RELATIVE OSCILLATION THEORY, WEIGHTED ZEROS OF THE WRONSKIAN, AND THE SPECTRAL SHIFT FUNCTION

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Abstract. We develop an analog of classical oscillation theory for Sturm–Liouville operators which, rather than measuring the spectrum of one single operator, measures the difference between the spectra of two different operators.

This is done by replacing zeros of solutions of one operator by weighted zeros of Wronskians of solutions of two different operators. In particular, we show that a Sturm-type comparison theorem still holds in this situation and demonstrate how this can be used to investigate the finiteness of eigenvalues in essential spectral gaps. Furthermore, the connection with Krein's spectral shift function is established.

1. Introduction

This paper is concerned with oscillation theory for Sturm–Liouville operators which was originated with Sturm's celebrated memoir [29] over a hundred and seventy years ago. Since then many extensions have been made, which are far too numerous to be listed here. Rather we refer to the recent survey by Simon [27] or the recent monograph by Zettl [41].

However, there are still open problems remaining: While classical oscillation theory works perfectly below the essential spectrum, its applicability in essential spectral gaps is still limited. The aim of the present paper is to fill this gap and provide a new powerful oscillation theory which works inside essential spectral gaps.

Around 1948 Hartman and others extended classical oscillation theory to singular Sturm–Liouville operators by showing the following in a series of papers ([7], [8], [9]): For a given Sturm–Liouville operator $H = -\frac{d^2}{dx^2} + q(x)$ on $L^2(0,\infty)$ with Dirichlet boundary condition, let $\psi(\lambda, x)$ be the solution of $H\psi = \lambda\psi$ satisfying the Dirichlet boundary condition. Then the dimension of the spectral projection $P_{(-\infty,\lambda)}(H)$ equals the number of zeros of $\psi(\lambda)$ in $(0,\infty)$. Of course a naive use of this result to count the number of eigenvalues inside an essential spectral gap would just give $\infty - \infty$. However, they also showed that this can be avoided by taking appropriate limits. More precisely, the dimension of $P_{(\lambda_0,\lambda_1)}(H)$ can be obtained by considering the difference of the number of zeros of the solutions $\psi(\lambda_1)$ and $\psi(\lambda_0)$ inside a finite interval $(0,b)$, and performing a limit $b \to \infty$. Further important extensions were later given by Weidmann in [37] (see also his lecture notes [38]). Only recently it was shown in [5] by Gesztesy, Simon, and one of us, that these limits can be avoided by using a renormalized version of oscillation theory, that is,

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by counting zeros of Wronskians of solutions instead (see again [27] for a pedagogical discussion respectively [23] for an interesting application to minimal surfaces).

So there are several tools to count eigenvalues in essential spectral gaps and one might ask what could be missing. To this end observe that all the previously mentioned results have one thing in common, they all work with only one operator H. One the other hand, a typical question inside an essential spectral gap is, what kind of perturbations H_1 of a given operator H_0 produce a finite/infinite number of eigenvalues. Such kind of problems clearly ask for an oscillation theory which compares the spectra of two given operators H_0 and H_1. To the best of our knowledge, such an oscillation theory was not available to date. We will demonstrate in this paper, that the right object to look at are zeros of Wronskians of suitable solutions of H_1 respectively H_0. In addition, since $H_1 - H_0$ is not necessarily of one sign, we will weight the zeros according to the sign of $H_1 - H_0$ at a zero (see Section 2 below). This will allow us to develop an analog of Sturm's oscillation theory comparing the spectra of two operators including an analog of Sturm's comparison theorem, which is applicable inside spectral gaps.

To further illustrate this, let us look at Kneser's result ([11], see also [4] for generalizations). Let

$$\tau = -\frac{d^2}{dx^2} + q(x) \tag{1.1}$$

be a Sturm–Liouville operator on the interval $(0, \infty)$ with $q(x) \to 0$ as $x \to \infty$. Then the bottom of the essential spectrum is an accumulation point of eigenvalues if

$$\liminf_{x\to\infty} -4x^2 q(x) > 1 \tag{1.2}$$

and it is no accumulation point if

$$\limsup_{x\to\infty} -4x^2 q(x) < 1. \tag{1.3}$$

The proof consists of the observation that the Euler equation

$$\tau_0 = -\frac{d^2}{dx^2} + \frac{\mu}{x^2} \tag{1.4}$$

is explicitly solvable plus an application of Sturm's comparison theorem between τ_0 and τ.

The corresponding result within spectral gaps is Rofe-Beketov's theorem for periodic operators ([20]). We state it in the version of [21], where a detailed proof is given. Suppose $\lim_{x\to\infty} x^2(q(x) - q_0(x))$ exists. Then a boundary point E_n of the essential spectrum is an accumulation points of eigenvalues if

$$\lim_{x\to\infty} \kappa_n x^2 (q(x) - q_0(x)) > 1 \tag{1.5}$$

and no accumulation point if

$$\lim_{x\to\infty} \kappa_n x^2 (q(x) - q_0(x)) < 1, \tag{1.6}$$

where the critical constant κ_n will be defined in Section 4 below.

Comparing this result with Kneser's result, there is an obvious mismatch: While Kneser's result covers all potentials which are above respectively below the critical case $q(x) = \frac{-1}{4x^2}$, the above result only covers the case where $q(x)$ has a precise asymptotic $q(x) = q_0(x) + \frac{c}{x^2} + o(x^{-2})$. As an application of our theory we will show how a comparison argument analogous to the one for Kneser's theorem, can be used to fill this gap.

We expect our results to be the beginning of a new branch in oscillation theory and hope that it will stimulate further research in this direction. Some possible extensions are listed at the end of Section 3.

2. Weighted zeros of Wronskians, Prüfer angles, and regular operators

The key ingredient of our new theory will be roughly speaking weighted zeros of Wronskians of solutions of different Sturm-Liouville operators. However, this naive definition has a few subtle problems which we need to clarify first.

To set the stage, we will consider Sturm-Liouville operators on $L^2((a,b), r\,dx)$ with $-\infty \le a < b \le \infty$ of the form

$$\tau = \frac{1}{r}\Big(- \frac{d}{dx} p \frac{d}{dx} + q \Big), \tag{2.1}$$

where the coefficients p, q, r are real-valued satisfying

$$p^{-1}, q, r \in L^1_{loc}(a,b), \quad p, r > 0. \tag{2.2}$$

We will use τ to describe the formal differentiation expression and H the operator given by τ with separated boundary conditions at a and/or b.

If a (resp. b) is finite and q, p^{-1}, r are in addition integrable near a (resp. b), we will say a (resp. b) is a *regular* endpoint. We will say τ respectively H is *regular* if both a and b are regular.

For every $z \in \mathbb{C}\backslash\sigma_{ess}(H)$ there is a unique (up to a constant) solution $\psi_-(z,x)$ of $\tau u = zu$ which is in L^2 near a and satisfies the boundary condition at a (if any). Similarly there is such a solution $\psi_+(z,x)$ near b.

One of our main objects will be the (modified) Wronskian

$$W_x(u_0, u_1) = u_0(x)\, p(x) u_1'(x) - p(x) u_0'(x)\, u_1(x) \tag{2.3}$$

of two functions u_0, u_1 and its zeros. Here we think of u_0 and u_1 as two solutions of two different Sturm–Liouville equations

$$\tau_j = \frac{1}{r}\Big(- \frac{d}{dx} p \frac{d}{dx} + q_j \Big), \qquad j = 0, 1. \tag{2.4}$$

Note that we have chosen $p_0 = p_1 \equiv p$ here. The case $p_0 \neq p_1$ will be given in [14].

Classical oscillation theory counts the zeros of solutions u_j, which are always simple since u and pu' cannot both vanish except if $u \equiv 0$. This is no longer true for the zeros of the Wronskian $W_x(u_0, u_1)$. In fact, it could even happen that the Wronskian vanishes on an entire interval if $q_0 = q_1$ on this interval (cf. (2.8)). Such a situation will be counted as just one zero, in other words, we will only count sign flips.

Furthermore, classical oscillation theory involves the spectrum of only one operator, but we want to measure the difference between the spectra of two operators. That is, we need a signed quantity. Hence we will weight the sign flips according to the sign of $q_1 - q_0$. Of course this *definition* is not good enough for the case $q_1 - q_0 \in L^1_{loc}$ considered here. Hence we will give a precise definition in terms of Prüfer variables next.

We begin by recalling the definition of Prüfer variables ρ_u, θ_u of an absolutely continuous function u:

$$u(x) = \rho_u(x)\sin(\theta_u(x)) \qquad p(x)u'(x) = \rho_u(x)\cos(\theta_u(x)). \tag{2.5}$$

If $(u(x), p(x)u'(x))$ is never $(0,0)$ and u, pu' are absolutely continuous, then ρ_u is positive and θ_u is uniquely determined once a value of $\theta_u(x_0)$ is chosen by requiring continuity of θ_u.

Notice that

$$W_x(u,v) = -\rho_u(x)\rho_v(x)\sin(\Delta_{v,u}(x)), \qquad \Delta_{v,u}(x) = \theta_v(x) - \theta_u(x). \tag{2.6}$$

Hence the Wronskian vanishes if and only if the two Prüfer angles differ by a multiple of π. We will call the total difference

$$\#_{(c,d)}(u_0,u_1) = \lceil \Delta_{1,0}(d)/\pi \rceil - \lfloor \Delta_{1,0}(c)/\pi \rfloor - 1 \tag{2.7}$$

the number of weighted sign flips in (c,d), where we have written $\Delta_{1,0}(x) = \Delta_{u_1,u_0}$ for brevity.

Next, let us show that this agrees with our considerations above. We take two solutions u_j, $j = 1,2$, of $\tau_j u_j = \lambda_j u_j$ and associated Prüfer variables ρ_j, θ_j. Since we can replace $q \to q - \lambda r$ it is no restriction to assume $\lambda_0 = \lambda_1 = 0$.

Under these assumptions $W_x(u_0,u_1)$ is absolutely continuous and satisfies

$$W_x'(u_0,u_1) = -(q_0(x) - q_1(x))u_0(x)u_1(x). \tag{2.8}$$

Lemma 2.1. *Abbreviate $\Delta_{1,0}(x) = \theta_1(x) - \theta_0(x)$ and suppose $\Delta_{1,0}(x_0) \equiv 0 \mod \pi$. If $q_0(x) - q_1(x)$ is (i) negative, (ii) zero, or (iii) positive for a.e. $x \in (x_0, x_0 + \varepsilon)$ respectively for a.e. $x \in (x_0 - \varepsilon, x_0)$, then the same is true for $(\Delta_{1,0}(x) - \Delta_{1,0}(x_0))/(x - x_0)$.*

Proof. By (2.8) we have

$$\begin{aligned} W_x(u_0,u_1) &= -\rho_0(x)\rho_1(x)\sin(\Delta_{1,0}(x)) \\ &= -\int_{x_0}^x (q_0(t) - q_1(t))u_0(t)u_1(t)dt \end{aligned} \tag{2.9}$$

and there are two cases to distinguish: Either $u_0(x_0)$, $u_1(x_0)$ are both different from zero or both equal to zero. If both are equal to zero, they must change sign at x_0 and hence in both cases the integrand is of one sign. Thus the result follows. □

Hence $\#_{(c,d)}(u_0,u_1)$ counts the weighted sign flips of the Wronskian $W_x(u_0,u_1)$, where a sign flip is counted as $+1$ if $q_0 - q_1$ is positive in a neighborhood of the sign flip, it is counted as -1 if $q_0 - q_1$ is negative in a neighborhood of the sign flip. If $q_0 - q_1$ changes sign (i.e., it is positive on one side and negative on the other) the Wronskian will not change its sign. In particular, we obtain

Lemma 2.2. *Let u_0, u_1 solve $\tau_j u_j = 0$, $j = 0,1$, $q_0 - q_1 \geq 0$. Then $\#_{(a,b)}(u_0,u_1)$ equals the number sign flips of $W(u_0,u_1)$ inside the interval (a,b).*

In the case $q_0 - q_1 \leq 0$ we get of course the corresponding negative number except for the fact that zeros at the boundary points are counted as well since $\lfloor -x \rfloor = -\lceil x \rceil$. That is, if $q_0 - q_1 > 0$, then $\#_{(c,d)}(u_0,u_1)$ equals the number of zeros of the Wronskian in (c,d) while if $q_0 - q_1 < 0$, it equals minus the number of zeros in $[c,d]$. In the next theorem we will see that this is quite natural. In addition, note that $\#(u,u) = -1$.

Finally, we establish the connection with the spectrum of regular operators. For this let H_0, H_1 be self-adjoint extensions of τ_0, τ_1 with the same separated boundary conditions.

Theorem 2.3. *Let H_0, H_1 be regular Sturm–Liouville operators associated with (2.4) and the same boundary conditions at a and b. Then*

(2.10)
$$\dim\operatorname{Ran} P_{(-\infty,\lambda_1)}(H_1) - \dim\operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#_{(a,b)}(\psi_{0,\pm}(\lambda_0), \psi_{1,\mp}(\lambda_1)).$$

The proof will be given in Section 5 employing interpolation between H_0 and H_1, using $H_\varepsilon = (1-\varepsilon)H_0 + \varepsilon H_1$ together with a careful analysis of Prüfer angles.

It is important to observe that in the special case $H_1 = H_0$, the left-hand side equals $\dim\operatorname{Ran} P_{(\lambda_1,\lambda_0)}(H_0)$ if $\lambda_1 > \lambda_0$ and $-\dim\operatorname{Ran} P_{[\lambda_0,\lambda_1]}(H_0)$ if $\lambda_1 < \lambda_0$. This is of course in accordance with our previous observation that $\#(\psi_{0,\pm}(\lambda_0), \psi_{1,\mp}(\lambda_1))$ equals the number of zeros in (a,b) if $\lambda_1 > \lambda_0$ while it equals minus the numbers of zeros in $[a,b]$ if $\lambda_1 < \lambda_0$.

3. Relative Oscillation Theory

After these preparations we are now ready to develop what we call relative oscillation theory and extend Theorem 2.3 to singular operators.

Definition 3.1. *For τ_0, τ_1 possibly singular Sturm–Liouville operators as in (2.4) on (a,b), we define*

$$\underline{\#}(u_0,u_1) = \liminf_{d\uparrow b,\, c\downarrow a} \#_{(c,d)}(u_0,u_1) \quad \text{and} \quad \overline{\#}(u_0,u_1) = \limsup_{d\uparrow b,\, c\downarrow a} \#_{(c,d)}(u_0,u_1), \tag{3.1}$$

where $\tau_j u_j = \lambda_j u_j$, $j = 0,1$.

We say that $\#(u_0,u_1)$ exists, if $\overline{\#}(u_0,u_1) = \underline{\#}(u_0,u_1)$, and write

$$\#(u_0,u_1) = \overline{\#}(u_0,u_1) = \underline{\#}(u_0,u_1). \tag{3.2}$$

in this case.

By Lemma 2.1 one infers that $\#(u_0,u_1)$ exists if $q_0 - \lambda_0 r - q_1 + \lambda_1 r$ has the same definite sign near the endpoints a and b. On the other hand, note that $\#(u_0,u_1)$ might not exist even if both a and b are regular, since the difference of Prüfer angles might oscillate around a multiple of π near an endpoint. Furthermore, even if it exists, one has $\#(u_0,u_1) = \#_{(a,b)}(u_0,u_1)$ only if there are no zeros at the endpoints (or if $q_0 - \lambda_0 r - q_1 + \lambda_1 r \geq 0$ at least near the endpoints).

Remark 3.2. *Note that cases like $\underline{\#}(u_0,u_1) = -\infty$ and $\overline{\#}(u_0,u_1) = +\infty$ can occur. To construct such a situation let $\tau_j = -\frac{d^2}{dx^2} + q_j$ on $(0,\infty)$ and $\lambda_0 = \lambda_1 = 0$. Let $u_j(x)$ be the solutions satisfying a Neumann boundary condition at $u_j'(0) = 0$.*

Choose $q_0(x) = 0$, $q_1(x) = 1$ on $(0,\pi)$ such that $\theta_1(\pi) - \theta_0(\pi) = \frac{3\pi}{2} - \frac{\pi}{2} = \pi$. Next, choose $q_0(x) = 1$, $q_1(x) = 0$ on $(\pi, 3\pi)$ such that $\theta_1(3\pi) - \theta_0(3\pi) = \frac{3\pi}{2} - \frac{5\pi}{2} = -\pi$. Next, choose $q_0(x) = 0$, $q_1(x) = 1$ on $(3\pi, 6\pi)$ such that $\theta_1(6\pi) - \theta_0(6\pi) = \frac{9\pi}{2} - \frac{5\pi}{2} = 2\pi$. Etc.

We begin with our analog of Sturm's comparison theorem for zeros of Wronskians. We will also establish a triangle-type inequality which will help us to provide streamlined proofs below. As with Sturm's comparison theorem, the proofs are elementary.

Theorem 3.3 (Comparison theorem for Wronskians)**.** *Suppose u_j satisfies $\tau_j u_j = \lambda_j u_j$, $j = 0, 1, 2$, where $\lambda_0 r - q_0 \le \lambda_1 r - q_1 \le \lambda_2 r - q_2$.*

If $c < d$ are two zeros of $W_x(u_0, u_1)$ such that $W_x(u_0, u_1)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0, u_2)$ in (c, d). Similarly, if $c < d$ are two zeros of $W_x(u_1, u_2)$ such that $W_x(u_1, u_2)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0, u_2)$ in (c, d).

Proof. Let c, d be two consecutive zeros of $W_x(u_0, u_1)$. We first assume that $W_c(u_0, v_2) = 0$ and consider $\tau_\varepsilon = (2-\varepsilon)\tau_1 + (\varepsilon - 1)\tau_2$, $\varepsilon \in [1, 2]$, restricted to (c, d) with boundary condition generated by the Prüfer angle of u_0 at c. Set $u_\varepsilon = \psi_{\varepsilon,-}$, then we have $\Delta_{u_0,u_\varepsilon}(c) = 0$ for all ε and $\Delta_{u,v_\varepsilon}(d)$ is increasing, implying that $W_x(u, v_\varepsilon)$ has at least one sign flip in (c, d) for $\varepsilon > 1$.

To finish our proof, let $\tilde{u}_2$ be a second linearly independent solution. Then, since $W(u_2, \tilde{u}_2)$ is constant, we can assume $0 < \Delta_{\tilde{u}_2,u_2}(x) < \pi$. This implies $\Delta_{\tilde{u}_2,u_0}(c) = \Delta_{\tilde{u}_2,u_2}(c) < \pi$ and $\Delta_{\tilde{u}_2,u_0}(d) = \Delta_{\tilde{u}_2,u_2}(d) + \Delta_{u_2,u_0}(d) > \pi$. Consequently $W_x(u_0, \tilde{u}_2)$ also has at least one sign flip in (c, d).

The second claim is proven analogously. □

Theorem 3.4 (Triangle inequality for Wronskians)**.** *Suppose u_j, $j = 0, 1, 2$ are given functions with u_j, pu_j' absolutely continuous and $(u_j(x), p(x)u_j'(x)) \ne (0, 0)$ for all x. Then*

$$\underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) - 1 \le \underline{\#}(u_0, u_2) \le \underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) + 1 \tag{3.3}$$

and similarly for $\underline{\#}$ replaced by $\overline{\#}$.

Proof. Take $a < c < d < b$. By definition

$$\#_{(c,d)}(u_0, u_2) = \lceil \Delta_{2,0}(d)/\pi \rceil - \lfloor \Delta_{2,0}(c)/\pi \rfloor - 1$$

and using $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$ respectively $\lceil x \rceil + \lceil y \rceil - 1 \le \lceil x + y \rceil \le \lceil x \rceil + \lceil y \rceil$ and $\Delta_{2,0} = \Delta_{2,1} + \Delta_{1,0}$, we obtain

$$\#_{(c,d)}(u_0, u_2) \le \#_{(c,d)}(u_0, u_1) + \#_{(c,d)}(u_1, u_2) + 1.$$

Thus the result follows by taking the limits $c \downarrow a$ and $d \uparrow b$. □

Having these results at our disposal, we will now develop what we call relative oscillation theory. We recall that in classical oscillation theory τ is called oscillatory if a solution of $\tau u = 0$ has infinitely many zeros.

Definition 3.5. *We call τ_1 relatively nonoscillatory with respect to τ_0, if the quantities $\underline{\#}(u_0, u_1)$ and $\overline{\#}(u_0, u_1)$ are finite for all solutions $\tau_j u_j = 0$, $j = 0, 1$.*

We call τ_1 relatively oscillatory with respect to τ_0, if one of the quantities $\underline{\#}(u_0, u_1)$ or $\overline{\#}(u_0, u_1)$ is infinite for some solutions $\tau_j u_j = 0$, $j = 0, 1$.

Note that this definition is in fact independent of the solutions chosen as a straightforward application of our triangle inequality (cf. Theorem 3.4) shows.

Corollary 3.6. *Let $\tau_j u_j = \tau_j v_j = 0$, $j = 0, 1$. Then*

$$|\underline{\#}(u_0, u_1) - \underline{\#}(v_0, v_1)| \le 4, \quad |\overline{\#}(u_0, u_1) - \overline{\#}(v_0, v_1)| \le 4. \tag{3.4}$$

Proof. By our comparison theorem we have $|\#(u_j, v_j)| \le 1$, $j = 0, 1$. Now use the triangle inequality, twice. □

The bounds can be improved using our comparison theorem for Wronskians to be ≤ 2 in the case of perturbations of definite sign.

If τ_0 is nonoscillatory our definition reduces to the classical one.

Lemma 3.7. *Suppose τ_0 is a nonoscillatory operator, then τ_1 is relatively nonoscillatory (resp. oscillatory) with respect to τ_0, if and only if τ_1 is nonoscillatory (resp. oscillatory).*

Proof. This follows by taking limits in

$$|\#_{(c,d)}(u,v) - \#_{(c,d)}(u) + \#_{(c,d)}(v)| \leq 2,$$

where $\#_{(c,d)}(u) = \lceil \theta_u(d)/\pi \rceil - \lfloor \theta_u(c)/\pi \rfloor - 1$ is the number of zeros of u inside (c,d). □

To demonstrate the usefulness of Definition 3.5, we now establish its connection with the spectra of self-adjoint operators associated with τ_j, $j = 0,1$.

Theorem 3.8. *Let H_j be self-adjoint operators associated with τ_j, $j = 0,1$. Then*

(i) *$\tau_0 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_1$ if and only if $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$.*

(ii) *Suppose $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$ and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for one $\lambda \in [\lambda_0, \lambda_1]$. Then it is relatively nonoscillatory for all $\lambda \in [\lambda_0, \lambda_1]$ if and only if $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) < \infty$.*

Proof. (i) This is item (i) of [5, Thm.7.5]. (ii) Let $\lambda, \tilde{\lambda} \in [\lambda_0, \lambda_1]$, $\tau_j u_j(\lambda) = \lambda u_j(\lambda)$, $j = 0,1$ and suppose we are relatively nonoscillatory at λ. Then applying our triangle inequality twice implies

$$\begin{aligned}\overline{\#}(u_0(\tilde{\lambda}), u_1(\tilde{\lambda})) &\leq \overline{\#}(u_0(\tilde{\lambda}), u_1(\lambda)) + \#(u_1(\lambda), u_1(\tilde{\lambda})) + 1 \\ &\leq \#(u_0(\tilde{\lambda}), u_0(\lambda)) + \overline{\#}(u_0(\lambda), u_1(\lambda)) + \#(u_1(\lambda), u_1(\tilde{\lambda})) + 2,\end{aligned}$$

and similar estimates with the roles of λ and $\tilde{\lambda}$ interchanged and $\overline{\#}$ replaced by $\underline{\#}$. Hence if $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) < \infty$ we are relatively nonoscillatory by (i). The converse direction is proven analogously. □

We remark that item (i), which corresponds to the case of equal operators $H_0 = H_1$ but different spectral parameters $\lambda_0 \neq \lambda_1$, is what was called renormalized oscillation theory by Gesztesy, Simon, and Teschl in [5]. It also follows from earlier results by Weidmann [37, Satz 4.1] (see also [38]), which was pointed out in the appendix of [21] and was called relative oscillation theorem there. An argument in the spirit of item (ii) was also one of the key ingredients in Rofe-Beketov's original work (cf. [20]).

For a practical application of this theorem one needs of course criteria when $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for λ inside an essential spectral gap.

Lemma 3.9. *Let $\lim_{x\to a} r(x)^{-1}(q_0(x) - q_1(x)) = 0$ if a is singular, and similarly, $\lim_{x\to b} r(x)^{-1}(q_0(x) - q_1(x)) = 0$ if b is singular. Then $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for $\lambda \in \mathbb{R} \backslash \sigma_{ess}(H_0)$.*

Proof. Since τ_1 can be written as $\tau_1 = \tau_0 + \tilde{q}_0 + \tilde{q}_1$, where $\tilde{q}_0$ has compact support near singular endpoints and $|\tilde{q}_1| < \varepsilon$, for arbitrarily small $\varepsilon > 0$, we infer that

$R_{H_1}(z) - R_{H_0}(z)$ is the norm limit of compact operators. Thus $R_{H_1}(z) - R_{H_0}(z)$ is compact and hence $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$.

Let $\delta > 0$ be the distance of λ to the essential spectrum and choose $a < c < d < b$, such that

$$|r^{-1}(q_1(x) - q_0(x))| \le \delta/2, \qquad x \notin (c,d).$$

Clearly $\#_{(c,d)}(u_0, u_1) < \infty$, since both operators are regular on (c,d). Moreover, observe that

$$q_0 - \lambda_+ r \le q_1 - \lambda r \le q_0 - \lambda_- r, \qquad \lambda_\pm = \lambda \pm \delta/2,$$

on $I = (a,c)$ or $I = (d,b)$. Then Theorem 3.8 (i) implies $\#_I(u_0(\lambda_-), u_0(\lambda_+)) < \infty$ and invoking Theorem 3.3 shows $\#_I(u_0(\lambda_\pm), u_1(\lambda)) < \infty$. From Theorem 3.4 and 3.8 (i) we infer

$$\overline{\#}_I(u_0(\lambda), u_1(\lambda)) < \#(u_0(\lambda), u_0(\lambda_+)) + \#_I(u_0(\lambda_+), u_1(\lambda)) + 1 < \infty$$

and similarly for $\underline{\#}_I(u_0(\lambda), u_1(\lambda))$. This shows that $\tau_1 - \lambda$ is relatively nonoscillatory with respect to τ_0. □

Our next task is to reveal the precise relation between the number of weighted sign flips and the spectra of H_1 and H_0. The special case $H_0 = H_1$ is covered by [5]:

Theorem 3.10 ([5]). *Let H_0 be a self-adjoint operator associated with τ_0 and suppose $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) = \#(\psi_{0,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_1)). \tag{3.5}$$

Combining this result with our triangle inequality already gives some rough estimates.

Lemma 3.11. *Let H_0, H_1 be self-adjoint operators associated with τ_0, τ_1, respectively, and separated boundary conditions. Suppose that $(\lambda_0, \lambda_1) \subseteq \mathbb{R}\backslash(\sigma_{ess}(H_0) \cup \sigma_{ess}(H_1))$, then*

$$\begin{aligned} &\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) \\ &\qquad \le \underline{\#}(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \overline{\#}(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)) + 2, \end{aligned} \tag{3.6}$$

respectively,

$$\begin{aligned} &\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) \\ &\qquad \ge \overline{\#}(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \underline{\#}(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)) - 2. \end{aligned} \tag{3.7}$$

Proof. By the triangle inequality (cf. Theorem 3.4) we have

$$\begin{aligned} &\#_{(c,d)}(\psi_{1,-}(\lambda_1), \psi_{1,+}(\lambda_0)) - \#_{(c,d)}(\psi_{0,-}(\lambda_1), \psi_{0,+}(\lambda_0)) \\ &\qquad \le \#_{(c,d)}(\psi_{1,-}(\lambda_1), \psi_{0,+}(\lambda_1)) + \#_{(c,d)}(\psi_{1,-}(\lambda_1), \psi_{0,+}(\lambda_1)) + 2. \end{aligned}$$

The result now follows by taking limits using that

$$\lim_{c\downarrow a, d\uparrow b} \#_{(c,d)}(\psi_{1,-}(\lambda_1), \psi_{1,+}(\lambda_0)) = \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1)$$

and

$$\lim_{c\downarrow a, d\uparrow b} \#_{(c,d)}(\psi_{0,-}(\lambda_0), \psi_{0,+}(\lambda_1)) = -\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0)$$

by the previous theorem. The second claim follows similarly. □

To turn the inequalities into equalities in Lemma 3.11 will be one of our remaining goals.

Observe that for semibounded operators can choose λ_0 below the spectra of H_0 and H_1 causing $\#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0))$ to vanish:

Lemma 3.12. *Let $r^{-1}(q_0 - q_1) \le \delta$ for some $\delta \in \mathbb{R}$. Furthermore, suppose that the operator H_0 associated with τ_0 is bounded from below, $H_0 \ge E_0$, and the form domains of H_0 and H_1 are equal. Then*

$$\#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)) = 0, \quad \lambda < E_0 - \delta. \tag{3.8}$$

Proof. Suppose there is $c \in (a, b)$ such that $W_c(\psi_{0,+}(\lambda), \psi_{1,-}(\lambda)) = 0$. Then, there is a γ such that

$$\varphi(x) = \begin{cases} \psi_{1,-}(\lambda, x), & x \le c, \\ \gamma\, \psi_{0,+}(\lambda, x), & x \ge c, \end{cases}$$

is continuous and hence in the form domain of H_0 (see the remark after Theorem A.3 in [5]). But then

$$\langle \varphi, H_0 \varphi \rangle \le (\lambda + \delta) \|\varphi\|^2 < E_0 \|\varphi\|^2,$$

contradicting $H_0 \ge E_0$. □

Our first approach will use approximation by regular problems. However, the standard approximation technique only implies strong convergence, which is not sufficient for our purpose. Hence our argument is based on a refined method by Stolz and Weidmann [28] which will provide convergence of spectral projections in the trace norm for suitably chosen regular operators (see [39] for a nice overview). The details will be given in Section 6.

Theorem 3.13. *Let H_0, H_1 be self-adjoint operators associated with τ_0, τ_1, respectively, and separated boundary conditions. Suppose*

(i) *$q_1 \le q_0$, near singular endpoints,*
(ii) *$\lim_{x\to a} r(x)^{-1}(q_0(x) - q_1(x)) = 0$ if a is singular and $\lim_{x\to b} r(x)^{-1}(q_0(x) - q_1(x)) = 0$ if b is singular,*
(iii) *H_0 and H_1 are associated with the same boundary conditions near a and b, that is, $\psi_{0,-}(\lambda)$ satisfies the boundary condition of H_1 at a (if any) and $\psi_{1,+}(\lambda)$ satisfies the boundary condition of H_0 at b (if any).*

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then

$$\dim \operatorname{Ran} P_{(-\infty,\lambda_0)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \tag{3.9}$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then $\tau_1 - \lambda_0$ is nonoscillatory with respect to $\tau_0 - \lambda_0$ and

$$\begin{aligned} &\dim \operatorname{Ran} P_{[\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1]}(H_0) \\ &\qquad = \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned} \tag{3.10}$$

Remark 3.14. *Note that condition (ii) implies $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ as pointed out in Lemma 3.9. In addition, (ii) implies that any function which is in $\mathfrak{D}(\tau_0)$ near a (or b) is also in $\mathfrak{D}(\tau_1)$ near a (or b), and vice versa. Hence condition (iii) is well-posed.*

Our second approach will connect our theory with Krein's spectral shift function. Given the regular case in Theorem 2.3, we can extend this result to operators whose resolvent difference is trace class by replacing the left-hand side in (2.10) by the spectral shift function. In order to fix the unknown constant in the spectral shift function, we will require that H_0 and H_1 are connected via a path within the set of operators whose resolvent difference with H_0 are trace class. Hence we will require

Hypothesis H. 3.15. *Suppose H_0 and H_1 are self-adjoint operators associated with τ_0 and τ_1 and separated boundary conditions. Abbreviate $\tilde{q} = r^{-1}|q_0 - q_1|$, and assume that:*

(i) *$\tilde{q}$ is relatively bounded with respect to H_0 with H_0-bound less than one or*
(i') *H_0 is bounded from below and $\tilde{q}$ is relatively form bounded with respect to H_0 with relative form bound less than one and*
(ii) *$\sqrt{\tilde{q}} R_{H_0}(z)$ is Hilbert–Schmidt for one (and hence for all) $z \in \rho(H_0)$.*

It will be shown in Section 8 that these conditions ensure that we can interpolate between H_0 and H_1 using operators H_ε, $\varepsilon \in [0,1]$, such that the resolvent difference of H_0 and H_ε is continuous in ε with respect to the trace norm. Hence we can fix $\xi(\lambda, H_1, H_0)$ by requiring $\varepsilon \mapsto \xi(\lambda, H_\varepsilon, H_0)$ to be continuous in $L^1(\mathbb{R}, (\lambda^2+1)^{-1} d\lambda)$, where we of course set $\xi(\lambda, H_0, H_0) = 0$ (see Lemma 8.5). While ξ is only defined a.e., it is constant on the intersection of the resolvent sets $\mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$, and we will require it to be continuous there. In particular, note that by Weyl's theorem the essential spectra of H_0 and H_1 are equal, $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$.

Theorem 3.16. *Let H_0, H_1 satisfy Hypothesis 3.15. Then for every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$ we have*

$$\xi(\lambda, H_1, H_0) = \#(\psi_{0,\pm}(\lambda), \psi_{1,\mp}(\lambda)). \tag{3.11}$$

In particular, this result implies that under these assumptions $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for every λ in an essential spectral gap.

The main idea is to interpolate between H_0 and H_1. Under proper assumptions it is then possible to control $\xi(\lambda, H_\varepsilon, H_0)$. However, it seems extremely hard to control the zeros of the Wronskian. To do this we will have to assume that $q_1 - q_0$ has compact support. We will then remove this restriction by extending the support first to one and then to the other side. The details will be given in Section 7.

Finally, we remark that since the results from [5] extend to one-dimensional Dirac operators [31] (see also [22]) and Jacobi operators [30] (compare also [32, Chap. 4]), similar results are expected to hold for these operators and will be given in [34] respectively [35]. Furthermore, it will of course be interesting to develop relative oscillation criteria for Sturm–Liouville operators! What are the analogs of some classical oscillation criteria? Is there an analog of [4]?

4. Applications

In this section we want to look at the classical problem of the number of eigenvalues in essential spectral gaps of perturbed periodic operators [1], [16] (see also [3]), [42]. The precise critical case was first determined by Rofe-Beketov in a series of papers [17]–[20] with later additions by Khryashchev [12] and Schmidt [21]. The purpose of this section is to show how the the results in [21] can be extended using our methods.

For convenience of the reader, we recall some basic facts of the theory of periodic differential operators first (see for example [2] or [38]).

Let p, q_0 be α-periodic, that is, $p(x+\alpha) = p(x)$, $q_0(x+\alpha) = q_0(x)$, and consider the corresponding Sturm–Liouville expressions

$$\tau_0 = -\frac{d}{dx} p \frac{d}{dx} + q_0. \tag{4.1}$$

Denote by $c(\lambda, x)$, $s(\lambda, x)$ a fundamental system of solutions corresponding to the initial conditions $c(\lambda, 0) = p(0)s'(\lambda, 0) = 1$, $s(\lambda, 0) = p(0)c'(\lambda, 0) = 0$. In particular, their Wronskian reads $W(c, s) = 1$. We then call

$$M(\lambda) = \begin{pmatrix} c(\lambda, \alpha) & s(\lambda, \alpha) \\ p(0)c'(\lambda, \alpha) & p(0)s'(\lambda, \alpha) \end{pmatrix} \tag{4.2}$$

the monodromy matrix. The discriminant $D(\lambda)$ is given by $D(\lambda) = \mathrm{tr}(M(\lambda))$.

Since we are only interested in the question wether the number of eigenvalues are finite or not, it suffices to look at the half-line case $(1, \infty)$ with a, for example, Dirichlet boundary condition at 1. Denote the corresponding self-adjoint operator by H_0. The essential spectrum of H_0 is given by

$$\sigma_{ess}(H_0) = \sigma_{ac}(H_0) = \{\lambda \,|\, |D(\lambda)| \le 2\} = \bigcup_{n=0}^{\infty} [E_{2n}, E_{2n+1}]. \tag{4.3}$$

The critical coupling constant at the endpoint E_n of an essential spectral gap introduced in Section 1 is given by ([21])

$$\kappa_n = \frac{\alpha^2}{4|D|'(E_n)}. \tag{4.4}$$

It is related to the effective mass $m(E_n)$ used in solid state physics via $\kappa_n = (8m(E_n))^{-1}$ (see [20]).

We note that $\kappa_{2n+1} > 0$ for a lower endpoint of a spectral gap, and $\kappa_{2n} < 0$ for an upper endpoint.

Before coming to our applications, we ensure that our hypothesis from the previous section are satisfied. We begin by computing the form domain of H_0.

Lemma 4.1. *Abbreviate*

$$\mathfrak{Q} = \{f \in L^2(1, \infty) \,|\, f \in AC[1, \infty),\ \sqrt{p} f' \in L^2(1, \infty)\}. \tag{4.5}$$

The form domain of H_0 is given by

$$\mathfrak{Q}(H_0) = \{f \in \mathfrak{Q} \,|\, f(1) = 0\}. \tag{4.6}$$

Moreover, for every $\varepsilon > 0$ there is some $C > 0$ such that

$$\sup_{x_0 \le x \le x_0+\alpha} |f(x)|^2 \le \varepsilon \int_{x_0}^{x_0+\alpha} p(x)|f'(x)|^2 dx + C \int_{x_0}^{x_0+\alpha} |f(x)|^2 dx, \quad f \in \mathfrak{Q}. \tag{4.7}$$

In particular, $q_1 - q_0$ is infinitesimally form bounded with respect to H_0 if

$$\int_{x_0}^{x_0+\alpha} |q_1(x) - q_0(x)| dx < C_0, \qquad x_0 \in (1, \infty), \tag{4.8}$$

where C_0 is independent of x_0.

Proof. First of all note that as in [5, Lem A.2], one can show that (4.7) holds.

Next, set

$$A = \sqrt{p}\frac{d}{dx}, \qquad \mathfrak{D}(A) = \{f \in \mathfrak{Q} \,|\, f(1) = 0\} \tag{4.9}$$

and note that A is then a closed operator with adjoint given by

$$\begin{aligned} A^* &= -\frac{d}{dx}\sqrt{p}, \\ \mathfrak{D}(A^*) &= \{f \in L^2(1,\infty) \,|\, \sqrt{p}f \in AC[1,\infty),\ (\sqrt{p}f)' \in L^2(1,\infty)\}. \end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned} A^*A &= -\frac{d}{dx}p\frac{d}{dx}, \\ \mathfrak{D}(A^*A) &= \{f \in L^2(1,\infty) \,|\, f, pf' \in AC[1,\infty),\ (pf')' \in L^2(1,\infty),\ f(1) = 0\} \end{aligned} \tag{4.11}$$

is self-adjoint with $\mathfrak{Q}(A^*A) = \mathfrak{D}(A)$ and by (4.7) q_0 is infinitesimally form bounded with respect to A^*A. Since the same is true for q_1 by assumption the lemma is proven. □

Lemma 4.2. *Let H_0 be an arbitrary Sturm–Liouville operator on (a,b). Then $\sqrt{q}R_{H_0}(z)$ is Hilbert–Schmidt if and only if*

$$\|\sqrt{q}R_{H_0}(z)\|_{\mathcal{J}_2}^2 = \frac{1}{\mathrm{Im}(z)}\int_a^b |q(x)|\mathrm{Im}(G_0(z,x,x))r(x)dx \tag{4.12}$$

is finite. Here $G_0(z,x,y) = (H_0 - z)^{-1}(x,y)$ denotes the Green's function of H_0.

Proof. From the first resolvent identity we have

$$G_0(z,x,y) - G_0(z',x,y) = (z - z')\int_a^b G_0(z,x,t)G_0(z',t,y)r(t)dt.$$

Setting $x = y$ and $z' = z^*$ we obtain

$$\mathrm{Im}(G_0(z,x,x)) = \mathrm{Im}(z)\int_a^b |G_0(z,x,t)|^2 r(t)dt.$$

Using this last formula to compute the Hilbert–Schmidt norm proves the lemma. □

We recall that in the case of periodic operators $G(z,x,x)$ is a bounded function of x. In fact, we have

$$|G(z,x,y)| \le const(z)\exp(-\gamma(z)|x-y|), \tag{4.13}$$

where $\gamma(z) > 0$ denotes the Floquet exponent.

Now we are ready to apply our theory:

Theorem 4.3. *Let p, q_0 α-periodic, D the discriminant of the periodic differential equation τ_0, and q_1 a perturbed potential which is regular at 1, such that either $q_0 - q_1 \in L^1(1,\infty)$ or $\lim_{x\to\infty}(q_0(x) - q_1(x)) = 0$. Define the differential expressions on $(1,\infty)$ by*

$$\tau_0 = -\frac{d}{dx}p(x)\frac{d}{dx} + q_0(x), \quad \tau_1 = -\frac{d}{dx}p(x)\frac{d}{dx} + q_1(x) \tag{4.14}$$

and let H_0, H_1 be the corresponding self-adjoint operators $L^2(1,\infty)$.

Let E_n be an endpoint of a gap in the essential spectrum of H_0. Then E_n is an accumulation point of eigenvalues of H_1 if

$$\liminf_{x\to\infty} \kappa_n x^2(q_1(x) - q_0(x)) > 1. \tag{4.15}$$

and E_n is no accumulation point of eigenvalues of H_1 if

$$\limsup_{x\to\infty} \kappa_n x^2(q_1(x) - q_0(x)) < 1 \tag{4.16}$$

and $\kappa_n(q_1 - q_0) \geq 0$ near infinity.

Proof. Lemma 4.2 together with (4.13) shows, that Hypothesis 3.15 is satisfied if $q_0 - q_1 \in L^1(1,\infty)$. Thus we can either apply Theorem 3.13 or Theorem 3.16 to conclude that $\tau_1 - \lambda$ is nonoscillatory with respect to $\tau_0 - \lambda$ for any $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H_0)$. Hence, Theorem 3.8 (ii) is applicable and it suffices to show that τ_1 is relatively oscillatory (resp. nonoscillatory) with respect to τ_0.

Without restriction, we assume $\kappa_n > 0$. For the first statement, note that we can find c, $\varepsilon > 0$ such that

$$q_0(x) < q_0(x) + \frac{1}{(\kappa_n - \varepsilon)x^2} < q_1(x), \qquad x > c.$$

Now since, perturbations with compact support only add finitely many eigenvalues, it is no restriction to assume $c = 1$.

Next, [21, Thm. 1] shows that $\tau_0 + (\kappa_n - \varepsilon)^{-1}x^{-2} - E_n$ is relatively oscillatory with respect to $\tau_0 - E_n$. Hence $\tau_1 - E_n$ being relatively oscillatory with respect to $\tau_0 - E_n$ now follows using our comparison theorem for Wronskians (cf. Theorem 3.3).

For the second statement, we first note that our conditions imply

$$q_0(x) \leq q_1(x) < q_0(x) + \frac{1}{(\kappa_n + \varepsilon)x^2}$$

near infinity, and then one proceeds as before. □

We note that even the second order term was also computed in [21]. So we obtain:

Theorem 4.4. *Assume*

$$\lim_{x\to\infty} \kappa_n x^2(q_1(x) - q_0(x)) = 1 \tag{4.17}$$

in addition to the assumptions in Theorem 4.3. Then E_n is an accumulation point of eigenvalues if

$$\liminf_{x\to\infty} \log^2(x)(\kappa_n x^2(q_1(x) - q_0(x)) - 1) > 1, \tag{4.18}$$

and E_n is not an accumulation point of eigenvalues if

$$\limsup_{x\to\infty} \log^2(x)(\kappa_n x^2(q_1(x) - q_0(x)) - 1) < 1. \tag{4.19}$$

Proof. Similarly to that of Theorem 4.3, except, one now uses [21, Thm. 2]. □

The main argument in [21] is a perturbation argument for the difference of Prüfer angles for the solution of the unperturbed and the perturbed equation. This corresponds exactly to calculating the asymptotics of the Prüfer angle of the Wronskian.

5. More on Prüfer angles and the case of regular operators

Now let us suppose that $\tau_{0,1}$ are both regular at a and b with boundary conditions

$$\cos(\alpha)f(a)-\sin(\alpha)p(a)f'(a)=0,\quad \cos(\beta)f(b)-\sin(\beta)p(b)f'(b)=0. \tag{5.1}$$

Hence we can choose $\psi_\pm(\lambda,x)$ such that $\psi_-(\lambda,a)=\sin(\alpha)$, $p(a)\psi_-'(\lambda,a)=\cos(\alpha)$ respectively $\psi_+(\lambda,b)=\sin(\beta)$, $p(b)\psi_+'(\lambda,b)=\cos(\beta)$. In particular, we may choose

$$\theta_-(\lambda,a)=\alpha\in[0,\pi),\quad -\theta_+(\lambda,b)=\pi-\beta\in[0,\pi). \tag{5.2}$$

Next we introduce

$$\tau_\varepsilon=\tau_0+\frac{\varepsilon}{r}(q_1-q_0) \tag{5.3}$$

and investigate the dependence with respect to $\varepsilon\in[0,1]$. If u_ε solves $\tau_\varepsilon u_\varepsilon=0$, then the corresponding Prüfer angles satisfy

$$\dot\theta_\varepsilon(x)=-\frac{W_x(u_\varepsilon,\dot u_\varepsilon)}{\rho_\varepsilon^2(x)}, \tag{5.4}$$

where the dot denotes a derivative with respect to ε.

Lemma 5.1. *We have*

$$W_x(\psi_{\varepsilon,\pm},\dot\psi_{\varepsilon,\pm})=\begin{cases}\int_x^b(q_0(t)-q_1(t))\psi_{\varepsilon,+}(t)^2dt\\ -\int_a^x(q_0(t)-q_1(t))\psi_{\varepsilon,-}(t)^2dt\end{cases}, \tag{5.5}$$

where the dot denotes a derivative with respect to ε and $\psi_{\varepsilon,\pm}(x)=\psi_{\varepsilon,\pm}(0,x)$.

Proof. Integrating (2.8) we obtain

$$W_x(\psi_{\varepsilon,\pm},\psi_{\tilde\varepsilon,\pm})=(\tilde\varepsilon-\varepsilon)\begin{cases}\int_x^b(q_0(t)-q_1(t))\psi_{\varepsilon,+}(t)\psi_{\tilde\varepsilon,+}(t)dt,\\ -\int_a^x(q_0(t)-q_1(t))\psi_{\varepsilon,-}(t)\psi_{\tilde\varepsilon,-}(,t)dt.\end{cases} \tag{5.6}$$

Now use this to evaluate the limit

$$\lim_{\tilde\varepsilon\to\varepsilon}W_x\Big(\psi_{\varepsilon,\pm},\frac{\psi_{\pm,\varepsilon}-\psi_{\tilde\varepsilon,\pm}}{\varepsilon-\tilde\varepsilon}\Big). \tag{5.7}$$

□

Denoting the Prüfer angles of $\psi_{\varepsilon,\pm}(x)=\psi_{\varepsilon,\pm}(0,x)$ by $\theta_{\varepsilon,+}(x)$, this result implies for $q_0-q_1\ge0$,

$$\dot\theta_{\varepsilon,+}(x)=-\frac{\int_x^b(q_0(t)-q_1(t))\psi_{\varepsilon,+}(t)^2dt}{\rho_{\varepsilon,+}(x)^2}\le0,$$

$$\dot\theta_{\varepsilon,-}(x)=\frac{\int_a^x(q_0(t)-q_1(t))\psi_{\varepsilon,-}(t)^2dt}{\rho_{\varepsilon,-}(x)^2}\ge0, \tag{5.8}$$

with strict inequalities if $q_1\not\equiv q_0$.

Now we are ready to investigate the associated operators H_0 and H_1. In addition, we will choose the same boundary conditions for H_ε as for H_0 and H_1.

Lemma 5.2. *Suppose $q_0-q_1\ge0$ (resp $q_0-q_1\le0$). Then the eigenvalues of H_ε are analytic functions with respect to ε and they are decreasing (resp. increasing).*

Proof. First of all the Prüfer angles $\theta_{\varepsilon,\pm}(x)$ are analytic with respect to ε since τ_ε is by a well-known result from ordinary differential equations (see e.g., [36, Thm. 13.III]). Moreover, $\lambda\in\sigma(H_\varepsilon)$ is equivalent to $\theta_{\varepsilon,+}(a)\equiv\alpha \mod \pi$ (respectively $\theta_{\varepsilon,-}(b)\equiv\beta \mod\pi$), where α (respectively β) generates the boundary condition (cf. (5.1)). □

In particular, this implies that $\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_\varepsilon)$ is continuous from below (resp. above) in ε if $q_0 - q_1 \geq 0$ (resp $q_0 - q_1 \leq 0$).

Now we are ready for the

Proof of Theorem 2.3. It suffices to prove the result for $\#(\psi_{0,+}, \psi_{\varepsilon,-})$. Again we can assume $\lambda_0 = \lambda_1 = 0$ without restriction. We split $q_0 - q_1$ according to

$$q_0 - q_1 = q_+ - q_-, \qquad q_+, q_- \geq 0,$$

and introduce the operator $\tau_- = \tau_0 - q_-/r$. Then τ_- is a negative perturbation of τ_0 and τ_1 is a positive perturbation of τ_-.

Furthermore define τ_ε by

$$\tau_\varepsilon = \begin{cases} \tau_0 + 2\varepsilon(\tau_- - \tau_0), & \varepsilon \in [0, 1/2], \\ \tau_- + 2(\varepsilon - 1/2)(\tau_1 - \tau_-), & \varepsilon \in [1/2, 1]. \end{cases}$$

Let us look at

$$N(\varepsilon) = \#(\psi_{0,+}, \psi_{\varepsilon,-}) = \lceil \Delta_\varepsilon(b)/\pi \rceil - \lfloor \Delta_\varepsilon(a)/\pi \rfloor - 1, \quad \Delta_\varepsilon(x) = \Delta_{\psi_{0,+}, \psi_{\varepsilon,-}}(x)$$

and consider $\varepsilon \in [0, 1/2]$. At the left boundary $\Delta_\varepsilon(a)$ remains constant whereas at the right boundary $\Delta_\varepsilon(b)$ is increasing by Lemma 5.1. Moreover, it hits a multiple of π whenever $0 \in \sigma(H_\varepsilon)$. So $N(\varepsilon)$ is a piecewise constant function which is continuous from below and jumps by one whenever $0 \in \sigma(H_\varepsilon)$. By Lemma 5.2 the same is true for

$$P(\varepsilon) = \dim \operatorname{Ran}\, P_{(-\infty,0)}(H_\varepsilon) - \dim \operatorname{Ran}\, P_{(-\infty,0]}(H_0)$$

and since we have $N(0) = P(0)$, we conclude $N(\varepsilon) = P(\varepsilon)$ for all $\varepsilon \in [0, 1/2]$. To see the remaining case $\varepsilon = [1/2, 1]$, simply replace increasing by decreasing and continuous from below by continuous from above. $\square$

6. Approximation by regular operators

Now we want to extend our results to singular operators. We will do so by approximating a singular operator by a sequence of regular ones following [38, Chap. 14].

Abbreviate in the following $L^2((c,d), r\,dx)$ as $L^2(c,d)$. Fix functions $u, v \in \mathfrak{D}(\tau)$ and pick $c_n \downarrow a$, $d_n \uparrow b$. Define $\tilde{H}_n$

$$\tilde{H}_n : \begin{array}{lcl} \mathfrak{D}(\tilde{H}_n) & \to & L^2(c_n, d_n) \\ f & \mapsto & \tau f \end{array}, \tag{6.1}$$

where

$$\mathfrak{D}(\tilde{H}_n) = \{f \in L^2(c_n, d_n) |\ \begin{array}{l} f, pf' \in AC(c_n, d_n),\ \tau f \in L^2(c_n, d_n), \\ W_{c_n}(u, f) = W_{d_n}(v, f) = 0\}. \end{array} \tag{6.2}$$

Take $H_n = \alpha \mathbb{1} \oplus \tilde{H}_n \oplus \alpha \mathbb{1}$ on $L^2(a,b) = L^2(a, c_n) \oplus L^2(c_n, d_n) \oplus L^2(d_n, b)$, where α is a fixed real constant. Then we have the following result:

Lemma 6.1. *Suppose that either H is limit point at a or that $u = \psi_-(\lambda_0)$ for some $\lambda_0 \in \mathbb{R}$ and similarly, that either H is limit point at b or $v = \psi_+(\lambda_1)$ for some $\lambda_1 \in \mathbb{R}$. Then H_m converges to H in strong resolvent sense as $m \to \infty$.*

Furthermore, if H is limit circle at a (resp. b), we can replace u (resp. v) with any function in $\mathfrak{D}(\tau)$, which generates the boundary condition.

However, strong resolvent convergence is not sufficient for our purpose here. In addition we will need the following result from [28] (see also [38]). We give a slightly refined analysis which allows eigenvalues at the boundary of the spectral intervals and the possibility of infinite-dimensional projections. We remark that for a self-adjoint projector P we have

$$\dim \operatorname{Ran}(P) = \operatorname{tr}(P) = \|P\|_{\mathcal{J}^1}, \tag{6.3}$$

where $\|.\|_{\mathcal{J}^1}$ denotes the trace class norm. If P is not finite-rank, all three numbers equal ∞. Then we have the following result ([33, Lem. 2], see also [28]):

Lemma 6.2. *Let $A_n \to A$ in strong resolvent sense and suppose* $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)) \le \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A))$.

Then,

$$\lim_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)) = \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)), \tag{6.4}$$

and if $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) < \infty$*, we have*

$$\lim_{n\to\infty} \|P_{(\lambda_0,\lambda_1)}(A_n) - P_{(\lambda_0,\lambda_1)}(A)\|_{\mathcal{J}^1} = 0. \tag{6.5}$$

Proof. This follows from (see e.g. [5, Lem. 5.2])

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) \le \liminf_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)), \tag{6.6}$$

together with Grümm's theorem ([25, Thm. 2.19]). □

The key result of Stolz and Weidmann is that this lemma is applicable if certain Weyl solutions $\psi_\pm(\lambda)$ are used to generate the boundary conditions of $\tilde{H}_n$. As already pointed out, the version below is slightly refined since it allows λ_0, λ_1 to be eigenvalues of H.

Lemma 6.3 ([28])**.** *Suppose $[\lambda_0,\lambda_1] \cap \sigma_{ess}(H) = \emptyset$ and let H_n be defined as in (6.1) with $u = \psi_-(\lambda_-)$, $v = \psi_+(\lambda_+)$ and $\lambda_\pm \in [\lambda_0,\lambda_1]$. Then,*

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(\tilde{H}_n)) \le \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H)). \tag{6.7}$$

Proof. Abbreviate $P = P_{(\lambda_0,\lambda_1)}(H)$, $P_n = P_{(\lambda_0,\lambda_1)}(\tilde{H}_n)$. For $\tilde{\psi}_1,\dots,\tilde{\psi}_k \in \operatorname{Ran} P_n$ being eigenfunctions of $\tilde{H}_n$, construct

$$\psi_j(x) = \begin{cases} \gamma_{j,u} u(x), & x < c_n, \\ \tilde{\psi}_j(x), & c_n \le x \le d_n, \\ \gamma_{j,v} v(x), & x > d_n, \end{cases}$$

where $\gamma_{j,u}$, $\gamma_{j,v}$ are chosen such that ψ_j and $p\psi_j'$ are continuous. A computation shows that

$$\|(H - \frac{\lambda_1+\lambda_0}{2})\psi\| < \frac{\lambda_1-\lambda_0}{2}\|\psi\|$$

for any ψ in the linear span of the ψ_j's, which yields the first result. □

This version is sufficient to give an alternative proof for the main theorem in [5].

Proof of Theorem 3.10. Approximate H_0 by regular operators H_n defined as in (6.1) with $u = \psi_{0,-}(\lambda_0)$, $v = \psi_{0,+}(\lambda_1)$. Denote by $\psi^n_{0,\pm}(\lambda)$ the solutions of the approximating problems. Then, by construction $\psi^n_{0,-}(\lambda_0,x) = \psi_{0,-}(\lambda_0,x)$ respectively $\psi^n_{0,+}(\lambda_1,x) = \psi_{0,+}(\lambda_1,x)$ for $x \in (c_n,d_n)$ and Theorem 2.3 in the special case $H_1 = H_0$ implies

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(\tilde{H}_n)) = \#_{(c_n,d_n)}(\psi_{0,-}(\lambda_0),\psi_{0,+}(\lambda_1)).$$

Letting $n \to \infty$ the left-hand side converges to $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H_0))$ by the first part of Lemma 6.3. Hence the right-hand side converges as well and, according to Definition 3.1, is given by $\#(\psi_{0,\mp}(\lambda_0),\psi_{0,\pm}(\lambda_1))$. □

However, the proof of our Theorem 3.13 requires some further extensions. In fact, in [28] Stolz and Weidmann point out that the Weyl functions of a different operator $\tilde{H}$ will also do, as long as $\tilde{H}$ is not too far away from H. Again the version below is slightly improved to allow for some border line cases.

Lemma 6.4 ([28])**.** *Suppose $[\lambda_0,\lambda_1] \cap \sigma_{ess}(H) = \emptyset$. Let $\tilde{\tau} = \tau + \tilde{q}$, where $\tilde{q}$ is bounded, and pick the same boundary conditions for $\tilde{H}$ as for H (if any). Abbreviate*

$$Q_a = [\liminf_{x\to a} \tilde{q}(x), \limsup_{x\to a} \tilde{q}(x)], \quad Q_b = [\liminf_{x\to b} \tilde{q}(x), \limsup_{x\to b} \tilde{q}(x)], \tag{6.8}$$

and choose λ_- such that one of following conditions holds:

(i) *$\lambda_- - Q_a \subseteq (\lambda_0,\lambda_1)$, or*
(ii) *$\lambda_- - Q_a \subseteq [\lambda_0,\lambda_1)$ and $\tilde{q} \le 0$ near a, or*
(iii) *$\lambda_- - Q_a \subseteq (\lambda_0,\lambda_1]$ and $\tilde{q} \ge 0$ near a.*

Similarly, choose λ_+ to satisfy one of these conditions with a replaced by b.

Then, H_n defined as in (6.1) with $u = \tilde{\psi}_-(\lambda_-)$, $v = \tilde{\psi}_+(\lambda_+)$, satisfies

$$\limsup_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(\tilde{H}_n)) \le \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H)). \tag{6.9}$$

Furthermore, if we just require

$$\lambda_- - Q_a \subseteq [\lambda_0,\lambda_1], \qquad \lambda_+ - Q_b \subseteq [\lambda_0,\lambda_1], \tag{6.10}$$

we at least have

$$\limsup_{n\to\infty} \operatorname{tr}(P_{[\lambda_0,\lambda_1]}(\tilde{H}_n)) \le \operatorname{tr}(P_{[\lambda_0,\lambda_1]}(H)). \tag{6.11}$$

Proof. Since any of our three conditions implies $|\lambda_- - \frac{\lambda_1+\lambda_0}{2} - \tilde{q}(x)| \le \frac{\lambda_1-\lambda_0}{2}$ for x sufficiently close to a and similarly $|\lambda_+ - \frac{\lambda_1+\lambda_0}{2} - \tilde{q}(x)| \le \frac{\lambda_1-\lambda_0}{2}$ for x sufficiently close to b, we can proceed as in the previous lemma to prove the first claim.

For the second claim choose n sufficiently large such that $|\lambda_\pm - \tilde{q}(x) - 2^{-1}(\lambda_1 + \lambda_0)| \le (2^{-1}(\lambda_1-\lambda_0)+\varepsilon)$ for $x < c_n$, respectively, $x > d_n$. Then, with the same argument as in the previous lemma, we have

$$\|(H - \frac{\lambda_1+\lambda_0}{2})\psi\| \le (\frac{\lambda_1-\lambda_0}{2} + \varepsilon)\|\psi\|$$

and hence the second claim follows. □

Since our results involve projections to half-open intervals, we need one further step.

Lemma 6.5. *Suppose $[\lambda_0,\lambda_1] \cap \sigma_{ess}(H) = \emptyset$. Let $\tilde{\tau} = \tau + \tilde{q}$, where $\lim_{x\to a} \tilde{q}(x) = 0$ if a is singular and $\lim_{x\to b} \tilde{q}(x) = 0$ if b is singular. Furthermore, pick the same boundary conditions for $\tilde{H}$ as for H (if any).*

Define H_n as in (6.1) with $u = \tilde{\psi}_-(\lambda)$, $v = \tilde{\psi}_+(\lambda)$, $\lambda \in \{\lambda_1, \lambda_2\}$. If $\lambda = \lambda_1$ and $\tilde{q} \le 0$ (near a and b), then

$$\lim_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1]}(\tilde{H}_n)) = \operatorname{tr}(P_{(\lambda_0,\lambda_1]}(H)), \tag{6.12}$$

and if $\lambda = \lambda_0$ and $\tilde{q} \ge 0$, then

$$\lim_{n\to\infty} \operatorname{tr}(P_{[\lambda_0,\lambda_1)}(\tilde{H}_n)) = \operatorname{tr}(P_{[\lambda_0,\lambda_1)}(H)). \tag{6.13}$$

Proof. Without restriction, we just proof the first claim. For a sufficiently small $\varepsilon > 0$ we still have $[\lambda_0, \lambda_1 + \varepsilon] \cap \sigma_{ess}(H) = \emptyset$ and thus by the previous lemma

$$\lim_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1+\varepsilon)}(\tilde{H}_n)) = \operatorname{tr}(P_{(\lambda_0,\lambda_1+\varepsilon)}(H))$$

and

$$\lim_{n\to\infty} \operatorname{tr}(P_{(\lambda_1,\lambda_1+\varepsilon)}(\tilde{H}_n)) = \operatorname{tr}(P_{(\lambda_1,\lambda_1+\varepsilon)}(H)).$$

Hence the result follows from $P_{(\lambda_0,\lambda_1]} = P_{(\lambda_0,\lambda_1+\varepsilon)} - P_{(\lambda_1,\lambda_1+\varepsilon)}$. □

Finally, we note:

Lemma 6.6. *Suppose $\tau_j u_j = \lambda_j u_j$, $j = 0, 1$, with $q_1 \le q_0$ near singular endpoints and $\lambda_0 < \lambda_1$. If $\tau_j u_{j,n} = \lambda_{j,n} u_{j,n}$, where $\lambda_{j,n} \to \lambda_j$ and $u_{j,n} \to u_j$, uniformly on compact sets $[c, d] \subseteq (a, b)$, then*

$$\liminf_{n\to\infty} \#(u_{0,n}, u_{1,n}) \ge \#(u_0, u_1). \tag{6.14}$$

Proof. Let $N \in \mathbb{N}_0$ be any finite number with $N \le \#(u_0, u_1)$. Choose a compact set $[c, d]$ containing N sign flips of $W(u_0, u_1)$. Then, for n sufficiently large, $W(u_{0,n}, u_{1,n})$ has N sign flips in $[c, d]$. Hence $\#(u_{0,n}, u_{1,n}) \ge \#_{(c,d)}(u_{0,n}, u_{1,n}) = N$ and the claim follows. □

Now, we are ready for the

Proof of Theorem 3.13. It suffices to show the $\#(\psi_{1,-}(\lambda_j), \psi_{0,+}(\lambda_j))$ case. Define $\tilde{H}_{j,n}$, $j = 0, 1$, as in (6.1) with $u = \psi_{1,-}(\lambda_0)$ and $v = \psi_{0,+}(\lambda_0)$.

Denote by $\psi^n_{j,\pm}(\lambda)$, $j = 0, 1$, the solutions of the approximating problems. Then, by Theorem 2.3,

$$\operatorname{tr}(P_{(-\infty,\lambda_0)}(\tilde{H}_{1,n})) - \operatorname{tr}(P_{(-\infty,\lambda_0]}(\tilde{H}_{0,n})) = \#_{(c_n,d_n)}(\psi^n_{1,-}(\lambda_0), \psi^n_{0,+}(\lambda_0))$$

and we need to investigate the limits as $n \to \infty$.

First of all $\psi^n_{1,-}(\lambda_0, x) = \psi_{1,-}(\lambda_0, x)$, $\psi^n_{0,+}(\lambda_0, x) = \psi_{0,+}(\lambda_0, x)$ for $x \in (c_n, d_n)$ implies

$$\begin{aligned}\lim_{n\to\infty} \#_{(c_n,d_n)}(\psi^n_{1,-}(\lambda_0), \psi^n_{0,+}(\lambda_0)) &= \lim_{n\to\infty} \#_{(c_n,d_n)}(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)) \\ &= \#(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)).\end{aligned}$$

This takes care of the number of sign flips and it remains to look at the spectral projections. Let $\lambda_0 < \sigma_{ess}(H_0)$, that is H_0 and hence also H_1 are bounded from below. Replacing $P_{(-\infty,\lambda_0)}(H_j)$ by $P_{(\lambda,\lambda_0)}(H_j)$ with some λ below the spectrum of both H_0 and H_1 we infer from Lemma 6.5

$$\lim_{n\to\infty} \operatorname{tr}(P_{(-\infty,\lambda_0)}(\tilde{H}_{1,n})) = \operatorname{tr}(P_{(-\infty,\lambda_0)}(H_1))$$

and

$$\lim_{n\to\infty} \operatorname{tr}(P_{(-\infty,\lambda_0]}(\tilde{H}_{0,n})) = \operatorname{tr}(P_{(-\infty,\lambda_0]}(H_0)).$$

This settles the first claim (3.9), where $\lambda_0 < \sigma_{ess}(H_0)$.

For the second claim (3.10), we first note that $\tau_1 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_0$ by Lemma 3.9. Next note that $\psi_{0,+}^n(\lambda_1, .) \to \psi_{0,+}(\lambda_1, .)$ pointwise, since

$$\psi_{0,+}^n(\lambda_1, x) = c_0(\lambda_1, x) + m_{0,+}^n(\lambda_1) s_0(\lambda_1, x), \tag{6.15}$$

where $c_0(\lambda, x)$, $s_0(\lambda, x)$ is a fundamental system of solutions for $\tau_0 - \lambda$, and $m_{0,+}^n(\lambda)$ are the corresponding Weyl–Titchmarsh m-functions. Next, strong resolvent convergence implies convergence of the Weyl m-function and hence uniform convergence of $\psi_{0,+}^n(\lambda_1, x) \to \psi_{0,+}(\lambda_1, x)$ on compact sets. Clearly the same applies to $\psi_{1,-}^n(\lambda_1, x) \to \psi_{1,-}(\lambda_1, x)$. Thus, by Lemma 6.2, Lemma 6.5, and Lemma 6.6,

$$\begin{aligned} &\operatorname{tr}(P_{[\lambda_0,\lambda_1)}(H_1)) - \operatorname{tr}(P_{(\lambda_0,\lambda_1]}(H_0)) \\ &\qquad \geq \#(\psi_{1,-}(\lambda_1), \psi_{0,+}(\lambda_1)) - \#(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)). \end{aligned} \tag{6.16}$$

Repeating the argument with $u = \psi_{1,-}(\lambda_1)$ and $v = \psi_{0,+}(\lambda_1)$ shows that

$$\begin{aligned} &\operatorname{tr}(P_{[\lambda_0,\lambda_1)}(H_1)) - \operatorname{tr}(P_{(\lambda_0,\lambda_1]}(H_0)) \\ &\qquad \leq \#(\psi_{1,-}(\lambda_1), \psi_{0,+}(\lambda_1)) - \#(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)). \end{aligned} \tag{6.17}$$

This proves the second claim. □

7. Approximation in trace norm

Now we begin with an alternative approach toward singular differential operators by proving the case where $q_1 - q_0$ has compact support. The next lemma would follow from Theorem 3.13, but to demonstrate that this approach is independent of the last, we will provide an alternative proof.

Lemma 7.1. *Let H_j, $j = 0, 1$, be Sturm–Liouville operators on (a, b) associated with τ_j, and suppose that $r^{-1}(q_1 - q_0)$ has support in a bounded interval $(c, d) \subseteq (a, b)$, where $a < c$ if a is singular and $d < b$ if b is singular. Moreover, suppose H_0 and H_1 have the same boundary conditions (if any).*

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then,

$$\dim \operatorname{Ran} P_{(-\infty,\lambda_0)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \tag{7.1}$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then,

$$\begin{aligned} &\dim \operatorname{Ran} P_{[\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1]}(H_0) \\ &\qquad = \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned} \tag{7.2}$$

Proof. By splitting $r^{-1}(q_1 - q_0)$ into a positive and negative part as in the proof of the regular case (Theorem 2.3), we can reduce it to the case where $r^{-1}(q_1 - q_0)$ is of one sign, say $r^{-1}(q_1 - q_0) \geq 0$. Define $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon) H_0$ and observe that $\psi_{\varepsilon,-}(z, x) = \psi_{0,-}(z, x)$ for $x \leq c$, respectively, $\psi_{\varepsilon,+}(z, x) = \psi_{0,+}(z, x)$ for $x \geq d$. Furthermore, $\psi_{\varepsilon,\pm}(z, x)$ is analytic with respect to ε and $\lambda \in \sigma_p(H_\varepsilon)$ if and only if $W_d(\psi_{0,+}(\lambda), \psi_{\varepsilon,-}(\lambda)) = 0$. Now the proof can be done as in the regular case. □

Lemma 7.2. *Suppose H_0, H_1 satisfy the same assumptions as in the previous lemma and set $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon) H_0$. Then,*

$$\| \sqrt{r^{-1}|q_0 - q_1|} R_{H_\varepsilon}(z) \|_{\mathcal{J}_2} \leq C(z), \qquad \varepsilon \in [0, 1]. \tag{7.3}$$

In particular, the resolvent difference of H_0 and H_1 is trace class and

$$\xi(\lambda, H_1, H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)) \tag{7.4}$$

for every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$. Here $\xi(H_1, H_0)$ is assumed to be constructed such that $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is a continuous mapping from $[0,1] \to L^1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda)$.

Proof. Denote by

$$G_\varepsilon(z,x,y) = (H_\varepsilon - z)^{-1}(x,y) = \frac{\psi_{\varepsilon,-}(z,x_<), \psi_{\varepsilon,+}(z,y_>)}{W(\psi_{\varepsilon,-}(z), \psi_{\varepsilon,+}(z))},$$

where $x_< = \min(x,y)$, $y_> = \max(x,y)$, the Green's function of H_ε. As pointed out in the proof of the previous lemma, $\psi_{\varepsilon,\pm}(z,x)$ is analytic with respect to ε and hence a simple estimate shows

$$\int_a^b \int_a^b |G_\varepsilon(z,x,y)|^2 |r(y)^{-1}(q_1(y) - q_0(y))| r(x) dx\, r(y) dy \le C(z)^2$$

for $\varepsilon \in [0,1]$, which establishes the first claim.

Furthermore, a straightforward calculation (using (2.8)) shows

$$G_{\varepsilon'}(z,x,y) = G_\varepsilon(z,x,y) + (\varepsilon - \varepsilon') \int_a^b G_{\varepsilon'}(z,x,t) r^{-1}(t)(q_1(t) - q_0(t)) G_\varepsilon(z,t,y) r(t) dt.$$

(Note that this does not follow from the second resolvent identity unless $r^{-1}(q_1 - q_0)$ is relatively bounded with respect to H_0.) Hence, $R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)$ can be written as the product of two Hilbert–Schmidt operators whose norm can be estimated by the first claim,

$$\|R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)\|_{\mathcal{J}_1} \le |\varepsilon' - \varepsilon| C(z)^2. \tag{7.5}$$

Thus $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is continuous by Lemma 8.3. The rest follows from (8.4). □

Now we come to the

Proof of Theorem 3.16. We first assume that we have compact support near one endpoint, say a. Furthermore, abbreviate $V = r^{-1}(q_0 - q_1)$ which satisfies Hypothesis 3.15 (and thus also Hypothesis 8.4).

Define by K_ε the operator of multiplication by $\chi_{(a,b_\varepsilon]}$ with $b_\varepsilon \uparrow b$ as $\varepsilon \uparrow 1$. Then K_ε satisfies the assumptions of Lemma 8.5. Introduce $H_\varepsilon = H_0 - K_\varepsilon V$, and denote by $\psi_{\varepsilon,-}(\lambda, x)$ the corresponding solutions satisfying the boundary condition at a.

By Lemma 8.5 we have $\xi(., H_\varepsilon, H_0) \to \xi(., H_1, H_0)$ as $\varepsilon \to 1$ in $L^1(\mathbb{R}, (\lambda^2 + 1)^{-1}d\lambda)$. Moreover, $H_\varepsilon \to H_1$ in (trace) norm resolvent sense and hence $\lambda \in \rho(H_1)$ implies $\lambda \in \rho(H_\varepsilon)$ for ε sufficiently close to 1. Since $\xi(\lambda, H_\varepsilon, H_0) \in \mathbb{Z}$ is constant near every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_\varepsilon)$, we must have $\xi(\lambda, H_\varepsilon, H_0) = \xi(\lambda, H_1, H_0)$ for $\varepsilon \ge \varepsilon_0$ with some ε_0 sufficiently close to 1.

Now let us turn to the Wronskians. We first prove the $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ case. By Lemma 7.2 we know

$$\xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)$$

for every $\varepsilon < 1$. Concerning the right-hand side observe that

$$W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = W_x(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$$

for $x \le b_\varepsilon$ and that $W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$ is constant for $x \ge b_\varepsilon$. This implies that for $\varepsilon \ge \varepsilon_0$ we have

$$\begin{aligned}\xi(\lambda, H_1, H_0) &= \xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))\\ &= \#_{(a,b_\varepsilon)}(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = \#_{(a,b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)).\end{aligned}$$

In particular, the last item $\#_{(a,b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ is eventually constant and thus has a limit which, by Definition 3.1, is $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$.

For the corresponding $\#(\psi_{1,+}(\lambda), \psi_{0,-}(\lambda))$ case one simply exchanges the roles of H_0 and H_1.

Hence the result holds if the perturbation has compact support near one endpoint. Now one repeats the argument to remove the compact support assumption near the other endpoint as well. □

8. Appendix: Some facts on the spectral shift function

In this appendix we collect some facts on Krein's spectral shift function which are of relevance to us. Most results are taken from [40] (see also [26] for an easy introduction).

Two operators H_0 and H_1 are called resolvent comparable, if

$$R_{H_1}(z) - R_{H_0}(z) \tag{8.1}$$

is trace class for one $z \in \rho(H_1) \cap \rho(H_0)$. By the first resolvent identity (8.1) then holds for all $z \in \rho(H_1) \cap \rho(H_0)$.

Theorem 8.1 (Krein [13])**.** *Let H_1 and H_0 be two resolvent comparable self-adjoint operators, then there exists a function*

$$\xi(\lambda, H_1, H_0) \in L^1(\mathbb{R}, (\lambda^2+1)^{-1} d\lambda) \tag{8.2}$$

such that

$$\operatorname{tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda, H_1, H_0) f'(\lambda) d\lambda \tag{8.3}$$

for every smooth function f with compact support.

Note: Equation (8.3) holds in fact for a much larger class of functions f. See [40, Thm. 8.7.1] for this and a proof of the last theorem.

The function $\xi(\lambda) = \xi(\lambda, H_1, H_0)$ is called Krein's spectral shift function and is unique up to an additive constant. Moreover, $\xi(\lambda)$ is constant on every interval $(\lambda_0, \lambda_1) \subset \rho(H_0) \cap \rho(H_1)$. Hence, if $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_j) < \infty$, $j = 0, 1$, then $\xi(\lambda)$ is a step function and

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) = \lim_{\varepsilon \downarrow 0} \Big(\xi(\lambda_1 - \varepsilon) - \xi(\lambda_0 + \varepsilon)\Big). \tag{8.4}$$

This formula explains the name spectral shift function.

Before investigating further the properties of the SSF, we will recall a few things about trace ideals (see for example [25]). First, for $1 \le p < \infty$ denote by $\mathcal{J}^p$ the Schatten p-class, and by $\|.\|_{\mathcal{J}^p}$ its norm. We will use $\|.\|$ for the usual operator norm. Using $\|A\|_{\mathcal{J}^p} = \infty$ if $A \notin \mathcal{J}^p$, we have the following inequalities for all operators:

$$\|AB\|_{\mathcal{J}^p} \le \|A\| \|B\|_{\mathcal{J}^p}, \quad \|AB\|_{\mathcal{J}^1} \le \|A\|_{\mathcal{J}^2} \|B|_{\mathcal{J}^2}. \tag{8.5}$$

Furthermore, we will use the notation of $\mathcal{J}^p$-converges to denote convergence in the respective $\|.\|_{\mathcal{J}^p}$-norm.

The following result from [6, Thm IV.11.3] will be needed.

Lemma 8.2. *Let $p > 0$, $A \in \mathcal{J}^p$, $T_n \xrightarrow{s} T$, $S_n \xrightarrow{s} S$ sequences of strongly convergent bounded linear operators in some separable Hilbert space, then*

$$\|T_n A S_n^* - T A S^*\|_{\mathcal{J}^p} \to 0. \tag{8.6}$$

We will also need the following continuity result for ξ. It will also allow us to fix the unknown constant.

Lemma 8.3. *Suppose H_ε, $\varepsilon \in [0,1]$, is a family of self-adjoint operators, which is continuous in the metric*

$$\rho(A,B) = \|R_A(z) - R_B(z)\|_{\mathcal{J}^1}, \tag{8.7}$$

for some fixed $z \in \mathbb{C}\backslash\mathbb{R}$ and abbreviate $\xi_\varepsilon = \xi(H_\varepsilon, H_0)$. Then there exists a unique choice of ξ_ε such that $\varepsilon \mapsto \xi_\varepsilon$ is a continuous map $[0,1] \to L^1(\mathbb{R}, (\lambda^2+1)^{-1} d\lambda)$ with $\xi_0 = 0$.

If $H_\varepsilon \geq \lambda_0$ is bounded from below, we can also allow $z = \lambda \in (-\infty, \lambda_0)$.

Proof. The first statement can be found in [40, Lem. 8.7.5]. To see the second statement, let $\lambda < \lambda_0$ and $|\lambda - z| < \lambda_0 - \lambda$ for some $z \in \mathbb{C}\backslash\mathbb{R}$. Abbreviate $R_\varepsilon(z) = R_{H_\varepsilon}(z)$. Now using the first resolvent identity gives

$$\begin{aligned}\|R_\varepsilon(z) - R_{\varepsilon'}(z)\|_{\mathcal{J}^1} \leq & \|R_\varepsilon(\lambda) - R_{\varepsilon'}(\lambda)\|_{\mathcal{J}^1} \\ & + |z-\lambda| \|R_\varepsilon(z)\| \|R_\varepsilon(\lambda) - R_{\varepsilon'}(\lambda)\|_{\mathcal{J}^1} \\ & + |z-\lambda| \|R_{\varepsilon'}(\lambda)\| \|R_\varepsilon(z) - R_{\varepsilon'}(z)\|_{\mathcal{J}^1}\end{aligned}$$

and our conditions imply

$$|z-\lambda| \|R_{\varepsilon'}(\lambda)\| \leq \frac{|z-\lambda|}{\lambda_0 - \lambda} < 1$$

and thus

$$\|R_\varepsilon(z) - R_{\varepsilon'}(z)\|_{\mathcal{J}^1} \leq \frac{1 + \frac{|z-\lambda|}{|\mathrm{Im}(z)|}}{1 - \frac{|z-\lambda|}{|\lambda_0 - \lambda|}} \|R_\varepsilon(\lambda) - R_{\varepsilon'}(\lambda)\|_{\mathcal{J}^1},$$

from which the statement follows. □

Our final aim is to find some conditions which allow us to verify the assumptions of this lemma. To do this, we derive some properties of relatively bounded operators multiplied by strongly continuous families of operators. The key example for these operators will be multiplication operators by characteristic functions strongly converging to the identity operator.

Hypothesis H. 8.4. *Suppose H_0 and V are self-adjoint such that:*

(i) *V is relatively bounded with respect to H_0 with H_0 bound less than one or*
(i') *H_0 is bounded from below and $|V|$ is relatively form bounded with respect to H_0 with relative form bound less than one and*
(ii) *$|V|^{1/2} R_{H_0}(z)$ is Hilbert–Schmidt for one (and hence for all) $z \in \rho(H_0)$.*

We recall that (i) means that $\mathfrak{D}(V) \supseteq \mathfrak{D}(H_0)$ and for some $a < 1$, $b \geq 0$,

$$\|V\psi\| \leq a\|H_0\psi\| + b\|\psi\|, \quad \forall \psi \in \mathfrak{D}(H_0). \tag{8.8}$$

and (i') means that $\mathfrak{Q}(V) \supseteq \mathfrak{Q}(H_0)$ and for some $a < 1$, $b \geq 0$,

$$\langle \psi, |V|\psi\rangle \leq a\langle \psi, H_0\psi\rangle + b\|\psi\|^2, \quad \forall \psi \in \mathfrak{Q}(H_0). \tag{8.9}$$

Note that in case (i') all sums of operators are meant as form sums.

Lemma 8.5. *Let $\varepsilon \ni [0,1] \to K_\varepsilon$ be a strongly continuous family of bounded self-adjoint operators which commute with V and $0 = K_0 \leq K_\varepsilon \leq K_1 = \mathbb{1}$.*

Assume Hypothesis 8.4. Then

$$H_\varepsilon = H_0 + K_\varepsilon V \tag{8.10}$$

are self-adjoint operators such that the assumptions of Lemma 8.3 hold.

Proof. We will abbreviate $V_\varepsilon = K_\varepsilon V$ and $R_\varepsilon(z) = R_{H_\varepsilon}(z)$.

We begin with the case where V is relatively bounded with respect to H_0. By the Kato–Rellich Theorem ([15, Thm.X.12]) H_ε is well-defined and self-adjoint. Moreover, there is a z with $\mathrm{Im}(z) \neq 0$ such that $\|VR_0(z)\| \leq a < 1$. Hence $\|V_\varepsilon R_0(z)\| \leq a$ and a straightforward calculation using the second resolvent identity,

$$VR_\varepsilon(z) = VR_0(z)(1 + V_\varepsilon R_0(z))^{-1},$$

shows that

$$\|VR_\varepsilon(z)\| \leq \frac{a}{1-a}.$$

Furthermore, again using the second resolvent identity, we have

$$|V|^{1/2}R_\varepsilon(z) = |V|^{1/2}R_0(z)(1 - V_\varepsilon R_\varepsilon(z)),$$

which shows that

$$\||V|^{1/2}R_\varepsilon(z)\|_{\mathcal{J}_2} \leq \frac{1}{1-a}\||V|^{1/2}R_0(z)\|_{\mathcal{J}_2}.$$

To show $\mathcal{J}^1$-continuity at some fixed $\varepsilon \in [0,1]$ observe

$$R_{\varepsilon'}(z) - R_\varepsilon(z) = R_{\varepsilon'}(z)|V|^{1/2}\Big((K_{\varepsilon'} - K_\varepsilon)\mathrm{sgn}(V)|V|^{1/2}R_\varepsilon\Big),$$

where the first term $R_{\varepsilon'}(z)|V|^{1/2} \subseteq (|V|^{1/2}R_{\varepsilon'}(z^*))^*$ is uniformly $\mathcal{J}^2$-bounded in ε' by our previous argument and the second term $\mathcal{J}^2$-converges to 0 as ε' to ε by Lemma 8.2.

Now we come to the case where V is relatively form bounded with respect to H_0. By the KLMN theorem ([15, Thm.X.17]), H_ε is self-adjoint since we have

$$|\langle \psi, V_\varepsilon\psi\rangle| = |\langle |V|^{1/2}\psi, K_\varepsilon \mathrm{sgn}(V)|V|^{1/2}\psi\rangle| \leq \langle \psi, |V|\psi\rangle, \quad \psi \in \mathfrak{Q}(V_\varepsilon) = \mathfrak{Q}(V).$$

Moreover, using (8.9) we have

$$\||V|^{1/2}R_\varepsilon(-\lambda)^{1/2}\|^2 \leq a, \qquad \frac{b}{a} < \lambda.$$

We abbreviate:

$$U = |V|^{1/2}, \quad W = \mathrm{sgn}(V)|V|^{1/2}.$$

For $\lambda > \frac{b}{a}$ we have (see [10, Sec. VI.3] or [24, Thm. II.12])

$$R_\varepsilon(-\lambda) = R_0(-\lambda)^{1/2}(1 + C_\varepsilon)^{-1}R_0(-\lambda)^{1/2},$$
$$C_\varepsilon = (UR_0(-\lambda)^{1/2})^*(K_\varepsilon W R_0(-\lambda)^{1/2}).$$

Hence, a straightforward calculation shows

$$\begin{aligned} R_\varepsilon(-\lambda) &= R_0(-\lambda) - (UR_0(-\lambda))^*(1+\tilde{C}_\varepsilon)^{-1}(K_\varepsilon W R_0(-\lambda)),\\ \tilde{C}_\varepsilon &= K_\varepsilon W R_0(-\lambda)^{1/2}(UR_0(-\lambda)^{1/2})^*. \end{aligned}$$

By $\|\tilde{C}_\varepsilon\| \le a < 1$, one concludes that $(1+\tilde{C}_\varepsilon)^{-1}$ exists as a bounded operator. Then

$$\begin{aligned} D_{\varepsilon,\varepsilon'}\psi &= (-\tilde{C}_\varepsilon(1+\tilde{C}_\varepsilon)^{-1} - \tilde{C}_{\varepsilon'}(1+\tilde{C}_{\varepsilon'})^{-1})\psi\\ &= (C_{\varepsilon'} - C_\varepsilon)(1+C_\varepsilon)^{-1}\psi - C_{\varepsilon'}D_{\varepsilon,\varepsilon'}\psi, \end{aligned}$$

where

$$D_{\varepsilon,\varepsilon'} = (1+\tilde{C}_\varepsilon)^{-1} - (1+\tilde{C}_{\varepsilon'})^{-1}.$$

Taking norms we obtain

$$\|D_{\varepsilon,\varepsilon'}\psi\| = \frac{1}{1-a}\|(C_{\varepsilon'} - C_\varepsilon)(1+C_\varepsilon)^{-1}\psi\|,$$

where the last term converges to 0 as $\varepsilon' \to \varepsilon$. This implies, that $(1+\tilde{C}_\varepsilon)^{-1}$ is strongly continuous. Now, we obtain for the difference of resolvents

$$R_\varepsilon(-\lambda) - R_{\varepsilon'}(-\lambda) = (UR_0(-\lambda))^*((1+\tilde{C}_\varepsilon)^{-1}K_\varepsilon - (1+\tilde{C}_{\varepsilon'})^{-1}K_{\varepsilon'}))(WR_0(-\lambda))$$

which $\mathcal{J}^1$-converges to 0 as $\varepsilon \to \varepsilon'$ by Lemma 8.2 and by $UR_0(-\lambda)$ and $WR_0(-\lambda)$ being Hilbert–Schmidt. □

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