(\Omega)$ into $L_2(\Gamma)$. More precisely, the following equalities for their ranges hold: $\mathcal{R}(\Gamma_0) = H^{3/2}(\Gamma)$ and $\mathcal{R}(\Gamma_1) = H^{1/2}(\Gamma)$, where $H^{3/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ are fractional Sobolev spaces on the surface Γ, see [19, 20]. For any vectors $u, v \in \mathcal{D}(A)$ the Green formula holds:

$$
(Au, v)_H - (u, Av)_H = (\Gamma_1 u, \Gamma_0 v)_E - (\Gamma_0 u, \Gamma_1 v)_E, \qquad u, v \in \mathcal{D}(A)
$$

The operator A_0 defined as a restriction of $A = \Delta$ to the domain $\mathcal{D}(A_0) := \{u \in H^2(\Omega) \mid \Gamma_0 u = 0\}$ is the operator of Dirichlet boundary value problem in the space $L_2(\Omega)$. It is a closed boundedly invertible selfadjoint operator with discrete spectrum. (See [7] for the proof of its closedness.)

The null set $\mathscr{H}$ of $A = \Delta$ consists of all harmonic functions h in Ω, $\Delta h = 0$ that belong to the space $H^2(\Omega)$. Note that $\Gamma_0\mathscr{H} \subset H^{3/2}(\Gamma)$ and $\Gamma_1\mathscr{H} \subset H^{1/2}(\Gamma)$, since $\mathscr{H} \subset H^2(\Omega)$. The closure of $\mathscr{H}$ in $L_2(\Omega)$ is wider that the Sobolev class $H^2(\Omega)$ and includes square integrable functions on Ω that don't possess boundary values on Γ and don't belong to the domain $\mathcal{D}(A)$.

The intersection $\mathcal{D}(A_0)\cap\mathscr{H}$ is trivial because any harmonic function from $\mathscr{H}$ with the zero boundary values on Γ is equal to zero almost everywhere in Ω. The operator Π is the left inverse to the mapping $h \mapsto h|_\Gamma$, $h \in \mathscr{H}$. It is the operator of harmonic continuation from the boundary Γ that maps a function $\varphi \in H^{3/2}(\Gamma)$ to the harmonic function h from $\mathscr{H}$ such that $h|_\Gamma = \varphi$. Noticing that, the decomposition $\mathcal{D}(A) = \mathcal{D}(A_0)\dot{+}\mathscr{H}$ becomes almost obvious. Indeed, for any $u \in H^2(\Omega)$ there exists a unique harmonic function $h = \Pi(u|_\Gamma)$ with the same boundary values $h|_\Gamma = u|_\Gamma$. Therefore, the difference $u - h$ belongs to $H^2(\Omega)$ and vanishes at the boundary: $(u-h)|_\Gamma = 0$. Hence, $u - h$ lies in the domain $\mathcal{D}(A_0)$. This decomposition of the space $H^2(\Omega)$ and the equality $\Gamma_0 H^2(\Omega) = H^{3/2}(\Gamma)$ given above show that $\Gamma_0\mathscr{H} = H^{3/2}(\Gamma)$.

Consequently, Assumptions 1 and 2 hold true for the collection $\{\Delta, u \mapsto u|_\Gamma, u \mapsto \partial_\nu u|_\Gamma, L_2(\Omega), L_2(\Gamma)\}$ and we can apply the theory developed in preceding sections to the semi-homogeneous spectral boundary value problem

$$\begin{cases} (\Delta - z)u = 0 \quad \text{in } \Omega, \\ u|_\Gamma = \varphi \end{cases} \tag{4.1}$$

where $\varphi \in H^{3/2}(\Gamma)$.

We start with the following representation of any vector $\mathfrak{u}_z \in H^2(\Omega)$ satisfying the equation $(\Delta - z)u = 0$, $z \in \mathbb{C}$ in Ω

$$\mathfrak{u}_z(x) = \int_\Gamma \left[G_z(x,y) \cdot \frac{\partial \mathfrak{u}_z}{\partial \nu} - \frac{\partial G_z(x,y)}{\partial \nu_y} \cdot \mathfrak{u}_z(y) \right] dS_y, \qquad x \in \Omega$$

where

$$G_z(x,y) := \frac{1}{4\pi} \frac{\exp(ik|x-y|)}{|x-y|}, \qquad x, y \in \mathbb{R}^3, \quad k := \sqrt{-z}, \quad \operatorname{Im} k \geq 0$$

is the fundamental solution to the Helmholz equation $(\Delta - z)u = 0$ in $\mathbb{R}^3$ and dS_y is the euclidian surface measure on Γ. This formula is well known for functions $\mathfrak{u}_z$ continuous in the closure of the domain Ω (see [9]). For the general case of $\mathfrak{u}_z \in H^1(\Omega)$ the relevant result can be found in [19]. We rewrite this representation in the form

$$\mathfrak{u}_z = \mathscr{S}_z(\partial_\nu \mathfrak{u}_z|_\Gamma) - \mathscr{D}_z(\mathfrak{u}_z|_\Gamma) = \mathscr{S}_z\Gamma_1\mathfrak{u}_z - \mathscr{D}_z\Gamma_0\mathfrak{u}_z \tag{4.2}$$

where operators $\mathscr{S}_z$ and $\mathscr{D}_z$ are the usual single and double layer potentials. For functions $w \in L_2(\Gamma)$ they are defined as mappings

$$(\mathscr{S}_z w)(x) := \int_\Gamma G_z(x,y) w(y) dS_y, \qquad (\mathscr{D}_z w)(x) := \int_\Gamma \frac{\partial G_z(x,y)}{\partial \nu_y} w(y) dS_y, \qquad x \in \Omega$$

Operators $\mathscr{S}_z$ and $\mathscr{D}_z$ acting from the space $L_2(\Gamma)$ into $L_2(\Omega)$ are compact for $\operatorname{Im} z \neq 0$, see [19]. Boundary values of (4.2) on the surface Γ can be calculated using well known formulae of potential theory. Namely, for any $w \in C^2(\overline{\Omega})$ the following equalities hold [9, 23]:

$$\begin{aligned} \Gamma_0\mathscr{S}_z w &= \mathrm{S}_z w, & \Gamma_0\mathscr{D}_z w &= -\frac{1}{2}(w - \mathrm{K}_z w), \\ \Gamma_1\mathscr{S}_z w &= \frac{1}{2}(w + \mathrm{K}'_z w), & \Gamma_1\mathscr{D}_z w &= \mathrm{R}_z w \end{aligned} \tag{4.3}$$

where for $x \in \partial\Omega$

$$(\mathrm{S}_z w)(x) := \int_\Gamma G_z(x,y) w(y) dS_y, \qquad (\mathrm{K}_z w)(x) := 2\int_\Gamma \frac{\partial G_z(x,y)}{\partial \nu_y} w(y) dS_y$$

are direct values of the single and double layer potentials on the surface Γ, and

$$(\mathrm{K}'_z w)(x) := 2\int_\Gamma \frac{\partial G_z(x,y)}{\partial \nu_x} w(y) dS_y, \qquad (\mathrm{R}_z w)(x) := \frac{\partial}{\partial \nu_x}\int_\Gamma \frac{\partial G_z(x,y)}{\partial \nu_y} w(y) dS_y$$

Operators K'_z and R_z are sometimes called the adjoint double layer potential and the hypersingular operator, respectively. It turns out that under our assumptions about the boundary Γ the mappings S_z, K_z, K'_z are compact operators on the space $L_2(\Gamma)$ ([13, 23]). Operator R_z is unbounded on $L_2(\Gamma)$. It can be shown, however, that $\mathrm{R}_z : H^1(\Gamma) \to L_2(\Gamma)$, see [19]. Consequently, values $\Gamma_0 \mathfrak{u}_z$ and $\Gamma_1 \mathfrak{u}_z$ for the vector $\mathfrak{u}_z$ from (4.2) belong to the domain of operators S_z, K_z, K'_z, and in addition, $\Gamma_0 \mathfrak{u}_z$ as an element of $H^{3/2}(\Gamma) \subset H^1(\Gamma)$, lies in the domain of R_z. Thus, we have

$$\begin{aligned} \Gamma_0 \mathfrak{u}_z &= \Gamma_0(\mathscr{S}_z \Gamma_1 - \mathscr{D}_z \Gamma_0)\mathfrak{u}_z = \mathrm{S}_z \Gamma_1 \mathfrak{u}_z + \frac{1}{2}(I - \mathrm{K}_z)\Gamma_0 \mathfrak{u}_z \\ \Gamma_1 \mathfrak{u}_z &= \Gamma_1(\mathscr{S}_z \Gamma_1 - \mathscr{D}_z \Gamma_0)\mathfrak{u}_z = \frac{1}{2}(I + \mathrm{K}'_z)\Gamma_1 \mathfrak{u}_z - \mathrm{R}_z \Gamma_0 \mathfrak{u}_z \end{aligned}$$

An elementary rearranging of summands yields:

$$(I + \mathrm{K}_z)\Gamma_0 \mathfrak{u}_z = 2\mathrm{S}_z \Gamma_1 \mathfrak{u}_z, \qquad (I - \mathrm{K}'_z)\Gamma_1 \mathfrak{u}_z = -2\mathrm{R}_z \Gamma_0 \mathfrak{u}_z$$

Recall now that operators K_z and K'_z are in fact analytic operator-functions of the complex parameter z. Since K_z and K'_z are compact, the operators $I + \mathrm{K}_z$ and $I - \mathrm{K}'_z$ are boundedly invertible for all $z \in \mathbb{C}$ except for a countable set of points accumulating to the infinity, provided they are invertible at least for one value of $z \in \mathbb{C}$. (See [6] for example.). More accurate result from [9] says that poles of the operator-function $(I + \mathrm{K}_z)^{-1}$ coincide with the eigenvalues of the Neumann boundary problem in $L_2(\Omega)$. This problem is defined as $(\Delta - z)u = 0$, $\partial_\nu u|_\Gamma = 0$. Analogously, poles of $(I - \mathrm{K}'_z)^{-1}$ are the eigenvalues of the Dirichlet boundary problem $(\Delta - z)u = 0$, $u|_\Gamma = 0$ in $L_2(\Omega)$, that is, the spectrum of operator A_0. Since all these eigenvalues are real, it follows that the Weyl function $M(\cdot)$ of the boundary value problem (4.1) and its inverse $M^{-1}(\cdot)$ are

$$M(z) = -2(I - \mathrm{K}'_z)^{-1}\mathrm{R}_z, \qquad M^{-1}(z) = 2(I + \mathrm{K}_z)^{-1}\mathrm{S}_z, \qquad z \in \mathbb{C}_\pm \tag{4.4}$$

Function $M(\cdot)$ is defined correctly on the domain $H^1(\Gamma)$, whereas values of its inverse $M^{-1}(\cdot)$ are compact operators on $L_2(\Gamma)$.

According to the theory above, the solution to problem (4.1) is the vector $(I - zA_0^{-1})^{-1}\Pi\varphi$, see formula (3.5). Now the representation (4.3) for $\Gamma_0 \mathscr{D}_z$ gives rise to the following Proposition.

Proposition 4.1. *Solutions $u_z^\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi$ of the problem (4.1) are given by the formula*

$$u_z^\varphi = -2\mathscr{D}_z(I - \mathrm{K}_z)^{-1}\varphi, \qquad z \in \rho(A_0) \tag{4.5}$$

where $\varphi \in H^{3/2}(\Gamma)$.

Proof. The fact that (4.5) is a solution to the problem (4.1) is the well known result of the potential theory. Taking into consideration (4.3) we need to show that $(I - \mathrm{K}_z)^{-1}\varphi$, $z \in \rho(A_0)$ is a vector from $H^{3/2}(\Gamma)$ for $\varphi \in H^{3/2}(\Gamma)$. This follows directly from the fact that K_z is a pseudodifferential operator of order not greater than -1 (see [1]). Hence, $(I - \mathrm{K}_z)^{-1}$ is an invertible pseudodifferential operator or order zero, so that $(I - \mathrm{K}_z)^{-1} H^{3/2}(\Gamma) = H^{3/2}(\Gamma)$. Finally, uniqueness of representation (4.5) for vectors $u_z^\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi$ is a consequence of Proposition 1.7 which guarantees that if two vectors u_1, u_2 from $\ker(A - zI)$ have the same boundary values $\Gamma_0 u_1 = \Gamma_0 u_2$ then necessarily $u_1 = u_2$.

The proof is complete. □

Remark 4.2. According to results above, the set of all solutions (4.5) coincides with $\ker(A - zI)$ and the mapping $\varphi \mapsto u_z^\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi$ establishes a one-to one correspondence between $\varphi \in H^{3/2}(\Gamma)$ and the respective solution u_z^φ. This result combined with formula (4.5) and properties of $(I - \mathrm{K}_z)^{-1}$ cited in the proof of Proposition 4.1 implies that the double layer potential $\mathscr{D}_z$ restricted to the set $H^{3/2}(\Gamma)$ is an invertible operator. More precisely, its range coincides with $\ker(A - zI)$ and the inverse mapping is given by the formula

$$(\mathscr{D}_z)^{-1}\mathfrak{u}_z = -2(I - \mathrm{K}_z)^{-1}\Gamma_0 \mathfrak{u}_z, \qquad \mathfrak{u}_z \in \ker(A - zI).$$

Remark 4.3. From the representation (4.5) we obtain for $\varphi = \Gamma_0 u_z^\varphi$

$$\Gamma_1 u_z^\varphi = -2\Gamma_1 \mathscr{D}_z (I - \mathrm{K}_z)^{-1}\varphi = -2\mathrm{R}_z (I - \mathrm{K}_z)^{-1}\varphi$$

since $(I - \mathrm{K}_z)^{-1}\varphi \in H^{3/2}(\Gamma)$ for $\varphi \in H^{3/2}(\Gamma)$. It means that the Weyl function $M(\cdot)$ can be rewritten in the form

$$M(z) = -2\mathrm{R}_z (I - \mathrm{K}_z)^{-1}, \qquad z \in \rho(A_0)$$

Remark 4.4. Operator $M(0) = -2\mathrm{R}_0(I - \mathrm{K}_0)^{-1} = -2(I - \mathrm{K}_0')^{-1}\mathrm{R}_0$ is the so-called Dirichlet-to-Neumann map for the Laplace equation $\Delta u = 0$. It maps a smooth function φ on the boundary Γ to the trace of the normal derivative on Γ of the solution u_0^φ to the Laplace equation $\Delta u = 0$ with the boundary condition $u|_\Gamma = \varphi$. In other words, $M(0)$ maps the boundary values of the Dirichlet boundary problem for the Laplacian into those of the Neumann problem, $M(0) : \varphi \mapsto \partial_\nu u_0^\varphi|_\Gamma$. Representation $M(0) = \Gamma_1 \Pi$ established above for the general case now becomes self-evident. Properties of the Dirichlet-to-Neumann map of various partial differential operators were investigated in a number of papers, starting from [27]. (See as well the survey [25].) In the case under consideration $M(0)$ is a positive selfadjoint pseudodifferential operator of order 1 (see [1, 27]). Further, since the Weyl function $M(z)$, $z \in \rho(A_0)$ maps φ to $u_z^\varphi|_\Gamma$ where u_z^φ is the solution to the equation $(\Delta - z)u = 0$, it can be said that the operator $M(z)$ is the Dirichlet-to-Neumann map for the equation $(\Delta - z)u = 0$, cf. [25].

Remark 4.5. Invertibility of the double layer potential $\mathscr{D}_z$, $z \in \rho(A_0)$ stated in Remark 4.2 shows that $\mathscr{D}_z$ maps $H^{3/2}(\Gamma)$ into a subset of $H^2(\Omega)$. Applying interpolation arguments [20] we conclude that $\mathscr{D}_z$ is the operator from $L^2(\Gamma)$ to the space $H^{1/2}(\Omega)$, therefore $\mathscr{D}_z$ is compact as a mapping from $E = L_2(\Gamma)$ to $H = L_2(\Omega)$. Thus, according to Proposition 4.1, the transformation $\varphi \mapsto (I - zA_0^{-1})^{-1}\Pi\varphi$, $\varphi \in E$ is compact. Furthermore, since $M(z) - M(\zeta) = (z-\zeta)\Pi^*(I - \zeta A_0^{-1})^{-1}(I - zA_0^{-1})^{-1}\Pi$ for $z, \zeta \in \rho(A_0)$, see (3.11), the difference $M(z) - M(\zeta)$ is a compact operator on E. In particular, it holds for $M(z) - M(0)$, thereby $M(z)$, $z \in \rho(A_0)$ is a compact perturbation of the Dirichlet-to-Neumann map $\Lambda = M(0)$ of the Laplacian.

Remark 4.6. Theorem 3.12 shows that if one knows the function $M(z)$, then one can determine the operator A and boundary maps Γ_0, Γ_1 up to isomorphism. We know from the case of the Sturm-Liouville operator in one dimension that this is the best possible result in the abstract setting. However for the multidimensional case of differential operators one can recover its coefficients knowing just $M(z_0)$ for a single point $z_0 \in \rho(A_0)$, see e. g. [24, 25]. In fact even weaker results are now known such as recovery of the coefficients knowing the function $M(z)$ on just a part of a smooth boundary, see [26] for the survey.

References

[1] M. S. Agranovich, *Elliptic Boundary Problems*, in Encyclopaedia of Mathematical Sciences, Vol. 79, Partial Differential Equations IX, Springer-Verlag, Berlin Heidelberg New York, 1997.

[2] A. Alonso, B. Simon, *The Birman-Krein-Vishik theory of selfadjoint extensions of semibounded operators*, J. Oper. Theory, 4 (1980), 251-270.

[3] W. O. Amrein, D. B. Pearson, *M-operators: a generalisation of Weyl-Titchmarsh theory*, J. Comput. Appl. Math., 171 (2004) 1-26.

[4] S. Bergman, M. Schiffer, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, Academic Press, New York, N.Y., 1953.

[5] M. S. Birman, M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht, Holland, 1987.

[6] M. S. Brodskiĭ, *Triangular and Jordan Representations of linear operators*, American Mathematical Society, Providence, Rhode Island, 1971.

[7] F. E. Browder, *On the spectral theory of elliptic diferential operators. I*, Math. Annalen, 142 (1961), 22-130.

[8] V.M. Bruk, *A certain class of boundary value problems with a spectral parameter in the boundary condition*, (Russian) Mat. Sb. (N.S.) 100 (142) (1976), 210-216.

[9] D. Colton, R. Kress, *Integral equation methods in scattering theory*, John Wiley & Sons, New York, 1983.

[10] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. 95 (1991), 1-95.

[11] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. 73, 2 (1995), 141-242.

[12] R. Duduchava, *The Green formula and layer potentials*, Int. Eq. Oper. Theory. 41 (2001), 127-178.

[13] E. B. Fabes, M. Jodeit Jr., and N. M. Rivière, *Potential Techniques for boundary value problems on C^1-domains*, Acta Math., 141 (1978), 165-186.

[14] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator differential equations*, Kluwer Academic Publishers Group, Dordrecht, 1991.

[15] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa, 22 (1968) 425-513. (From Stanford University dissertation 1966).

[16] G. Grubb, *Known and unknown results on elliptic boundary problems*, Bull. Amer. Math. Soc., 43 (2006), 227-230.

[17] A.N.Kočubeĭ, *On extension of symmetric operators and symmetric binary relations*, (Russian) Mat. Zametki, 17 (1975), 41-48.

[18] N. D. Kopachevskii, S. G. Krein, *Abstract Green Formula for a Triple of Hilbert Spaces, Abstract Boundary-Value and Spectral Problems*, Ukr. Math. Bull. 1 (2004), 77-105.

[19] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[20] J. L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, Berlin Heidelberg New York, 1973.

[21] M. S. Livŝiç, *Operators, Oscillations, Waves (Open Systems)*, American Mathematical Society, Providence, Rhode Island, 1973.

[22] V. E. Lyantse, O. G. Storozh, *Methods of the theory of unbounded operators*, Naukova Dumka, Kiev, 1983.

[23] V. G. Maz'ya, *Boundary Integral Equations*, in Encyclopaedia of Mathematical Sciences, Vol. 27, Analysys IV, Springer-Verlag, Berlin Heidelberg New York, 1991.

[24] A. Nachman, J. Sylvester and G. Uhlmann *An n-Dimensional Borg-Levinson Theorem*, Comm. Math. Phys. 115 (1988), 595-605.

[25] J. Sylvester, G. Uhlmann, *The Dirichlet to Neumann map and applications*, In: Proceedings of the Conference "Inverse problems in partial differential equations (Arcata,1989)", SIAM, Philadelphia, 1990, 101-139.

[26] G. Uhlmann, *Commentary on Alberto P. Calderón's paper: On an inverse boundary value problem*, Selecta (to appear) (A. Bellow, C. E. Kenig, and P. Malliavin, eds.).

[27] M. Vishik, *On general boundary conditions for elliptic differential equations*, AMS Translations, 24 (1963), 107-172.

[28] H. Weyl, *The method of orthogonal projection in potential theory* , Duke Math. J., 7 (1940), 411-444.

Vladimir Ryzhov
67-6400 Spencer Road
Kelowna, BC
V1X 7T6
Canada
vryzhov@bridges.com