EXPONENTIAL DECAY OF EIGENFUNCTIONS AND GENERALIZED EIGENFUNCTIONS OF A NON SELF-ADJOINT MATRIX SCHRÖDINGER OPERATOR RELATED TO NLS

DIRK HUNDERTMARK AND YOUNG-RAN LEE

ABSTRACT. We study the decay of eigenfunctions of the non self-adjoint matrix operator $\mathcal{H} = \left(\begin{smallmatrix} -\Delta+\mu+U & W \\ -W & \Delta-\mu-U \end{smallmatrix}\right)$, for $\mu > 0$, corresponding to eigenvalues in the strip $-\mu < \mathrm{Re}E < \mu$.

1. INTRODUCTION

For some positive μ, we consider the system

$$\mathcal{H} := \begin{pmatrix} -\Delta + \mu + U & W \\ -W & \Delta - \mu - U \end{pmatrix}, \tag{1.1}$$

with real-valued functions U and W. We will impose some weak conditions on U and W which insure that $\mathcal{H}$ is a closed operator on the domain $\mathcal{D}(\mathcal{H}) = H^2(\mathbb{R}^d, \mathbb{C}^2)$. The unperturbed operator $\mathcal{H}_0$, where $U = W = 0$, is given by

$$\mathcal{H}_0 := \begin{pmatrix} -\Delta + \mu & 0 \\ 0 & \Delta - \mu \end{pmatrix}.$$

Note that $\mathcal{H}_0$ is a self-adjoint operator on the domain $H^2(\mathbb{R}^d, \mathbb{C}^2)$ and, by inspection, the spectrum of $\mathcal{H}_0$ equals $\sigma(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty)$.

Our assumptions on U and W are

A. U and W are $-\Delta$-bounded with relative bound zero. That is, the domains $\mathcal{D}(U)$ and $\mathcal{D}(W)$, as multiplication operators with real-valued functions, contain the Sobolev space $H^2(\mathbb{R}^d) = \mathcal{D}(-\Delta)$ and for all $\varepsilon > 0$ there exists a C_ε such that

$$\|Xg\|_2 \leq \varepsilon \| - \Delta g\|_2 + C_\varepsilon \|g\|_2 \quad \text{for } X = U, W.$$

B. U and W decay to zero at infinity.

Date: 22 March 2007, revised version exponential13.tex .

2000 *Mathematics Subject Classification.* 35B20, 35B40, 35P30 .

Key words and phrases. Non-linear Schrödinger equation, decay of eigenfunctions.

D.H. supported in part by NSF grant DMS–0400940.

Let us explain how such a non-symmetric system naturally arises in the stability/instability study of solutions of the non-linear Schrödinger equation (NLS): The NLS is a non-linear evolution equation of the form

$$i\partial_t\psi = -\Delta\psi - F(|\psi|^2)\psi \quad \text{on } \mathbb{R}^d \tag{1.2}$$

for some real-valued non-negative function F, for example, $F(s) = s^\sigma$ with $\sigma > 0$. Making the ansatz $\psi(t,x) = e^{it\mu}\phi(x)$, with $\mu > 0$, one gets the time independent NLS

$$(-\Delta + \mu - F(|\phi|^2))\phi = 0. \tag{1.3}$$

Now let ϕ be a solution of (1.3). If $\phi > 0$ one often calls it the non-linear ground state, but we will not make this requirement. Perturbing ψ a bit, one makes the ansatz $\psi = e^{it\mu}(\phi + R)$ and gets the equation

$$i\partial_t R = \big[-\Delta + \mu - (F(|\phi|^2) + F'(|\phi|^2)|\phi|^2)\big]R - F'(|\phi|^2)\phi^2\overline{R} + N$$

where N is a term quadratic in R. Note that ϕ can always be chosen to be real-valued, however, we do not need to make this assumption. The above can be written as the system,

$$i\partial_t \begin{pmatrix} R \\ \overline{R} \end{pmatrix} = \mathcal{H}\begin{pmatrix} R \\ \overline{R} \end{pmatrix} + \mathcal{N} \tag{1.4}$$

where $\mathcal{N}$ is a term quadratic in R, $\mathcal{H}$ is as in (1.1), and the potentials are given by $U = -F(|\phi|^2) - F'(|\phi|^2)|\phi|^2$ and $W = -F'(|\phi|^2)\phi^2$. Hence the non-linear system (1.4) describes the time behavior of a perturbation R of the NLS around a stationary solution ϕ.

Since $\mathcal{N}$ is quadratic in R, one sees that to first order in the perturbation R, the spectral properties of systems like the one in (1.1) determine the linear stability/instability properties of (1.4) and hence the linear stability/instability of stationary solutions of the NLS (1.2). For this reason, the study of the spectral properties of operators given by (1.1) has received renewed interest in recent years, see, for example, [1, 3, 4, 7, 17, 20, 21].

Remark 1.1. (i) Assumption A is equivalent to

$$\lim_{\lambda\to\infty} \|X(-\Delta + i\lambda)^{-1}\| = 0$$

for $X = U, W$, see, for example, [5, 12, 15]. Note that because of assumption A, the operator $\mathcal{H}$ is a closed operator on its domain $\mathcal{D}(\mathcal{H}) = \mathcal{D}(\mathcal{H}_0) = H^2(\mathbb{R}^d, \mathbb{C}^2)$.

(ii) Assumption A is fulfilled, if U and W obey certain local uniform L^p-conditions, $U, W \in L^p_{\text{loc,unif}}(\mathbb{R}^d)$ with $p = 2$ for $d \le 3$ and $p > d/2$ if $d \ge 4$, or, slightly more generally, if they are in the Stummel class S_d, see [5, 18, 19].

(iii) We want to stress the fact that we do not make any assumptions on how fast U and W decay, only that they tend to zero at infinity.

Our first result deals with the essential spectrum of systems like (1.1). Since there are several non-equivalent definitions for the essential spectrum of non-selfadjoint operators, let us discuss these a little bit in more detail: Let T be an arbitrary closed operator on a Hilbert space. Its resolvent set $\rho(T)$ consists of all $z \in \mathbb{C}$ such that $T - z$ is boundedly invertible. Its spectrum is given by $\sigma(T) = \mathbb{C} \setminus \rho(T)$. A closed operator T is Fredholm, if its range is closed and both the kernel and co-kernel, the orthogonal complement of its range, are finite-dimensional. Its index is the difference of the dimensions of its kernel and co-kernel, $\mathrm{ind}(T) = \mathrm{nul}(T) - \mathrm{def}\, T$, where $\mathrm{nul}(T) = \dim \ker(T)$ and $\mathrm{def}(T) = \dim \mathrm{ran}(T)^{\perp}$. T is semi-Fredholm, if its range is closed and either its kernel or co-kernel is finite-dimensional. Consider the following sets

- $\Delta_1(T) = \{z \in \mathbb{C} |\, T - z \text{ is semi-Fredholm}\}$.
- $\Delta_2(T) = \{z \in \mathbb{C} |\, T - z \text{ is semi-Fredholm and } \mathrm{nul}(T - z) < \infty\}$.
- $\Delta_3(T) = \{z \in \mathbb{C} |\, T - z \text{ is Fredholm}\}$.
- $\Delta_4(T) = \{z \in \mathbb{C} |\, T - z \text{ is Fredholm with } \mathrm{ind}(T - z) = 0\}$.
- $\Delta_5(T) = \{z \in \Delta_4(T) |\text{ a deleted neighborhood of } z \text{ is in } \rho(T)\}$.

Note that $\rho(T) = \{z \in \mathbb{C} |\, T-z \text{ is Fredholm with } \mathrm{nul}(T-z) = \mathrm{def}(T-z) = 0\}$. Thus all sets defined above contain the resolvent set $\rho(T)$ and possible definitions for the essential spectrum, in terms of Fredholm properties, are given by

$$\sigma_{\mathrm{ess},j}(T) = \mathbb{C} \setminus \Delta_j(T), \quad j = 1, \dots, 5.$$

Remark 1.2. (i) These definitions are taken from page 40 in [6], see also [11]. The first one is the one used by Kato, see page 243 in [14], the fifth was introduced by Browder, [2], see also the discussion in Appendix B.

(ii) Theorem IX-1.5 in [6] shows that $\sigma_{\mathrm{ess},5}(T)$ is the union of $\sigma_{\mathrm{ess},1}(T)$ with all components of $\mathbb{C} \setminus \sigma_{\mathrm{ess},1}(T)$ which do not intersect the resolvent set.

(iii) For self-adjoint operators, all definitions above coincide. In general, one has the inclusions

$$\sigma_{\mathrm{ess},1}(\mathcal{H}) \subset \sigma_{\mathrm{ess},2}(\mathcal{H}) \subset \sigma_{\mathrm{ess},3}(\mathcal{H}) \subset \sigma_{\mathrm{ess},4}(\mathcal{H}) \subset \sigma_{\mathrm{ess},5}(\mathcal{H}) \subset \sigma(\mathcal{H})$$

since $\Delta_j(T)$ is a decreasing sequence of sets containing the resolvent set $\rho(T)$. All of the above inclusion can be strict, see the discussion in [11].

(iv) Another natural definition, in the spirit of the essential spectrum for self-adjoint operators, is to define the essential spectrum as the complement (in the spectrum) of the discrete spectrum. More precisely, if we denote by $\sigma_{\mathrm{disc}}(T)$ the set of all isolated points $\lambda \in \sigma(T)$ with finite *algebraic* multiplicity. Then the essential spectrum should be given by $\sigma(T) \setminus \sigma_{\mathrm{disc}}(T)$.

This definition of essential spectrum is introduced on page 106 in [16]. In fact, it coincides with the fifth one,

$$\sigma_{\mathrm{ess},5}(T) = \sigma(T) \setminus \sigma_{\mathrm{disc}}(T). \tag{1.5}$$

We could not find any proof of this in the literature and, for the convenience of the reader, give a proof of this in Appendix B.

(v) It might be surprising that in our case all of the above five definitions of essential spectrum coincide, as the following theorem shows.

Theorem 1.3. *Under the above conditions on U and W, one has*

$$\sigma_{\mathrm{ess},j}(\mathcal{H}) = \sigma(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty) \qquad \text{for } j = 1, 2, 3, 4, 5,$$

and the spectrum of $\mathcal{H}$ outside of its essential spectrum consists of a discrete set of eigenvalues of finite algebraic multiplicity.

Moreover, $\sigma(\mathcal{H})$ is symmetric under reflection along the real and imaginary axes, that is, $\sigma(\mathcal{H}) = -\sigma(\mathcal{H})$ and $\sigma(\mathcal{H}^*) = \sigma(\mathcal{H})$.

Remark 1.4. (i) The only condition needed on $V = \left(\begin{smallmatrix} U & W \\ -W & -U \end{smallmatrix}\right)$ is that V is relatively $\mathcal{H}_0$-compact, which is the case if U and W are relatively Laplacian compact.

(ii) Because of Theorem 1.3, there is no need to distinguish between the different definitions for the essential spectrum in the following.

(iii) Since $\mathcal{H}$ is not self-adjoint, it can happen that, for some eigenvalue z, $\ker((\mathcal{H} - z)^2) \neq \ker(\mathcal{H} - z)$, that is, $\mathcal{H}$ can possess generalized eigenspaces (a non-trivial Jordan normal form). However, the generalized eigenspace stabilizes, that is, for any $z \in \sigma(\mathcal{H}) \setminus \sigma_{\mathrm{ess}}(\mathcal{H})$ there is a $k \in \mathbb{N}$ with $\ker(\mathcal{H} - z)^{m+1} = \ker(\mathcal{H} - z)^m$ for all $m \geq k$, see the proof of Theorem 1.3. In the application to the non-linear Schrödinger equation, this typically happens at $z = 0$, see [22, 23, 20, 17].

(iv) Under the so-called positivity condition,

$$L_- := \, -\Delta + \mu + U - W \geq 0,$$

one has $\sigma(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$ and each eigenvalue with $z \neq 0$ has trivial Jordan form, $\ker((\mathcal{H} - z)^2) = \ker(\mathcal{H} - z)$. That is, the generalized eigenspace for non-zero eigenvalues coincides with the eigenspace. Under the positivity condition, only $z = 0$ can possess a generalized eigenspace. This is, for example, shown in [1, 17] and the proof carries over to our assumptions on U and W.

Our main goal in this paper is to prove that the generalized eigenfunctions of the above system with energies in the gap of the essential spectrum decay exponentially. This is the content of the next theorem.

Theorem 1.5. *Let E be an eigenvalue of $\mathcal{H}$ with $-\mu < \mathrm{Re}(E) < \mu$. Then under the above assumptions on U and W, every eigenfunction and generalized eigenfunction corresponding to E decays exponentially. More precisely, if $(\mathcal{H} - E)^k \varphi = 0$ for some $k \in \mathbb{N}$, we have the L^2-decay estimate*

$$e^{(\sqrt{\mu - |\mathrm{Re}E| - 2\delta})|x|}\varphi \in L^2(\mathbb{R}^d, \mathbb{C}^2) \qquad \text{for all positive } \delta < \frac{1}{2}(\mu - |\mathrm{Re}E|). \tag{1.6}$$

Remark 1.6. This result improves the exponential decay estimates of [17] in two directions. First, the authors of [17] need much stronger condition on the off-diagonal part W, namely some exponential decay of W. Secondly, they considered only (generalized) eigenfunctions corresponding to real eigenvalues within the gap $(-\mu, \mu)$. However, as shown in [8, 9] and [10], certain supercritical non-linearities lead to linearizations of NLS around the ground state which have a pair of purely imaginary eigenvalues in addition to their generalized eigenspace at zero, see also Lemma 17 in [20]. Our result shows that no a-priori decay rate for the matrix potential has to be specified for this. Moreover, the decay rate is uniform in the imaginary part of the eigenvalues and explicitly depends only on the positivity of $\mu - |\mathrm{Re}E|$.

In the next section we give the proof of Theorem 1.3. Theorem 1.5 is proved in Sections 3 and 4. Exponential decay of eigenfunctions is given in Section 3 and exponential decay of generalized eingefunctions in Section 4.

Acknowledgement: It is a pleasure to thank Wilhelm Schlag for bringing this type of spectral problem to our attention. Young-Ran Lee thanks the School of Mathematics at the University of Birmingham, England, for their warm hospitality. Furthermore, we would like to thank Des Evans for some discussions and the unknown referee for numerous comments which, in particular, helped to improve Theorem 1.3.

2. Proof of theorem 1.3

Using, for example, the Fourier transform, one sees that the unperturbed operator $\mathcal{H}_0$ is self-adjoint on $H^2(\mathbb{R}^d, \mathbb{C}^2)$ and that its spectrum is given by $\sigma(\mathcal{H}_0) = \sigma_{\mathrm{ess}}(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty)$. Recall that all different notions of essential spectrum coincide due to the self-adjointness of $\mathcal{H}_0$. Since U and W are Laplacian bounded with relative bound zero and go to zero at infinity, they are Laplacian compact. In particular, this implies for $V = \left(\begin{smallmatrix} U & W \\ -W & -U \end{smallmatrix}\right)$ that

$$V(\mathcal{H}_0 - z)^{-1} \quad \text{is compact}$$

for all $z \in \mathbb{C} \setminus \sigma(\mathcal{H}_0) = \mathbb{C} \setminus ((-\infty, -\mu] \cup [\mu, \infty))$. Thus, by Theorem IX-2.1 in [6],

$$\sigma_{\mathrm{ess},j}(\mathcal{H}) = \sigma_{\mathrm{ess}}(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty) \quad \text{for } j = 1, 2, 3, 4.$$

To prove the claim for $\sigma_{\mathrm{ess},5}(\mathcal{H})$ we need to know a bit more about the resolvent set of $\mathcal{H}$. Let us first prove that the eigenvalues of $\mathcal{H}$ in $\mathbb{C} \setminus \sigma(\mathcal{H}_0)$ form a discrete set with only $\sigma(\mathcal{H}_0)$ are possible accumulation points. By Remark 1.1.i and the assumptions on V, we see that

$$\lim_{\lambda \to \infty} \|V(\mathcal{H}_0 + i\lambda)^{-1}\| = 0. \tag{2.1}$$

Using (2.1), one sees that $1 + V(\mathcal{H}_0 - z)^{-1}$ is invertible for some complex z and hence the analytic Fredholm alternative, see [15], shows that there exists a discrete subset D of $\mathbb{C} \setminus \sigma(\mathcal{H}_0)$ such that $1 + V(\mathcal{H}_0 - z)^{-1}$ is invertible for all $z \in \mathbb{C} \setminus \sigma(\mathcal{H}_0)$ which are not in D.

This set D is precisely the set of all eigenvalues of $\mathcal{H}$ in $\mathbb{C} \setminus \sigma(\mathcal{H}_0)$. Indeed, by the compactness of $V(\mathcal{H}_0 - z)^{-1}$, $1 + V(\mathcal{H}_0 - z)^{-1}$ is not invertible if and only if $-1 \in \sigma(V(\mathcal{H}_0 - z)^{-1})$. So there exists a non-trivial $\phi \in L^2(\mathbb{R}^d, \mathbb{C}^2)$ with

$$V(\mathcal{H}_0 - z)^{-1}\phi = -\phi.$$

With $\psi = (\mathcal{H}_0 - z)^{-1}\phi$, we can rewrite this as

$$(\mathcal{H}_0 + V)\psi = z\psi,$$

so z is an eigenvalue of $\mathcal{H}$ with eigenvector ψ. In addition, reversing the above argument, one sees that if $z \notin \mathbb{C} \setminus \sigma(\mathcal{H}_0)$ is an eigenvalue of $\mathcal{H}$, then $-1 \in \sigma(V(\mathcal{H}_0 - z)^{-1})$ and hence $z \in D$. So the set D consists of all eigenvalues of $\mathcal{H}$ in $\mathbb{C} \setminus \sigma(\mathcal{H}_0)$.

As a second step, let us show that the resolvent set of $\mathcal{H}$ is quite big, it contains $\mathbb{C} \setminus (\sigma(\mathcal{H}_0) \cup D)$. Indeed, for any $z \notin \sigma(\mathcal{H}_0) \cup D$,

$$\begin{aligned} &(\mathcal{H} - z)(\mathcal{H}_0 - z)^{-1}\big(1 + V(\mathcal{H}_0 - z)^{-1}\big)^{-1} = \\ &(\mathcal{H}_0 - z)(\mathcal{H}_0 - z)^{-1}\big(1 + V(\mathcal{H}_0 - z)^{-1}\big)^{-1} + V(\mathcal{H}_0 - z)^{-1}\big(1 + V(\mathcal{H}_0 - z)^{-1}\big)^{-1} = \\ &\qquad \big(1 + V(\mathcal{H}_0 - z)^{-1}\big)\big(1 + V(\mathcal{H}_0 - z)^{-1}\big)^{-1} = I. \end{aligned}$$

Thus, for those values of z, $\mathcal{H} - z$ is surjective. A similar calculation shows

$$(\mathcal{H}_0 - z)^{-1}\big(1 + V(\mathcal{H}_0 - z)^{-1}\big)^{-1}(\mathcal{H} - z) = I,$$

so $\mathcal{H} - z$ is also injective, and hence a bijection if $z \notin \sigma(\mathcal{H}_0) \cup D$. Thus, by the closed graph theorem, $\mathcal{H} - z$ is boundedly invertible for those values of z with inverse

$$(\mathcal{H} - z)^{-1} = (\mathcal{H}_0 - z)^{-1}(1 + V(\mathcal{H}_0 - z)^{-1})^{-1}. \tag{2.2}$$

In particular, the resolvent set of $\mathcal{H}$ contains at least the set $\mathbb{C}\setminus(\sigma(\mathcal{H}_0)\cup D)$, where the discrete set D is the set of eigenvalues of $\mathcal{H}$ in $\mathbb{C}\setminus\sigma(\mathcal{H}_0)$.

Coming back to $\sigma_{\text{ess},5}(\mathcal{H})$, we simply note that, due to the above, $\mathbb{C}\setminus\sigma_{\text{ess},1}(\mathcal{H})$ is a connected set which intersects the resolvent set of $\mathcal{H}$. Hence, by Remark 1.2.ii,

$$\sigma_{\text{ess},5}(\mathcal{H}) = \sigma_{\text{ess},1}(\mathcal{H}) = (-\infty, -\mu] \cup [\mu, \infty)$$

also.

Now we show that the generalized eigenspace corresponding to eigenvalues $z_0 \in D$ of $\mathcal{H}$ is finite-dimensional. Let P_{z_0} be the corresponding Riesz projection. See chapter 6 in [12], chapter III-6.4 in [14], or chapter XII.2 in [16] for a definition and a discussion of the general properties of Riesz projections. By the discussion on page 178 in [14] one knows that $\operatorname{ran}(P_{z_0})$ is a reducing subspace for $\mathcal{H}$ and one knows, see III-6.5 in [14], that $\mathcal{H}-z_0$ restricted to $\operatorname{ran}(P_{z_0})$ is quasi-nilpotent, that is, its spectral radius is zero. Again, from formula III-6.32 in [14] one knows that P_{z_0} is the residue of $(\mathcal{H}-z)^{-1}$ at $z = z_0$.

By the analytic Fredholm theorem, the residues of $(1+V(\mathcal{H}_0-z)^{-1})^{-1}$ at $z = z_0$ are finite rank and, using (2.2), we then know in addition that P_{z_0} is a finite rank operator. In particular, $\mathcal{H}-z_0$ restricted to $\operatorname{ran}(P_{z_0})$ is nilpotent, since every finite rank quasi-nilpotent operator is nilpotent, see problem I-5.6 on page 38 in [14]. That is, there is an $m \in \mathbb{N}$ such that $\ker(\mathcal{H}-z_0)^m = \operatorname{ran}(P_{z_0})$.

The symmetry of the spectrum around the real and imaginary axis is well-known. It follows from the fact that $\mathcal{H}$ is unitary equivalent to its adjoint $\mathcal{H}^*$ and to $-\mathcal{H}$. Indeed, writing $L = -\Delta + \mu + U$, that is, $\mathcal{H} = \left(\begin{smallmatrix} L & W \\ -W & -L \end{smallmatrix}\right)$, one has

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} L & W \\ -W & -L \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} L & -W \\ W & -L \end{pmatrix} = \mathcal{H}^*$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} L & W \\ -W & -L \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -L & -W \\ W & L \end{pmatrix} = -\mathcal{H}.$$

∎

Remark 2.1. There is an alternative way to show that $\sigma_{\text{ess},5}(\mathcal{H}) = \sigma(\mathcal{H})$. Once one knows that $\rho(\mathcal{H}) \neq \emptyset$ one can use Remark 1.2.iv and the fact that the unperturbed operator $\mathcal{H}_0$ is self-adjoint with a gap in its spectrum to argue as follows: A simple calculation, using (2.2), gives

$$(\mathcal{H}-z)^{-1} - (\mathcal{H}_0-z)^{-1} = -(\mathcal{H}_0-z)^{-1}(1+V(\mathcal{H}_0-z)^{-1})^{-1}V(\mathcal{H}_0-z)^{-1},$$

for $z \in \rho(\mathcal{H}) \cap \rho(\mathcal{H}_0)$. Since the right hand side is a compact operator, a version of Weyl's criterion for suitable non-self-adjoint operators, Theorem

XIII.14 in [16], shows $\sigma(\mathcal{H}) \setminus \sigma_{\mathrm{disc}}(T) = \sigma(\mathcal{H}_0)$). In addition, this immediately gives that the spectrum of $\mathcal{H}$ outside of $\sigma(\mathcal{H}_0)$ has finite algebraic multiplicity since it is the discrete spectrum.

3. Exponential decay of eigenfunctions

We show in this section that every eigenfunction of $\mathcal{H}$ corresponding to an eigenvalue E with $-\mu < \mathrm{Re}E < \mu$ decays exponentially.

Let $\varphi = \binom{\varphi_1}{\varphi_2}$ be an eigenfunction with eigenvalue E, i.e., $\mathcal{H}\varphi = E\varphi$, or,

$$\begin{aligned} L\varphi_1 + W\varphi_2 &= E\varphi_1 \\ -W\varphi_1 - L\varphi_2 &= E\varphi_2. \end{aligned}$$

This can be rewritten as

$$\begin{pmatrix} L - E & W \\ W & L + E \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0.$$

Thus we are led to study the zero energy eigenfunctions of the energy dependent operator

$$\widehat{H}_E := \begin{pmatrix} L - E & W \\ W & L + E \end{pmatrix}.$$

Recalling $L = -\Delta + \mu + U$, we can write $\widehat{H}_E = \widehat{H}_{0,E} + \widehat{V}$ with

$$\widehat{H}_{0,E} := \begin{pmatrix} -\Delta + \mu - E & 0 \\ 0 & -\Delta + \mu + E \end{pmatrix} \quad \text{and} \quad \widehat{V} := \begin{pmatrix} U & W \\ W & U \end{pmatrix}.$$

Note that the eigenvalue E need not be real, since $\mathcal{H}$ is not a self-adjoint operator. This corresponds to the fact that, for complex E, the energy dependent operator $\widehat{H}_E$ will also not be self-adjoint. In this case, we have $\widehat{H}_E = \mathrm{Re}\widehat{H}_E + i\mathrm{Im}\widehat{H}_E$, where

$$\mathrm{Re}\widehat{H}_E = \begin{pmatrix} -\Delta + \mu - \mathrm{Re}E + U & W \\ W & -\Delta + \mu + \mathrm{Re}E + U \end{pmatrix}$$

and

$$\mathrm{Im}\widehat{H}_E = \begin{pmatrix} -\mathrm{Im}E & 0 \\ 0 & \mathrm{Im}E \end{pmatrix}.$$

To prove exponential decay of φ_1 and φ_2, we apply a modification of the Agmon method from the theory of Schrödinger operators, see, for example, [13], to the operator $\widehat{H}_E$. We need the following three preparatory lemmas.

Lemma 3.1. *Let $B_R^c = \{x \in \mathbb{R}^d : |x| \geq R\}$. Then*

$$\Sigma := \lim_{R\to\infty} \inf \left\{ \frac{\langle \varphi, \mathrm{Re}(\widehat{H}_E)\varphi \rangle}{\|\varphi\|^2} : \varphi \in H^2(\mathbb{R}^d, \mathbb{C}^2),\ \mathrm{supp}(\varphi) \subset B_R^c \right\} \geq \mu_E, \tag{3.1}$$

where we put $\mu_E := \mu - |\mathrm{Re}E|$.

Proof. Since $-\Delta \geq 0$, we obtain $\mathrm{Re}\widehat{H}_{0,E} \geq \mu_E$. Indeed, for any $\varphi \in Dom(\widehat{H}_0) = H^2(\mathbb{R}^d, \mathbb{C}^2)$,

$$\begin{aligned}\langle \varphi, \mathrm{Re}\widehat{H}_{0,E}\varphi\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^2)} &= \langle \varphi_1, (-\Delta + \mu - \mathrm{Re}E)\varphi_1\rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \langle \varphi_2, (-\Delta + \mu + \mathrm{Re}E)\varphi_2\rangle_{L^2(\mathbb{R}^d)} \\ &\geq (\mu - \mathrm{Re}E)\|\varphi_1\|_{L^2}^2 + (\mu + \mathrm{Re}E)\|\varphi_2\|_{L^2}^2 \\ &\geq \mu_E \|\varphi\|^2_{L^2(\mathbb{R}^d,C^2)}. \end{aligned} \tag{3.2}$$

To estimate $\langle \varphi, \widehat{V}\varphi\rangle$, note that the matrix $\widehat{V} = \left(\begin{smallmatrix} U & W \\ W & U \end{smallmatrix}\right)$ has eigenvalues $U \pm |W|$. Thus

$$\langle \varphi, \widehat{V}\varphi\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^2)} \geq \int_{\mathbb{R}^d} (U(x) - |W(x)|)(|\varphi_1(x)|^2 + |\varphi_2(x)|^2)\, dx.$$

By assumption A, the two functions U and W tend to zero at infinity, so for any $\varepsilon > 0$, there exists $R_\epsilon > 0$ such that $U(x) \geq -\varepsilon/2$ and $|W(x)| < \varepsilon/2$ whenever $|x| > R_\epsilon$. Using this and the above lower bound, one immediately gets for any φ with $\mathrm{supp}(\varphi) \subset B^c_{R_\epsilon}$,

$$\langle \varphi, \widehat{V}\varphi\rangle \geq -\epsilon\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^2)}. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\mathrm{Re}\langle \varphi, \widehat{H}_E\varphi\rangle = \langle \varphi, \mathrm{Re}\widehat{H}_{0,E}\varphi\rangle + \langle \varphi, \widehat{V}\varphi\rangle \geq (\mu_E - \epsilon)\|\varphi\|^2, \tag{3.4}$$

for any $\varphi \in Dom(\widehat{H})$ with $\mathrm{supp}(\varphi) \subset B^c_{R_\epsilon}$. Since the infimum in the right-hand side of (3.1) is increasing in R, (3.4) gives

$$\Sigma \geq \mu_E - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude (3.1). ∎

For the next lemma, we need a cut-off function j_R. Let $0 \leq j \leq 1$ with $j \in \mathcal{C}^\infty(\mathbb{R}_+)$ and $j(t) = 1$ for $0 \leq t \leq 1$ and $j = 0$ for $t \geq 2$ and put $j_R(t) = j(t/R)$. Moreover, let $\langle x\rangle = \sqrt{1 + |x|^2}$.

Lemma 3.2. *Let* $-\mu < \mathrm{Re}E < \mu$. *Then for any positive* $\delta < \mu_E/2$, *there exists* $R = R(\delta) > 0$ *such that with the cut-off function* $j = j_R$ *and uniformly in* $\varepsilon > 0$

$$\langle j\varphi, \big(\mathrm{Re}\widehat{H}_E - |\nabla f_\epsilon|^2\big) j\varphi\rangle \geq \delta\langle j\varphi, j\varphi\rangle,\ \varphi \in H^2(\mathbb{R}^d, \mathbb{C}^2),$$

where $f_\varepsilon(x) = \dfrac{\beta\langle x\rangle}{1 + \varepsilon\langle x\rangle}$ *with* $\beta = \sqrt{\mu_E - 2\delta}$.

Proof. By assumption, $\mu_E = \mu - |\mathrm{Re}E| > 0$. Pick any $0 < \delta < \mu_E/2$. By Lemma 3.1, there exists $R_\delta > 0$ such that for all $\varphi \in Dom(\widehat{H}_E)$ with $\mathrm{supp}(\varphi) \subset B^c_{R_\delta}$,

$$\langle \varphi, \mathrm{Re}\widehat{H}_E\varphi\rangle \geq (\mu_E - \delta)\langle \varphi, \varphi\rangle.$$

Thus, with the cut-off function $j_R = J_R(\delta)$, we obtain, for any $\varphi \in H^2(\mathbb{R}^d, \mathbb{C}^2)$,

$$\langle j_R\varphi, \mathrm{Re}\widehat{H}_E j_R\varphi\rangle \geq (\mu_E - \delta)\langle j_R\varphi, j_R\varphi\rangle.$$

Since $|\nabla f_\epsilon| \leq \beta$, we get

$$\langle j_R\varphi, \big(\mathrm{Re}\widehat{H}_E - |\nabla f_\epsilon|^2\big) j_R\varphi\rangle \geq (\mu_E - \delta - \beta^2)\langle j_R\varphi, j_R\varphi\rangle = \delta\langle j_R\varphi, j_R\varphi\rangle.$$

∎

Lemma 3.3. *If, in addition to the hypothesis of Lemma 3.2, $\widehat{H}_E\varphi = 0$, then*

$$\|j_R e^{f_\epsilon}\varphi\| \leq \delta^{-1}\|e^{f_\epsilon}[\widehat{H}_0, j_R]\varphi\| \tag{3.5}$$

where $[\widehat{H}_0, j_R] = \widehat{H}_0 j_R - j_R\widehat{H}_0$ and $\widehat{H}_0 = \left(\begin{smallmatrix} -\Delta+\mu & 0 \\ 0 & -\Delta+\mu \end{smallmatrix}\right)$.

Proof. Let $\mathcal{C}_b^\infty(\mathbb{R}^d)$ be the set of bounded, infinitely often differentiable functions. Note $e^{\pm g}\mathcal{D}(\widehat{H}_E) = \mathcal{D}(\widehat{H}_E)$ and, since $e^g\widehat{H}_E e^{-g} = e^g\mathrm{Re}\widehat{H}_E e^{-g} + i\mathrm{Im}\widehat{H}_E$, also

$$\mathrm{Re}\langle \psi, e^g\widehat{H}_E e^{-g}\psi\rangle = \langle \psi, \big(\mathrm{Re}\widehat{H}_E - |\nabla g|^2\big)\psi\rangle \tag{3.6}$$

for any $\psi \in Dom(\widehat{H}_E)$ and any real valued function $g \in \mathcal{C}_b^\infty(\mathbb{R}^d)$, see Appendix A.

Let $\widehat{H}_E\varphi = 0$. Since $f_\varepsilon \in \mathcal{C}_b^\infty(\mathbb{R}^d)$, the product $e^{f_\varepsilon}\varphi$ is in the domain of $\widehat{H}_E$. So we can apply Lemma 3.2 with φ replaced by $e^{f_\epsilon}\varphi$. Using (3.6), we obtain

$$\begin{aligned}\delta\|j_R e^{f_\epsilon}\varphi\|^2 &\leq \langle j_R e^{f_\epsilon}\varphi, \big(\mathrm{Re}\widehat{H}_E - |\nabla f_\epsilon|^2\big) j_R e^{f_\epsilon}\varphi\rangle \\ &= \mathrm{Re}\langle j_R e^{f_\epsilon}\varphi, e^{f_\epsilon}\widehat{H}_E e^{-f_\epsilon} j_R e^{f_\epsilon}\varphi\rangle \\ &= \mathrm{Re}\langle j_R e^{f_\epsilon}\varphi, e^{f_\epsilon}\widehat{H}_E j_R\varphi\rangle. \end{aligned} \tag{3.7}$$

As $\widehat{H}_E\varphi = 0$, the right hand side of (3.7) is equal to $\mathrm{Re}\langle j e^{f_\epsilon}\varphi, e^{f_\epsilon}[\widehat{H}_E, j]\varphi\rangle$. Then, by the Cauchy-Schwarz inequality and $[\widehat{H}_E, j] = [\widehat{H}_0, j]$, we conclude (3.5). ∎

The following corollary finishes the proof of Theorem 1.5 for eigenfunctions.

Corollary 3.4 (=Theorem 1.5 for eigenfunctions). *Let* $-\mu < \mathrm{Re}E < \mu$ *and* φ *an eigenfunction of zero energy for* $\widehat{H}_E$, *i.e.*, $\mathcal{H}\varphi = E\varphi$. *Then* φ *decays exponentially. More precisely, for all positive* $\delta < \frac{1}{2}(\mu - |\mathrm{Re}E|)$,

$$e^{(\sqrt{\mu-|\mathrm{Re}E|-2\delta})|x|}\varphi(x) \in L^2(\mathbb{R}^d, \mathbb{C}^2).$$

Proof. Simply note that $[\widehat{H}_0, j]$ is a first order differential operator concentrated on the annulus $R \le |x| \le 2R$. Indeed,

$$[\widehat{H}_0, j_R] = \begin{pmatrix} [-\Delta+\mu, j_R] & 0 \\ 0 & [-\Delta+\mu, j_R] \end{pmatrix} = \begin{pmatrix} [-\Delta, j_R] & 0 \\ 0 & [-\Delta, j_R] \end{pmatrix}$$
$$= \begin{pmatrix} (-\Delta j_R) - \nabla j_R \cdot \nabla & 0 \\ 0 & (-\Delta j_R) - \nabla j_R \cdot \nabla \end{pmatrix}$$

and j_R is constant outside the annulus $R \le |x| \le 2R$. Thus $e^{f_\epsilon}[-\Delta, j]$ is a bounded operator in $H^2(\mathbb{R}^d)$ with a uniform bound in ϵ. Hence also

$$\limsup_{\varepsilon\to 0} \|e^{f_\epsilon}[\widehat{H}_0, j]\varphi\| < \infty$$

for any $\varphi \in H^2(\mathbb{R}^d, \mathbb{C}^2)$. Since $f_\varepsilon \uparrow f$ as $\varepsilon \to 0$ we can use dominated convergence and (3.5) to conclude for any eigenfunction of $\mathcal{H}$ with energy E

$$\|e^{\sqrt{\mu_E-2\delta}\langle x\rangle} j\varphi\| = \lim_{\varepsilon\to 0} \|e^{f_\varepsilon} j\varphi\ \| < \infty.$$

for any $0 < \delta < \mu_E/2$, where $\mu_E = \mu - |\mathrm{Re}E| > 0$, by assumption. Since $j = 1$ outside a compact set, $e^{\sqrt{\mu_E-2\delta}\langle x\rangle}\varphi$ is square integrable on all of $\mathbb{R}^d$. ∎

4. Exponential decay of generalized eigenfunctions

The method in the previous section can be used to show that all generalized eigenfunctions decay exponentially. We need a little extension of Lemma 3.3. But first some more notation: For $\varphi \in L^2(\mathbb{R}^d, \mathbb{C}^2)$ let $\widetilde{\varphi} = \left(\begin{smallmatrix} \varphi_1 \\ -\varphi_2 \end{smallmatrix}\right)$. With this, we have the following

Lemma 4.1. *Let* $-\mu < \mathrm{Re}E < \mu$. *Assume that for some* $k \in \mathbb{N}$, $\psi_{l-1} \in Dom(\widehat{H}_E)^{k-(l-1)}$ *for* $l = 1, \ldots, k$ *with* $\tilde{\psi}_l = \widehat{H}_E\psi_{l-1}$. *Then for all positive* $\delta < \mu_E/2$ *we have*

$$\|je^{f_\varepsilon}\psi_0\| \le \sum_{l=0}^{k-1} \delta^{-(l+1)}\|e^{f_\varepsilon}[\widehat{H}_0, j]\psi_l\| + \delta^{-k}\|e^{f_\varepsilon} j\psi_k\| \tag{4.1}$$

Proof. It is enough to show that

$$\|je^{f_\varepsilon}\psi_{l-1}\| \le \delta^{-1}\big(\|e^{f_\varepsilon}[\widehat{H}_0, j]\psi_{l-1}\| + \|e^{f_\varepsilon} j\psi_l\|\big) \tag{4.2}$$

for $l = 1, \ldots, k$. Then (4.1) follows from iterating this bound.

Using Lemma 3.2 and the assumption $\widehat{H}_E\psi_{l-1} = \widetilde{\psi}_l$ we see

$$\begin{aligned}
\delta\|je^{f_\epsilon}\psi_{l-1}\|^2 &\le \langle je^{f_\epsilon}\psi_{l-1}, \big(\mathrm{Re}\widehat{H}_E - |\nabla f_\epsilon|^2\big)je^{f_\epsilon}\psi_{l-1}\rangle \\
&= \mathrm{Re}\langle je^{f_\epsilon}\psi_{l-1}, e^{f_\epsilon}\widehat{H}_E e^{-f_\epsilon} je^{f_\epsilon}\psi_{l-1}\rangle \\
&= \mathrm{Re}\langle je^{f_\epsilon}\psi_{l-1}, e^{f_\epsilon}\widehat{H}_E j\psi_{l-1}\rangle \\
&= \mathrm{Re}\langle je^{f_\epsilon}\psi_{l-1}, e^{f_\epsilon}[\widehat{H}_E, j]\psi_{l-1} + e^{f_\epsilon} j\widetilde{\psi}_l\rangle \\
&= \mathrm{Re}\langle je^{f_\epsilon}\psi_{l-1}, e^{f_\epsilon}[\widehat{H}_0, j]\psi_{l-1}\rangle + \mathrm{Re}\langle je^{f_\epsilon}\psi_{l-1}, je^{f_\epsilon}\widetilde{\psi}_l\rangle \\
&\le \|je^{f_\epsilon}\psi_{l-1}\|\Big\{\big\|e^{f_\epsilon}[\widehat{H}_0, j]\psi_{l-1}\big\| + \|je^{f_\epsilon}\widetilde{\psi}_l\|\Big\},
\end{aligned}$$

which gives (4.2), since $\|je^{f_\epsilon}\widetilde{\psi}_l\| = \|je^{f_\epsilon}\psi_l\|$. ∎

Corollary 4.2 (=Theorem 1.5 for generalized eigenfunctions)**.** *Let $E \in \mathbb{C}$ with $\mu < \mathrm{Re}E < \mu$ and φ be a generalized eigenfunction of $\mathcal{H}$ with eigenvalue E. Then, for all positive $\delta < \frac{1}{2}(\mu - |\mathrm{Re}E|)$,*

$$e^{(\sqrt{\mu-|\mathrm{Re}E|-2\delta})|x|}\varphi \in L^2(\mathbb{R}^d, \mathbb{C}^2).$$

Proof. Using Theorem 1.3, see also Remark 1.4.iii, we know that the generalized eigenspace corresponding to E is finite dimensional. Thus there is a $k \in \mathbb{N}$ such that $\ker(\mathcal{H} - E)^{m+1} = \ker(\mathcal{H} - E)^m$ for all $m \ge k$. So fix this k and assume that $(\mathcal{H} - E)^k\varphi = 0$. Put $\psi_l = (\mathcal{H} - E)^l\varphi$ and $\psi_0 = \varphi$. Then $\psi_l = (\mathcal{H} - E)\psi_{l-1}$. Or, in terms of the operator $\widehat{H}_E$,

$$\widetilde{\psi}_l = \widehat{H}_E\psi_{l-1}.$$

Note that in this case $\psi_k = 0$. So for all $0 < \delta < \mu_E/2$ and large enough R Lemma 4.1 gives for all $\varepsilon > 0$

$$\|je^{f_\varepsilon}\psi_0\| \le \sum_{l=0}^{k-1} \delta^{-(l+1)}\|e^{f_\varepsilon}[\widehat{H}_0, j]\psi_l\|.$$

Letting $\varepsilon \to 0$, as in the proof of Corollary 3.4, finishes the proof. ∎

Appendix A. Proof of equation (3.6)

Here we prove equation (3.6), that is, $\mathrm{Re}\langle\psi, e^g\widehat{H}_E e^{-g}\psi\rangle = \langle\psi, \big(\mathrm{Re}\widehat{H}_E - |\nabla g|^2\big)\psi\rangle$ for any $\psi \in Dom(\widehat{H}_E)$ and any real valued function $g \in \mathcal{C}_b^\infty(\mathbb{R}^d)$.

Proof. Since $\nabla(e^{-g}\psi) = e^{-g}\big(\nabla\psi - (\nabla g)\psi\big)$, we have $e^g\nabla e^{-g} = \nabla - \nabla g$. Thus,

$$\begin{aligned}
e^g(-\Delta)e^{-g} &= -(e^g\nabla e^{-g})^2 = -(\nabla - \nabla g)^2 = -\Delta + \nabla\cdot\nabla g + \nabla g\cdot\nabla - (\nabla g)^2 \\
&= -\Delta - |\nabla g|^2 + iB
\end{aligned}$$

where the operator $B = -i(\nabla \cdot \nabla g + \nabla g \cdot \nabla)$ is self-adjoint. Therefore,

$$\begin{aligned} e^g \widehat{H}_E e^{-g} &= \begin{pmatrix} e^g(-\Delta)e^{-g} + \mu - E & 0 \\ 0 & e^g(-\Delta)e^{-g} + \mu + E \end{pmatrix} + e^g \widehat{V} e^{-g} \\ &= \begin{pmatrix} -\Delta - |\nabla g|^2 + iB + \mu - E & 0 \\ 0 & -\Delta - |\nabla g|^2 + iB + \mu - E \end{pmatrix} + \widehat{V} \\ &= \widehat{H}_E - |\nabla g|^2 + iB. \end{aligned}$$

Taking the real part, one arrives at (3.6). ∎

Appendix B. On the equality of certain essential spectra

Let us now prove (1.5), that is, $\sigma(T) \setminus \sigma_{\mathrm{disc}}(T) = \sigma_{\mathrm{ess},5}(T)$ for any closed operator T in a Banach space X. Here $\sigma(T)$ is the complement of the resolvent set $\rho(T)$ and the discrete spectrum $\sigma_{\mathrm{disc}}(T)$ is the set of all isolated points in $\sigma(T)$ with finite *algebraic* multiplicity. Recall that in this case, the nullity and deficiency are given by

$$\mathrm{nul}(T) = \dim \ker(T)$$

and

$$\mathrm{def}(T) = \dim \big(X/\mathrm{ran}(T)\big).$$

From the definition of resolvent set, the set of all $z \in \mathbb{C}$ for which $T - z$ is a bijection (and hence boundedly invertible, by the inverse theorem), we have

$$\rho(T) = \{z \in \mathbb{C} |\, T - z \text{ is Fredholm with } \mathrm{nul}(T-z) = \mathrm{def}(T-z) = 0\}.$$

Recall that $\sigma_{\mathrm{ess},5}(T) = \mathbb{C} \setminus \Delta_5(T)$ with

$$\begin{aligned} \Delta_5(T) = \{z \in \mathbb{C} |\, T - z &\text{ is Fredholm with } \mathrm{ind}(T-z) = 0 \\ &\text{and a deleted neighborhood of } z \text{ is in } \rho(T)\}. \end{aligned}$$

A straightforward rewriting of this condition shows that

$$\begin{aligned} \Delta_5(T) = \rho(T) \cup \{\lambda |\, \lambda &\text{ is an isolated point in } \sigma(T) \text{ such that} \\ &T - \lambda \text{ is Fredholm with } \mathrm{ind}(T-\lambda) = 0\}. \end{aligned}$$

Thus to show (1.5) it is enough to prove

Lemma B.1. *Let T be a closed operator on some Banach space X. Then*

$$\begin{aligned} \sigma_{\mathrm{disc}}(T) = \{\lambda |\, \lambda &\textit{ is an isolated point in } \sigma(T) \textit{ such that} \\ &T - \lambda \textit{ is Fredholm with } \mathrm{ind}(T-\lambda) = 0\}. \end{aligned}$$

Proof. Let λ be an isolated point in $\sigma(T)$ such that $T-\lambda$ is Fredholm with index zero. Since $\lambda \notin \rho(T)$ and $\operatorname{ind}(T-z)=0$, we must have $0<\operatorname{nul}(T-\lambda)<\infty$, which implies λ is an eigenvalue of T with finite *geometric* multiplicity. Since $\operatorname{ran}(T-\lambda)$ is closed, Theorem IV-5.10 in conjunction with Theorem IV-5.28 in [14] shows that the algebraic multiplicity of λ must also be finite. Hence $\lambda \in \sigma_{\text{disc}}(T)$.

Conversely, let $\lambda \in \sigma_{\text{disc}}(T)$, i.e., λ be an isolated point in $\sigma(T)$ with finite *algebraic* multiplicity. We need to show that $T-\lambda$ is Fredholm with index zero. By Theorem III-6.17, together with Section III-6.5 in [14], there is a decomposition of $X = M' \oplus M''$ such that M' and M'' are reducing subspaces for T and $M' \cap M'' = 0$. In fact, if P_λ is the Riesz projection corresponding to λ, then $M' = \operatorname{ran}(P_\lambda)$ and $M'' = \operatorname{ran}(1-P_\lambda)$. Furthermore, $T-\lambda$ restricted to M' is bounded and quasi-nilpotent and $T-\lambda$ restricted to M'' is bijective. Note that λ having finite algebraic multiplicity is equivalent to M' being finite-dimensional. In particular, $\operatorname{ran}(T-\lambda) = \operatorname{ran}((T-\lambda)|_{M'}) \oplus M''$ is closed. One has

$$\ker(T-\lambda) = \ker((T-\lambda)|_{M'}) \oplus \ker((T-\lambda)|_{M''})$$

and

$$X/(T-\lambda) = M'/(T-\lambda)|_{M'} \oplus M''/(T-\lambda)|_{M''}.$$

Hence,

$$\operatorname{nul}(T-\lambda) = \operatorname{nul}((T-\lambda)|_{M'}) + \operatorname{nul}((T-\lambda)|_{M''})$$

and

$$\operatorname{def}(T-\lambda) = \operatorname{def}((T-\lambda)|_{M'}) + \operatorname{def}((T-\lambda)|_{M''}).$$

Since $(T-\lambda)|_{M''}$ is a bijection, we know that the second terms above are zero, that is, $\operatorname{nul}(T-\lambda) = \operatorname{nul}((T-\lambda)|_{M'})$ and $\operatorname{def}(T-\lambda) = \operatorname{def}((T-\lambda)|_{M'})$. Moreover, both are finite, since M' is finite dimensional, and the well-known dimension formula from finite dimensional linear algebra shows that

$$\operatorname{nul}((T-\lambda)|_{M'}) = \operatorname{def}((T-\lambda)|_{M'}).$$

Thus $T-\lambda$ is indeed Fredholm with index zero.

∎

Remark B.2. Let us also remark on the original definition of essential spectrum by Browder, see Definition 11 on page 107 in [2]. Browder defines $\sigma_{\text{ess},B}(T) = \mathbb{C} \setminus \Delta_B(T)$ with

$$\Delta_B(T) = \{z \in C |\, \operatorname{ran}(T-z) \text{ is closed, } z \text{ is of finite algebraic multiplicity, and } z \text{ is not a limit point of } \sigma(T)\}.$$

One can rewrite this as

$$\Delta_B(T) = \rho(T) \cup \{z \in C \,|\, \mathrm{ran}(T - z) \text{ is closed and } z \text{ is an isolated point in } \sigma(T) \text{ with finite algebraic multiplicity}\}.$$

As shown in the proof of Lemma B.1, for any isolated point $z \in \sigma(T)$ with finite algebraic multiplicity, $\mathrm{ran}(T - z)$ is always closed. Thus, in fact,

$$\begin{aligned}\Delta_B(T) &= \rho(T) \cup \{z \in C \,|\, z \text{ is an isolated point in } \sigma(T) \\ &\qquad\qquad \text{with finite algebraic multiplicity}\} \\ &= \rho(T) \cup \sigma_{\mathrm{disc}}(T),\end{aligned}$$

where the last equality is due to Lemma B.1. Hence Browder's original definition indeed gives the same essential spectrum as $\sigma_{\mathrm{ess},5}(T)$.

References

[1] V. S. Buslaev and G. S. Perel′man, *Scattering for the nonlinear Schrödinger equation: states that are close to a soliton,* (Russian) Algebra i Analiz **4** (1992), no. 6, 63–102; translation in St. Petersburg Math. J. **4** (1993), no. 6, 1111–1142

[2] F. E. Browder, *On the spectral theory of elliptic differential operators,* Math. Ann. **142** (1961), 22–130.

[3] S.-M. Chang, S. Gustafson, K. Nakanishi, and T.-P. Tsai, *Spectra of linearized operators for NLS solitary waves,* preprint math.AP/0611483 (2006) at xxx.lanl.gov.

[4] S. Cuccagna, D. Pelinovsky, and V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem,* Comm. Pure Appl. Math. **58** (2005), no. 1, 1–29.

[5] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry,* Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin, 1987.

[6] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators,* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1987.

[7] M. B. Erdogan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: II,* to appear in Journal d'Analyse Math.

[8] M. Grillakis, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system.* In Integrable systems and applications (Ile d'Oléron, 1988), 154–191, Lecture Notes in Phys., **342**, Springer, Berlin, 1989.

[9] M. Grillakis, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system.* Comm. Pure Appl. Math. *43* (1990), no. 3, 299–333.

[10] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I.* J. Funct. Anal. **74** (1987), no. 1, 160–197.
[11] K. Gustafson and J. Weidmann *On the essential spectrum,* J. Math. Anal. Appl. **25** (1969), 121–127.
[12] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory. With applications to Schrödinger operators,* Applied Mathematical Sciences, **113**. Springer-Verlag, New York, 1996.
[13] W. Hunziker and I. M. Sigal, *The quantum N-body problem,* J. Math. Phys. **41** (2000), no. 6, 3448–3510.
[14] T. Kato, *Perturbation theory for linear operators.* Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
[15] M. Reed and B. Simon, *Methods of modern mathematical physics I. Functional Analysis,* Revised and enlarged edition. Academic Press, San Diego, 1980.
[16] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, New York-London, 1978.
[17] I. Rodnianski, W. Schlag, and A. Soffer, *Asymptotic stability of N-Soliton states of NLS,* preprint math.AP/0309114 (2003) at xxx.lanl.gov.
[18] M. Schechter, *Spectra of Partial Differential Operators,* North Holland, Amsterdam 1971.
[19] F. Stummel, *Singuläre elliptische Differentialoperatoren in Hilbertschen Räumen,* Math. Ann. **132** (1956), 150–176.
[20] W. Schlag, *Stable manifolds for an orbitally unstable NLS,* to appear in Annals of Math.
[21] V. Vougalter and D. Pelinovsky, *Eigenvalues of zero energy in the linearized NLS problem,* Journal of Mathematical Physics **47** (2006), no. 6, 062701, 13pp.
[22] M. I. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations,* SIAM J. Math. Anal. **16** (1985), no. 3, 472–491.
[23] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations,* Comm. Pure Appl. Math. **39** (1986), no. 1, 51–67.

School of Mathematics, Watson Building, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK. On leave from Department of Mathematics, Altgeld Hall, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801.
E-mail address: `hundertd@maths.bham.ac.uk and dirk@math.uiuc.edu`

Department of Mathematics, Altgeld Hall, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801.
E-mail address: `yrlee4@math.uiuc.edu`