Geometric modification of quaternionic quantum mechanics

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Abstract

We present an axiomatic modification of quaternionic quantum mechanics with a *possible worlds* semantics capable of *predicting* essential "nonquantum" features of an observable universe model - the *topology* and *dimensionality* of spacetime, the *existence*, the *signature* and a *specific form* of a metric on it, and certain naturally distinguished directions (*vistas*) in spacetime unrelated to its metric properties.

1 Introduction

The technical purpose of the paper is to provide a formal definition of the notion of an *observer* and related constructs in order to make the description of quantum systems more compatible with the kinematic structure of general relativity (GR). We consider the following kinematic axioms of GR:

Axiom 1. Spacetime of the universe is a smooth manifold.

Axiom 2. The dimensionality of spacetime is four.

Axiom 3. Spacetime is equipped with a lorentzian metric.

The fact that these *still are* axioms is somewhat troublesome, at least to the author, who would rather see them as corrolaries of a single assertion, which is often a good motivation for changes in a theory.

We shall consider changes to the kinematic, dynamic and semantic structure of quaternionic quantum mechanics (QQM, [2]) that deal with notions pertaining to the above axioms and the following assertion:

Assertion 1. The logic of the observer is bivalent boolean.

We shall show that (technically accurate versions of) the statements of the axioms are derivable from (a technically accurate version of) this assertion. We start with three simple Observations.

Observation 1. In QQM the quaternionic hilbert space $\mathbb{V}$ contains a natural principal bundle, $(\mathcal{V}^\circ)$, with the following components:

1. The total space is the set $\mathcal{V}^\circ := \mathbb{V}\backslash\{\mathbf{0}\}$ of nonzero vectors of $\mathbb{V}$, with the natural manifold structure canonically generated by the linear structure of $\mathbb{V}$.

2. The base space is a quaternionic projective space $\mathcal{PV}^\circ$ whose points are quaternionic rays in $\mathbb{V}$.

3. The standard fiber is the set $\mathcal{H}^\circ = \mathbb{H}\backslash\{\mathbf{0}\}$ of nonzero quaternions which is a four dimensional real manifold and a lie group $\mathcal{H}^\circ \cong SU(2) \times \mathbb{R}^+$.

4. The structure group is also $\mathcal{H}^\circ$. It acts on $\mathcal{V}^\circ$ from the left, and for each $\psi \in \mathcal{V}^\circ$ its orbit (the ray through ψ) is a copy of $\mathcal{H}^\circ$ via the fiber diffeomorphism.

The recent discovery [16] of natural relativistic structure on $\mathcal{H}^\circ$ turns each fiber of $(\mathcal{V}^\circ)$ into a lorentzian manifold. We shall refer to this bundle as the *hyperquantum bundle over* $\mathbb{V}$. This principal bundle is a generalization of a principal bundle $(\mathcal{V}^\circ_{\mathbb{C}})$ used in geometric quantum mechanics (GQM), the *quantum bundle over* a *complex* hilbet space $\mathbb{V}_{\mathbb{C}}$ (see [14] and references therein). The hyperquantum bundle forms a basis for a modification of kinematics of QQM.

Observation 2. The total space $\mathcal{V}^\circ$ has the structure of a hyperkähler manifold with the riemannian metric and symplectic forms induced by the decomposition of values of the quaternionic hermitian form on $\mathbb{V}$ in the canonical basis $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ of quaternions. Hence vector fields and flows on $\mathcal{V}^\circ$ are subject to the hyperhamiltonian formalism [8], roughly a superposition of three hamiltonian evolutions on $\mathcal{V}^\circ$, which forms a basis for the modification of dynamics of QQM.

Observation 3. The observer theory sketched in [15] supplies material for a rigorous definition of an *observer* and *perceptible* analogues of standard physical constructs such as *time*, *spacetime* and a *dynamical system*, and study the dependence of their properties on the type of logic of the observer. The *perceptible spacetime* acquires a group structure, and a *perceptible dynamical system* becomes a set with a left action of a monoid. The main result of [15] asserts that if the logic of an observer is bivalent boolean then the perceptible spacetime is isomorphic to the lie group of nonzero quaternions, $\mathcal{H}^\circ$, and a perceptible physical system is a set with a left action of $\mathcal{H}^\circ$. A modest category-theoretic generalization of this schema provides a semantic modification of QQM.

In this paper we restate and combine these observations in a coherent way. It should be stressed that the proposed modification is not a complete theory because the treatment of certain important aspects such as quantum measurement and probability is too sketchy, the connections with dynamics of GR are not discussed, and we give no examples. Our primary goal is *complete unambiguity*, i. e., axiomatic consistency rather than completeness. The paper is organized as follows.

Section 2 - Basic constructs and notations. The technical material is given very selectively, the purpose being an introduction of notational conventions rather than education. For example, the reader is assumed to have some familiarity with birkhoff categories and hyperkähler geometry. $\mathbb{N}$ denotes natural numbers and zero, $\mathbb{C}$ an $\mathbb{R}$ are the fields of complex numbers and real numbers, respectively, and $\mathbb{R}$ is assumed taken with its standard linear order and euclidean topology. Small Greek indices, α, β, γ and small Latin indices p, q *always* run 0 to 3 and 1 to 3, respectively. Summation is assumed on repeated indices of different levels unless explicitly indicated otherwise. We use the $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ device to denote tensor ranks. For example, a one-form is a $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$-tensor. Given a smooth manifold $\mathcal{M}$, the set of $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$-tensor fields on $\mathcal{M}$ is denoted $\mathcal{M}\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$.

Section 3 - An unconventional look at linear algebras over a field $\mathbb{F}$ ($\mathbb{F}$-*algebras*) introduced categorically in *Section* 2. $\mathbb{F}$-algebras are defined as vector spaces over $\mathbb{F}$ equipped with a rank three tensor (*structure tensor*). We introduce the notion of a *principal inner product*, and for $\mathbb{F} = \mathbb{R}$ a *principal metric* on the set of invertible elements of an $\mathbb{R}$-algebra.

Section 4 - We compute the principal inner products and principal metrics of the *quaternion algebra*, which turn out to be minkowski and closed FLRW,

respectively.
Section 5 - A slightly nonstandard description of hyperkäher manifolds.
Section 6 - Using the techniques of *Section* 5, we outline basic notions associated with quaternionic hilbert spaces and quaternion linear maps.
Section 7 - The modification of semantics of QQM. We introduce the notions of an *experient* and a *reality.* The reader is warned that we shall neither discuss the philosophical issues involved, nor use the terminology and notation that usually accompany them (see for example [3], [5]). Ours is a simpler and more pragmatic task – to *formally* redefine certain technical constructs of physics in terms of *elementary experiences* of an observer, considered as *primitive entities.*
Section 8 - We study a particular species of experients called $\mathbb{F}$-*observers* and the associated notions of a *temporal reality* and a *phenomenon.*
Section 9 - A specialization of some of the above notions, namely an *observer* and a *robust reality.*
Section 10 - We define the notion of a *dynamical system*, its *evolution* and its *perceptibles.* The important notion of *propensity* is defined.
Section 11 - A special case of a robust reality, a *cosmology.* This Section contains one of the central results of the paper, namely the essential uniqueness of the cosmology of the observer. We compute several characteristics of the cosmology such as topology and dimensionality of its spacetime, as well as properties of its metric and naturally distinguished directions in the spacetime.
Section 12 - We define *physical systems*, *observables* and their *measurements.* It is shown that standard quantum systems of complex quantum mechanics correspond to a *degenerate*, in a strictly defined sense, kind of physical systems, and the notions of a measurement and propensity seem to reflect certain aspects of quantum measurement and probability, respectively, although they are not equivalent.
Section 13 - We conclude the paper with an informal summary of the results.

2 Technical preliminaries

Definition 2.1. A *signature*, Σ, is an ordered pair (S, s), where S is a set of *elementary symbols* and $\mathsf{s} : \mathsf{S} \to \mathbb{N}$ is the *arity map*, assigning to each elementary symbol $s \in \mathsf{S}$ a natural number $\mathsf{s}(s)$, called the *arity* of s.

Definition 2.2. For a category $\mathcal{E}$ with products and coproducts and a signature $\Sigma = (\mathsf{S}, \mathsf{s})$, an endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ is called a *Σ-functor* (*on* $\mathcal{E}$) if for each $\mathcal{E}$-object A,

$$\Gamma(\mathsf{A}) = \coprod_{s \in \mathsf{S}} \mathsf{A}^{\mathsf{s}(s)}, \tag{1}$$

where $\coprod$ denotes coproduct of $\mathcal{E}$-objects, and $\mathsf{A}^{\mathsf{s}(s)}$ is a product of $\mathsf{s}(s)$ copies of A.

Example 2.1. Given a signature $\Sigma = (\mathsf{S}, \mathsf{s})$, an example of a Σ-functor on the category of sets, $\boldsymbol{Set}$, is a functor that sends a set A to the set $\mathsf{S}_0 + (\mathsf{S}_1 \times \mathsf{A}) + (\mathsf{S}_2 \times \mathsf{A} \times \mathsf{A}) + \cdots$, where $\mathsf{S}_n \subseteq \mathsf{S}$ is the set of elementary symbols of arity n; $+$ and $\times$ denote disjoint union (coproduct) and cartesian product of sets, respectively. Regarding cartesian products, our conventions are as follows. A cartesian product $\prod_{i \in I} \mathsf{A}_i$ is empty iff there is an empty set among A_i. For a nonempty set A, its *zeroth* cartesian power, A^0, is a singleton whose element is called an *empty tuple*.

Definition 2.3. Given an endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ on a category $\mathcal{E}$, an *algebra*, $\mathbb{A}$, *for* Γ is an ordered pair (A, a), where A is an $\mathcal{E}$-object, called the *carrier*, and $\mathsf{a} : \Gamma(\mathsf{A}) \to \mathsf{A}$ is an $\mathcal{E}$-arrow, called the *structure map* of the algebra. Let $\mathbb{A} = (\mathsf{A}, \mathsf{a})$ and $\mathbb{B} = (\mathsf{B}, \mathsf{b})$ be algebras for Γ. A *Γ-morphism*, $\mathbb{A} \to \mathbb{B}$, is a map $f : \mathsf{A} \to \mathsf{B}$, such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(\mathsf{A}) & \xrightarrow{\Gamma(f)} & \Gamma(\mathsf{B}) \\ {\scriptstyle \mathsf{a}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{b}} \\ \mathsf{A} & \xrightarrow{f} & \mathsf{B} \end{array} . \tag{2}$$

Example 2.2. Let $\mathbb{F}$ be a field, and $\mathsf{S} = \{\mathbf{0}, +\} \cup \mathbb{F}$ and $\mathsf{s}(\mathbf{0}) = 0$, $\mathsf{s}(+) = 2$, $\mathsf{s}(r) = 1, \forall r \in \mathbb{F}$. A *vector space*, $V = (\mathsf{V}, \mathsf{v})$, *over a field* $\mathbb{F}$ is an algebra for the Σ-functor $\Gamma : \boldsymbol{Set} \to \boldsymbol{Set}$ on the category of sets for the signature $\Sigma = (\mathsf{S}, \mathsf{s})$.

Remark 2.1. Dually, *coalgebras for an endofunctor* $\Gamma : \mathcal{E} \to \mathcal{E}$ are defined by reversal of structure maps. Algebras and coalgebras for an endofunctor Γ and their Γ-morphisms form categories denoted $\mathcal{E}^\Gamma$ and $\mathcal{E}_\Gamma$, respectively, (see, e. g., [10]).

Example 2.3. A monoid is an example of an algebra, $M = (\mathsf{M}, \mathsf{m})$, for a Σ-functor on $\boldsymbol{Set}$ with $\mathsf{S} = \{\imath, *\}, \mathsf{s}(\imath) = 0, \mathsf{s}(*) = 2$. As with every Σ-functor, the structure map m can be split into the constituents, giving the more familiar notation $(\mathsf{M}, \imath, *)$, where $\imath$ is understood as a preferred element, the identity of M, and $*$ is a binary operation on M.

Definition 2.4. An endofunctor Γ on $\mathcal{E}$ is called a *monad* if there exist two natural transformations, $\flat : id(\mathcal{E}) \to \Gamma$ and $\natural : \Gamma^2 \to \Gamma$ such that the following diagrams commute:

$$\begin{tikzcd} \Gamma^3 \arrow[r, "\Gamma\natural"] \arrow[d, "\natural\Gamma"'] & \Gamma^2 \arrow[d, "\natural"] \\ \Gamma^2 \arrow[r, "\natural\Gamma"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, "\Gamma\flat"] \arrow[dr, equal] & \Gamma^2 \arrow[d, "\natural"] & \Gamma \arrow[l, "\flat\Gamma"'] \arrow[dl, equal] \\ & \Gamma & \end{tikzcd}, \tag{3}$$

where Γ^n denotes n iterations of the functor.

Definition 2.5. The *category of algebras*, $\bar{\mathcal{E}}^\Gamma$, *for the monad* $\Gamma : \mathcal{E} \to \mathcal{E}$ is a subcategory of $\mathcal{E}^\Gamma$ such that the following diagrams commute for each object $\mathbb{A} = (\mathsf{A}, \mathsf{a})$ of $\bar{\mathcal{E}}^\Gamma$:

$$\begin{tikzcd} \Gamma^2(\mathbb{A}) \arrow[r, "\Gamma(\mathsf{a})"] \arrow[d, "\natural(\mathbb{A})"'] & \Gamma(\mathbb{A}) \arrow[d, "\mathsf{a}"] \\ \Gamma(\mathbb{A}) \arrow[r, "\mathsf{a}"] & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "\flat(\mathbb{A})"] \arrow[dr, "id(\mathbb{A})"'] & \Gamma(\mathbb{A}) \arrow[d, "\mathsf{a}"] \\ & \mathbb{A} \end{tikzcd} \tag{4}$$

Example 2.4. For each monoid $M = (\mathsf{M}, \mathsf{m})$, an endofunctor Γ on ***Set*** sending a set X to the set $\mathsf{M} \times \mathsf{X}$ is a monad. An object, $\mathbb{X} = (\mathsf{X}, \mathsf{x})$, in the category of algebras, M***Set***, for this monad is a set, X, with a left action, x, of the monoid M, also referred to as an *M-set*. For each $a \in \mathsf{M}$, we can define a map $\mathsf{x}_a : \mathsf{X} \to \mathsf{X}$ by $\mathsf{x}_a(x) := \mathsf{x}(a, x), \forall x \in \mathsf{X}$. The arrows $(\mathsf{X}, \mathsf{x}) \to (\mathsf{X}', \mathsf{x}')$ are *equivariant* (i. e., preserving the action) functions $f : \mathsf{X} \to \mathsf{X}'$, making the following diagram commute:

$$\begin{tikzcd} \mathsf{X} \arrow[r, "f"] \arrow[d, "\mathsf{x}_a"'] & \mathsf{X}' \arrow[d, "\mathsf{x}'_a"] \\ \mathsf{X} \arrow[r, "f"] & \mathsf{X}' \end{tikzcd}. \tag{5}$$

The category of M-sets is of utmost importance in our exposition.

Example 2.5. An $\mathbb{F}$-*algebra* (a linear algebra over the field $\mathbb{F}$) is an example of an algebra for a Σ-functor on ***Set*** with $\mathsf{S} = \{\mathbf{0}, +, \cdot\} \cup \mathbb{F}$ and $\mathsf{s}(\mathbf{0}) = 0, \mathsf{s}(+) = 2, \mathsf{s}(\cdot) = 2, \mathsf{s}(r) = 1, \forall r \in \mathbb{F}$. In the next Section we shall describe $\mathbb{F}$-algebras from a different point of view.

Definition 2.6. A complete regularly co-well-powered category with regular epi-mono factorizations is called *birkhoff* if it has enough regular projectives.

Definition 2.7. A full subcategory of a Birkhoff category is called a *birkhoff variety* if it is closed under products, subobjects and quotients.

Definition 2.8. An endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ is called a *varietor* if it preserves regular epis, and the forgetful functor $U : \mathcal{E}^\Gamma \to \mathcal{E}$ has a left adjoint.

3 $\mathbb{F}$-algebras

Definition 3.1. An $\mathbb{F}$-*algebra*, $\mathbb{A}$, is an ordered pair (A, A), where A is a vector space over a field $\mathbb{F}$, referred to as the *underlying vector space* of $\mathbb{A}$, and A is a $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$-tensor on A, called the *structure tensor* of $\mathbb{A}$. Each vector $\boldsymbol{a}$ of A is called an *element* of $\mathbb{A}$, denoted $\boldsymbol{a} \in \mathbb{A}$. The *dimensionality* of $\mathbb{A}$ is that of A.

Remark 3.1. The tensor A induces a binary operation $A \times A \to A$, called the *multiplication* of $\mathbb{A}$: to each pair of vectors $(\boldsymbol{a}, \boldsymbol{b})$ the tensor A associates a vector $\boldsymbol{ab} : A^* \to \mathbb{F}$, such that $(\boldsymbol{ab})(\tilde{\boldsymbol{\tau}}) = \mathsf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}), \forall \tilde{\boldsymbol{\tau}} \in A^*$. An $\mathbb{F}$-algebra with an associative multiplication is called *associative*. An element $\boldsymbol{\imath}$, such that $\boldsymbol{a\imath} = \boldsymbol{\imath a} = \boldsymbol{a}, \forall \boldsymbol{a} \in \mathbb{A}$ is called an *identity* of $\mathbb{A}$. For a fixed $\mathbb{F}$, the $\mathbb{F}$-algebras form a category denoted $\mathsf{Alg}\{\mathbb{F}\}$ in which arrows $\mathbb{A} \to \mathbb{B}$ are algebra homomorphisms.

Definition 3.2. For an $\mathbb{F}$-algebra $\mathbb{A}$ and a nonzero one-form $\tilde{\boldsymbol{\tau}} \in A^*$, a *principal inner product* is a $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$-tensor, $\mathsf{A}\,[\tilde{\boldsymbol{\tau}}]$, on A, assigning to each ordered pair $(\boldsymbol{a}, \boldsymbol{b})$ a number $\mathsf{A}\,[\tilde{\boldsymbol{\tau}}](\boldsymbol{a}, \boldsymbol{b}) := \mathsf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}$, just in case it is symmetric, $\mathsf{A}\,[\tilde{\boldsymbol{\tau}}](\boldsymbol{a}, \boldsymbol{b}) = \mathsf{A}\,[\tilde{\boldsymbol{\tau}}](\boldsymbol{b}, \boldsymbol{a}), \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}$. The one-form $\tilde{\boldsymbol{\tau}}$ is referred to as the *generating form* of $\mathsf{A}\,[\tilde{\boldsymbol{\tau}}]$

Remark 3.2. In other words, a principal inner product is the contraction of a one-form with the structure tensor.

Definition 3.3. For each $\mathbb{F}$-algebra $\mathbb{A} = (A, \mathsf{A})$, the $\mathbb{F}$-algebra $[\mathbb{A}] = (A, [\mathsf{A}])$, with the structure tensor defined by

$$[\mathsf{A}](\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) := \mathsf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) - \mathsf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{b}, \boldsymbol{a}), \forall \tilde{\boldsymbol{\tau}} \in A^*, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{A},$$

is called the *commutator* algebra of $\mathbb{A}$.

Definition 3.4. A finite dimensional associative $\mathbb{R}$-algebra with an identity is called a *unital* algebra.

Remark 3.3. The set $\mathcal{A}^\bullet$ of invertible elements of a unital algebra $\mathbb{A}$ is a lie group with respect to the multiplication of $\mathbb{A}$, with $[\mathbb{A}]$ as its lie algebra.

Remark 3.4. Each real vector space V induces a natural manifold structure on its carrier. This manifold which we denote $\mathcal{V}$, is referred to as the *linear manifold canonically generated by V*. Since V and $\mathcal{V}$ have the same carrier, there is a bijection $\mathfrak{J}_V : \mathcal{V} \to V$. We use the normal $(a, u, ...)$ and bold $(\boldsymbol{a}, \boldsymbol{u}...)$ fonts, to denote the elements of $\mathcal{V}$ and V, respectively, e. g., $\mathfrak{J}_V(a) = \boldsymbol{a}$. The tangent space $T_a\mathcal{V}$ is identified with V at each point $a \in \mathcal{V}$ via an isomorphism $\mathfrak{J}^*_a : T_a\mathcal{V} \to V$ sending a tangent vector to the curve $\mu : \mathbb{R} \to \mathcal{V}$, $\mu(t) = a+tu$, at the point $\mu(0) = a \in \mathcal{V}$, to the vector $\boldsymbol{u} \in V$, with the "total" map $\mathfrak{J}^*_V : T\mathcal{V} \to V$. The set of nonzero vectors of V constitutes a submanifold of $\mathcal{V}$, referred to as the *punctured manifold* (*of* V), denoted $\mathcal{V}^\circ$.

Definition 3.5. For a real vector space V and a linear map $\boldsymbol{F} : V \to V$, a vector field $\boldsymbol{f} : \mathcal{V} \to T\mathcal{V}$ on $\mathcal{V}$, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\boldsymbol{f}} & T\mathcal{V} \\ {\scriptstyle \mathfrak{J}_V}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{J}^*_V} \\ V & \xrightarrow{\boldsymbol{F}} & V \end{array}, \tag{6}$$

is called the *vector field canonically generated* by $\boldsymbol{F}$.

Remark 3.5. In particular, for a unital algebra $\mathbb{A}$, the linear manifold canonically generated by its underlying vector space A is denoted by $\mathcal{A}$ and the punctured manifold by $\mathcal{A}^\circ$.

Definition 3.6. For real vector spaces U, V, and a map $\boldsymbol{F} : U \to V$, the map $F : \mathcal{U} \to \mathcal{V}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & \mathcal{V} \\ {\scriptstyle \mathfrak{J}_U}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{J}_V} \\ U & \xrightarrow{\boldsymbol{F}} & V \end{array}, \tag{7}$$

is called the *linear induction* of $\boldsymbol{F}$.

Remark 3.6. For each basis $(\boldsymbol{e}_j)$ on the vector space A of a unital algebra, there is a natural basis field on $\mathcal{A}^\bullet$, namely the basis $(\hat{\boldsymbol{e}}_j)$ of left invariant vector fields generated by $(\boldsymbol{e}_j)$. We call $(\hat{\boldsymbol{e}}_j)$ a *lie frame generated by* $(\boldsymbol{e}_j)$. The value, $(\hat{\boldsymbol{e}}_j)(a)$, of $(\hat{\boldsymbol{e}}_j)$ at a is a basis on the tangent space $T_a\mathcal{A}^\bullet$; it is referred to as a *lie basis* (at a) generated by $(\boldsymbol{e}_j)$. In particular, $(\hat{\boldsymbol{e}}_j)(\imath)$, the lie basis at the identity generated by $(\boldsymbol{e}_j)$ coincides with $(\boldsymbol{e}_j)$.

Definition 3.7. For a unital algebra $\mathbb{A}$, let $(\hat{\boldsymbol{e}}_j)$ be a lie frame on $\mathcal{A}^\bullet$, generated by a basis $(\boldsymbol{e}_j)$ on A. The *structure field* of the lie group $\mathcal{A}^\bullet$ is a tensor field $\boldsymbol{\mathcal{A}}$ on $\mathcal{A}^\bullet$, assigning to each point $a \in \mathcal{A}$ a $[{}^1_2]$-tensor $\boldsymbol{\mathcal{A}}(a)$ on $T_a\mathcal{A}^\bullet$, with components $\mathcal{A}^i_{jk}(a) := (\mathcal{A}(a))^i_{jk}$ in the basis $(\hat{\boldsymbol{e}}_j)(a)$, defined by

$$\mathcal{A}^i_{jk}(a) := \mathsf{A}^i_{jk}, \quad \forall a \in \mathcal{A}^\bullet,$$

where A^i_{jk} are the components of the structure tensor $\boldsymbol{\mathsf{A}}$ in the basis $(\boldsymbol{e}_j)$.

Remark 3.7. Intuitively, the structure field is the constant extension of the structure tensor along the left invariant vector fields.

Definition 3.8. For a unital algebra $\mathbb{A}$ and each $a \in \mathcal{A}^\bullet$, an $\mathbb{R}$-algebra $\mathbb{A}\{a\} = (A\{a\}, \boldsymbol{\mathsf{A}}\{a\})$, where $A\{a\} := T_a\mathcal{A}^\bullet$, and $\boldsymbol{\mathsf{A}}\{a\} := \boldsymbol{\mathcal{A}}(a)$, is called the *tangent algebra* of the lie group $\mathcal{A}^\bullet$ at a.

Remark 3.8. It is easy to see that for each $a \in \mathcal{A}^\bullet$, the tangent algebra $\mathbb{A}\{a\}$ is isomorphic to $\mathbb{A}$; in particular, each $\mathbb{A}\{a\}$ is unital.

Definition 3.9. For a unital algebra $\mathbb{A}$ and a twice differentiable real function $\mathfrak{T}$ on the lie group $\mathcal{A}^\bullet$, a *principal metric on* $\mathcal{A}^\bullet$ is a $[{}^0_2]$-tensor field $\boldsymbol{\mathfrak{T}}$ on $\mathcal{A}^\bullet$, such that $\boldsymbol{\mathfrak{T}}(a) = \boldsymbol{\mathsf{A}}\{a\}[\tilde{\boldsymbol{a}}], \forall a \in \mathcal{A}^\bullet$, where $\tilde{\boldsymbol{a}} := d\mathfrak{T}(a)$ is the value of the gradient of $\mathfrak{T}$ at a. We refer to $\mathfrak{T}$ as the *generating function* of $\boldsymbol{\mathfrak{T}}$.

Remark 3.9. In other words, a principal metric is the contraction of a one-form field on $\mathcal{A}^\bullet$ with the structure field of $\mathcal{A}^\bullet$. For each $a \in \mathcal{A}^\bullet$, the value, $\boldsymbol{\mathfrak{T}}(a)$, of $\boldsymbol{\mathfrak{T}}$ is a principal inner product on the tangent algebra $\mathbb{A}\{a\}$.

4 Quaternion algebra

Definition 4.1. A four dimensional $\mathbb{R}$-algebra, $\mathbb{H} = (H, \boldsymbol{\mathsf{H}})$, is called a *quaternion algebra* (with *quaternions* as its elements), if there is a basis on H, in which the components of the structure tensor H are given by the entries of the following matrices,

$$\mathsf{H}^0_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \mathsf{H}^1_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\mathsf{H}^2_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \mathsf{H}^3_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (8)$$

We refer to such a basis as *canonical.*

Remark 4.1. The vectors of the canonical basis are denoted $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. A quaternion algebra is unital, with the first vector of the canonical basis, $\mathbf{1}$, as its identity. Since $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is a basis on a real vector space, any quaternion $\boldsymbol{a}$ can be presented as $a^0\mathbf{1}+a^1\boldsymbol{i}+a^2\boldsymbol{j}+a^3\boldsymbol{k}$, $a^\beta \in \mathbb{R}$. We refer to $a^0\mathbf{1}$ and $a^p\boldsymbol{i}_p$ as the *real* and *imaginary parts* of $\boldsymbol{a}$, respectively, denoted $\mathfrak{Re}(\boldsymbol{a})$ and $\mathfrak{Im}(\boldsymbol{a})$, respectively. For each quaternion $\boldsymbol{a}$ the quaternion $\bar{\boldsymbol{a}} := \mathfrak{Re}(\boldsymbol{a}) - \mathfrak{Im}(\boldsymbol{a})$ is called *conjugate* to $\boldsymbol{a}$. Quaternions $\boldsymbol{a}$ with $\mathfrak{Im}(\boldsymbol{a}) = \mathbf{0}$ are in one-to-one correspondence with real numbers, which is often denoted, with certain notational abuse, as $\mathbb{R} \subset \mathbb{H}$. Quaternions $\boldsymbol{a}$ with $\mathfrak{Re}(\boldsymbol{a}) = \mathbf{0}$ are called *pure imaginary.* They constitute a three dimensional subspace of H.

Remark 4.2. Since $\mathbb{H}$ has no zero divisors, the punctured manifold $\mathcal{H}^\circ$, together with the multiplication of $\mathbb{H}$, constitute a lie group isomorphic to the lie group $\mathcal{H}^\bullet$ of invertible elements of $\mathbb{H}$.

Remark 4.3. A linear transformation $H \to H$ with the following components in the canonical basis,

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathsf{B} \end{pmatrix}, \mathsf{B} \in SO(3),$$

takes $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ to a basis $(\boldsymbol{i}_\beta)$ in which the components (8) of the structure tensor will *not* change. Thus $(\boldsymbol{i}_\beta)$ is a canonical basis, and we have a class $\{\boldsymbol{i}\}$ of canonical bases parameterized by elements of $SO(3)$, whose members differ from one another by a rotation in the hyperplane of pure imaginary quaternions. We shall call such a bijection $\Xi : SO(3) \to \{\boldsymbol{i}\}$ an *array.* Given an array Ξ, the basis $(\boldsymbol{i}_\beta) = \Xi(\mathbf{1})$, where $\mathbf{1}$ is the identity of $SO(3)$, is called the *standard basis* (*with respect to* Ξ).

Definition 4.2. An inner product on a four dimensional real vector space V is called *minkowski* if there exists a basis $(\boldsymbol{e}_\beta)$ on V, such that the components of the inner product in $(\boldsymbol{e}_\beta)$ are given by the entries of the following matrix:

$$\pm\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9}$$

Lemma 4.1 (Trifonov, 1995)**.** *A one-form $\tilde{\boldsymbol{\tau}} \in H^*$ is a generating form of a principal inner product on $\mathbb{H}$ iff there exists a canonical basis in which $\tilde{\boldsymbol{\tau}}$ has the components $(\tilde{\tau}_0, 0, 0, 0)$. For each generating form $\tilde{\boldsymbol{\tau}} \in H^*$, the principal inner product $\mathbf{H}[\tilde{\boldsymbol{\tau}}]$ is minkowski.*

Remark 4.4. Each canonical basis $(\boldsymbol{i}_\beta)$ induces a *canonical* coordinate system $(w,\ x,\ y,\ z)$ on the linear manifold $\mathcal{H}$ canonically generated by H, and therefore also on the punctured manifold $\mathcal{H}^\circ$: a point $a \in \mathcal{H}^\circ$ such that $\mathcal{J}(a) = \boldsymbol{a} = a^\beta \boldsymbol{i}_\beta$ is assigned coordinates $(w = a^0,\ x = a^1,\ y = a^2,\ z = a^3)$. This coordinate system covers both $\mathcal{H}$ and $\mathcal{H}^\circ$ with a single patch. Since $0 := \mathcal{I}^{-1}(\mathbf{0}) \notin \mathcal{H}^\circ$, at least one of the coordinates is always nonzero for any point $a \in \mathcal{H}^\circ$. For a differentiable function $R : \mathbb{R} \to \mathbb{R}\setminus\{0\}$ there is a system of natural spherical coordinates $(\eta,\ \chi,\ \theta,\ \varphi)$ on $\mathcal{H}^\circ$, related to the canonical coordinates by

$$w = R(\eta)\cos(\chi), \quad x = R(\eta)\sin(\chi)\sin(\theta)\cos(\varphi),$$
$$y = R(\eta)\sin(\chi)\sin(\theta)\sin(\varphi), \quad z = R(\eta)\sin(\chi)\cos(\theta).$$

Remark 4.5. Given a canonical basis $(\boldsymbol{i}_\beta)$ on H, values of a quaternion valued map $f : \mathsf{X} \to \mathbb{H}$ can be decomposed in $(\boldsymbol{i}_\beta)$ producing an ordered quadruple (f_β) of real-valued maps called the *constituents* of f *in* the basis $(\boldsymbol{i}_\beta)$. We call f_0 the *temporal* constituent, and each f_p is referred to as a *spatial* constituent of f. We shall be careful not to confuse these entities with *components* of tensorial objects in the basis $(\boldsymbol{i}_\beta)$.

Remark 4.6. There are three natural basis fields on $\mathcal{H}^\circ$ induced by each canonical basis $(\boldsymbol{i}_\beta)$: the lie frame $(\hat{\boldsymbol{i}}_\beta)$ (see Remark 3.6), which is a noncoordinate basis field, and two coordinate basis fields, the *canonical frame*, $(\partial_w,\ \partial_x,\ \partial_y,\ \partial_z)$ and the corresponding *spherical frame* $(\partial^R_\eta,\ \partial^R_\chi,\ \partial^R_\theta,\ \partial^R_\varphi)$.

Remark 4.7. A left invariant vector field $\hat{\boldsymbol{u}}$ on $\mathcal{H}^\circ$, generated by a vector $\boldsymbol{u} \in H$ with the components (u^β) in a canonical basis, associates to each point $a \in \mathcal{H}^\circ$ with the coordinates $(w,\, x,\, y,\, z)$ a vector $\hat{\boldsymbol{u}}(a) \in T_a\mathcal{H}^\circ$ with the components $\hat{u}^\beta(a) = (\boldsymbol{a}\boldsymbol{u})^\beta$ in the basis $(\partial_w,\, \partial_x,\, \partial_y,\, \partial_z)(a)$ on $T_a\mathcal{H}^\circ$:

$$\hat{u}^0(a) = wu^0 - xu^1 - yu^2 - zu^3, \quad \hat{u}^1(a) = wu^1 + xu^0 + yu^3 - zu^2,$$
$$\hat{u}^2(a) = wu^2 - xu^3 + yu^0 + zu^1, \quad \hat{u}^3(a) = wu^3 + xu^2 - yu^1 + zu^0. \quad (10)$$

Definition 4.3. A lorentzian metric on a four dimensional manifold is called *closed FLRW* (Friedmann-Lemaître-Robertson-Walker) if there is a coordinate system (x^β), such that in the corresponding coordinate frame the components of the metric are given by the entries of the following matrix:

$$\pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathsf{a}^2 & 0 & 0 \\ 0 & 0 & -\mathsf{a}^2\sin^2(x^1) & 0 \\ 0 & 0 & 0 & -\mathsf{a}^2\sin^2(x^1)\sin^2(x^2) \end{pmatrix}, \quad (11)$$

where $\mathsf{a} : \mathbb{R} \to \mathbb{R}$, referred to as the *scale factor*, is a function of x^0 only.

Theorem 4.1 (Trifonov, 2007)**.** *Every principal metric on $\mathcal{H}^\circ$ is closed FLRW with the generating function, Υ, a monotonous function of η for each spherical frame. The components of each $\boldsymbol{\Upsilon}$ have the form* (11) *in a spherical frame $(\partial^R_\eta,\, \partial^R_\chi,\, \partial^R_\theta,\, \partial^R_\varphi)$ with R satisfying*

$$R(\eta) = \exp \int \frac{d\eta}{\pm\sqrt{|\,\dot{\Upsilon}\,|}} \quad , \quad (12)$$

and the scale factor $\mathsf{a} = \sqrt{|\,\dot{\Upsilon}\,|}$, *where* $\dot{\Upsilon} := \frac{d\Upsilon}{d\eta}$.

5 Hyperkähler manifolds

Let $\mathcal{M} = (M, \boldsymbol{g}, (\tilde{\boldsymbol{\omega}}^p))$ be hyperkähler manifold. Then, for an array $\Xi : SO(3) \to \{\boldsymbol{i}\}$, we can consider the ordered quadruple $(\boldsymbol{g}, \tilde{\boldsymbol{\omega}}^1, \tilde{\boldsymbol{\omega}}^2, \tilde{\boldsymbol{\omega}}^3))$ as constituents $(\tilde{\boldsymbol{\omega}}^\beta)$, where $\tilde{\boldsymbol{\omega}}^0 := \boldsymbol{g}$, of a quaternion valued map $\tilde{\boldsymbol{\omega}} : T\mathcal{M} \times T\mathcal{M} \to \mathbb{H}$ in the standard basis $\Xi(\mathbf{1})$, assigning to each ordered pair $(\boldsymbol{u}, \boldsymbol{v})$ of tangent vectors at each point $\phi \in \mathcal{M}$ a quaternion

$$\tilde{\boldsymbol{\omega}}^0(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{i}_0 + \tilde{\boldsymbol{\omega}}^1(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{i}_1 + \tilde{\boldsymbol{\omega}}^2(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{i}_2 + \tilde{\boldsymbol{\omega}}^3(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{i}_3, \quad (13)$$

which allows us to define the spatial constituents $\tilde{\boldsymbol{\omega}}^p$ in every canonical basis $(\boldsymbol{i}_\beta)$. Similarly, we can assign the ordered quadruple $(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$, where $\mathcal{J}_0$ is the identity map on $T\mathcal{M}$, and $\mathcal{J}_p$ is the complex structure corresponding to $\tilde{\boldsymbol{\omega}}^p$, to each canonical basis $(\boldsymbol{i}_\beta)$, as constituents of the hypercomplex structure $\boldsymbol{\mathcal{J}}$.

Definition 5.1. For an array Ξ and a hyperkähler manifold $\mathcal{M}$, the ordered pair $\boldsymbol{\Omega} : (\tilde{\boldsymbol{\omega}}, \boldsymbol{\mathcal{J}})$ is called a *hyperkähler structure* on $\mathcal{M}$ *generated by* Ξ. For a canonical basis $(\boldsymbol{i}_\beta)$ the maps $\tilde{\boldsymbol{\omega}}^p$ and $\mathcal{J}_p$ are called the *symplectic* and *complex constituents* of $\boldsymbol{\Omega}$, respectively, *in* the basis $(\boldsymbol{i}_\beta)$.

Definition 5.2. Let $\mathcal{M}$ and $\mathcal{N}$ be hyperkähler manifolds with hyperkähler structures $\boldsymbol{\Omega}^{\mathcal{M}}$ and $\boldsymbol{\Omega}^{\mathcal{N}}$, respectively, generated by an array Ξ. A smooth map $f : \mathcal{M} \to \mathcal{N}$ is called *quaternionic* (*regular*) if there exists an $SO(3)$ matrix B such that

$$\sum_{p,q=1}^{3} \mathsf{B}_{pq} \mathcal{J}_q^{\mathcal{N}} \circ df \circ \mathcal{J}_p^{\mathcal{M}} = df, \tag{14}$$

where $\mathcal{J}_p^{\mathcal{N}}$ and $\mathcal{J}_p^{\mathcal{N}}$ are the complex constituents, in a canonical basis $(\boldsymbol{i}_\beta)$, of the hyperkähler structures of $\mathcal{M}$ and $\mathcal{N}$, respectively, and df is the differential of f.

Remark 5.1. Quaternionic maps generalize holomorphic functions of complex analysis (see e. g., [6]).

Definition 5.3. For a quaternionic map $f : \mathcal{M} \to \mathcal{H}$, its *epifield* is a vector field $\boldsymbol{f}$ on $\mathcal{M}$, such that

$$(df_0)(\boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{f}, \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \mathcal{M}[\begin{smallmatrix}1\\0\end{smallmatrix}], \tag{15}$$

where $\boldsymbol{g}$ is the riemannian metric on $\mathcal{M}$, and df_0 is the differential of f_0. The maps f and f_0 are referred to as the *generating map* and the *main generator* of $\boldsymbol{f}$, respectively.

Remark 5.2. Since the temporal constituent f_0 is invariant under a canonical basis change, so is the definition of an epifield.

6 Quaternionic hilbert space

Definition 6.1. A left module V over $\mathbb{H}$ is called a *(left) quaternionic vector space.*

Remark 6.1. A *right quaternionic vector space* is defined similarly, as well as the right versions of constructs based on it.

Definition 6.2. A map $\widehat{\boldsymbol{F}} : \mathsf{V} \to \mathsf{V}$ is called a *(left) quaternion linear operator*, if

$$\widehat{\boldsymbol{F}}(\boldsymbol{a}\boldsymbol{\phi}) = \boldsymbol{a}\widehat{\boldsymbol{F}}(\boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} \in \mathsf{V}, \quad \forall \boldsymbol{a} \in \mathbb{H}.$$

Definition 6.3. A *(left) quaternionic hilbert space*, $\mathbb{V}$, is a quaternionic vector space, V, together with a map $\langle \cdot \mid \cdot \rangle : \mathsf{V} \times \mathsf{V} \to \mathbb{H}$, called a *quaternionic hermitian inner product*, such that

$$\begin{aligned}
&\langle \boldsymbol{\phi} \mid \boldsymbol{\psi} + \boldsymbol{\xi} \rangle = \langle \boldsymbol{\phi} \mid \boldsymbol{\psi} \rangle + \langle \boldsymbol{\phi} \mid \boldsymbol{\xi} \rangle, \\
&\qquad \langle \boldsymbol{\phi} \mid \boldsymbol{\psi} \rangle = \overline{\langle \boldsymbol{\psi} \mid \boldsymbol{\phi} \rangle}, \quad \langle \boldsymbol{a}\boldsymbol{\phi} \mid \boldsymbol{\psi} \rangle = \boldsymbol{a}\langle \boldsymbol{\phi} \mid \boldsymbol{\psi} \rangle, \\
&\qquad \| \boldsymbol{\phi} \|^2 := \langle \boldsymbol{\phi} \mid \boldsymbol{\phi} \rangle \in \mathbb{R}, \| \boldsymbol{\phi} \|^2 > 0, \quad \forall \boldsymbol{\phi} \neq \boldsymbol{0}, \\
&\qquad\qquad \forall \boldsymbol{\phi}, \boldsymbol{\psi}, \boldsymbol{\xi} \in \mathsf{V}, \forall \boldsymbol{a} \in \mathbb{H},
\end{aligned}$$

and the diagonal $\| \cdot \|$ induces a topology on V, relative to which V is separable and complete.

Definition 6.4. For a quaternionic hilbert space $\mathbb{V}$ and a quaternion linear operator $\widehat{\boldsymbol{F}}$ on $\mathbb{V}$, its *quaternionic adjoint* (*with respect to* $\langle \cdot \mid \cdot \rangle$) is a quaternion linear operator $\widehat{\boldsymbol{F}}^{\dagger}$ on V, such that

$$\langle \boldsymbol{\phi} \mid \widehat{\boldsymbol{F}}(\boldsymbol{\psi}) \rangle = \langle \widehat{\boldsymbol{F}}^{\dagger}(\boldsymbol{\phi}) \mid \boldsymbol{\psi} \rangle, \quad \forall \boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbb{V}.$$

Definition 6.5. We refer to a quaternion linear operator $\widehat{\boldsymbol{F}}$ as *quaternionic (anti)hermitian* if it coincides with (the negative of) its quaternionic adjoint.

Remark 6.2. Given a basis, $(\boldsymbol{e}_j)$, on an n dimensional quaternionic hilbert space $\mathbb{V}$, and a canonical basis, $(\boldsymbol{i}_\beta)$, on H, $\mathbb{V}$ induces a real $4n$ dimensional vector space, V, and the latter canonically generates a real linear $4n$ dimensional manifold $\mathcal{V}$ and the punctured manifold $\mathcal{V}^\circ$.

Definition 6.6. For an operator $\widehat{\boldsymbol{F}} : \mathbb{V} \to \mathbb{V}$ on a quaternionic hilbert space $\mathbb{V}$ its *expectation operator* is a map $\widehat{F} : \mathbb{V} \to \mathbb{H}$ assigning to each $\boldsymbol{\phi} \in \mathbb{V}$ a quaternion $\langle \boldsymbol{\phi} \mid \widehat{\boldsymbol{F}}(\boldsymbol{\phi}) \rangle$.

Lemma 6.1. *For each quaternionic antihermitian operator, $\widehat{\boldsymbol{F}}$, its expectation operator $\widehat{F}$ has the following property:*

$$\widehat{F}(\boldsymbol{\phi}) = \mathfrak{Im}(\widehat{F}(\boldsymbol{\phi})), \quad \forall \boldsymbol{\phi} \in \mathbb{V}. \tag{16}$$

Proof.

$$\langle \boldsymbol{\phi} \mid \widehat{\boldsymbol{F}}(\boldsymbol{\phi})\rangle = \langle -\widehat{\boldsymbol{F}}(\boldsymbol{\phi}) \mid \boldsymbol{\phi}\rangle = -\langle \widehat{\boldsymbol{F}}(\boldsymbol{\phi}) \mid \boldsymbol{\phi}\rangle = -\overline{\langle \boldsymbol{\phi} \mid \widehat{\boldsymbol{F}}(\boldsymbol{\phi})\rangle}. \tag{17}$$

Thus, for each $\boldsymbol{\phi} \in \mathbb{V}$ the quaternion $\langle \boldsymbol{\phi} \mid \widehat{\boldsymbol{F}}(\boldsymbol{\phi})\rangle$ coincide with the negative of its adjoint, which means that its real part is zero. □

Definition 6.7. For a quaternionic antihermitian operator $\widehat{\boldsymbol{F}} : \mathbb{V} \to \mathbb{V}$ the linear induction $F : \mathcal{V} \to \mathcal{H}$ of its expectation operator $\widehat{F}$ is called the *expectation* of $\widehat{\boldsymbol{F}}$.

Definition 6.8. For a quaternion linear operator $\widehat{\boldsymbol{F}} : \mathbb{V} \to \mathbb{V}$ a vector field $\boldsymbol{f} : \mathcal{V} \to T\mathcal{V}$ canonically generated by $-\widehat{\boldsymbol{F}}$ is called the *hyperfield* of $\widehat{\boldsymbol{F}}$.

Remark 6.3. The set of the integral curves of the hyperfield of $\widehat{\boldsymbol{F}}$ can be formally represented by the following differential equation

$$\dot{\psi} = -\widehat{\boldsymbol{F}}(\psi). \tag{18}$$

Epifields are a generalization of *schrödinger vector fields* of geometric quantum mechanics [14]. Indeed, due to the simple relationship between eigenvalues of hermitian and antihermitian operators in the complex case we can use either kind to represent observables. If we take antihermitian operators, the Schrödinger equation has the form (18), with $\widehat{\boldsymbol{F}} = \widehat{\boldsymbol{H}}$, which defines the set of integral curves of the schrödinger vector field of the hamiltonian operator $\widehat{\boldsymbol{H}}$.

Given an array Ξ, the quaternionic hermitian product $\langle\cdot \mid \cdot\rangle$ induces a map $T\mathcal{V}\times T\mathcal{V} \to \mathbb{H}$ whose spatial constituents in a canonical basis $(\boldsymbol{i}_\beta)$ can be identified with the symplectic constituents in $(\boldsymbol{i}_\beta)$ of a hyperkähler structure $\boldsymbol{\Omega}$ on $\mathcal{V}$ generated by Ξ. Similarly, the maps $(\boldsymbol{i}'_\beta)$ defined by

$$\boldsymbol{i}'_\beta(\boldsymbol{\phi}) := \boldsymbol{i}_\beta\boldsymbol{\phi}, \quad \forall \boldsymbol{\phi} \in \mathbb{V}$$

produce the complex constituents of $\boldsymbol{\Omega}$.

Example 6.1. The quaternion algebra $\mathbb{H}$ is a quaternionic vector space. Together with a natural quaternionic hermitian inner product defined by

$$\langle \boldsymbol{a} \mid \boldsymbol{b}\rangle := \boldsymbol{a}\overline{\boldsymbol{b}}, \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}, \tag{19}$$

it is a quaternionic hilbert space. Therefore $\mathcal{H}$ and $\mathcal{H}^\circ$ possess natural hyperkähler structures.

Theorem 6.1. *For an array Ξ and a quaternionic hilbert space $\mathbb{V}$ let $\boldsymbol{f}$ be the epifield of a quaternionic map $f : \mathcal{V}^\circ \to \mathcal{H}^\circ$, such that $\mathfrak{Im}(f)$ is the expectation of a quaternion antihermitian operator $\widehat{\boldsymbol{F}}$ on $\mathbb{V}$. Then $\boldsymbol{f}$ is the hyperfield of $\widehat{\boldsymbol{F}}$, and for each canonical basis $(\boldsymbol{i}_\beta)$ there exists an ordered triple $(\boldsymbol{f}_p)$ of vector fields on $\mathbb{V}$ such that*

$$\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2 + \boldsymbol{f}_3, \quad (df_p)(\boldsymbol{u}) = \tilde{\boldsymbol{\omega}}^p(\boldsymbol{f}_p, \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \mathcal{X}[\begin{smallmatrix}1\\0\end{smallmatrix}], \tag{20}$$

(no summation on p) where df_p is the differential of f_p, and $\tilde{\boldsymbol{\omega}}^p$ is a symplectic constituent of the hyperkähler structure on $\mathcal{V}^\circ$ generated by Ξ, in the basis $(\boldsymbol{i}_\beta)$.

Proof. Given a canonical basis $(\boldsymbol{i}_\beta)$, the hyperfield of $\widehat{\boldsymbol{F}}$ is the hyperhamiltonian vector field whose generator is (f_p) [8]. It was shown in [8] and [13] that the conditions (20) are satisfied for the hyperhamiltonian vector field and hence for $\boldsymbol{f}$. □

7 Semantics

The observer theory outlined in [15] describes an observer as an *experient*, as opposed to an *occupant*, of the environment. This can be presented within a framework similar to what is known as the *initial algebra* and *final coalgebra* approach to syntax and semantics of formal languages ([1], [17]). The underlying idea is that perception and comprehension are somehow dual to each other, and each *experience* can be considered as both a *percept* and a *concept*. This idea has its origin in logic and theoretical computer science where similar dualities are considered (a symbol and its meaning, syntax of a formal language and its semantics). We start with a category $\mathcal{E}$, the *metaenvironment*, representing the totality of experiences, and an endofunctor Γ on $\mathcal{E}$, the *construer*, interpreted as the *language of thought* of the experient (see [7] for a philosophical discussion). The categories of algebras, $\mathcal{E}^\Gamma$, and coalgebras, $\mathcal{E}_\Gamma$, for this endofunctor represent *perception* and *comprehension* of the experient, respectively. If the perception, $\mathcal{E}^\Gamma$, has an initial object, $\mathbb{I}$, the latter serves as the *syntax* of the experient's language of thought. Then an algebra, $\mathbb{A}$, for Γ is considered a *paradigm* or a *model* for the language of thought, and the unique arrow $\mathbb{I} \to \mathbb{A}$ as a *meaning function* or an *interpretation*.

Definition 7.1. An *experient*, $\boldsymbol{\mathcal{E}}$, is an ordered pair $(\mathcal{E}, \Gamma)$, where $\mathcal{E}$ is a birkhoff category, called the *metauniverse*, and $\Gamma : \mathcal{E} \to \mathcal{E}$ is a varietor called the *construer*. The objects and arrows of $\mathcal{E}$ are called *metaphenomena* and *metalinks*, respectively. $\mathcal{E}$-elements of metaphenomena are called *reflexors*. The category of algebras, $\mathcal{E}^{\Gamma}$ is called the *perception category*. Objects and arrows of $\mathcal{E}^{\Gamma}$ are referred to as *paradigms* and *shifts*, respectively. An experient is *coherent* if the metauniverse is a topos. An experient is called *boolean* if the metauniverse is a boolean topos.

Remark 7.1. It follows from the Definition 7.1 that $\mathcal{E}^{\Gamma}$ is also a birkhoff category (see [10], p. 125).

Remark 7.2. Intuitively, metaphenomena are *complex percepts* composed out of elementary ones (reflexors). For a coherent experient $\boldsymbol{\mathcal{E}} = (\mathcal{E}, \Gamma)$, we are to think of the internal logic of $\mathcal{E}$ as the *metalogic* of the experient. In particular, the metalogic of a boolean experient is boolean.

Definition 7.2. For a paradigm $\mathbb{A}$ of an experient $\boldsymbol{\mathcal{E}}$, and a natural number n, a *reality of rank* n is an ordered pair $\mathcal{R} = (\mathbb{A}, \boldsymbol{R})$, where $\boldsymbol{R}$ is a subobject of $\mathbb{A}^n$. We refer to $\mathbb{A}$ as the *underlying paradigm*, and $\boldsymbol{R}$ is called the *ontology* of $\mathcal{R}$.

Remark 7.3. Intuitively, a reality of rank n is an n-ary relation on a collection of reflexors. From this point on we are interested exclusively in realities of rank 2, henceforth referred to simply as *realities*.

Definition 7.3. An *existence mode*, $\boldsymbol{\mathcal{E}}(\mathfrak{V})$, of an experient $\boldsymbol{\mathcal{E}} = (\mathcal{E}, \Gamma)$ is an ordered pair $(\boldsymbol{\mathcal{E}}, \mathfrak{V})$, where $\mathfrak{V}$ is a birkhoff variety of $\mathcal{E}^{\Gamma}$. A paradigm $\mathbb{A}$ such that it is (not) a $\mathfrak{V}$-object is called *(non)existent with respect to* $\boldsymbol{\mathcal{E}}(\mathfrak{V})$. A shift $\mathbb{A} \to \mathbb{B}$ such that it is (not) a $\mathfrak{V}$-arrow is called *(im)possible with respect to* $\boldsymbol{\mathcal{E}}(\mathfrak{V})$.

Remark 7.4. It should be noted that we thus consider every paradigm $\mathbb{A}$ of an experient also a paradigm of the experient in a certain existence mode, although $\mathbb{A}$ may be nonexistent with respect to the latter. Given an existence mode, $\boldsymbol{\mathcal{E}}(\mathfrak{V})$, of an experient $\boldsymbol{\mathcal{E}}$, we say that the experient *is in the mode* $\mathfrak{V}$, or that the experient *is* the $\mathfrak{V}$-*observer*.

8 $\mathbb{F}$-observers

Different existence modes possess different amount of structure to allow for a definition of notions that we normally associate with observers. We shall focus on a class of boolean experients whose metauniverse is $\boldsymbol{Set}$, and the construer is a Σ-functor for the following signature:

$$\Sigma = (\mathsf{S}, \mathsf{s}); \quad \mathsf{S} = \{+, \cdot, \mathbf{0}\} \cup \mathbb{F}, \quad \mathsf{s}(+) = 2, \\ \mathsf{s}(\cdot) = 2, \quad \mathsf{s}(\mathbf{0}) = 0, \quad \mathsf{s}(r) = 1, \quad \forall r \in \mathbb{F}, \quad (21)$$

where $\mathbb{F}$ is a field. They are introduced implicitly in [15], where a particular existence mode, an $\mathsf{Alg}\{\mathbb{F}\}$-observer, where $\mathsf{Alg}\{\mathbb{F}\}$ is the category of $\mathbb{F}$-algebras, is studied. $\mathsf{Alg}\{\mathbb{F}\}$-observers are sufficiently fine structured to define the fundamental notion of a *temporal reality*, and at the same time they are relatively simple: their metaphenomena and reflexors are sets and set elements, respectively. For the rest of the paper we shall deal exclusively with $\mathsf{Alg}\{\mathbb{F}\}$-observers, to whom we henceforth refer simply as $\mathbb{F}$-*observers*.

Definition 8.1. For an existent paradigm $\mathbb{A}$ of an $\mathbb{F}$-observer the underlying vector space A is called the *sensory domain* of $\mathbb{A}$, with the principal inner products of $\mathbb{A}$ referred to as *sensory forms* of the paradigm. The dual vector space A^* is called the *ether domain* of $\mathbb{A}$, with the elements called *ether forms*. The *motor domain* of $\mathbb{A}$ is the multiplicative subgroupoid, $M_{\mathbb{A}} = (\mathsf{M}_{\mathbb{A}}, \imath, *)$, of $\mathbb{A}$, generated by the set of nonzero reflexors of the paradigm $\mathbb{A}$. We refer to elements of $M_{\mathbb{A}}$ as *effectors*. An existent finite dimensional paradigm, $\mathbb{A}$, is called *rational* if its motor domain is a monoid; otherwise $\mathbb{A}$ is called *irrational*. A paradigm is called *transient* if it is neither rational nor irrational. The set $\mathcal{A}^\bullet$ of invertible reflexors of a rational paradigm $\mathbb{A}$ is called the *perception domain* of $\mathbb{A}$.

Remark 8.1. For a rational paradigm $\mathbb{A}$ of an $\mathbb{F}$-observer, the perception domain $\mathcal{A}^\bullet$ is a group. For each reality $\mathcal{R} = (\mathbb{A}, \boldsymbol{R})$ of an $\mathbb{F}$-observer, the ontology, $\boldsymbol{R}$, is just a binary relation on the carrier of $\mathbb{A}$.

Definition 8.2. A reality of an $\mathbb{F}$-observer is called *immanent* if its underlying paradigm is existent and its ontology is a partial order. Otherwise the reality is called *transcendent*.

Definition 8.3. For an $\mathbb{F}$-observer, an immanent reality $(\mathbb{F}, \preceq)$ is called a *temporal template* if its ontology $\preceq$ is a linear order on $\mathbb{F}$.

Remark 8.2. Not every existence mode has a temporal template. For instance, complex numbers, $\mathbb{C}$, do not admit linear orders ([12], p. 304). Therefore the $\mathbb{C}$-observer has no temporal templates.

Definition 8.4. For a rational paradigm $\mathbb{A}$ of an $\mathbb{F}$-observer the $\mathbb{A}$-*universe* is the category of $M_{\mathbb{A}}$-sets, $M_{\mathbb{A}}\boldsymbol{Set}$ (see Example 2.4), with objects and arrows called $\mathbb{A}$-*phenomena* and $\mathbb{A}$-*links*, respectively. For an $\mathbb{A}$-phenomenon $\mathbb{X} = (\mathsf{X}, \mathsf{x})$, the metaphenomenon X is called the *propensity realm*, with elements called *propensity modes* of $\mathbb{X}$. Each metalink $\mathsf{f} : \mathsf{X} \to A$, where A is the carrier of $\mathbb{A}$, is referred to as an *attribute* of $\mathbb{X}$.

Remark 8.3. We refer to the internal logic of the topos $M_{\mathbb{A}}\boldsymbol{Set}$ as the *operational logic* of the $\mathbb{F}$-observer *with respect to* $\mathbb{A}$, or as the *logic* of the $\mathbb{A}$-universe.

Definition 8.5. For each $\mathbb{A}$-phenomenon $\mathbb{X} = (\mathsf{X}, \mathsf{x})$, the $\mathcal{A}^{\bullet}$-set $(\mathsf{X}, \bar{\mathsf{x}})$, where $\bar{\mathsf{x}} : \mathcal{A}^{\bullet} \times \mathsf{X} \to \mathsf{X}$ is the restriction of x to $\mathcal{A}^{\bullet}$, is called the *perceptible part* of $\mathbb{X}$. For each $\phi \in \mathsf{X}$ the orbits, W_ϕ and $\overline{W}_\phi$, of ϕ, with respect to the actions of $M_{\mathbb{A}}$ and $\mathcal{A}^{\bullet}$, respectively, are called an *existence mode* and a *presence mode* of $\mathbb{X}$, respectively.

Remark 8.4. In other words, W_ϕ and $\overline{W}_\phi$ are the sets

$$W_\phi = \{\psi \in \mathsf{X} \quad : \quad \psi = \mathsf{x}(\xi, \phi), \quad \forall \xi \in M_{\mathbb{A}}\}$$
$$\overline{W}_\phi = \{\psi \in \mathsf{X} \quad : \quad \psi = \bar{\mathsf{x}}(\xi, \phi), \quad \forall \xi \in \mathcal{A}^{\bullet}\}. \quad (22)$$

Intuitively, each presence mode of an $\mathbb{A}$-phenomenon is the perceptible part of one of its existence modes.

Definition 8.6. An $\mathbb{A}$-phenomenon $\mathbb{X}$ is called *stable* if there is a bijection $\sigma : \overline{W} \to \mathcal{A}^{\bullet}$, called a *proper view*, for each presence mode $\overline{W}$ of $\mathbb{X}$. For a stable phenomenon, the propensity modes $\phi \in (W_\phi \backslash)\overline{W}_\phi$ are called *(im)perceptible.*

Definition 8.7. For an $\mathbb{F}$-observer, a rational paradigm $\mathbb{A}$ is *consistent* if the $\mathbb{A}$-universe is a boolean topos; a consistent paradigm of maximal dimensionality is called a *home* paradigm of the $\mathbb{F}$-observer.

Remark 8.5. Intuitively, the consistency condition requires the logic of the $\mathbb{A}$-universe to match the metalogic of an $\mathbb{F}$-observer. $\mathbb{F}$-observers without home paradigms, referred to as *Wanderers*, may or may not be of interest, but our main concern will be precisely with home paradigms, and more specifically with home paradigms of the $\mathbb{R}$-observer and their $\mathbb{A}$-universes, due to the following result.

Theorem 8.1 (Trifonov, 1995)**.** *Every home paradigm of the* $\mathbb{R}$*-observer is isomorphic to the quaternion algebra* $\mathbb{H}$ *with a family of minkowski sensory forms.*

Definition 8.8. For an $\mathbb{F}$-observer, a reality $\mathcal{R} = (\mathbb{A}, \boldsymbol{R})$ is called *stable* if it is immanent and $\mathbb{A}$ is a rational paradigm; otherwise the reality is called *unstable* (or *virtual*).

Remark 8.6. Given a stable realty $\mathcal{R} = (\mathbb{A}, \boldsymbol{R})$ we refer to the logic of the $\mathbb{A}$-universe also as the *logic of* $\mathcal{R}$.

Definition 8.9. For an $\mathbb{F}$-observer with a temporal template $(\mathbb{F}, \preceq)$, a reality $\mathcal{R} = (\mathbb{A}, \boldsymbol{R})$ together with an $\mathbb{F}$-valued map $\mathsf{T} : \mathbb{A} \to \mathbb{F}$, is called a *temporal reality* if it is stable and

$$a\boldsymbol{R}b \iff (\mathsf{T}(a) \preceq \mathsf{T}(b) \wedge a \neq b) \vee (a = b), \forall a, b \in \mathbb{A}. \tag{23}$$

Otherwise the reality is called *atemporal.* The ontology of a temporal reality is called its *temporal order*, and the map T is referred to as the *global time* of the reality. The *perceptible time* of $\mathcal{R}$ is the restriction, $\mathcal{T} : \mathcal{A}^\bullet \to \mathbb{F}$, of T to the perception domain.

Remark 8.7. It should be emphasized that a temporal reality is defined only with respect to a certain temporal template. Some $\mathbb{F}$-observers may have several temporal templates, and some may have none. For example, since the $\mathbb{C}$-observer has no temporal templates, all realities of such an observer are atemporal. It is easy to see that a temporal template and a global time uniquely determine the temporal order of $\mathcal{R}$.

Definition 8.10. For an $\mathbb{F}$-observer and a presence mode, $\overline{W}$, of his stable $\mathbb{A}$-phenomenon, the pullback, $\mathcal{T}_{\overline{W}} := \mathcal{T} \circ \sigma$, of the perceptible time $\mathcal{T}$ under the proper view σ is referred to as the *perceptible time of* $\overline{W}$.

Definition 8.11. For an $\mathbb{F}$-observer, let $\mathbb{X}$ be a stable $\mathbb{A}$-phenomenon, $\mathcal{R} = (\mathbb{A}, \boldsymbol{R})$ a temporal reality, and T a map $\mathsf{X} \to \mathbb{F}$. An ordered triple $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$ is called a *realization* (*of* $\mathbb{X}$ *in* $\mathcal{R}$) if the following diagram commutes for each $\overline{W}$,

$$\begin{tikzcd}
\overline{W} \arrow[r, "\sigma"] \arrow[d, "j"'] \arrow[dr, "\mathcal{T}_{\overline{W}}"] & \mathcal{A}^\bullet \arrow[d, "\mathcal{T}"] \\
\mathsf{X} \arrow[r, "T"] & \mathbb{F}
\end{tikzcd}$$

where j is the inclusion map. We refer to $\mathbb{X}$, $\mathcal{R}$ and T as the *underlying phenomenon*, the *background reality* and the *ambient time* of the realization, respectively.

9 Observers

Due to the results of [15] and [16], in the remainder of the paper we shall deal exclusively with the $\mathbb{R}$-observer, henceforth referred to simply as the *observer*. If not mentioned explicitly, it is assumed in the following that the constructs under consideration always refer to the observer.

Lemma 9.1. *Any rational paradigm of the observer is a unital algebra.*

Proof. Since nonzero elements of $\mathbb{A}$ obey associative multiplication, nonassociativity can occur only in the permutations of $(\boldsymbol{ab})\mathbf{0}$, which is impossible since $\boldsymbol{b}\mathbf{0} = \mathbf{0}, \forall \boldsymbol{b} \in \mathbb{A}$. Thus, $\mathbb{A}$ is associative, finite dimensional, and the identity of the motor monoid is the identity of $\mathbb{A}$. Therefore it is unital. $\square$

Corollary 9.1. *The perception domain $\mathcal{A}^\bullet$ of a rational paradigm $\mathbb{A}$ is a lie group with respect to the multiplication of $\mathbb{A}$ (see Remark 3.3).*

Definition 9.1. For a rational paradigm $\mathbb{A}$, the linear manifold $\mathcal{A}$, canonically generated by the sensory domain A is called the *sensory manifold* of $\mathbb{A}$, and each reflexor $a \in \mathcal{A}$ is called a *viewpoint*. Viewpoints $a \in (\mathcal{A}\backslash)\mathcal{A}^\bullet$ are called *(im)proper*. A basis, $(\boldsymbol{e}_\beta)$, on the sensory domain A, is called a *sensory basis* of the paradigm. For a sensory basis $(\boldsymbol{e}_\beta)$ the lie frame $(\hat{\boldsymbol{e}}_\beta)$, on the perception domain $\mathcal{A}^\bullet$, is called a *perception frame* of the paradigm.

Definition 9.2. For a rational paradigm $\mathbb{A}$, a reflexor $\boldsymbol{u} \in A$, and a proper viewpoint $a \in \mathcal{A}^\bullet$, the integral curve, through a, of the left invariant vector field generated by $\boldsymbol{u}$ is called a $(\boldsymbol{u}, a)$-*vista*.

Remark 9.1. Intuitively, $(\boldsymbol{u}, a)$-vistas indicate naturally distinguished directions within the perception domain.

Lemma 9.2. *The observer has a unique temporal template.*

Proof. Indeed, there is a unique linear order on $\mathbb{R}$, namely the standard order $\leq$ (see, e. g., [12]). Thus, $(\mathbb{R}, \leq)$ is unique. $\square$

Remark 9.2. The previous result makes it unnecessary to mention the temporal order explicitly, and we use the simplified notation $\mathcal{R} = (\mathbb{A}, \mathsf{T})$ for a temporal reality of the observer.

Remark 9.3. It is shown in [15] that besides the home paradigm, the observer has exactly two (up to an $\mathbb{R}$-algebra isomorphism) consistent paradigms, namely the one dimensional $\mathbb{R}$-algebra of reals, $\mathbb{R}$, and the two dimensional $\mathbb{R}$-algebra of complex numbers, $\mathbb{C}$, both subalgebras of $\mathbb{H}$.

Definition 9.3. A reality is called *robust* if it is temporal and there exists a principal metric, $\boldsymbol{\mathfrak{T}}$, on the perception domain $\mathcal{A}^\bullet$ of the underlying paradigm with the perceptible time $\mathfrak{T}$ as its generating function. Given a robust reality $\mathcal{R} = (\mathbb{A}, \mathsf{T})$, the structure field of $\mathbb{A}$ is called the *structure (field)* of the reality, and we refer to $d\mathfrak{T}$ and $\boldsymbol{\mathfrak{T}}$ as the *ether (field)*, and the *metric* of $\mathcal{R}$, respectively. The ordered pair $\mathcal{S} = (\mathcal{A}^\bullet, \boldsymbol{\mathfrak{T}})$ is called the *spacetime* of $\mathcal{R}$.

Definition 9.4. A realization $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$ is called *robust* if the background reality is robust, and the perceptible part of the underlying phenomenon $\mathbb{X}$ is a principal $\mathcal{A}^\bullet$-bundle (X), such that the propensity realm X is its total space and each proper view $\sigma : \overline{W} \to \mathcal{A}^\bullet$ is a fiber diffeomorphism. The dimensionality of X is referred to as the *rank* of $\mathbf{X}$. For a presence mode, $\overline{W}$, of the underlying phenomenon the pullback $\boldsymbol{\mathfrak{T}}_{\overline{W}}$, of the metric of the background reality under the proper view σ is called the *metric* of $\overline{W}$. The ordered pair $\mathcal{W} = (\overline{W}, \boldsymbol{\mathfrak{T}}_{\overline{W}})$ is referred to as a *(possible) world* of $\mathbf{X}$. The bundle (X) is called the *monocosm* of the realization.

10 Dynamical systems

Definition 10.1. A realization $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$ is called a *dynamical system* if it is robust and its propensity realm is a riemannian manifold $\mathcal{X} = (\mathsf{X}, \boldsymbol{g})$. The riemannian metric $\boldsymbol{g}$ is referred to as the *propensity metric* of $\mathbf{X}$.

Remark 10.1. There is a natural connection on the monocosm $(\mathcal{X})$ of a physical system: the horizontal space at any point ψ is defined as the set of tangent vectors orthogonal to the world $\mathcal{W}_\psi$ with respect to the propensity metric. We refer to this connection as the *fundamental connection* of the dynamical system.

Definition 10.2. For a dynamical system $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$ a *perceptible* is a smooth map $f : \mathcal{X} \to \mathcal{A}^\bullet$. A *temporal evolution* of a dynamical system $\mathbf{X}$

is an integral curve of a vector field $\boldsymbol{f}_T$ on $\mathcal{X}$, called the *temporal evolution vector field* of $\mathbf{X}$, such that

$$(dT)(\boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{f}_T, \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \mathcal{X}[\begin{smallmatrix}1\\0\end{smallmatrix}], \tag{24}$$

The propensity realm of a dynamical system is referred to as its *state space*, and propensity modes are called *states*. A state ψ such that the vector $\boldsymbol{f}_T(\psi)$ is vertical is called the *proper* state of $\mathbf{X}$, and the possible world $\mathcal{W}_\psi$ is called an *accessible world*.

Remark 10.2. A perceptible is a smooth restriction of an attribute of the underlying phenomenon to the perception domain. Intuitively, a temporal evolution of a dynamical system is the motion of the observer's proper viewpoint across possible worlds of the system along its temporal evolution vector field. At each point of an evolution the observer encounters a possible world, a diffeomorphically perturbed copy of the spacetime of the background reality, which contains perceptible information about the system.

Definition 10.3. For a dynamical system $\mathbf{X}$, an ordered triple (f, ϕ, ψ), where f is a perceptible and ϕ, ψ are states, is called an *f-observation* (*of ϕ relative to ψ*). The states ϕ and ψ are called the *initial* and the *final* states, and the worlds $\mathcal{W}_\phi$ and $\mathcal{W}_\psi$ are called the *source* and the *target* worlds of the f-observation, respectively. The *propensity* of an f-observation (f, ϕ, ψ) is the length $\rho(\phi, \psi)$ of the shortest geodesics between ϕ and ψ.

11 Cosmologies

Definition 11.1. For the home paradigm, $\mathbb{H}$, a sensory basis $(\boldsymbol{i}_\beta)$ and the corresponding perception frame $(\hat{\boldsymbol{\imath}}_\beta)$ are called *canonical*. Each array $\Xi : SO(3) \to \{\boldsymbol{i}\}$ is called a *sensory array*.

Definition 11.2. A robust reality is called a *cosmology* if its underlying paradigm is isomorphic to the home paradigm of the observer.

Remark 11.1. As follows from Theorem 4.1, the choice of cosmologies of the observer is extremely limited. In fact, there is a unique, up to the functional variable T, cosmology, $(\mathbb{H}, \mathsf{T})$. Let us review its basic properties.

1. The perception domain $\mathcal{H}^\bullet$ of the underlying paradigm $\mathbb{H}$ is the lie group of nonzero quaternions with the $\mathbb{R} \times \mathbf{S}^3$ topology, the product of a real line and a three-sphere.

2. The spacetime $\mathcal{S} = (\mathcal{H}^\bullet, \boldsymbol{\mathcal{T}})$ of the cosmology is a smooth four dimensional manifold with a closed FLRW metric (see Theorem 4.1).

3. Given a canonical sensory basis $(\boldsymbol{i}_\beta)$ the metric has the following components in the spherical frame $(\partial^R_\eta, \partial^R_\chi, \partial^R_\theta, \partial^R_\varphi)$

$$\mathcal{T}_{\alpha\beta} = diag(-1, -\mathsf{a}^2, -\mathsf{a}^2\sin^2\chi, -\mathsf{a}^2\sin^2\chi\sin^2\theta), \quad \mathsf{a} := \sqrt{|\,\dot{\mathcal{T}}\,|},$$
$$\text{with} \qquad R = \exp\int \frac{d\eta}{\pm\sqrt{|\,\dot{\mathcal{T}}\,|}}. \tag{25}$$

4. The perceptible time $\mathcal{T}$ is a monotonous function of η in the spherical frame $(\partial^R_\eta, \partial^R_\chi, \partial^R_\theta, \partial^R_\varphi)$.

5. The structure field of the cosmology has the following components in the frame $(\partial^R_\eta, \partial^R_\chi, \partial^R_\theta, \partial^R_\varphi)$:

$$\mathcal{H}^0_{\alpha\beta} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda^{-1} & 0 & 0 \\ 0 & 0 & -\lambda^{-1}\sin^2\chi & 0 \\ 0 & 0 & 0 & -\lambda^{-1}\sin^2\chi\sin^2\theta \end{pmatrix},$$
$$\mathcal{H}^1_{\alpha\beta} = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^2\chi\sin\theta \\ 0 & 0 & -\sin^2\chi\sin\theta & 0 \end{pmatrix},$$
$$\mathcal{H}^2_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\sin\theta \\ \lambda & 0 & 0 & 0 \\ 0 & \sin\theta & 0 & 0 \end{pmatrix}, \ \mathcal{H}^3_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 1/\sin\theta & 0 \\ 0 & -1/\sin\theta & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix},$$

where $\lambda := \dot{R}/R$.

6. The ether field of the cosmology has the following components in the spherical frame $(\partial^R_\eta, \partial^R_\chi\colon \partial^R_\theta, \partial^R_\varphi)$:

$$d\mathcal{T} = (\dot{\mathcal{T}}, 0, 0, 0).$$

7. For a canonical sensory basis $(\boldsymbol{i}_\beta)$ and the corresponding canonical coordinate system (w, x, y, z), let $\boldsymbol{u}$ be a reflexor with the components (u^β) in $(\boldsymbol{i}_\beta)$, and a is a proper viewpoint with coordinates $(\bar{w}, \bar{x}, \bar{y}, \bar{z})$. Then the $(\boldsymbol{u}, a)$-vista can be easily computed by solving the system of differential equations (10) with a parameter t:

$$\begin{aligned} w(t) &= \exp(u^0 t)(\frac{-u^1\bar{x} - u^2\bar{y} - u^3\bar{z}}{\omega} \sin\omega t + \bar{w}\cos\omega t) \\ x(t) &= \exp(u^0 t)(\frac{u^1\bar{w} - u^2\bar{z} + u^3\bar{y}}{\omega} \sin\omega t + \bar{x}\cos\omega t) \\ y(t) &= \exp(u^0 t)(\frac{u^1\bar{z} + u^2\bar{w} - u^3\bar{x}}{\omega} \sin\omega t + \bar{y}\cos\omega t) \\ z(t) &= \exp(u^0 t)(\frac{-u^1\bar{y} + u^2\bar{x} + u^3\bar{w}}{\omega} \sin\omega t + \bar{z}\cos\omega t), \end{aligned} \tag{26}$$

where $\omega := \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}$.

Remark 11.2. The theory of the $\mathbb{R}$-observer we have developed so far is left invariant (utilizing left invariant vector fields on perception domains). It is easy to show that the metric of the right invariant cosmology coincide with (25), but the $(\boldsymbol{u}, a)$-vistas are different. This can be used, in principle, by the observer to determine which, out of two universes he "lives in".

12 Physical systems

Definition 12.1. A dynamical system $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$ is called a *physical system* if $\mathcal{R}$ is a cosmology and the state space is a hyperkähler manifold.

Definition 12.2. For a physical system $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$, a perceptible f, together with a sensory array Ξ, is called an *observable* of $\mathbf{X}$ if for each canonical sensory basis $\boldsymbol{i}_\beta$ there exits an ordered triple $(\boldsymbol{f}_p)$ of vector fields on the state space $\mathcal{X}$, such that

$$\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2 + \boldsymbol{f}_3, \quad (df_p)(\boldsymbol{u}) = \tilde{\boldsymbol{\omega}}^p(\boldsymbol{f}_p, \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \mathcal{X}[{}^1_0], \tag{27}$$

(no summation on p) where $\boldsymbol{f}$ is defined by

$$(df_0)(\boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{f}, \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \mathcal{X}[{}^1_0], \tag{28}$$

$\tilde{\boldsymbol{\omega}}^p$ is a symplectic constituent of a hyperkähler structure on $\mathcal{X}$ generated by Ξ, and f_p is a spatial constituent of f in the basis $(\boldsymbol{i}_\beta)$. We refer to $\boldsymbol{f}$ as the *f-field.* A state ψ such that the vector $\boldsymbol{f}(\psi)$ is vertical is called an *f-proper* state of $\mathbf{X}$. A possible world $\mathcal{W}$ is called *f-(in)accessible* iff there is (not) an f-proper state ψ, such that $\mathcal{W} = \mathcal{W}_\psi$. For each f-proper state ψ the value $f(\psi)$ is called a *relative perceptible property* of $\mathbf{X}$.

Remark 12.1. The above definition generalizes the notions of GQM, where points at which the schrödinger vector field of an observable becomes vertical, and the corresponding value of f parametrize the eigenvectors and eigenvalues of $\widehat{\boldsymbol{F}}$, respectively [14].

Definition 12.3. For a physical system $\mathbf{X} = (\mathbb{X}, \mathcal{R}, T)$, its *hamiltonian* is an observable h such that the temporal evolution vector field of $\mathbf{X}$ coincides with the h-field.

Remark 12.2. A physical system can be thought of as a *sufficiently smooth fine-graining* of a cosmology. It is a natural generalization and modification of the notion of a quantum system of GQM.

Definition 12.4. For a physical system, an f-observation (f, ϕ, ψ) is called *successful* if the propensity $\rho(\phi, \psi)$ exists, and its target world $\mathcal{W}_\psi$ is f-accessible, in which case we refer to $\mathcal{W}_\psi$ as the *actual world* of the f-observation. Otherwise the f-observation is called *failed* (or *unsuccessful*), and the world $\mathcal{W}_\psi$ is called *virtual.*

Definition 12.5. For a physical system, a successful f-observation (f, ϕ, ψ) is called an *f-measurement* if f is an observable. For an f-measurement (f, ϕ, ψ) its *result* is an ordered pair $(f(\psi), \rho(\phi, \psi))$, where $f(\psi)$ is a relative perceptible property of the physical system, and $\rho(\phi, \psi)$ is the propensity.

Remark 12.3. Intuitively, the final state of an f-measurement marks the "landing spot" of the observer (or, more correctly, of his proper viewpoint), and the observer finds himself in a (target) world whose properties may differ, in a strictly defined sense, from the respective properties of the source world.

Definition 12.6. A physical system is called a *hyperquantum system* if its monocosm is a hyperquantum bundle $(\mathcal{V}^\circ)$ over a quaternionic hilbert space $\mathbb{V}$, and the imaginary part $\mathfrak{Im}(h)$ of its hamiltonian coincides with the restriction to $\mathcal{V}^\circ$ of the expectation of a quaternionic antihermitian operator $\widehat{\boldsymbol{H}}$ on $\mathbb{V}$.

Remark 12.4. It seems tempting to obtain quantum systems of complex quantum mechanics (CQM) by demanding the existence of a canonical sensory basis in which the hamiltonian has a unique nonzero spatial constituent. However, this would not be quite correct technically because within our framework quantum systems as they described in CQM are not even physical systems - their background reality is not a cosmology since the underlying paradigm is the two dimensional consistent (boolean) paradigm $\mathbb{C}$. Therefore possible worlds of quantum systems are two dimensional. In this sense hyperquantum systems are a nontrivial generalization of quantum systems, and the latter are a *degenerate* case of the former: two out of four dimensions are collapsed in each possible world.

Definition 12.7. A dynamical system is called a *quantum system* if its background reality is $(\mathbb{C}, \mathsf{T})$, its monocosm is a quantum bundle $(\mathcal{V}^{\circ}_{\mathbb{C}})$ over a complex hilbert space $\mathbb{V}_{\mathbb{C}}$, the complex imaginary part $\mathfrak{Im}(h)$ of its hamiltonian coincides with the restriction to $\mathcal{V}^{\circ}_{\mathbb{C}}$ of the expectation of a complex antihermitian operator $\widehat{\boldsymbol{H}}$ on $\mathbb{V}_{\mathbb{C}}$.

Remark 12.5. It is a standard result in GQM that a schrödinger evolution of a quantum system is also a hamiltonian evolution with the expectation of $\widehat{\boldsymbol{H}}$ as its generating function [14], so the above definition is equivalent to the description of a quantum system in GQM. There is a unique equivalent of a canonical sensory basis, $(\mathbf{1}, \boldsymbol{i})$, on the sensory domain C of the paradigm $\mathbb{C}$, hence we do not have to specify a (complex) sensory array. For an f-measurement (f, ϕ, ψ) of a quantum system the propensity $\rho(\phi, \psi)$ can be expressed in terms of probability of obtaining a particular result ([4] and references therein), and so propensity is a generalization of the notion of quantum probability.

13 Summary

As we mention in the introduction, the technical purpose of the paper is to provide *formal definitions* of observer related notions which are normally considered too ambiguous for constructive discussion within mainstream physics. Below we list some of them, and venture to offer an interpretation which is in no way exclusive.

The *observer* is represented by an *existence mode* of a *boolean experient*, and is capable of perceiving various *realities*, each based on a *paradigm*. In some realities the observer tends to distinguish *dynamical systems*, collections of experiences stable in a strictly defined sense. Dynamical systems spend most of their *ambient time* roaming their *possible worlds* according to the evolution equation (24). An *observation* of a dynamical system perturbs its evolution resulting occasionally in a creation of a *perceptible property* of the system with respect to its *actual world*. For reasons that are beyond the scope of this theory, the contemporary *operational logic* of the observer seems to be bivalent boolean *(Assertion 1)*, and hence the largest immediate environment conforming to this requirement corresponds to a *robust reality* of his *home paradigm*. Then the kinematic axioms of GR follow: the spacetime of the reality is a smooth manifold, because it is a lie group, its dimensionality is indeed four, and it has a lorentzian metric of a very special type (closed FLRW). It is curious that the requirement of booleanity alone is sufficient, and bivalence follows (see [9]).

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