On the Spectral Zeta Function for the Non-commutative Harmonic Oscillator

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Abstract

The spectral zeta function for the so-called non-commutative harmonic oscillator is able to be meromorphically extended to the whole complex plane, having only one simple pole at the same point $s = 1$ where Riemann's zeta function $\zeta(s)$ has, and possesses a trivial zero at each non-positive even integer. The essential part of its proof is sketched. A new result is also given on the lower and upper bounds of the eigenvalues of the non-commutative harmonic oscillator.

Keywords: spectral zeta functions; Riemann's zeta function; harmonic oscillator; non-commutative harmonic oscillators; Weyl's law; Bernoulli's number.

1 Introduction and Results

The aim of this note is to give a survey of our recent results in [1] (cf. [2]) on the spectral zeta function for — the zeta function associated with the spectrum of — the non-commutative harmonic oscillator $Q := Q(x, D_x)$, improving a previous result there on the lower and upper bounds of the eigenvalues of Q, and also to mention some recent results. This Q was introduced by A. Parmeggiani and the second author ([16, 17]) as a differential operator

$$Q(x, D_x) \;:=\; A\Big(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\Big) + J\Big(x\partial_x + \frac{1}{2}\Big)$$

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$$= \begin{pmatrix} \alpha\Big(-\frac{\partial_x^2}{2}+\frac{x^2}{2}\Big) & -\Big(x\partial_x+\frac{1}{2}\Big) \\ x\partial_x+\frac{1}{2} & \beta\Big(-\frac{\partial_x^2}{2}+\frac{x^2}{2}\Big) \end{pmatrix}, \qquad x \in \mathbf{R}, \quad (1)$$

with $D_x := -i\partial_x$ and $\partial_x := \frac{d}{dx}$, acting on the Hilbert space $L^2(\mathbf{R}, \mathbf{C}^2) = L^2(\mathbf{R}) \otimes \mathbf{C}^2$, which is dependent on two positive parameters α and β with $\alpha\beta > 1$ through

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2)$$

It may not be an operator being a Hamiltonian coming from some existing system in quantum mechanics in the present stage. In the papers [16] and [17], among others, the following basic facts for $Q = Q(x, D_x)$ were already observed: Q is a selfadjoint operator in $L^2(\mathbf{R}, \mathbf{C}^2)$, and essentially selfadjoint on $\mathcal{S}(\mathbf{R}, \mathbf{C}^2)$. It has only discrete spectrum consisting of the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \quad \uparrow \infty$, with uniformly bounded multiplicity. As a result, the first eigenvalue λ_1 is positive, so that Q has a bounded inverse Q^{-1}. However, the value of any eigenvalue λ_n is hardly computed explicitly (A numerical study of the spectrum was carried out in [9]). The two relations of non-commutativity, i.e. the one for matrices $[A,\, J] = (\beta - \alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq 0$ when $\alpha \neq \beta$ and the canonical commutation relation $[\partial_x,\, x] = 1$, make highly non-trivial the spectrum of the non-commutative harmonic oscillator, which thus in general is a non-trivial pair of the harmonic oscillators.

We define the spectral zeta function associated with Q as

$$\zeta_Q(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n{}^s}. \quad (3)$$

Then it was shown in [2] that the series on the right-hand side converges absolutely in the half plane $\mathrm{Re}\, s > 1$. $\zeta_Q(s)$ has dependence on the two parameters α and β (essentially, on the ratio α/β), since each eigenvalue λ_n does. Notice here that this spectral zeta function is giving a deformation of the Riemann zeta function $\zeta(s)$, which in turn, we may think, is essentially associated with the spectrum of the harmonic oscillator $-\frac{\partial_x^2}{2} + \frac{x^2}{2}$. Indeed, when $\alpha = \beta$, the non-commutative harmonic oscillator becomes (equivalent with) a couple of two usual harmonic oscillators whose eigenvalues are given by $(n + \frac{1}{2})\sqrt{\alpha^2 - 1}$ $(n = 0, 1, 2, \ldots)$ with multiplicity two (see [17]).

Although no eigenvalues of Q can be calculated explicitly, we would expect to find with this zeta function $\zeta_Q(s)$ another way to investigate the spectral properties of Q and, for instance, some knowledge of its n-th eigenvalue to be gained through the analytic property of the spectral zeta function.

We have shown in [1] the following theorem and corollary on $\zeta_Q(s)$.

Theorem 1. *There exist constants $C_{Q,j}$ $(j = 1, 2, \ldots)$ such that $\zeta_Q(s)$ is,* *for every positive integer n, represented as*

$$\zeta_Q(s) = \frac{1}{\Gamma(s)}\left[\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\,\frac{1}{s-1} + \sum_{j=1}^{n}\frac{C_{Q,j}}{s+2j-1} + H_{Q,n}(s)\right], \qquad (4)$$

where $\Gamma(s)$ is the gamma function and $H_{Q,n}(s)$ is a holomorphic function in $\mathrm{Re}\, s > -2n$. Consequently, the spectral zeta function $\zeta_Q(s)$ is meromorphic in the whole complex plain $\mathbf{C}$ with a simple pole at $s = 1$ and has zeros for $s = 0, -2, -4, \ldots$ (non-positive even integers).

Here, note that $\frac{1}{\Gamma(s)}$ is holomorphic in the whole complex plane $\mathbf{C}$ and has zeros at $s = 0, -1, -2, \cdots$. In this sense, one says that the non-positive even integers are the trivial zeros of $\zeta_Q(s)$.

Corollary. (Weyl's law)

$$\sum_{\lambda_n < x} 1 \sim \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\, x \qquad (x \to \infty). \qquad (5)$$

A meaning of the coefficient of the main term is investigated in [14], Theorem 5.2 with Remark 5.1. It is quite interesting to obtain the error estimate of this Weyl law; both the order and its coefficient which should be expressed by the invariants of the system.

Let $K(t) = K(t,x,y) := e^{-tQ}(x,y)$ be the heat kernel for Q. But we cannot get its explicit expression by solving the heat equation

$$[\partial_t + Q(x, D_x)]K(t,x,y) = 0\,. \qquad (6)$$

By the Mellin transform we have

$$Q^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tQ}dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t)\,dt, \qquad \mathrm{Re}\, s > 1,$$

so that

$$\zeta_Q(s) = \mathrm{Tr}\, Q^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\mathrm{Tr}\, K(t). \qquad (7)$$

As mentioned before, we cannot obtain any of the eigenvalues λ_n explicitly, but estimate their lower and upper bounds. The following theorem gives an improvement of the result in [1], where we treated only the case for the first eigenvalue λ_1 by use of a Lieb–Thirring inequality [8].

Theorem 2. *Let λ_{2j-1}, λ_{2j}, $j = 1, 2, \ldots,$ be the $(2j-1)$-th and $2j$-th eigenvalues of Q. Then*

$$\big(j - \frac{1}{2}\big)\min\{\alpha,\beta\}\sqrt{1-\frac{1}{\alpha\beta}} \le \lambda_{2j-1} \le \lambda_{2j} \le \big(j-\frac{1}{2}\big)\max\{\alpha,\beta\}\sqrt{1-\frac{1}{\alpha\beta}}. \quad (8)$$

Note that the two closed intervals made of the bounds (8) for two pairs of the eigenvalues $\{\lambda_{2j-1},\, \lambda_{2j}\}$ and $\{\lambda_{2k-1},\, \lambda_{2k}\}$ with $j < k$

$$\Big[\big(j - \frac{1}{2}\big) \min\{\alpha, \beta\} \sqrt{1 - 1/(\alpha\beta)},\ \big(j - \frac{1}{2}\big) \max\{\alpha, \beta\} \sqrt{1 - 1/(\alpha\beta)}\Big]$$

and

$$\Big[\big(k - \frac{1}{2}\big) \min\{\alpha, \beta\} \sqrt{1 - 1/(\alpha\beta)},\ \big(k - \frac{1}{2}\big) \max\{\alpha, \beta\} \sqrt{1 - 1/(\alpha\beta)}\Big]$$

do not intersect with each other or do, according as $(k - \frac{1}{2})/(j - \frac{1}{2})$ is greater or not greater than $\frac{\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}$. Hence, in particular, we see the eigenvalue λ_{2j-1} or λ_{2j} has a multiplicity not greater than 2 if $j < \frac{\frac{\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}+1}{2(\frac{\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}-1)} = \frac{|\alpha+\beta|}{2|\alpha-\beta|}$. Otherwise, it will possibly happen to have a multiplicity greater than 2. It was shown in [9] that the first eigenvalue λ_1 is simple for sufficiently large α, β. This fact was also observed by Parmeggiani [13] with perturbation theory. Moreover, he studies in [14] and [15] clustering properties and multiplicity of general eigenvalues through the dynamics along the periodic trajectories of the bicharacteristics associated with the eigenvalues of the symbol of the system.

We recall some of the properties of the Riemann zeta function $\zeta(s)$ to check whether $\zeta_Q(s)$ has analogous ones. By definition, we have $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}\, s > 1$, of which the series on the right converges absolutely in $\mathrm{Re}\, s > 1$. This $\zeta(s)$ has only one simple pole at $s = 1$, and has also zeros at $s = -2, -4, \ldots$ (negative even integers), and satisfies the functional equation

$$\pi^{-\frac{s}{2}} \Gamma\Big(\frac{s}{2}\Big) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\Big(\frac{1-s}{2}\Big) \zeta(1-s), \tag{9}$$

and the Euler product expression

$$\zeta(s) = \prod_{p:\text{prime}} \frac{1}{1 - p^{-s}} \qquad (\mathrm{Re}\, s > 1). \tag{10}$$

Note that $\alpha = \beta$ if and only if J and A commute, and hence if and only if Q (or Q^{-1}) and A commute if and only if Q (or Q^{-1}) and J commute. In this case, Q is unitarily equivalent to a couple of the ordinary harmonic oscillators $\sqrt{\alpha^2 - 1} Q_0$, where $Q_0 \equiv \Big(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\Big) \otimes I$ with I being the 2×2 identity matrix. In particular, when $\alpha = \beta = \sqrt{2}$, Q and Q_0 are unitarily equivalent to each other, whence it follows that

$$\zeta_Q(s) = \zeta_{Q_0}(s) = 2 \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^s} = 2(2^s - 1)\, \zeta(s). \tag{11}$$

Note here also that when $\alpha = \beta$, the inequalities (8) become equalities; it implies that $\lambda_{2j-1} = \lambda_{2j} = \big(j - \frac{1}{2}\big) \sqrt{\alpha^2 - 1}$ in this case.

But it is hard to expect $\zeta_Q(s)$ to have some functional equation and/or Euler product expression. There are many spectral zeta functions in geometry, dynamical systems,.., but not many seem to be known to be meromorphically extended to the whole complex plain $\mathbf{C}$. Among those studies, D. Robert [19] considered, for a general class of elliptic systems of pseudo-differential operators P, their power P^{-s} through the Dunford integral (only) to do a meromorphic continuation of $\operatorname{Tr} P^{-s}$ to the whole complex plane $\mathbf{C}$.

In Section 2 we give an outline of the proof of Theorem 1 and a proof of Theorem 2, and in Section 3 some remarks and open questions.

2 Proofs of Theorem 1 and Theorem 2

Outline of Proof of Theorem 1.

The actual proof is a little lengthy because of non-commutativity. We have denoted by $K(t) = K(t, x, y) := e^{-tQ}(x, y)$ the heat kernel for Q.

From the expression (7) of $\zeta_Q(s)$ through the Mellin transform, we write

$$\zeta_Q(s) = \frac{1}{\Gamma(s)}\Big[\int_0^1 + \int_1^\infty\Big] dt\, t^{s-1} \operatorname{Tr} K(t) =: Z_0(s) + Z_\infty(s). \tag{12}$$

Then it is easy to see the second term $Z_\infty(s)$ in the last member is holomorphic in $\mathbf{C}$. So we must study $Z_0(s)$. To do so, we need an asymptotic expansion for small $t > 0$ like

$$\operatorname{Tr} K(t) \sim c_{-1}t^{-1} + c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots, \quad 0 < t << 1.$$

Define the operator $K_1(t)$ for $f \in \mathcal{S}(\mathbf{R}, \mathbf{C}^2) = \mathcal{S}(\mathbf{R}) \otimes \mathbf{C}^2$, by the pseudo-differential operator

$$\begin{aligned} K_1(t)f(x) &= \int K_1(t, x, y) f(y) dy \\ &= \frac{1}{2\pi}\int\int e^{i(x-y)\xi} \exp\Big[-t\Big(A\frac{\xi^2 + y^2}{2} + Jyi\xi\Big)\Big] f(y)dyd\xi. \end{aligned} \tag{13}$$

Put

$$R_2(t) := K(t) - K_1(t), \tag{14}$$

so that $K(t) = K_1(t) + R_2(t)$.

Proposition 1.

$$\begin{aligned} Z_0(s) &= \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1} \operatorname{Tr} K(t) \\ &= \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1} \operatorname{Tr} K_1(t) + \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1} \operatorname{Tr} R_2(t) \\ &= \frac{\alpha + \beta}{\sqrt{\alpha\beta(\alpha\beta - 1)}} \frac{1}{\Gamma(s)} \frac{1}{s-1} + \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1} \operatorname{Tr} R_2(t). \end{aligned}$$

It is hence sufficient to treat only the second term $\frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1}\mathrm{Tr}\, R_2(t)$ in the last member of the equation in Proposition 1. Since by (6)

$$0 = [\partial_t + Q]K(t,x,y) = [\partial_t + Q]K_1(t,x,y) + [\partial_t + Q]R_2(t,x,y),$$

we have

$$[\partial_t + Q]R_2(t,x,y) = -[\partial_t + Q]K_1(t,x,y) =: F(t,x,y). \tag{15}$$

By Duhamel's principle, we can solve (15) to get

$$\begin{aligned} R_2(t) &= \int_0^t e^{-(t-u)Q}F(u)du \\ &= \int_0^t du \int K_1(t-u,x,z)F(u,z,y)dz + \int_0^t du \int R_2(t-u,x,z)F(u,z,y)dz \\ &=: K_2(t) + R_3(t), \end{aligned}$$

so that $K(t) = K_1(t) + K_2(t) + R_3(t)$.

Repeating this iteration yields

$$K(t) = K_1(t) + K_2(t) + \cdots + K_n(t) + R_{n+1}(t), \tag{16}$$

$$K_m(t) = \int_0^t du_1 \int_0^{t-u_1} du_2 \cdots \int_0^{t-u_1-u_2\cdots-u_{m-2}} du_{m-1}$$
$$\times K_1(t-u_1-\cdots-u_{m-1})F(u_{m-1})\cdots F(u_2)F(u_1), \quad 2 \le m \le n, \tag{17}$$

$$R_{n+1}(t) = \int_0^t du_1 \int_0^{t-u_1} du_2 \cdots \int_0^{t-u_1-u_2\cdots-u_{m-1}}$$
$$\times K(t-u_1-\cdots-u_m)F(u_m)F(u_{m-1})\cdots F(u_2)F(u_1)du_m. \tag{18}$$

So we have to study the traces of $K_m(t)$ and $R_{n+1}(t)$ for small $t > 0$.

Lemma 1. *For every small $\varepsilon > 0$ one has*

$$|\mathrm{Tr}\, R_2(t)| \le C(\varepsilon)t^{-\varepsilon},$$
$$|\mathrm{Tr}\, R_{n+1}(t)| \le C^n \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} t^{n/2}, \quad n \ge 2,$$

where $C(\varepsilon)$ is a positive constant independent of t but dependent on ε, and C a positive constant independent of t and n.

This lemma implies that the Mellin transform of $\mathrm{Tr}\, R_2(t)$ is holomorphic in $\mathrm{Re}\, s > \varepsilon$ and hence in $\mathrm{Re}\, s > 0$ because ε is arbitrary, and that of $\mathrm{Tr}\, R_{n+1}(t)$ $(n \ge 2)$ in $\mathrm{Re}\, s > -\frac{n}{2}$.

A little patient, lengthy calculation yields the following key asymptotics of $K_m(t)$ as $t \downarrow 0$.

Lemma 2. *For $m = 2, 3, \ldots,$ one has for $t \downarrow 0$,*

$$\mathrm{Tr}\, K_m(t) \sim \sum_{j=0}^{\infty} c_{m,j} t^j\,,$$

with $c_{m,j} = 0$ for $0 \le j < m-2$ and $j = 2\ell$ being positive even integers.

Hence we have by Proposition 1, Lemma 1 and Lemma 2

$$\begin{aligned}
\mathrm{Tr}K_1(t) &= \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}t^{-1}\,, \\
\mathrm{Tr}K_2(t) &\sim c_{2,1}t + c_{2,3}t^3 + c_{2,5}t^5 + c_{2,7}t^7 + c_{2,9}t^9 + \cdots\,, \\
\mathrm{Tr}K_3(t) &\sim c_{3,1}t + c_{3,3}t^3 + c_{3,5}t^5 + c_{3,7}t^7 + c_{3,9}t^9 + \cdots\,, \\
\mathrm{Tr}K_4(t) &\sim c_{4,3}t^3 + c_{4,5}t^5 + c_{4,7}t^7 + c_{4,9}t^9 + \cdots\,, \\
\mathrm{Tr}K_5(t) &\sim c_{5,3}t^3 + c_{5,5}t^5 + c_{5,7}t^7 + c_{5,9}t^9 + \cdots\,, \\
\mathrm{Tr}K_6(t) &\sim c_{6,5}t^5 + c_{6,7}t^7 + c_{6,9}t^9 + \cdots\,, \\
\mathrm{Tr}K_7(t) &\sim \cdots\cdots\cdots\cdots\cdots\cdots\,, \\
\cdots &\sim \cdots\cdots\cdots\cdots\cdots\,, \\
\mathrm{Tr}R_{n+1}(t) &= O(t^{n/2})\,.
\end{aligned}$$

Thus with the Mellin transform we can conclude

$$\begin{aligned}
\zeta_Q(s) &= \frac{1}{\Gamma(s)}\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\frac{1}{s-1} + (c_{2,1}+c_{3,1})\frac{1}{\Gamma(s)}\frac{1}{s+1} \\
&\quad +(c_{2,3}+c_{3,3}+c_{4,3}+c_{5,3})\frac{1}{\Gamma(s)}\frac{1}{s+3} \\
&\quad +(c_{2,5}+c_{3,5}+c_{4,5}+c_{5,5}+c_{6,5})\frac{1}{\Gamma(s)}\frac{1}{s+5} + \cdots\cdots + \frac{1}{\Gamma(s)}H_{Q,n}(s),
\end{aligned}$$

showing Theorem 1.

Proof of Theorem 2.

We develop a method used in [1], *Remark 3*, p.714, to give an alternative proof of the assertion for λ_1, not appealing to a Lieb–Thirring inequality.

Though we cannot solve (6) explicitly, we can solve the modified equation

$$[\partial_t + Q'(x, D_x)]K'(t,x,y) = 0\,, \tag{19}$$

where

$$\begin{aligned}
Q' &:= A^{-1/2}QA^{-1/2} = \frac{1}{2}(-\partial_x^2 + x^2) + \gamma J(x\partial_x + \frac{1}{2}), \\
&= \frac{1}{2}(-i\partial_x + i\gamma Jx)^2 + \frac{1-\gamma^2}{2}x^2, \qquad \gamma := \frac{1}{\sqrt{\alpha\beta}}\,.
\end{aligned} \tag{20}$$

In fact, we can show

$$\begin{aligned}
K'(t,x,y) &= e^{-tQ'} = (1-\gamma^2)^{1/4}\exp[\frac{1}{2}\gamma(x^2-y^2)J] \\
&\quad \times p((1-\gamma^2)^{1/2}t, (1-\gamma^2)^{1/2}x, (1-\gamma^2)^{1/2}y),
\end{aligned} \tag{21}$$

where $p(t,x,y) := \exp[-\frac{1}{2}(-\partial_x^2 + x^2)](x,y)$ is the heat kernel of the ordinary harmonic oscillator. Indeed, it was shown in [17] that Q' is unitarily equivalent to $\frac{\gamma}{2}[-\partial_x^2 + (\alpha\beta - 1)x^2] \otimes I$, and also

$$\zeta_{Q'}(s) = \operatorname{Tr} Q'^{-s} = (1-\gamma^2)^{-s/2}\zeta_{Q_0}(s) = (1-\gamma^2)^{-s/2} 2(2^s - 1)\zeta(s), \qquad (22)$$

with $\zeta_{Q_0}(s)$ in (11). Hence we know that every eigenvalue λ'_n of Q' has multiplicity two, so that its $(2j-1)$-th and $2j$-th eigenvalues λ'_{2j-1} and λ'_{2j} coincide and are equal to

$$\lambda'_{2j-1} = \lambda'_{2j} = \big(j - \frac{1}{2}\big)\sqrt{1 - 1/(\alpha\beta)}. \qquad (23)$$

Note that $Q = A^{1/2}Q'A^{1/2}$. Using this fact and the min-max principle (e.g. [18]), we have with $n := 2j-1$ or $n := 2j$, $j = 1, 2, \ldots$,

$$\begin{aligned}
\lambda_n &= \sup_{u_1,\cdots,u_{n-1}} \inf_{u\neq 0;\, u\perp[u_1,\cdots,u_{n-1}]} \frac{(u,Qu)}{\|u\|^2} \\
&= \sup_{u_1,\cdots,u_{n-1}} \inf_{u\neq 0;\, u\perp[u_1,\cdots,u_{n-1}]} \frac{(A^{1/2}u, Q'A^{1/2}u)}{\|u\|^2} \\
&= \sup_{u_1,\cdots,u_{n-1}} \inf_{u\neq 0;\, u\perp[A^{-1/2}u_1,\cdots,A^{-1/2}u_{n-1}]} \frac{(A^{1/2}u, Q'A^{1/2}u)}{\|u\|^2} \\
&= \sup_{u_1,\cdots,u_{n-1}} \inf_{u\neq 0;\, A^{1/2}u\perp[u_1,\cdots,u_{n-1}]} \frac{(A^{1/2}u, Q'A^{1/2}u)}{\|u\|^2} \\
&= \sup_{u_1,\cdots,u_{n-1}} \inf_{v\neq 0;\, v\perp[u_1,\cdots,u_{n-1}]} \frac{(v, Q'v)}{\|A^{-1/2}v\|^2} \qquad (v := A^{1/2}u) \\
&= \sup_{u_1,\cdots,u_{n-1}} \inf_{v\neq 0;\, v\perp[u_1,\cdots,u_{n-1}]} \frac{(v, Q'v)}{\|v\|^2}\frac{\|v\|^2}{\|A^{-1/2}v\|^2}.
\end{aligned}$$

Here we have denoted by $[u_1, \ldots, u_{n-1}]$ the subspace spanned by the vectors $u_1, \ldots, u_{n-1}$ in the domain of Q; note that Q and Q' have the same domain. It is crucial above that the third equality holds, because if the vectors $\{u_1, \ldots, u_{n-1}\}$ are linearly independent, so are the vectors $\{A^{-1/2}u_1, \ldots, A^{-1/2}u_{n-1}\}$, since A is one-to-one and has a bounded inverse A^{-1}, and because a vector w belongs to the domain of Q or Q' if and only if $A^{-1/2}w$ does. We note that with $m := \min\{\alpha, \beta\}$, $M := \max\{\alpha, \beta\}$,

$$\frac{1}{M}\|v\|^2 \leq \|A^{-1/2}v\|^2 = (v, A^{-1}v) \leq \frac{1}{m}\|v\|^2, \quad v \in L^2(\mathbf{R}, \mathbf{C}^2),$$

whence $m \leq \|v\|^2/\|A^{-1/2}v\|^2 \leq M$ for $v \neq 0$.

It follows that

$$\begin{aligned}
\lambda_{2j-1} \leq \lambda_{2j} &\leq M \sup_{u_1,\cdots,u_{2j-1}} \inf_{v\neq 0;\, v\perp[u_1,\cdots,u_{2j-1}]} \frac{(v, Q'v)}{\|v\|^2} = M\lambda'_{2j} = M\lambda'_{2j-1}\,, \\
\lambda_{2j} \geq \lambda_{2j-1} &\geq m \sup_{u_1,\cdots,u_{(2j-1)-1}} \inf_{v\neq 0;\, v\perp[u_1,\cdots,u_{(2j-1)-1}]} \frac{(v, Q'v)}{\|v\|^2} = m\lambda'_{2j-1}\,.
\end{aligned}$$

Thus we have shown the lower and upper bounds of λ_n as in Theorem 2.

3 Some Remarks and Open Questions

We have shown one of the basic analytic properties of the spectral zeta function $\zeta_Q(s)$ for the non-commutative harmonic oscillator $Q = Q(x, D_x)$ in (1). There still remain a number of open questions.

1. Can one take the limit $n \to \infty$ in the Theorem 1 ? In other words, does the series $\sum_{j=1}^{n} \frac{C_{Q,j}}{s+2j-1}$ converge when $n \to \infty$? If $\alpha = \beta$, then one knows that this is true. For the precise situation, see the last discussion given in [1], pp.735–738.

2. In view of the Euler product expression (10), one recognizes easily that Riemann's zeta function $\zeta(s)$ has no zeros in $\operatorname{Re} s > 1$. We have proved that $\zeta_Q(s)$ has no zeros in $\operatorname{Re} s > c$ for sufficiently large c, though. Can one determine the zero-free region of $\zeta_Q(s)$ in the half plane $\operatorname{Re} s > 1$ where the defining series of $\zeta_Q(s)$ converges absolutely ?

3. $\zeta(s)$ has an infinite number of nontrivial zeros on the line $\operatorname{Re} s = \frac{1}{2}$. Is there any such a critical line or curve for $\zeta_Q(s)$? If so, what is it ? Also, does $\zeta_Q(s)$ possess any functional equation or Euler product expression such as $\zeta(s)$ does (though it is hardly expected) ? See also the very final remark in [1].

4. Compute the values of $\zeta_Q(s)$ for $s = n$ being positive integers greater than 1. For Riemann's zeta function $\zeta(s)$, the Bernouille numbers B_{2m} ($m \in \mathbf{N}$) describe such special values: $\zeta(2m) = \frac{(-1)^{m-1}}{2} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}$.

In [2], we have computed $\zeta_Q(2)$ and $\zeta_Q(3)$ and discussed a connection of these values with (singly confluent) Heun's differential equations. Here a Heun differential equation is a Fuchsian ordinary equation with four regular singular points in a complex domain. Actually, $\zeta_Q(2)$ is represented as a contour integral for a holomorphic solution in the unit disk of this equation, and also $\zeta_Q(3)$ as one for the holomorphic solution of the same equation but with an extra inhomogeneous term. Based on these results, H. Ochiai [12] has obtained a beautiful explicit expression of $\zeta_Q(2)$ in terms of the complete elliptic integral of the first kind. In fact, by suitable change of the variable, he could show that the singly confluent Heun equation in question turns to be a Gaussian hypergeometric differential equation. Exploiting a similar idea, in [5], K. Kimoto and the second author have obtained an analogous explicit expression for the value $\zeta_Q(3)$ by using a Gauss hypergeometric function and solving the aforementioned inhomogeneous equation, and shown that the coefficients of the Taylor expansion of the holomorphic solutions of these Heun equations possess some remarkable congruence relations such as the Apéry numbers have. We note that the Apéry numbers are known to be used for proving the irrationality of $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(3)$ (see [20]). Moreover, in [6], one finds that the above Heun equation (for describing the value $\zeta_Q(2)$) is the Picard–Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion and the parameter for a family of such curves can be regarded as a modular function of the congruent subgroup $\Gamma_0(4)$.

According to the result [10] (see also [11]), the spectral problem of the non-commutative harmonic oscillator is known to be equivalent to solving a certain Heun's equation (but not confluent).

5. Understand clearly the situation or intrinsic reason why our special values are so related with the confluent Heun equations as in view of the results in [10]. Explore, moreover, any relation between the values of $\zeta_Q(s)$ at $s = -2m+1$ and those at $s = 2m$ via Heun's equation to compensate for a missing state of such a functional equation as $\zeta(s)$ has in (9).

6. Calculate the 'characteristic polynomial' of Q. To be more precise, let us recall the determinant of an operator $A = \mathrm{diag}(a_1, a_2, a_3, \ldots)$ defined by

$$\det A := \exp\Big(-\operatorname{Res}_{s=0} \frac{\zeta(s,A)}{s^2}\Big). \tag{24}$$

Here $\zeta(s,A)$ is a Dirichlet series defined by $\zeta(s,A) = \sum_{n=1}^{\infty} {a_n}^{-s}$ ($\operatorname{Re} s \gg 1$) and assumed to be meromorphic around $s=0$ (see [3], [4] for this definition of the zeta regularized determinants and products). A similar technique we developed in [1] and [2] allows us to state a conjecture that the Hurwitz type spectral zeta function defined by

$$\zeta_Q(s,z) := \sum_{n=1}^{\infty} (\lambda_n - z)^{-s} \qquad (z \notin \{\lambda_n\}_{n=1,2,3\ldots}) \tag{25}$$

can be meromorphically extended to the whole complex plane. If this is true, the determinant $\det(Q - zI) := \exp\big(-\operatorname{Res}_{s=0}\frac{\zeta_Q(s,z)}{s^2}\big)$ can be defined. For instance, for the particular operator Q_0, i.e. the Q with $\alpha = \beta = \sqrt{2}$, mentioned in (11), one actually shows from the Lerch formula [7] that

$$\det(Q_0 - zI) = 2\pi \Gamma\Big(\frac{1}{2} - z\Big)^{-2}.$$

We close the present note by asking the following fundamental but probably a difficult question.

7. For Q, is there a pair of *annihilation* and *creation operators* ?

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