THE ABLOWITZ–LADIK HIERARCHY REVISITED

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Abstract. We provide a detailed recursive construction of the Ablowitz–Ladik (AL) hierarchy and its zero-curvature formalism. The two-coefficient AL hierarchy under investigation can be considered a complexified version of the discrete nonlinear Schrödinger equation and its hierarchy of nonlinear evolution equations.

Specifically, we discuss in detail the stationary Ablowitz–Ladik formalism in connection with the underlying hyperelliptic curve and the stationary Baker–Akhiezer function and separately the corresponding time-dependent Ablowitz–Ladik formalism.

1. Introduction

The prime example of an integrable nonlinear differential-difference system to be discussed in this paper is the Ablowitz–Ladik system,

$$\begin{aligned} &-i\alpha_t - (\alpha^+ - 2\alpha + \alpha^-) + \alpha\beta(\alpha^+ + \alpha^-) = 0, \\ &-i\beta_t + (\beta^+ - 2\beta + \beta^-) - \alpha\beta(\beta^+ + \beta^-) = 0 \end{aligned} \tag{1.1}$$

with $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here we used the notation $f^{\pm}(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, for complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}}$. It arose in the mid-seventies when Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3], [18]), used inverse scattering methods to analyze certain integrable differential-difference systems. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\overline{\alpha}$, and in the defocusing case, where $\beta = \overline{\alpha}$, (1.1) yields the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t + 2\alpha - (1 \pm |\alpha|^2)(\alpha^+ + \alpha^-) = 0. \tag{1.2}$$

We will refer to (1.1) as the Ablowitz–Ladik system. The principal theme of this paper will be to derive a detailed recursive construction of the Ablowitz–Ladik hierarchy, a completely integrable sequence of systems of nonlinear evolution equations on the lattice $\mathbb{Z}$ whose first nonlinear member is the Ablowitz–Ladik system (1.1). In addition, we discuss the zero-curvature formalism for the Ablowitz–Ladik (AL) hierarchy in detail.

Since the original discovery of Ablowitz and Ladik in the mid-seventies, there has been great interest in the area of integrable differential-difference equations. Two principal directions of research are responsible for this development: Originally, the development was driven by the theory of completely integrable systems and

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its applications to fields such as nonlinear optics, and more recently, it gained additional momentum due to its intimate connections with the theory of orthogonal polynomials. In the following we first briefly recall the development in connection with integrable systems and subsequently turn to the one influenced by research on orthogonal polynomials.

The first systematic discussion of the Ablowitz–Ladik (AL) hierarchy appears to be due to Schilling [47] (cf. also [56], [60], [63]); infinitely many conservation laws are derived, for instance, by Ding, Sun, and Xu [22]; the bi-Hamiltonian structure of the AL hierarchy is considered by Ercolani and Lozano [24]; connections between the AL hierarchy and the motion of a piecewise linear curve have been established by Doliwa and Santini [23]; Bäcklund and Darboux transformations were studied by Geng [25] and Vekslerchik [61]; the Hirota bilinear formalism, AL τ-functions, etc., were considered by Vekslerchik [60]. The initial value problem for half-infinite AL systems was discussed by Common [20], for an application of the inverse scattering method to (1.2) we refer to Vekslerchik and Konotop [62]. This just scratches the surface of these developments and the interested reader will find much more material in the references cited in these papers and the ones discussed below. Algebro-geometric (and periodic) solutions of the AL system (1.1) have briefly been studied by Ahmad and Chowdhury [8], [9], Bogolyubov, Prikarpatskii, and Samoilenko [15], Bogolyubov and Prikarpatskii [16], Chow, Conte, and Xu [19], Geng, Dai, and Cao [26], Vaninsky [58].

In an effort to analyze models describing oscillations in nonlinear dispersive wave systems, Miller, Ercolani, Krichever, and Levermore [43] (see also [42]) gave a detailed analysis of algebro-geometric solutions of the AL system (1.1). Introducing

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad V(z) = i \begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ \beta^- z - \beta & 1 + \alpha^-\beta - z^{-1} \end{pmatrix} \tag{1.3}$$

with $z \in \mathbb{C}$ a spectral parameter, the authors in [43] relied on the fact that the Ablowitz–Ladik system (1.1) is equivalent to the zero-curvature equations

$$U_t + UV - V^+U = 0. \tag{1.4}$$

Miller, Ercolani, Krichever, and Levermore [43] then derived the theta function representations of α, β satisfying the AL system (1.1). Vekslerchik [59] also studied finite-genus solutions for the AL hierarchy by establishing connections with Fay's identity for theta functions. Recently, an in depth study of algebro-geometric solutions for the entire AL hierarchy has been provided in [34].

The connection between the Ablowitz–Ladik system (1.1) and orthogonal polynomials comes about as follows: Let $\{\alpha(n)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers subject to the condition $|\alpha(n)| < 1$, $n \in \mathbb{N}$, and define the transfer matrix

$$T(z) = \begin{pmatrix} z & \alpha \\ \overline{\alpha} z & 1 \end{pmatrix}, \quad z \in \mathbb{T}, \tag{1.5}$$

with spectral parameter z on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Consider the system of difference equations

$$\Phi(z, n) = T(z, n)\Phi(z, n-1), \quad (z, n) \in \mathbb{T} \times \mathbb{N}, \tag{1.6}$$

with initial condition $\Phi(z,0)=\begin{pmatrix}1\\1\end{pmatrix}$, where

$$\Phi(z,n)=\begin{pmatrix}\varphi(z,n)\\ z^n\overline{\varphi}(1/z,n)\end{pmatrix}, \quad (z,n)\in\mathbb{T}\times\mathbb{N}_0. \tag{1.7}$$

(Here $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.) Then $\varphi(\,\cdot\,,n)$ are monic polynomials of degree n first introduced by Szegő in the 1920's in his seminal work on the asymptotic distribution of eigenvalues of sections of Toeplitz forms [53], [54] (see also [55, Ch. XI]). Szegő's point of departure was the trigonometric moment problem and hence the theory of orthogonal polynomials on the unit circle. Indeed, given a probability measure $d\sigma$ supported on an infinite set on the unit circle, one is interested in finding monic polynomials $\chi(\,\cdot\,,n)$ of degree $n\in\mathbb{N}_0$ in $z=e^{i\theta}$, $\theta\in[0,2\pi]$, such that

$$\int_0^{2\pi} d\sigma(e^{i\theta})\,\overline{\chi(e^{i\theta},m)}\chi(e^{i\theta},n)=\gamma(n)^{-2}\delta_{m,n}, \quad m,n\in\mathbb{N}_0, \tag{1.8}$$

where $\gamma(0)^2=1$, $\gamma(n)^2=\prod_{j=1}^n\big(1-|\alpha(j)|^2\big)^{-1}$, $n\in\mathbb{N}$. Szegő showed that the corresponding polynomials (1.7) with φ replaced by χ satisfy the recurrence formula (1.6). Early work in this area includes important contributions by Akhiezer, Geronimus, Krein, Tomčuk, Verblunsky, Widom, and others, and is summarized in the books by Akhiezer [10], Geronimus [31], Szegő [55], and especially in the recent two-volume treatise by Simon [50].

Under appropriate ergodicity assumptions on α and the hypothesis of a vanishing Lyapunov exponent on prescribed spectral arcs on the unit circle $\mathbb{T}$, Geronimo and Johnson [29] (cf. also [28], [30]) studied (1.6) and developed the corresponding spectral theory associated with (1.6) and the unitary operator it generates in $\ell^2(\mathbb{Z})$. More recently, the defocusing case with periodic and quasi-periodic coefficients was studied in great detail by Deift [21], Golinskii and Nevai [39], Killip and Nenciu [40], Li [41], Nenciu [44]–[46], Simon [48], [50, Ch. 11], [51], and [52].

An important extension of (1.6) was developed by Baxter in a series of papers on Toeplitz forms [11]–[14] in 1960–63. In these papers the transfer matrix T in (1.5) is replaced by the more general (complexified) transfer matrix $U(z)=\left(\begin{smallmatrix} z & \alpha\\ \beta z & 1\end{smallmatrix}\right)$ in (1.3), that is, precisely the matrix U responsible for the spatial part in the Ablowitz–Ladik system in its zero-curvature formulation (1.4). Studying the following extension of (1.6),

$$\Psi(z,n)=U(z,n)\Psi(z,n-1), \quad (z,n)\in\mathbb{T}\times\mathbb{N}, \tag{1.9}$$

Baxter was led to biorthogonal polynomials on the unit circle with respect to a complex-valued measure on $\mathbb{T}$. Baxter's U matrix in (1.3) led to a new hierarchy of nonlinear difference equations, called the Szegő–Baxter (SB) hierarchy in [27]. The latter reference also contains an in depth study of algebro-geometric solutions of (1.6). The algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data, that is, the construction of α and β by starting from a set of initial data (nonspecial divisors) of full measure, will be presented in [35]. The Hamiltonian and Lax formalisms for the AL hierarchy will be revisited in [36].

In addition to these recent developments on the AL system and the AL hierarchy, we offer a variety of results in this paper apparently not covered before. These include:

• An effective recursive construction of the AL hierarchy using Laurent polynomials.

• The detailed connection between the AL hierarchy and a "complexified" version of transfer matrices first introduced by Baxter.

• A detailed treatment of the stationary and time-dependent Ablowitz–Ladik formalism.

The structure of this paper is as follows: In Section 2 we describe our zero-curvature formalism for the Ablowitz–Ladik (AL) hierarchy. Extending a recursive polynomial approach discussed in great detail in [32] in the continuous case and in [17], [33, Ch. 4], [57, Chs. 6, 12] in the discrete context to the case of Laurent polynomials with respect to the spectral parameter, we derive the AL hierarchy of systems of nonlinear evolution equations whose first nonlinear member is the Ablowitz–Ladik system (1.1). Section 3 is devoted to a detailed study of the stationary AL hierarchy. We employ the recursive Laurent polynomial formalism of Section 2 to describe nonnegative divisors of degree $2p-1$ on a hyperelliptic curve $\mathcal{K}_{2p-1}$ of genus $2p-1$ associated with the pth equation in the stationary AL hierarchy. The corresponding time-dependent results for the AL hierarchy are presented in detail in Section 4. Finally, Appendix A is of a technical nature and summarizes expansions of various key quantities related to the Laurent polynomial recursion formalism as the spectral parameter tends to zero or to infinity.

2. The Ablowitz–Ladik Hierarchy, Recursion Relations, Zero-Curvature Pairs, and Hyperelliptic Curves

In this section we provide the construction of the Ablowitz–Ladik hierarchy employing a polynomial recursion formalism and derive the associated sequence of Ablowitz–Ladik zero-curvature pairs. Moreover, we discuss the Burchnall–Chaundy polynomial in connection with the stationary Ablowitz–Ladik hierarchy and the underlying hyperelliptic curve.

We denote by $\mathbb{C}^{\mathbb{Z}}$ the set of complex-valued sequences indexed by $\mathbb{Z}$.

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \neq 1, \ n \in \mathbb{Z}. \tag{2.1}$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}}, \ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,), \beta(n,\,\cdot\,) \in C^1(\mathbb{R}), \ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \neq 1, \ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{2.2}$$

Actually, up to Remark 2.11 our analysis will be time-independent and hence only the lattice variations of α and β will matter.

We denote by $S^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm}f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}. \tag{2.3}$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm}f, \quad f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.4}$$

To construct the Ablowitz–Ladik hierarchy we will try to generalize (1.3) by considering the 2×2 matrix

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \quad z \in \mathbb{C}, \tag{2.5}$$

and making the ansatz

$$V_p(z) = i \begin{pmatrix} G_p^-(z) & -F_p^-(z) \\ H_p^-(z) & -K_p^-(z) \end{pmatrix}, \quad p \in \mathbb{N}_0, \tag{2.6}$$

where G_p, K_p, F_p, and H_p are chosen as Laurent polynomials (suggested by the appearance of z^{-1} in the matrix V in (1.3))

$$\begin{aligned} G_p(z) &= \sum_{j=1}^p z^{-j} g_{p-j,-} + \sum_{j=0}^p z^j g_{p-j,+}, \quad F_p(z) = \sum_{j=1}^p z^{-j} f_{p-j,-} + \sum_{j=0}^p z^j f_{p-j,+}, \\ K_p(z) &= \sum_{j=1}^p z^{-j} k_{p-j,-} + \sum_{j=0}^p z^j k_{p-j,+}, \quad H_p(z) = \sum_{j=1}^p z^{-j} h_{p-j,-} + \sum_{j=0}^p z^j h_{p-j,+}. \end{aligned} \tag{2.7}$$

Without loss of generality we will only look at the time-independent case and add time later on. Then the stationary zero-curvature equation,

$$0 = UV_p - V_p^+ U, \tag{2.8}$$

is equivalent to the following relationships between the Laurent polynomials

$$UV_p - V_p^+U = i \begin{pmatrix} z(G_p^- - G_p) + z\beta F_p + \alpha H_p^- & F_p - zF_p^- - \alpha(G_p + K_p^-) \\ z\beta(G_p^- + K_p) - zH_p + H_p^- & -z\beta F_p^- - \alpha H_p + K_p - K_p^- \end{pmatrix}, \tag{2.9}$$

respectively

$$z(G_p^- - G_p) + z\beta F_p + \alpha H_p^- = 0, \tag{2.10}$$

$$z\beta F_p^- + \alpha H_p - K_p + K_p^- = 0, \tag{2.11}$$

$$-F_p + zF_p^- + \alpha(G_p + K_p^-) = 0, \tag{2.12}$$

$$z\beta(G_p^- + K_p) - zH_p + H_p^- = 0. \tag{2.13}$$

Lemma 2.2. *Suppose the Laurent polynomials defined in* (2.7) *satisfy the zero-curvature equation* (2.8)*, then*

$$\begin{gathered} f_{0,+} = 0, \qquad h_{0,-} = 0, \qquad g_{0,\pm} - g_{0,\pm}^- = 0, \\ k_{\ell,\pm} - k_{\ell,\pm}^- = g_{\ell,\pm} - g_{\ell,\pm}^-, \qquad l = 0, \dots, p-1. \end{gathered}$$

Proof. Comparing coefficients at the highest order of z in (2.11) and the lowest in (2.10) immediately yields $f_{0,+} = 0$, $h_{0,-} = 0$. Then $g_{0,+}$, $k_{0,-}$ are necessarily constant by (2.10), (2.11). Inserting the coefficients of z^ℓ in (2.12) and $z^{\ell+1}$ in (2.13) into (2.11) at z^ℓ and using (2.10) at $z^{\ell+1}$ (we abbreviate $\gamma = 1 - \alpha\beta$) one obtains

$$\begin{aligned} k_{p-\ell,+} - k_{p-\ell,+}^- &= \alpha h_{p-\ell,+} + \beta f_{p-\ell+1,+}^- \\ &= \alpha h_{p-\ell-1,+}^- + \beta f_{p-\ell,+} + \alpha\beta(k_{p-\ell,+} - k_{p-\ell,+}^-) - \alpha\beta(g_{p-\ell,+} - g_{p-\ell,+}^-) \\ &= \alpha\beta(k_{p-\ell,+} - k_{p-\ell,+}^-) + \gamma(g_{p-\ell,+} - g_{p-\ell,+}^-), \end{aligned}$$

that is, $\gamma(k_{p-\ell,+} - k^-_{p-\ell,+}) = \gamma(g_{p-\ell,+} - g^-_{p-\ell,+})$. The same analysis at $z^{-\ell}$ (starting with (2.10)) shows $k_{p-\ell,-} - k^-_{p-\ell,-} = g_{p-\ell,-} - g^-_{p-\ell,-}$. In particular, $g_{0,-}$ is constant as well. □

In particular, this lemma shows that we can choose $k_{\ell,\pm} = g_{\ell,\pm}$, $0 \le \ell \le p-1$, without restriction (since this can always be achieved by adding a Laurent polynomial times the identity to V_p, which does not affect the zero-curvature equation). Hence the ansatz can be refined as follows (it is more convenient to re-label $h_{p,+} = h_{p-1,-}$)

$$F_p(z) = \sum_{\ell=1}^{p} f_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+} z^{\ell}, \tag{2.14}$$

$$G_p(z) = \sum_{\ell=1}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+} z^{\ell}, \tag{2.15}$$

$$H_p(z) = \sum_{\ell=0}^{p-1} h_{p-1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+} z^{\ell}, \tag{2.16}$$

$$K_p(z) = \sum_{\ell=0}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} g_{p-\ell,+} z^{\ell} = G_p(z) + g_{p,-} - g_{p,+}. \tag{2.17}$$

Plugging this ansatz into the zero-curvature equation (2.8) and comparing coefficients then yields the following result.

Lemma 2.3. *The sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ are recursively defined by*

$$g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \tag{2.18}$$

$$g_{\ell+1,+} - g^-_{\ell+1,+} = \alpha h^-_{\ell,+} + \beta f_{\ell,+}, \quad 0 \le \ell \le p-1, \tag{2.19}$$

$$f^-_{\ell+1,+} = f_{\ell,+} - \alpha(g_{\ell+1,+} + g^-_{\ell+1,+}), \quad 0 \le \ell \le p-2, \tag{2.20}$$

$$h_{\ell+1,+} = h^-_{\ell,+} + \beta(g_{\ell+1,+} + g^-_{\ell+1,+}), \quad 0 \le \ell \le p-2, \tag{2.21}$$

and

$$g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \tag{2.22}$$

$$g_{\ell+1,-} - g^-_{\ell+1,-} = \alpha h_{\ell,-} + \beta f^-_{\ell,-}, \quad 0 \le \ell \le p-1, \tag{2.23}$$

$$f_{\ell+1,-} = f^-_{\ell,-} + \alpha(g_{\ell+1,-} + g^-_{\ell+1,-}), \quad 0 \le \ell \le p-2, \tag{2.24}$$

$$h^-_{\ell+1,-} = h_{\ell,-} - \beta(g_{\ell+1,-} + g^-_{\ell+1,-}), \quad 0 \le \ell \le p-2. \tag{2.25}$$

Here $c_{0,\pm} \in \mathbb{C}$ are given constants. In addition, (2.9) *reads*

$$\begin{aligned} &UV_p - V_p^+ U \\ &= i\begin{pmatrix} 0 & -\alpha(g_{p,+} + g^-_{p,-}) + f_{p-1,+} - f^-_{p-1,-} \\ z(\beta(g^-_{p,+} + g_{p,-}) - h_{p-1,-} + h^-_{p-1,+}) & 0 \end{pmatrix}. \end{aligned} \tag{2.26}$$

For later use we also introduce

$$f_{-1,\pm} = h_{-1,\pm} = 0. \tag{2.27}$$

Remark 2.4. *The sequences* $\{f_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, *and* $\{h_{\ell,+}\}_{\ell\in\mathbb{N}_0}$ *can be computed recursively as follows: Assume that* $f_{\ell,+}$, $g_{\ell,+}$, *and* $h_{\ell,+}$ *are known. Equation* (2.19) *is a first-order difference equation in* $g_{\ell+1,+}$ *that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by* $c_{\ell+1,+}\in\mathbb{C}$. *Relations* (2.20) *and* (2.21) *then determine* $f_{\ell+1,+}$ *and* $h_{\ell+1,+}$, *etc. The sequences* $\{f_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, *and* $\{h_{\ell,-}\}_{\ell\in\mathbb{N}_0}$ *are determined similarly.*

Upon setting
$$\gamma = 1-\alpha\beta, \tag{2.28}$$
one explicitly obtains
$$\begin{aligned}
f_{0,+} &= c_{0,+}(-\alpha^+),\\
f_{1,+} &= c_{0,+}\big(-\gamma^+\alpha^{++}+(\alpha^+)^2\beta\big)+c_{1,+}(-\alpha^+),\\
g_{0,+} &= \tfrac12 c_{0,+},\\
g_{1,+} &= c_{0,+}(-\alpha^+\beta)+\tfrac12 c_{1,+},\\
g_{2,+} &= c_{0,+}\big((\alpha^+\beta)^2-\gamma^+\alpha^{++}\beta-\gamma\alpha^+\beta^-\big)+c_{1,+}(-\alpha^+\beta)+\tfrac12 c_{2,+},\\
h_{0,+} &= c_{0,+}\beta,\\
h_{1,+} &= c_{0,+}\big(\gamma\beta^- -\alpha^+\beta^2\big)+c_{1,+}\beta,\\
f_{0,-} &= c_{0,-}\alpha,\\
f_{1,-} &= c_{0,-}\big(\gamma\alpha^- -\alpha^2\beta^+\big)+c_{1,-}\alpha,\\
g_{0,-} &= \tfrac12 c_{0,-},\\
g_{1,-} &= c_{0,-}(-\alpha\beta^+)+\tfrac12 c_{1,-},\\
g_{2,-} &= c_{0,-}\big((\alpha\beta^+)^2-\gamma^+\alpha\beta^{++}-\gamma\alpha^-\beta^+\big)+c_{1,-}(-\alpha\beta^+)+\tfrac12 c_{2,-},\\
h_{0,-} &= c_{0,-}(-\beta^+),\\
h_{1,-} &= c_{0,-}\big(-\gamma^+\beta^{++}+\alpha(\beta^+)^2\big)+c_{1,-}(-\beta^+), \text{ etc.}
\end{aligned} \tag{2.29}$$
Here $\{c_{\ell,\pm}\}_{\ell\in\mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.19), (2.23).

In the special case $c_{0,+}=2$, $c_{0,-}=0$ (describing the stationary Baxter–Szegő hierarchy, cf. Remark 2.9 (ii)), the precise structure of the sequences $f_{\ell,+}, g_{\ell,+}, h_{\ell,+}$ is described in [27, Lemma 2.2].

In particular, the stationary zero-curvature relation (2.8), $0=UV_p-V_p^+U$, is equivalent to
$$-\alpha(g_{p,+}+g_{p,-}^-)+f_{p-1,+}-f_{p-1,-}^-=0, \tag{2.30}$$
$$\beta(g_{p,+}^-+g_{p,-})+h_{p-1,+}^- -h_{p-1,-}=0. \tag{2.31}$$
Thus, varying $p\in\mathbb{N}_0$, equations (2.30) and (2.31) give rise to the stationary Ablowitz–Ladik (AL) hierarchy which we introduce as follows
$$\text{s-AL}_p(\alpha,\beta)=\begin{pmatrix}-\alpha(g_{p,+}+g_{p,-}^-)+f_{p-1,+}-f_{p-1,-}^-\\ \beta(g_{p,+}^-+g_{p,-})+h_{p-1,+}^- -h_{p-1,-}\end{pmatrix}=0,\quad p\in\mathbb{N}_0. \tag{2.32}$$
Explicitly (recalling $\gamma=1-\alpha\beta$),
$$\text{s-AL}_0(\alpha,\beta)=\begin{pmatrix}-c_0\alpha\\ c_0\beta\end{pmatrix}=0,$$

$$\text{s-AL}_1(\alpha,\beta) = \begin{pmatrix} -\gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_1\alpha \\ \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_1\beta \end{pmatrix} = 0,$$

$$\text{s-AL}_2(\alpha,\beta) = \begin{pmatrix} -\gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix}$$
$$+ \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_2\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_2\beta \end{pmatrix} = 0, \text{ etc.}, \tag{2.33}$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_p = (c_{p,+} + c_{p,-})/2, \quad p \in \mathbb{N}_0. \tag{2.34}$$

By definition, the set of solutions of (2.32), with p ranging in $\mathbb{N}_0$ and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k = 1,\dots,\ell$, and choosing $c_{0,\pm} = 1$,

$$\hat{f}_{0,+} = -\alpha^+, \quad \hat{f}_{0,-} = \alpha, \quad \hat{f}_{\ell,\pm} = f_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.35}$$

$$\hat{g}_{0,\pm} = \tfrac{1}{2}, \quad \hat{g}_{\ell,\pm} = g_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.36}$$

$$\hat{h}_{0,+} = \beta, \quad \hat{h}_{0,-} = -\beta^+, \quad \hat{h}_{\ell,\pm} = h_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{2.37}$$

By induction one infers that

$$f_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{f}_{k,\pm}, \quad g_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{g}_{k,\pm}, \quad h_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{h}_{k,\pm}. \tag{2.38}$$

In a slight abuse of notation we will occasionally stress the dependence of $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ on α,β by writing $f_{\ell,\pm}(\alpha,\beta)$, $g_{\ell,\pm}(\alpha,\beta)$, and $h_{\ell,\pm}(\alpha,\beta)$.

Remark 2.5. *Using the nonlinear recursion relations* (A.14)–(A.19) *in Theorem* A.2*, one infers inductively that all homogeneous elements $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, $\ell \in \mathbb{N}_0$, are polynomials in α,β, and some of their shifts.* (*Alternatively, one can prove directly by induction that the nonlinear recursion relations* (A.14)–(A.19) *are equivalent to that in* (2.18)–(2.25) *with all summation constants put equal to zero, $c_{\ell,\pm} = 0$, $\ell \in \mathbb{N}$.*)

Remark 2.6. *As an efficient tool to later distinguish between homogeneous and nonhomogeneous quantities $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, and $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, $\hat{h}_{\ell,\pm}$, respectively, we now introduce the notion of degree as follows. Denote*

$$f^{(r)} = S^{(r)}f, \quad f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad S^{(r)} = \begin{cases} (S^+)^r, & r \geq 0, \\ (S^-)^{-r}, & r < 0, \end{cases} \quad r \in \mathbb{Z}, \tag{2.39}$$

and define

$$\deg\big(\alpha^{(r)}\big) = r, \quad \deg\big(\beta^{(r)}\big) = -r, \quad r \in \mathbb{Z}. \tag{2.40}$$

This then results in

$$\begin{aligned} &\deg\big(\hat{f}^{(r)}_{\ell,+}\big) = \ell + 1 + r, \quad \deg\big(\hat{f}^{(r)}_{\ell,-}\big) = -\ell + r, \quad \deg\big(\hat{g}^{(r)}_{\ell,\pm}\big) = \pm\ell, \\ &\deg\big(\hat{h}^{(r)}_{\ell,+}\big) = \ell - r, \quad \deg\big(\hat{h}^{(r)}_{\ell,-}\big) = -\ell - 1 - r, \quad \ell \in \mathbb{N}_0,\ r \in \mathbb{Z}, \end{aligned} \tag{2.41}$$

using induction in the linear recursion relations (2.18)–(2.25).

In accordance with our notation introduced in (2.35)–(2.37), (2.49), and (2.51), the corresponding homogeneous stationary Ablowitz–Ladik equations are defined by

$$\text{s-}\widehat{\mathrm{AL}}_p(\alpha,\beta) = \text{s-AL}_p(\alpha,\beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p}, \quad p\in\mathbb{N}_0. \tag{2.42}$$

In the special case $p=1$ one obtains *the* stationary Ablowitz–Ladik equation

$$\text{s-}\widehat{\mathrm{AL}}_1(\alpha,\beta) = \begin{pmatrix} -\gamma(\alpha^- + \alpha^+) \\ \gamma(\beta^- + \beta^+) \end{pmatrix} = 0. \tag{2.43}$$

We also note the following useful result.

Lemma 2.7. *Assume* (2.1). *Then,*

$$\begin{aligned} g_{\ell,+} - g_{\ell,+}^- &= \alpha h_{\ell,+} + \beta f_{\ell,+}^-, \quad \ell\in\mathbb{N}_0, \\ g_{\ell,-} - g_{\ell,-}^- &= \alpha h_{\ell,-}^- + \beta f_{\ell,-}, \quad \ell\in\mathbb{N}_0. \end{aligned} \tag{2.44}$$

Moreover, we note the following symmetries,

$$\hat f_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat h_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \hat g_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat g_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \ell\in\mathbb{N}_0. \tag{2.45}$$

Proof. The relations (2.44) are derived as follows:

$$\begin{aligned} \alpha h_{\ell+1,+} + \beta f_{\ell+1,+}^- &= \alpha h_{\ell,+}^- + \alpha\beta(g_{\ell+1,+} + g_{\ell+1,+}^-) + \beta f_{\ell,+} - \alpha\beta(g_{\ell+1,+} + g_{\ell+1,+}^-) \\ &= \alpha h_{\ell,+}^- + \beta f_{\ell,+} = g_{\ell+1,+} - g_{\ell+1,+}^-, \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} \alpha h_{\ell+1,-}^- + \beta f_{\ell+1,-} &= \alpha h_{\ell,-} - \alpha\beta(g_{\ell+1,-} + g_{\ell+1,-}^-) + \beta f_{\ell,-}^- + \alpha\beta(g_{\ell+1,-} + g_{\ell+1,-}^-) \\ &= \alpha h_{\ell,-} + \beta f_{\ell,-}^- = g_{\ell+1,-} + g_{\ell+1,-}^-. \end{aligned} \tag{2.47}$$

The statement (2.45) follows by showing that $\hat h_{\ell,\mp}(\beta,\alpha)$ and $\hat g_{\ell,\mp}(\beta,\alpha)$ satisfy the same recursion relations as those of $\hat f_{\ell,\pm}(\alpha,\beta)$ and $\hat g_{\ell,\pm}(\alpha,\beta)$, respectively. That the recursion constants are the same, follows from the observation that the corresponding coefficients have the proper degree. □

Next we turn to the Laurent polynomials F_p, G_p, H_p, and K_p defined in (2.14)–(2.17). Explicitly, one obtains

$$
\begin{aligned}
F_0 &= 0,\\
F_1 &= c_{0,-}\alpha z^{-1} + c_{0,+}(-\alpha^+),\\
F_2 &= c_{0,-}\alpha z^{-2} + \big(c_{0,-}\big(\gamma\alpha^- - \alpha^2\beta^+\big) + c_{1,-}\alpha\big)z^{-1}\\
&\quad + c_{0,+}\big(-\gamma^+\alpha^{++} + (\alpha^+)^2\beta\big) + c_{1,+}(-\alpha^+) + c_{0,+}(-\alpha^+)z,\\
G_0 &= \tfrac{1}{2}c_{0,+},\\
G_1 &= \tfrac{1}{2}c_{0,-}z^{-1} + c_{0,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{1,+} + \tfrac{1}{2}c_{0,+}z,\\
G_2 &= \tfrac{1}{2}c_{0,-}z^{-2} + \big(c_{0,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{1,-}\big)z^{-1}\\
&\quad + c_{0,+}\big((\alpha^+\beta)^2 - \gamma^+\alpha^{++}\beta - \gamma\alpha^+\beta^-\big) + c_{1,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{2,+}\\
&\quad + \big(c_{0,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{1,+}\big)z + \tfrac{1}{2}c_{0,+}z^2,\\
H_0 &= 0,\\
H_1 &= c_{0,-}(-\beta^+) + c_{0,+}\beta z,\\
H_2 &= c_{0,-}(-\beta^+)z^{-1} + c_{0,-}\big(-\gamma^+\beta^{++} + \alpha(\beta^+)^2\big) + c_{1,-}(-\beta^+)\\
&\quad + \big(c_{0,+}(\gamma\beta^- - \alpha^+\beta^2) + c_{1,+}\beta\big)z + c_{0,+}\beta z^2,\\
K_0 &= \tfrac{1}{2}c_{0,-},\\
K_1 &= \tfrac{1}{2}c_{0,-}z^{-1} + c_{0,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{1,-} + \tfrac{1}{2}c_{0,+}z,\\
K_2 &= \tfrac{1}{2}c_{0,-}z^{-2} + \big(c_{0,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{1,-}\big)z^{-1}\\
&\quad + c_{0,-}\big((\alpha\beta^+)^2 - \gamma^+\alpha\beta^{++} - \gamma\alpha^-\beta^+\big) + c_{1,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{2,-}\\
&\quad + \big(c_{0,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{1,+}\big)z + \tfrac{1}{2}c_{0,+}z^2, \text{ etc.}
\end{aligned}
\tag{2.48}
$$

The corresponding homogeneous quantities are defined by ($\ell \in \mathbb{N}_0$)

$$
\begin{aligned}
\widehat F_0(z) &= \widehat F_{0,\pm}(z) = 0,\\
\widehat F_\ell(z) &= \sum_{k=1}^{\ell} \hat f_{\ell-k,-} z^{-k} + \sum_{k=0}^{\ell-1} \hat f_{\ell-1-k,+} z^k = \widehat F_{\ell,-} + \widehat F_{\ell,+},\\
\widehat G_0(z) &= \widehat G_{0,+}(z) = \frac{1}{2}, \quad \widehat G_{0,-}(z) = 0,\\
\widehat G_\ell(z) &= \sum_{k=1}^{\ell} \hat g_{\ell-k,-} z^{-k} + \sum_{k=0}^{\ell} \hat g_{\ell-k,+} z^k = \widehat G_{\ell,-} + \widehat G_{\ell,+},\\
\widehat H_0(z) &= \widehat H_{0,\pm}(z) = 0,\\
\widehat H_\ell(z) &= \sum_{k=0}^{\ell-1} \hat h_{\ell-1-k,-} z^{-k} + \sum_{k=1}^{\ell} \hat h_{\ell-k,+} z^k = \widehat H_{\ell,-} + \widehat H_{\ell,+},\\
\widehat K_0(z) &= \widehat K_{0,-}(z) = \frac{1}{2}, \quad \widehat K_{0,+}(z) = 0,\\
\widehat K_\ell(z) &= \sum_{k=0}^{\ell} \hat g_{\ell-k,-} z^{-k} + \sum_{k=1}^{\ell} \hat g_{\ell-k,+} z^k = \widehat G_\ell(z) + \hat g_{\ell,-} - \hat g_{\ell,+}\\
&= \widehat K_{\ell,-} + \widehat K_{\ell,+},
\end{aligned}
\tag{2.49}
$$

where $\widehat{F}_{\ell,+}$, $\widehat{G}_{\ell,+}$, $\widehat{H}_{\ell,+}$, and $\widehat{K}_{\ell,+}$ denote the polynomial parts of $\widehat{F}_\ell$, $\widehat{G}_\ell$, $\widehat{H}_\ell$, and $\widehat{K}_\ell$, respectively, and $\widehat{F}_{\ell,-}$, $\widehat{G}_{\ell,-}$, $\widehat{H}_{\ell,-}$, and $\widehat{K}_{\ell,-}$ denote the Laurent parts of $\widehat{F}_\ell$, $\widehat{G}_\ell$, $\widehat{H}_\ell$, and $\widehat{K}_\ell$. Similarly, with $F_{\ell,+}$, $G_{\ell,+}$, $H_{\ell,+}$, and $K_{\ell,+}$ denoting the polynomial parts of F_ℓ, G_ℓ, H_ℓ, and K_ℓ, respectively, and $F_{\ell,-}$, $G_{\ell,-}$, $H_{\ell,-}$, and $K_{\ell,-}$ denoting the Laurent parts of F_ℓ, G_ℓ, H_ℓ, and K_ℓ such that

$$\begin{aligned} F_\ell(z) &= F_{\ell,-}(z) + F_{\ell,+}(z), \quad G_\ell(z) = G_{\ell,-}(z) + G_{\ell,+}(z), \\ H_\ell(z) &= H_{\ell,-}(z) + H_{\ell,+}(z), \quad K_\ell(z) = K_{\ell,-}(z) + K_{\ell,+}(z), \end{aligned} \tag{2.50}$$

one finds that

$$\begin{aligned} F_{\ell,\pm} &= \sum_{k=1}^{\ell} c_{\ell-k,\pm} \widehat{F}_{k,\pm}, \quad H_{\ell,\pm} = \sum_{k=1}^{\ell} c_{\ell-k,\pm} \widehat{H}_{k,\pm}, \\ G_{\ell,-} &= \sum_{k=1}^{\ell} c_{\ell-k,-} \widehat{G}_{k,-}, \quad G_{\ell,+} = \sum_{k=0}^{\ell} c_{\ell-k,+} \widehat{G}_{k,+}, \\ K_{\ell,-} &= \sum_{k=0}^{\ell} c_{\ell-k,-} \widehat{K}_{k,-}, \quad K_{\ell,+} = \sum_{k=1}^{\ell} c_{\ell-k,+} \widehat{K}_{k,+}. \end{aligned} \tag{2.51}$$

In addition, one immediately obtains the following relations from (2.45):

Lemma 2.8. *Assume* (2.1) *and let* $\ell \in \mathbb{N}_0$. *Then,*

$$\widehat{F}_{\ell,\pm}(\alpha,\beta,z,n) = \widehat{H}_{\ell,\mp}(\beta,\alpha,z^{-1},n), \tag{2.52}$$

$$\widehat{H}_{\ell,\pm}(\alpha,\beta,z,n) = \widehat{F}_{\ell,\mp}(\beta,\alpha,z^{-1},n), \tag{2.53}$$

$$\widehat{G}_{\ell,\pm}(\alpha,\beta,z,n) = \widehat{G}_{\ell,\mp}(\beta,\alpha,z^{-1},n), \tag{2.54}$$

$$\widehat{K}_{\ell,\pm}(\alpha,\beta,z,n) = \widehat{K}_{\ell,\mp}(\beta,\alpha,z^{-1},n). \tag{2.55}$$

Now we return to the stationary Ablowitz–Ladik hierarchy. In the following we will frequently assume that α, β satisfy the pth stationary Ablowitz–Ladik equations supposing a particular choice of summation constants $c_{\ell,\pm} \in \mathbb{C}$, $\ell = 0, \dots, p$, $p \in \mathbb{N}$, has been made.

If one assumes in addition to (2.1) that

$$\gamma(n) = 1 - \alpha(n)\beta(n) \neq 0, \quad n \in \mathbb{Z}, \tag{2.56}$$

one can show that $g_{p,+} = g_{p,-}$ up to a lattice constant. Indeed, inserting (2.32), that is,

$$f_{p-1,+} = f^-_{p-1,-} + \alpha(g_{p,+} + g^-_{p,-}), \quad h^-_{p-1,+} = h_{p-1,-} - \beta(g^-_{p,+} + g_{p,-}), \tag{2.57}$$

into (2.19) at $\ell = p-1$, and using (2.23) at $\ell = p-1$, one concludes

$$\begin{aligned} g_{p,+} - g^-_{p,+} &= \alpha h^-_{p-1,+} + \beta f_{p-1,+} \\ &= \alpha h_{p-1,-} + \beta f^-_{p-1,-} - \alpha\beta(g^-_{p,+} + g_{p,-}) + \alpha\beta(g_{p,+} + g^-_{p,-}) \\ &= (1-\alpha\beta)(g_{p,-} - g^-_{p,-}) + \alpha\beta(g_{p,+} - g^-_{p,+}), \end{aligned} \tag{2.58}$$

and hence,

$$\gamma\big((g_{p,+} - g_{p,-}) - (g^-_{p,+} - g^-_{p,-})\big) = 0. \tag{2.59}$$

Thus, $g_{p,+} - g_{p,-}$ is a lattice constant which can be set equal to zero without loss of generality since one can always add a multiple of the identity to V_p. Thus we will henceforth assume that

$$g_{p,+} = g_{p,-}, \tag{2.60}$$

which in turn implies that

$$K_p = G_p \tag{2.61}$$

and hence renders V_p in (2.6) traceless in the stationary context. (We note that equations (2.60) and (2.61) cease to be valid in the time-dependent context, though.)

Remark 2.9. (*i*) *The particular choice* $c_{0,+} = c_{0,-} = 1$ *yields the stationary Ablowitz–Ladik equation. Scaling* $c_{0,\pm}$ *with the same constant then amounts to scaling* V_p *with this constant which drops out in the stationary zero-curvature equation* (2.8).
(*ii*) *Different ratios between* $c_{0,+}$ *and* $c_{0,-}$ *will lead to different stationary hierarchies. In particular, the choice* $c_{0,+} = 2$, $c_{0,-} = 0$ *yields the stationary Baxter–Szegő hierarchy considered in detail in* [27]. *However, in this case some parts from the recursion relation for the negative coefficients still remain. In fact,* (2.25) *reduces to* $g_{p,-} - g_{p,-}^- = \alpha h_{p-1,-}$, $h_{p-1,-} = 0$ *and thus requires* $g_{p,-}$ *to be a constant in addition to* (2.80) *below. This renders the system* (2.80) *overdetermined except in the stationary case.*
(*iii*) *Finally, by Lemma* 2.8*, the choice* $c_{0,+} = 0$, $c_{0,-} = 2$ *again yields the Baxter–Szegő hierarchy, but with* α *and* β *interchanged.*

Next, still assuming (2.1) and (2.56), and taking into account (2.61), one infers that the expression R_{2p}, defined as

$$R_{2p} = G_p^2 - F_p H_p, \tag{2.62}$$

is a lattice constant, that is, $R_{2p} - R_{2p}^- = 0$, since taking determinants in the stationary zero-curvature equation (2.8) immediately yields

$$\gamma\big(-(G_p^-)^2 + F_p^- H_p^- + G_p^2 - F_p H_p\big) z = 0. \tag{2.63}$$

Hence, $R_{2p}(z)$ only depends on z, and assuming in addition to (2.1) and (2.56) that

$$c_{0,\pm} \in \mathbb{C} \setminus \{0\}, \tag{2.64}$$

one may write R_{2p} as[1]

$$R_{2p}(z) = (c_{0,+}^2/4) z^{-2p} \prod_{m=0}^{4p-1} (z - E_m), \quad \{E_m\}_{m=0}^{4p-1} \subset \mathbb{C} \setminus \{0\}. \tag{2.65}$$

Moreover, (2.62) also implies

$$\lim_{z\to 0} 4z^{2p} R_{2p}(z) = c_{0,+}^2 \prod_{m=0}^{4p-1} (-E_m) = c_{0,-}^2, \tag{2.66}$$

and hence,

$$\prod_{m=0}^{4p-1} E_m = \frac{c_{0,-}^2}{c_{0,+}^2}. \tag{2.67}$$

Relation (2.62) allows one to introduce a hyperelliptic curve $\mathcal{K}_{2p-1}$ of (arithmetic) genus $2p-1$ (possibly with a singular affine part), where

$$\mathcal{K}_{2p-1} \colon \mathcal{F}_{2p-1}(z,y) = y^2 - 4c_{0,+}^{-2} z^{2p} R_{2p}(z) = 0. \tag{2.68}$$

[1] We use the convention that a product over an empty index set is to be interpreted equal to 1.

Remark 2.10. *In the special case* $c_{\ell,+} = c_{\ell,-}$, $\ell = 0, \dots, p$, *the symmetries of Lemma* 2.8 *also hold for* F_p, G_p, *and* H_p *and thus* $R_{2p}(1/z) = R_{2p}(z)$, *that is, the numbers* E_m, $m = 0, \dots, 4p-1$, *come in pairs* $(E_k, 1/E_k)$, $k = 1, \dots, 2p$.

Equations (2.10)–(2.13) permit one to derive nonlinear difference equations for F_p, G_p, and H_p separately. One obtains

$$\begin{aligned}
&\big((\alpha^+ + z\alpha)^2 F_p - z(\alpha^+)^2\gamma F_p^-\big)^2 - 2z\alpha^2\gamma^+\big((\alpha^+ + z\alpha)^2 F_p + z(\alpha^+)^2\gamma F_p^-\big)F_p^+ \\
&\quad + z^2\alpha^4(\gamma^+)^2(F_p^+)^2 = 4(\alpha\alpha^+)^2(\alpha^+ + \alpha z)^2 R_{2p},
\end{aligned} \tag{2.69}$$

$$\begin{aligned}
&(\alpha^+ + z\alpha)(\beta + z\beta^+)(z + \alpha^+\beta)(1 + z\alpha\beta^+)G_p^2 \\
&\quad + z(\alpha^+\gamma G_p^- + z\alpha\gamma^+ G_p^+)(z\beta^+\gamma G_p^- + \beta\gamma^+ G_p^+) \\
&\quad - z\gamma\big((\alpha^+\beta + z^2\alpha\beta^+)(2-\gamma^+) + 2z(1-\gamma^+)(2-\gamma)\big)G_p^- G_p \\
&\quad - z\gamma^+\big(2z(1-\gamma)(2-\gamma^+) + (\alpha^+\beta + z^2\alpha\beta^+)(2-\gamma)\big)G_p^+ G_p \\
&\quad = (\alpha^+\beta - z^2\alpha\beta^+)^2 R_{2p},
\end{aligned} \tag{2.70}$$

$$\begin{aligned}
&z^2\big((\beta^+)^2\gamma H_p^- - \beta^2\gamma^+ H_p^+\big)^2 - 2z(\beta + z\beta^+)^2\big((\beta^+)^2\gamma H_p^- + \beta^2\gamma^+ H_p^+\big)H_p \\
&\quad + (\beta + z\beta^+)^4 H_p^2 = 4z^2(\beta\beta^+)^2(\beta + \beta^+ z)^2 R_{2p}.
\end{aligned} \tag{2.71}$$

Equations (2.69)–(2.71) can be used to derive nonlinear recursion relations for the homogeneous coefficients $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, and $\hat h_{\ell,\pm}$ (i.e., the ones satisfying (2.35)–(2.37) in the case of vanishing summation constants) as proved in Theorem A.2 in Appendix A. This then yields a proof that $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, and $\hat h_{\ell,\pm}$ are polynomials in α, β, and some of their shifts (cf. Remark 2.5). In addition, as proven in Theorem A.2, (2.69) leads to an explicit determination of the summation constants $c_{1,\pm}, \dots, c_{p,\pm}$ in (2.32) in terms of the zeros $E_0, \dots, E_{4p-1}$ of the associated Laurent polynomial R_{2p} in (2.65). In fact, one can prove (cf. (A.31))

$$c_{\ell,\pm} = c_{0,\pm} c_\ell\big(\underline{E}^{\pm 1}\big), \quad \ell = 0, \dots, p, \tag{2.72}$$

where

$$\begin{aligned}
&c_0\big(\underline{E}^{\pm 1}\big) = 1, \\
&c_k\big(\underline{E}^{\pm 1}\big) \\
&= -\sum_{\substack{j_0,\dots,j_{4p-1}=0 \\ j_0+\cdots+j_{4p-1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{4p-1})!}{2^{2k}(j_0!)^2\cdots(j_{4p-1}!)^2(2j_0-1)\cdots(2j_{4p-1}-1)} E_0^{\pm j_0}\cdots E_{4p-1}^{\pm j_{4p-1}}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \dots, p,
\end{aligned} \tag{2.73}$$

are symmetric functions of $\underline{E}^{\pm 1} = (E_0^{\pm 1}, \dots, E_{4p-1}^{\pm 1})$ defined in (A.6).

Remark 2.11. *If* α, β *satisfy one of the stationary Ablowitz–Ladik equations in* (2.32) *for a particular value of* p, $\text{s-AL}_p(\alpha, \beta) = 0$, *then they satisfy infinitely many such equations of order higher than* p *for certain choices of summation constants* $c_{\ell,\pm}$. *In fact, they satisfy certain stationary Ablowitz–Ladik equations* $\text{s-AL}_q(\alpha, \beta) = 0$ *for every* $q \geq p + 1$. *This can be shown as in* [32, Remark I.1.5].

Finally we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each equation in the hierarchy, that is, for each p, we introduce a deformation (time) parameter $t_p \in \mathbb{R}$ in α, β, replacing $\alpha(n), \beta(n)$ by $\alpha(n, t_p), \beta(n, t_p)$. Moreover, the definitions (2.5), (2.6), and (2.14)–(2.17) of U, V_p, and F_p, G_p, H_p, K_p, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_p} + UV_p - V_p^+ U = 0, \quad p \in \mathbb{N}_0, \tag{2.74}$$

then results in the equations

$$\begin{aligned}
0 &= U_{t_p} + UV_p - V_p^+ U \\
&= i \begin{pmatrix} z(G_p^- - G_p) + z\beta F_p + \alpha H_p^- & -i\alpha_{t_p} + F_p - zF_p^- - \alpha(G_p + K_p^-) \\ -iz\beta_{t_p} + z\beta(G_p^- + K_p) - zH_p + H_p^- & -z\beta F_p^- - \alpha H_p + K_p - K_p^- \end{pmatrix} \\
&= i \begin{pmatrix} 0 & \begin{matrix} -i\alpha_{t_p} - \alpha(g_{p,+} + g_{p,-}^-) \\ +f_{p-1,+} - f_{p-1,-}^- \end{matrix} \\ \begin{matrix} z\big(- i\beta_{t_p} + \beta(g_{p,+}^- + g_{p,-}) \\ -h_{p-1,-} + h_{p-1,+}^-\big) \end{matrix} & 0 \end{pmatrix},
\end{aligned} \tag{2.75}$$

or equivalently,

$$\alpha_{t_p} = i\big(zF_p^- + \alpha(G_p + K_p^-) - F_p\big), \tag{2.76}$$

$$\beta_{t_p} = -i\big(\beta(G_p^- + K_p) - H_p + z^{-1}H_p^-\big), \tag{2.77}$$

$$0 = z(G_p^- - G_p) + z\beta F_p + \alpha H_p^-, \tag{2.78}$$

$$0 = z\beta F_p^- + \alpha H_p + K_p^- - K_p. \tag{2.79}$$

Varying $p \in \mathbb{N}_0$, the collection of evolution equations

$$\begin{aligned}
\mathrm{AL}_p(\alpha, \beta) \begin{pmatrix} -i\alpha_{t_p} - \alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ -i\beta_{t_p} + \beta(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0, \\
(n, t_p) \in \mathbb{Z} \times \mathbb{R}, \ p \in \mathbb{N}_0,
\end{aligned} \tag{2.80}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly,

$$\begin{aligned}
&\mathrm{AL}_0(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_0} - c_0\alpha \\ -i\beta_{t_0} + c_0\beta \end{pmatrix} = 0, \\
&\mathrm{AL}_1(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_1} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_1\alpha \\ -i\beta_{t_1} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_1\beta \end{pmatrix} = 0, \\
&\mathrm{AL}_2(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_2} - \gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ -i\beta_{t_2} + \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\qquad\qquad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_2\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_2\beta \end{pmatrix} = 0, \ \text{etc.},
\end{aligned} \tag{2.81}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy.

The corresponding homogeneous equations are then defined by

$$\widehat{\mathrm{AL}}_p(\alpha, \beta) = \mathrm{AL}_p(\alpha, \beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p} = 0, \quad p \in \mathbb{N}_0. \tag{2.82}$$

The special case $p = 1$, that is,

$$\widehat{\mathrm{AL}}_1(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_1} - \gamma(\alpha^- + \alpha^+) \\ -i\beta_{t_1} + \gamma(\beta^- + \beta^+) \end{pmatrix} = 0, \tag{2.83}$$

represents *the* Ablowitz–Ladik equation.

By (2.80), (2.19), and (2.23), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_p} = i\gamma\big((g_{p,+} - g_{p,+}^-) - (g_{p,-} - g_{p,-}^-)\big). \tag{2.84}$$

For instance, if α, β satisfy $\mathrm{AL}_1(\alpha,\beta) = 0$, then

$$\gamma_{t_1} = i\gamma\big(\alpha(c_{0,-}\beta^+ + c_{0,+}\beta^-) - \beta(c_{0,+}\alpha^+ + c_{0,-}\alpha^-)\big). \tag{2.85}$$

Remark 2.12. *From* (2.10)–(2.13) *and the explicit computations of the coefficients* $f_{\ell,\pm}$*,* $g_{\ell,\pm}$*, and* $h_{\ell,\pm}$*, one concludes that the zero-curvature equation* (2.75) *and hence the Ablowitz–Ladik hierarchy is invariant under the scaling transformation*

$$\alpha \to \alpha_c = \{c\,\alpha(n)\}_{n\in\mathbb{Z}}, \quad \beta \to \beta_c = \{\beta(n)/c\}_{n\in\mathbb{Z}}, \quad c \in \mathbb{C}\setminus\{0\}. \tag{2.86}$$

Moreover, $R_{2p} = G_p^2 - H_pF_p$ *and hence* $\{E_m\}_{m=0}^{4p-1}$ *are invariant under this transformation. Furthermore, choosing* $c = \mathrm{e}^{ic_pt}$*, one verifies that it is no restriction to assume* $c_p = 0$*. This also shows that stationary solutions* α, β *can only be constructed up to a constant.*

3. The Stationary Ablowitz–Ladik Formalism

This section is devoted to a detailed study of the stationary Ablowitz–Ladik hierarchy. Our principal tools are derived from combining the polynomial recursion formalism introduced in Section 2 and a fundamental meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_{2p-1}$. With the help of ϕ we study the Baker–Akhiezer vector Ψ, Dubrovin-type equations governing the motion of auxiliary divisors on $\mathcal{K}_{2p-1}$, and trace formulas for α and β.

Unless explicitly stated otherwise, we suppose throughout this section that

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0,1\},\ n \in \mathbb{Z}, \tag{3.1}$$

and assume (2.5)–(2.27), (2.6)–(2.8), (2.14)–(2.17), (2.32), (2.61), (2.62), (2.65), keeping $p \in \mathbb{N}_0$ fixed.

We recall the hyperelliptic curve

$$\begin{aligned} &\mathcal{K}_{2p-1}\colon \mathcal{F}_{2p-1}(z,y) = y^2 - 4c_{0,+}^{-2}z^{2p}R_{2p}(z) = 0, \\ &R_{2p}(z) = (c_{0,+}^2/4)z^{-2p}\prod_{m=0}^{4p-1}(z - E_m), \quad \{E_m\}_{m=0}^{4p-1} \subset \mathbb{C}\setminus\{0\}, \end{aligned} \tag{3.2}$$

as introduced in (2.68). Throughout this section we assume the affine part of $\mathcal{K}_{2p-1}$ to be nonsingular, that is, we suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 4p-1. \tag{3.3}$$

$\mathcal{K}_{2p-1}$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_{2p-1}$. Points P on $\mathcal{K}_{2p-1}\setminus\{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z,y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_{2p-1}$ satisfying $\mathcal{F}_{2p-1}(z,y) = 0$. The complex structure on $\mathcal{K}_{2p-1}$ is then defined in the usual manner. Hence, $\mathcal{K}_{2p-1}$ becomes a two-sheeted hyperelliptic Riemann surface of genus $2p-1$ in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_{2p-1}$ (i.e., by fixing $E_0, \dots, E_{4p-1}$), the summation constants $c_{1,\pm}, \dots, c_{p,\pm}$ in $f_{p,\pm}$, $g_{p,\pm}$, and $h_{p,\pm}$ (and hence in the corresponding stationary s-AL_p equations) are uniquely determined as is clear from (2.72), (2.73), which establish the summation constants $c_{\ell,\pm}$ as symmetric functions of $E_0^{\pm 1}, \dots, E_{4p-1}^{\pm 1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$. (The trivial case $p = 0$ is explicitly treated in Example 3.5.)

We denote by $\{\mu_j(n)\}_{j=1,\dots,2p-1}$ and $\{\nu_j(n)\}_{j=1,\dots,2p-1}$ the zeros of $(\,\cdot\,)^p F_p(\,\cdot\,,n)$ and $(\,\cdot\,)^{p-1} H_p(\,\cdot\,,n)$, respectively; they will play a special role in the following. Thus, we may write

$$F_p(z) = -c_{0,+}\alpha^+ z^{-p} \prod_{j=1}^{2p-1} (z - \mu_j), \tag{3.4}$$

$$H_p(z) = c_{0,+}\beta z^{-p+1} \prod_{j=1}^{2p-1} (z - \nu_j), \tag{3.5}$$

and we recall that (cf. (2.62))

$$R_{2p} - G_p^2 = -F_p H_p. \tag{3.6}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_j of F_p and H_p from the complex plane $\mathbb{C}$ to the curve $\mathcal{K}_{2p-1}$. From (3.6) one infers that

$$R_{2p}(z) - G_p(z)^2 = 0, \quad z \in \{\mu_j, \nu_k\}_{j,k=1,\dots,2p-1}. \tag{3.7}$$

We now introduce $\{\hat\mu_j\}_{j=1,\dots,2p-1} \subset \mathcal{K}_{2p-1}$ and $\{\hat\nu_j\}_{j=1,\dots,2p-1} \subset \mathcal{K}_{2p-1}$ by

$$\hat\mu_j(n) = (\mu_j(n), (2/c_{0,+})\mu_j(n)^p G_p(\mu_j(n), n)), \quad j = 1, \dots, 2p-1,\ n \in \mathbb{Z}, \tag{3.8}$$

and

$$\hat\nu_j(n) = (\nu_j(n), -(2/c_{0,+})\nu_j(n)^p G_p(\nu_j(n), n)), \quad j = 1, \dots, 2p-1,\ n \in \mathbb{Z}. \tag{3.9}$$

We also introduce the points $P_{0,\pm}$ by

$$P_{0,\pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_{2p-1}, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{4p-1} E_m. \tag{3.10}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_{2p-1}$.

Next, we briefly recall our conventions used in connection with divisors on $\mathcal{K}_{2p-1}$. A map, $\mathcal{D}\colon \mathcal{K}_{2p-1} \to \mathbb{Z}$, is called a divisor on $\mathcal{K}_{2p-1}$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_{2p-1}$. The set of divisors on $\mathcal{K}_{2p-1}$ is denoted by $\operatorname{Div}(\mathcal{K}_{2p-1})$. We shall employ the following (additive) notation for divisors,

$$\begin{gathered}\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ \underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_{2p-1}, \quad Q_0 \in \mathcal{K}_{2p-1},\ m \in \mathbb{N},\end{gathered} \tag{3.11}$$

where for any $Q \in \mathcal{K}_{2p-1}$,

$$\mathcal{D}_Q\colon \mathcal{K}_{2p-1} \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_{2p-1} \setminus \{Q\}, \end{cases} \tag{3.12}$$

and $\operatorname{Sym}^n \mathcal{K}_{2p-1}$ denotes the nth symmetric product of $\mathcal{K}_{2p-1}$. In particular, $\operatorname{Sym}^m \mathcal{K}_{2p-1}$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_{2p-1})$ of degree m. Moreover, for a nonzero, meromorphic function f on $\mathcal{K}_{2p-1}$, the divisor of f is denoted by (f). Two divisors $\mathcal{D}, \mathcal{E} \in \operatorname{Div}(\mathcal{K}_{2p-1})$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_{2p-1}) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_{2p-1}) \,|\, \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ f \in \mathcal{M}(\mathcal{K}_{2p-1}) \setminus \{0\}, \tag{3.13}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P\in\mathcal{K}_{2p-1}} \mathcal{D}(P)$.

Next we introduce the fundamental meromorphic function on $\mathcal{K}_{2p-1}$ by

$$\phi(P,n) = \frac{(c_{0,+}/2)z^{-p}y + G_p(P,n)}{F_p(P,n)} \tag{3.14}$$

$$= \frac{-H_p(P,n)}{(c_{0,+}/2)z^{-p}y - G_p(P,n)}, \tag{3.15}$$

$$P = (z,y) \in \mathcal{K}_{2p-1},\ n \in \mathbb{Z},$$

with divisor $(\phi(\,\cdot\,,n))$ of $\phi(\,\cdot\,,n)$ given by

$$(\phi(\,\cdot\,,n)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{3.16}$$

using (3.4) and (3.5). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_{2p-1}\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_{2p-1}\} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}). \tag{3.17}$$

(The function ϕ is closely related to one of the variants of Weyl–Titchmarsh functions discussed in [37], [38], [49] in the special defocusing case $\beta = \bar{\alpha}$.) Given $\phi(\,\cdot\,,n)$, the meromorphic stationary Baker–Akhiezer vector $\Psi(\,\cdot\,,n,n_0)$ on $\mathcal{K}_{2p-1}$ is then defined by

$$\Psi(P,n,n_0) = \begin{pmatrix} \psi_1(P,n,n_0) \\ \psi_2(P,n,n_0) \end{pmatrix},$$

$$\psi_1(P,n,n_0) = \begin{cases} \prod_{m=n_0+1}^{n} (z + \alpha(m)\phi^-(P,m)), & n \geq n_0+1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0} (z + \alpha(m)\phi^-(P,m))^{-1}, & n \leq n_0-1, \end{cases} \tag{3.18}$$

$$\psi_2(P,n,n_0) = \phi(P,n_0) \begin{cases} \prod_{m=n_0+1}^{n} (z\beta(m)\phi(z,m-1)^{-1} + 1), & n \geq n_0+1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0} (z\beta(m)\phi(z,m-1)^{-1} + 1)^{-1}, & n \leq n_0-1. \end{cases} \tag{3.19}$$

Basic properties of ϕ and Ψ are summarized in the following result.

Lemma 3.1. *Suppose that α, β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.32). *Moreover, assume* (3.2) *and* (3.3) *and let $P = (z,y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n,n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{3.20}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_p(z)}{F_p(z)}, \tag{3.21}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_p(z)}{F_p(z)}, \tag{3.22}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p}\frac{y(P)}{F_p(z)}. \tag{3.23}$$

The vector Ψ satisfies

$$U(z,n)\Psi^-(P,n,n_0) = \Psi(P,n,n_0), \tag{3.24}$$

$$V_p(z,n)\Psi^-(P,n,n_0) = -(i/2)c_{0,+}z^{-p}y\Psi^-(P,n,n_0), \tag{3.25}$$

$$\psi_2(P,n,n_0) = \phi(P,n)\psi_1(P,n,n_0), \tag{3.26}$$

$$\psi_1(P,n,n_0)\psi_1(P^*,n,n_0) = z^{n-n_0}\frac{F_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \tag{3.27}$$

$$\psi_2(P,n,n_0)\psi_2(P^*,n,n_0) = z^{n-n_0}\frac{H_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \tag{3.28}$$

$$\begin{aligned}&\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) + \psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \\ &\quad = 2z^{n-n_0}\frac{G_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0),\end{aligned} \tag{3.29}$$

$$\begin{aligned}&\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) - \psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \\ &\quad = -c_{0,+}z^{n-n_0-p}\frac{y}{F_p(z,n_0)}\Gamma(n,n_0),\end{aligned} \tag{3.30}$$

where we used the abbreviation

$$\Gamma(n,n_0) = \begin{cases} \prod_{m=n_0+1}^{n}\gamma(m) & n \geq n_0+1, \\ 1 & n = n_0, \\ \prod_{m=n+1}^{n_0}\gamma(m)^{-1} & n \leq n_0-1. \end{cases} \tag{3.31}$$

Proof. To prove (3.20) one uses the definition (3.14) of ϕ and equations (2.10), (2.12), and (2.62) to obtain

$$\begin{aligned}&\alpha\phi(P)\phi^-(P) - \phi(P)^- + z\phi(P) - z\beta \\ &\quad = \frac{1}{F_pF_p^-}\Big(\alpha G_pG_p^- + (c_{0,+}/2)z^{-p}y\alpha(G_p+G_p^-) + \alpha R_{2p} \\ &\qquad - (G_p^- + (c_{0,+}/2)z^{-p}y)F_p + z(G_p + (c_{0,+}/2)z^{-p}y)F_p^- - z\beta F_pF_p^-\Big) \\ &\quad = \frac{1}{F_pF_p^-}\Big(\alpha G_p(G_p+G_p^-) + F_p(-\alpha H_p - G_p^- - z\beta F_p^-) + zF_p^-G_p\Big) = 0.\end{aligned} \tag{3.32}$$

Equations (3.21)–(3.23) are clear from the definitions of ϕ and y. By definition of ψ, (3.26) holds for $n = n_0$. By induction,

$$\frac{\psi_2(P,n,n_0)}{\psi_1(P,n,n_0)} = \frac{z\beta(n)\phi^-(P,n)^{-1}+1}{z+\alpha(n)\phi^-(P,n)}\frac{\psi_2^-(P,n,n_0)}{\psi_1^-(P,n,n_0)} = \frac{z\beta(n)+\phi^-(P,n)}{z+\alpha(n)\phi^-(P,n)}, \tag{3.33}$$

and hence ψ_2/ψ_1 satisfies the Riccati-type equation (3.20)

$$\alpha(n)\phi^-(P,n)\frac{\psi_2(P,n,n_0)}{\psi_1(P,n,n_0)} - \phi^-(P,n) + z\frac{\psi_2(P,n,n_0)}{\psi_1(P,n,n_0)} - z\beta(n) = 0. \tag{3.34}$$

This proves (3.26).

The definition of ψ implies

$$\begin{aligned}\psi_1(P,n,n_0) &= (z+\alpha(n)\phi^-(P,n))\psi_1^-(P,n,n_0) \\ &= z\psi_1^-(P,n,n_0) + \alpha(n)\psi_2^-(P,n,n_0),\end{aligned} \tag{3.35}$$

$$\begin{aligned}\psi_2(P,n,n_0) &= (z\beta(n)\phi^-(P,n)^{-1}+1)\psi_2^-(P,n,n_0) \\ &= z\beta(n)\psi_1^-(P,n,n_0) + \psi_2^-(P,n,n_0),\end{aligned} \tag{3.36}$$

which proves (3.24). Property (3.25) follows from (3.26) and the definition of ϕ. To prove (3.27) one can use (2.10) and (2.12)

$$\psi_1(P)\psi_1(P^*) = (z+\alpha\phi^-(P))(z+\alpha\phi^-(P^*))\psi_1^-(P)\psi_1^-(P^*)$$

$$\begin{aligned}
&= \frac{1}{F_p^-}(z^2F_p^- + 2z\alpha G_p^- + \alpha^2 H_p^-)\psi_1^-(P)\psi_1^-(P^*) \\
&= \frac{1}{F_p^-}(z^2F_p^- - z\alpha\beta F_p + z\alpha(G_p + G_p^-))\psi_1^-(P)\psi_1^-(P^*) \\
&= z\gamma\frac{F_p}{F_p^-}\psi_1^-(P)\psi_1^-(P^*). \qquad (3.37)
\end{aligned}$$

Equation (3.28) then follows from (3.22) and (3.24). Finally, equation (3.29) (resp. (3.30)) is proved by combining (3.22) and (3.26) (resp. (3.23) and (3.26)). ☐

Combining the polynomial recursion approach of Section 2 with (3.4) and (3.5) readily yields trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of F_p and H_p, respectively. For simplicity we just record the simplest cases.

Lemma 3.2. *Suppose that* α, β *satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.32)*. Then,*

$$\frac{\alpha}{\alpha^+} = \prod_{j=1}^{2p-1}\mu_j\bigg(\prod_{m=0}^{4p-1}E_m\bigg)^{-1/2}, \qquad (3.38)$$

$$\frac{\beta^+}{\beta} = \prod_{j=1}^{2p-1}\nu_j\bigg(\prod_{m=0}^{4p-1}E_m\bigg)^{-1/2}, \qquad (3.39)$$

$$\sum_{j=1}^{2p-1}\mu_j = \alpha^+\beta - \gamma^+\frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \qquad (3.40)$$

$$\sum_{j=1}^{2p-1}\nu_j = \alpha^+\beta - \gamma\frac{\beta^-}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \qquad (3.41)$$

Proof. We compare coefficients in (2.14) and (3.4)

$$\begin{aligned}
z^pF_p(z) &= f_{0,-} + \cdots + z^{2p-2}f_{1,+} + z^{2p-1}f_{0,+} \\
&= c_{0,+}\alpha^+\bigg(\prod_{j=1}^{2p-1}\mu_j + \cdots + z^{2p-2}\sum_{j=1}^{2p-1}\mu_j - z^{2p-1}\bigg) \qquad (3.42)
\end{aligned}$$

and use $f_{0,-} = c_{0,-}\alpha$ and $f_{1,+} = c_{0,+}\big((\alpha^+)^2\beta - \gamma^+\alpha^{++}\big) - \alpha^+c_{1,+}$ which yields (3.38) and (3.40). Similarly, one employs $h_{0,-} = -c_{0,-}\beta^+$ and $h_{1,+} = c_{0,+}\big(\gamma\beta^- - \alpha^+\beta^2\big) + \beta c_{1,+}$ for the remaining formulas (3.39) and (3.41). ☐

Next we turn to asymptotic properties of ϕ and Ψ in a neighborhood of $P_{\infty_\pm}$ and $P_{0,\pm}$.

Lemma 3.3. *Suppose that* α, β *satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.32)*. Moreover, let* $P = (z,y) \in \mathcal{K}_{2p-1}\backslash\{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$*,* $(n, n_0) \in \mathbb{Z}^2$*. Then* ϕ *has the asymptotic behavior*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \beta + \beta^-\gamma\zeta + O(\zeta^2), & P \to P_{\infty_+}, \\ -(\alpha^+)^{-1}\zeta^{-1} + (\alpha^+)^{-2}\alpha^{++}\gamma^+ + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \qquad (3.43)$$

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \alpha^{-1} - \alpha^{-2}\alpha^{-}\gamma\zeta + O(\zeta^2), & P \to P_{0,+}, \\ -\beta^{+}\zeta - \beta^{++}\gamma^{+}\zeta^2 + O(\zeta^3), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.44}$$

The components of the Baker–Akhiezer vector Ψ *have the asymptotic behavior*

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}(1+O(\zeta)), & P \to P_{\infty_+}, \\ \frac{\alpha^+(n)}{\alpha^+(n_0)}\Gamma(n,n_0) + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.45}$$

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \frac{\alpha(n)}{\alpha(n_0)} + O(\zeta), & P \to P_{0,+}, \\ \zeta^{n-n_0}\Gamma(n,n_0)(1+O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z, \tag{3.46}$$

$$\psi_2(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \beta(n)\zeta^{n_0-n}(1+O(\zeta)), & P \to P_{\infty_+}, \\ -\frac{1}{\alpha^+(n_0)}\Gamma(n,n_0)\zeta^{-1}(1+O(\zeta)), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.47}$$

$$\psi_2(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \frac{1}{\alpha(n_0)} + O(\zeta), & P \to P_{0,+}, \\ -\beta^+(n)\Gamma(n,n_0)\zeta^{n+1-n_0}(1+O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.48}$$

The divisors (ψ_j) *of* ψ_j, $j = 1,2$, *are given by*

$$(\psi_1(\,\cdot\,,n,n_0)) = \mathcal{D}_{\underline{\hat\mu}(n)} - \mathcal{D}_{\underline{\hat\mu}(n_0)} + (n-n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}), \tag{3.49}$$

$$(\psi_2(\,\cdot\,,n,n_0)) = \mathcal{D}_{\underline{\hat\nu}(n)} - \mathcal{D}_{\underline{\hat\mu}(n_0)} + (n-n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}) + \mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_-}}. \tag{3.50}$$

Proof. The existence of the asymptotic expansion of ϕ in terms of the local coordinate $\zeta = 1/z$ near $P_{\infty_\pm}$, respectively, $\zeta = z$ near $P_{0,\pm}$ is clear from the explicit form of ϕ in (3.14) and (3.15). Insertion of the Laurent polynomials F_p into (3.14) and H_p into (3.15) then yields the explicit expansion coefficients in (3.43) and (3.44). Alternatively, and more efficiently, one can insert each of the following asymptotic expansions

$$\begin{aligned} \phi(P) &\underset{z\to\infty}{=} \phi_{-1}z + \phi_0 + \phi_1 z^{-1} + O(z^{-2}), \\ \phi(P^*) &\underset{z\to\infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2}), \\ \phi(P) &\underset{z\to 0}{=} \phi_0 + \phi_1 z + O(z^2), \\ \phi(P^*) &\underset{z\to 0}{=} \phi_1 z + \phi_2 z^2 + O(z^3) \end{aligned} \tag{3.51}$$

into the Riccati-type equation (3.20) and, upon comparing coefficients of powers of z, which determines the expansion coefficients ϕ_k in (3.51), one concludes (3.43) and (3.44).

Next we compute the divisor of ψ_1. By (3.18) it suffices to compute the divisor of $z + \alpha\phi^-(P)$. First of all we note that

$$z + \alpha\phi^-(P) = \begin{cases} z + O(1), & P \to P_{\infty_+}, \\ \frac{\alpha^+}{\alpha}\gamma + O(z^{-1}), & P \to P_{\infty_-}, \\ \frac{\alpha}{\alpha^-} + O(z), & P \to P_{0,+}, \\ \gamma z + O(z^2), & P \to P_{0,-}, \end{cases} \tag{3.52}$$

which establishes (3.45) and (3.46). Moreover, the poles of the function $z+\alpha\phi^-(P)$ in $\mathcal{K}_{2p-1} \setminus \{P_{0,\pm}, P_{\infty_\pm}\}$ coincide with the ones of $\phi^-(P)$, and so it remains to compute the missing $2p-1$ zeros in $\mathcal{K}_{2p-1} \setminus \{P_{0,\pm}, P_{\infty\pm}\}$. Using (2.12), (2.61), (2.62), and $y(\hat{\mu}_j) = (2/c_{0,+})\mu_j^p G_p(\mu_j)$ (cf. (3.8)) one computes

$$\begin{aligned} z + \alpha\phi^-(P) &= z + \alpha \frac{(c_{0,+}/2)z^{-p}y + G_p^-}{F_p^-} \\ &= \frac{F_p + \alpha((c_{0,+}/2)z^{-p}y - G_p)}{F_p^-} \\ &= \frac{F_p}{F_p^-} + \alpha \frac{(c_{0,+}/2)^2 z^{-2p}y^2 - G_p^2}{F_p^-((c_{0,+}/2)z^{-p}y + G_p)} \\ &= \frac{F_p}{F_p^-}\bigg(1 + \frac{\alpha H_p}{(c_{0,+}/2)z^{-p}y + G_p}\bigg) \underset{P\to\hat{\mu}_j}{=} \frac{F_p(P)}{F_p^-(P)}O(1). \end{aligned} \tag{3.53}$$

Hence the sought after zeros are at $\hat{\mu}_j$, $j = 1, \dots, 2p-1$ (with the possibility that a zero at $\hat{\mu}_j$ is cancelled by a pole at $\hat{\mu}_j^-$).

Finally, the behavior of ψ_2 follows immediately using $\psi_2 = \phi\psi_1$. □

In addition to (3.43), (3.44) one can use the Riccati-type equation (3.20) to derive a convergent expansion of ϕ around $P_{\infty\pm}$ and $P_{0,\pm}$ and recursively determine the coefficients as in Lemma 3.3. Since this is not used later in this section, we omit further details at this point.

Since nonspecial divisors play a fundamental role in the derivation of theta function representations of algebro-geometric solutions of the AL hierarchy in [34], we now take a closer look at them.

Lemma 3.4. *Suppose that α, β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.32). *Moreover, assume* (3.2) *and* (3.3) *and let $n \in \mathbb{Z}$. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_{2p-1}\}$, and $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\underline{\hat{\nu}}_1, \dots, \underline{\hat{\nu}}_{2p-1}\}$, be the pole and zero divisors of degree $2p-1$, respectively, associated with α, β, and ϕ defined according to* (3.8) *and* (3.9)*, that is,*

$$\begin{aligned} \hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+})\mu_j(n)^p G_p(\mu_j(n), n)), \quad j = 1, \dots, 2p-1,\ n \in \mathbb{Z}, \\ \hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+})\nu_j(n)^p G_p(\nu_j(n), n)), \quad j = 1, \dots, 2p-1,\ n \in \mathbb{Z}. \end{aligned} \tag{3.54}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial for all $n \in \mathbb{Z}$.

Proof. We provide a detailed proof in the case of $\mathcal{D}_{\underline{\hat{\mu}}(n)}$. By [33, Thm. A.31] (see also [32, Thm. A.30]), $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is special if and only if $\{\hat{\mu}_1(n), \dots, \hat{\mu}_p(n)\}$ contains at least one pair of the type $\{\hat{\mu}(n), \hat{\mu}(n)^*\}$. Hence $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is certainly nonspecial as long as the projections $\mu_j(n)$ of $\hat{\mu}_j(n)$ are mutually distinct, $\mu_j(n) \neq \mu_k(n)$ for $j \neq k$. On the other hand, if two or more projections coincide for some $n_0 \in \mathbb{Z}$, for instance,

$$\mu_{j_1}(n_0) = \cdots = \mu_{j_N}(n_0) = \mu_0, \quad N \in \{2, \dots, p\}, \tag{3.55}$$

then $G_p(\mu_0, n_0) \neq 0$ as long as $\mu_0 \notin \{E_0, \dots, E_{2p+1}\}$. This fact immediately follows from (2.62) since $F_p(\mu_0, n_0) = 0$ but $R_{2p}(\mu_0) \neq 0$ by hypothesis. In particular, $\hat{\mu}_{j_1}(n_0), \dots, \hat{\mu}_{j_N}(n_0)$ all meet on the same sheet since

$$\hat{\mu}_{j_r}(n_0) = (\mu_0, (2/c_{0,+})\mu_0^p G_p(\mu_0, n_0)), \quad r = 1, \dots, N, \tag{3.56}$$

and hence no special divisor can arise in this manner. Remaining to be studied is the case where two or more projections collide at a branch point, say at $(E_{m_0}, 0)$

for some $n_0 \in \mathbb{Z}$. In this case one concludes $F_p(z,n_0) \underset{z\to E_{m_0}}{=} O\big((z-E_{m_0})^2\big)$ and

$$G_p(E_{m_0},n_0) = 0 \tag{3.57}$$

using again (2.62) and $F_p(E_{m_0},n_0) = R_{2p}(E_{m_0}) = 0$. Since $G_p(\,\cdot\,,n_0)$ is a Laurent polynomial, (3.57) implies $G_p(z,n_0) \underset{z\to E_{m_0}}{=} O((z-E_{m_0}))$. Thus, using (2.62) once more, one obtains the contradiction,

$$\begin{aligned} O\big((z-E_{m_0})^2\big) &\underset{z\to E_{m_0}}{=} R_{2p}(z) \\ &\underset{z\to E_{m_0}}{=} \bigg(\frac{c_{0,+}}{2E_{m_0}^p}\bigg)^2 (z-E_{m_0})\Bigg(\prod_{\substack{m=0\\ m\neq m_0}}^{4p-1} (E_{m_0}-E_m) + O(z-E_{m_0})\Bigg). \end{aligned} \tag{3.58}$$

Consequently, at most one $\hat\mu_j(n)$ can hit a branch point at a time and again no special divisor arises. Finally, by our hypotheses on α, β, $\hat\mu_j(n)$ stay finite for fixed $n \in \mathbb{Z}$ and hence never reach the points $P_{\infty_\pm}$. (Alternatively, by (3.43), $\hat\mu_j$ never reaches the point P_{∞_+}. Hence, if some $\hat\mu_j$ tend to infinity, they all necessarily converge to P_{∞_-}.) Again no special divisor can arise in this manner.

The proof for $\mathcal{D}_{\underline{\hat\nu}(n)}$ is analogous (replacing F_p by H_p and noticing that by (3.43), ϕ has no zeros near $P_{\infty_\pm}$), thereby completing the proof. $\square$

We conclude this section with the trivial case $p = 0$ excluded thus far.

Example 3.5. *Assume $p = 0$ and $c_{0,+} = c_{0,-} = c_0 \neq 0$ (cf. (2.60)). Then,*

$$\begin{aligned} &F_0 = \widehat{F}_0 = H_0 = \widehat{H}_0 = 0, \quad G_0 = K_0 = \frac{1}{2}c_0, \quad \widehat{G}_0 = \widehat{K}_0 = \frac{1}{2}, \quad R_0 = \frac{1}{4}c_0^2, \\ &\alpha = \beta = 0, \\ &U = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad V_0 = \frac{ic_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{3.59}$$

Introducing

$$\Psi_+(z,n,n_0) = \begin{pmatrix} z^{n-n_0} \\ 0 \end{pmatrix}, \quad \Psi_-(z,n,n_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad n, n_0 \in \mathbb{Z}, \tag{3.60}$$

one verifies the equations

$$U\Psi_\pm^- = \Psi_\pm, \quad V_0\Psi_\pm^- = \pm\frac{ic_0}{2}\Psi_\pm^-. \tag{3.61}$$

4. The Time-Dependent Ablowitz–Ladik Formalism

In this section we extend the algebro-geometric analysis of Section 3 to the time-dependent Ablowitz–Ladik hierarchy.

For most of this section we assume the following hypothesis.

Hypothesis 4.1. *(i) Suppose that α, β satisfy*

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}}, \ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,), \ \beta(n,\,\cdot\,) \in C^1(\mathbb{R}), \ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\}, \ (n,t) \in \mathbb{Z}\times\mathbb{R}. \end{aligned} \tag{4.1}$$

(ii) Assume that the hyperelliptic curve $\mathcal{K}_{2p-1}$ satisfies (3.2) *and* (3.3).

The basic problem in the analysis of algebro-geometric solutions of the Ablowitz–Ladik hierarchy consists of solving the time-dependent rth Ablowitz–Ladik flow with initial data a stationary solution of the pth equation in the hierarchy. More precisely, given $p \in \mathbb{N}_0$ we consider a solution $\alpha^{(0)}$, $\beta^{(0)}$ of the pth stationary Ablowitz–Ladik equation $\text{s-AL}_p(\alpha^{(0)}, \beta^{(0)}) = 0$, associated with the hyperelliptic curve $\mathcal{K}_{2p-1}$ and a corresponding set of summation constants $\{c_{\ell,\pm}\}_{\ell=1,\dots,p} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution α, β of the rth Ablowitz–Ladik flow $\text{AL}_r(\alpha, \beta) = 0$ with $\alpha(t_{0,r}) = \alpha^{(0)}$, $\beta(t_{0,r}) = \beta^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Ablowitz–Ladik equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{V}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_{s,\pm}$, $\tilde{g}_{s,\pm}$, $\tilde{h}_{s,\pm}$, $\tilde{c}_{s,\pm}$, in order to distinguish them from V_p, F_p, G_p, H_p, $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, $c_{\ell,\pm}$, in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth Ablowitz–Ladik flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are interested in solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned}
&\widetilde{\text{AL}}_r(\alpha, \beta) \begin{pmatrix} -i\alpha_{t_r} - \alpha(\tilde{g}_{r,+} + \tilde{g}_{r,-}^-) + \tilde{f}_{r-1,+} - \tilde{f}_{r-1,-}^- \\ -i\beta_{t_r} + \beta(\tilde{g}_{r,+}^- + \tilde{g}_{r,-}) - \tilde{h}_{r-1,-} + \tilde{h}_{r-1,+}^- \end{pmatrix} = 0, \\
&(\alpha, \beta)\big|_{t=t_{0,r}} = \big(\alpha^{(0)}, \beta^{(0)}\big),
\end{aligned} \tag{4.2}$$

$$\text{s-AL}_p\left(\alpha^{(0)}, \beta^{(0)}\right) = \begin{pmatrix} -\alpha^{(0)}(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta^{(0)}(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0 \tag{4.3}$$

for some $t_{0,r} \in \mathbb{R}$, where $\alpha = \alpha(n, t_r)$, $\beta = \beta(n, t_r)$ satisfy (4.1) and a fixed curve $\mathcal{K}_{2p-1}$ is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ in (4.3). In terms of the zero-curvature formulation this amounts to solving

$$U_{t_r}(z, t_r) + U(z, t_r)\widetilde{V}_r(z, t_r) - \widetilde{V}_r^+(z, t_r)U(z, t_r) = 0, \tag{4.4}$$

$$U(z, t_{0,r})V_p(z, t_{0,r}) - V_p^+(z, t_{0,r})U(z, t_{0,r}) = 0. \tag{4.5}$$

One can show (cf. [35]) that the stationary Ablowitz–Ladik system (4.5) is actually satisfied for all times $t_r \in \mathbb{R}$: Thus, we actually impose

$$U_{t_r}(z, t_r) + U(z, t_r)\widetilde{V}_r(z, t_r) - \widetilde{V}_r^+(z, t_r)U(z, t_r) = 0, \tag{4.6}$$

$$U(z, t_r)V_p(z, t_r) - V_p^+(z, t_r)U(z, t_r) = 0, \tag{4.7}$$

instead of (4.4) and (4.5). For further reference, we recall the relevant quantities here (cf. (2.5), (2.6), (2.14)–(2.16), (2.61)):

$$\begin{aligned}
U(z) &= \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \\
V_p(z) &= i \begin{pmatrix} G_p^-(z) & -F_p^-(z) \\ H_p^-(z) & -G_p^-(z) \end{pmatrix}, \\
\widetilde{V}_r(z) &= i \begin{pmatrix} \widetilde{G}_r^-(z) & -\widetilde{F}_r^-(z) \\ \widetilde{H}_r^-(z) & -\widetilde{K}_r^-(z) \end{pmatrix},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
F_p(z) &= \sum_{\ell=1}^{p} f_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+} z^{\ell} = -c_{0,+}\alpha^{+} z^{-p} \prod_{j=1}^{2p-1}(z-\mu_j),\\
G_p(z) &= \sum_{\ell=1}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+} z^{\ell},\\
H_p(z) &= \sum_{\ell=0}^{p-1} h_{p-1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+} z^{\ell} = c_{0,+}\beta z^{-p+1} \prod_{j=1}^{2p-1}(z-\nu_j),\\
\widetilde F_r(z) &= \sum_{s=1}^{r} \tilde f_{r-s,-} z^{-s} + \sum_{s=0}^{r-1} \tilde f_{r-1-s,+} z^{s}, \qquad (4.9)\\
\widetilde G_r(z) &= \sum_{s=1}^{r} \tilde g_{r-s,-} z^{-s} + \sum_{s=0}^{r} \tilde g_{r-s,+} z^{s},\\
\widetilde H_r(z) &= \sum_{s=0}^{r-1} \tilde h_{r-1-s,-} z^{-s} + \sum_{s=1}^{r} \tilde h_{r-s,+} z^{s},\\
\widetilde K_r(z) &= \sum_{s=0}^{r} \tilde g_{r-s,-} z^{-s} + \sum_{s=1}^{r} \tilde g_{r-s,+} z^{s} = \widetilde G_r(z) + \tilde g_{r,-} - \tilde g_{r,+} \qquad (4.10)
\end{aligned}$$

for fixed $p, r \in \mathbb{N}_0$. Here $f_{\ell,\pm}$, $\tilde f_{s,\pm}$, $g_{\ell,\pm}$, $\tilde g_{s,\pm}$, $h_{\ell,\pm}$, and $\tilde h_{s,\pm}$ are defined as in (2.18)–(2.25) with appropriate sets of summation constants $c_{\ell,\pm}$, $\ell \in \mathbb{N}$, and $\tilde c_{k,\pm}$, $k \in \mathbb{N}$. Explicitly, (4.6) and (4.7) are equivalent to (cf. (2.10)–(2.13), (2.76)–(2.79)),

$$\alpha_{t_r} = i\big(z\widetilde F_r^- + \alpha(\widetilde G_r + \widetilde K_r^-) - \widetilde F_r\big), \tag{4.11}$$
$$\beta_{t_r} = -i\big(\beta(\widetilde G_r^- + \widetilde K_r) - \widetilde H_r + z^{-1}\widetilde H_r^-\big), \tag{4.12}$$
$$0 = z(\widetilde G_r^- - \widetilde G_r) + z\beta\widetilde F_r + \alpha\widetilde H_r^-, \tag{4.13}$$
$$0 = z\beta\widetilde F_r^- + \alpha\widetilde H_r + \widetilde K_r^- - \widetilde K_r, \tag{4.14}$$
$$0 = z(G_p^- - G_p) + z\beta F_p + \alpha H_p^-, \tag{4.15}$$
$$0 = z\beta F_p^- + \alpha H_p - G_p + G_p^-, \tag{4.16}$$
$$0 = -F_p + zF_p^- + \alpha(G_p + G_p^-), \tag{4.17}$$
$$0 = z\beta(G_p + G_p^-) - zH_p + H_p^-, \tag{4.18}$$

respectively. In particular, (2.62) holds in the present t_r-dependent setting, that is,

$$G_p^2 - F_pH_p = R_{2p}. \tag{4.19}$$

As in the stationary context (3.8), (3.9) we introduce

$$\begin{aligned}
\hat\mu_j(n,t_r) = (\mu_j(n,t_r), (2/c_{0,+})\mu_j(n,t_r)^p G_p(\mu_j(n,t_r),n,t_r)) \in \mathcal{K}_{2p-1},&\\
j = 1,\dots,2p-1,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R},&
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
\hat\nu_j(n,t_r) = (\nu_j(n,t_r), -(2/c_{0,+})\nu_j(n,t_r)^p G_p(\nu_j(n,t_r),n,t_r)) \in \mathcal{K}_{2p-1},&\\
j = 1,\dots,2p-1,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R},&
\end{aligned} \tag{4.21}$$

and note that the regularity assumptions (4.1) on α, β imply continuity of μ_j and ν_k with respect to $t_r \in \mathbb{R}$ (away from collisions of these zeros, μ_j and ν_k are of course C^∞).

In analogy to (3.14), (3.15), one defines the following meromorphic function $\phi(\,\cdot\,, n, t_r)$ on $\mathcal{K}_{2p-1}$,

$$\begin{aligned}\phi(P,n,t_r) &= \frac{(c_{0,+}/2)z^{-p}y + G_p(z,n,t_r)}{F_p(z,n,t_r)} &(4.22)\\ &= \frac{-H_p(z,n,t_r)}{(c_{0,+}/2)z^{-p}y - G_p(z,n,t_r)}, &(4.23)\\ P = (z,y) &\in \mathcal{K}_{2p-1},\ (n,t_r) \in \mathbb{Z}\times\mathbb{R},\end{aligned}$$

with divisor $(\phi(\,\cdot\,,n,t_r))$ of $\phi(\,\cdot\,,n,t_r)$ given by

$$(\phi(\,\cdot\,,n,t_r)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_r)}. \tag{4.24}$$

The time-dependent Baker–Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P,n,n_0,t_r,t_{0,r}) = \begin{pmatrix}\psi_1(P,n,n_0,t_r,t_{0,r})\\ \psi_2(P,n,n_0,t_r,t_{0,r})\end{pmatrix}, \tag{4.25}$$

$$\begin{aligned}\psi_1(P,n,n_0,t_r,t_{0,r}) &= \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde{G}_r(z,n_0,s) - \widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\big)\Big) &(4.26)\\ &\quad\times\begin{cases}\prod_{m=n_0+1}^{n}(z+\alpha(m,t_r)\phi^-(P,m,t_r)), & n\ge n_0+1,\\ 1, & n=n_0,\\ \prod_{m=n+1}^{n_0}(z+\alpha(m,t_r)\phi^-(P,m,t_r))^{-1}, & n\le n_0-1,\end{cases}\\ \psi_2(P,n,n_0,t_r,t_{0,r}) &= \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde{G}_r(z,n_0,s) - \widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\big)\Big) &(4.27)\\ &\quad\times\phi(P,n_0,t_r)\begin{cases}\prod_{m=n_0+1}^{n}\Big(\frac{z\beta(m,t_r)}{\phi^-(P,m,t_r)}+1\Big), & n\ge n_0+1,\\ 1, & n=n_0,\\ \prod_{m=n+1}^{n_0}\Big(\frac{z\beta(m,t_r)}{\phi^-(P,m,t_r)}+1\Big)^{-1}, & n\le n_0-1,\end{cases}\\ P = (z,y) &\in \mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,t_r)\in\mathbb{Z}\times\mathbb{R}.\end{aligned}$$

One observes that

$$\begin{aligned}&\psi_1(P,n,n_0,t_r,\tilde{t}_r) = \psi_1(P,n_0,n_0,t_r,\tilde{t}_r)\psi_1(P,n,n_0,t_r,t_r),\\ &P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,n_0,t_r,\tilde{t}_r)\in\mathbb{Z}^2\times\mathbb{R}^2.\end{aligned} \tag{4.28}$$

The following lemma records basic properties of ϕ, ψ, and Ψ in analogy to the stationary case discussed in Lemma 3.1.

Lemma 4.2. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. In addition, let* $P = (z,y) \in \mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-}\}$, $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$, *and* $r\in\mathbb{N}_0$. *Then* ϕ *satisfies*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{4.29}$$

$$\phi_{t_r}(P) = i\widetilde{F}_r\phi^2(P) - i(\widetilde{G}_r(z)+\widetilde{K}_r(z))\phi(P) + i\widetilde{H}_r(z), \tag{4.30}$$

$$\phi(P)\phi(P^*) = \frac{H_p(z)}{F_p(z)}, \tag{4.31}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_p(z)}{F_p(z)}, \tag{4.32}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p}\frac{y(P)}{F_p(z)}. \tag{4.33}$$

Moreover, assuming $P = (z, y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, then Ψ satisfies

$$\psi_2(P, n, n_0, t_r, t_{0,r}) = \phi(P, n, t_r)\psi_1(P, n, n_0, t_r, t_{0,r}), \tag{4.34}$$

$$U(z, n, t_r)\Psi^-(P, n, n_0, t_r, t_{0,r}) = \Psi(P, n, n_0, t_r, t_{0,r}), \tag{4.35}$$

$$V_p(z, n, t_r)\Psi^-(P, n, n_0, t_r, t_{0,r}) = -(i/2)c_{0,+}z^{-p}y\Psi^-(P, n, n_0, t_r, t_{0,r}), \tag{4.36}$$

$$\Psi_{t_r}(P, n, n_0, t_r, t_{0,r}) = \widetilde{V}_r^+(n, t_r)\Psi(P, n, n_0, t_r, t_{0,r}), \tag{4.37}$$

$$\psi_1(P, n, n_0, t_r, t_{0,r})\psi_1(P^*, n, n_0, t_r, t_{0,r}) = z^{n-n_0}\frac{F_p(z, n, t_r)}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r), \tag{4.38}$$

$$\psi_2(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) = z^{n-n_0}\frac{H_p(z, n, t_r)}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r), \tag{4.39}$$

$$\begin{aligned}&\psi_1(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) + \psi_1(P^*, n, n_0, t_r, t_{0,r})\psi_2(P, n, n_0, t_r, t_{0,r})\\ &\quad = 2z^{n-n_0}\frac{G_p(z, n, t_r)}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r),\end{aligned} \tag{4.40}$$

$$\begin{aligned}&\psi_1(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) - \psi_1(P^*, n, n_0, t_r, t_{0,r})\psi_2(P, n, n_0, t_r, t_{0,r})\\ &\quad = -c_{0,+}z^{n-n_0-p}\frac{y}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r).\end{aligned} \tag{4.41}$$

In addition, as long as the zeros $\mu_j(n_0, s)$ of $F_p(\,\cdot\,, n_0, s)$ are all simple and distinct from zero for $s \in \mathcal{I}_\mu$, $\mathcal{I}_\mu \subseteq \mathbb{R}$ an open interval, $\Psi(\,\cdot\,, n, n_0, t_r, t_{0,r})$ is meromorphic on $\mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ for $(n, t_r, t_{0,r}) \in \mathbb{Z} \times \mathcal{I}_\mu^2$.

Proof. Equations (4.29), (4.31)–(4.36), and (4.38)–(4.41) are proved as in the stationary case, see Lemma 3.1. Thus, we turn to the proof of (4.30) and (4.37): Differentiating the Riccati-type equation (4.29) yields

$$\begin{aligned}0 &= \big(\alpha\phi\phi^- - \phi^- + z\phi - z\beta\big)_{t_r}\\ &= \alpha_{t_r}\phi\phi^- + (\alpha\phi^- + z)\phi_{t_r} + (\alpha\phi - 1)\phi^-_{t_r} - z\beta_{t_r}\\ &= \big((\alpha\phi^- + z) + (\alpha\phi - 1)S^-\big)\phi_{t_r} + i\phi\phi^-\big(\alpha(\widetilde{G}_r + \widetilde{K}_r^-) + z\widetilde{F}_r^- - \widetilde{F}_r\big)\\ &\quad + iz\beta(\widetilde{G}_r^- + \widetilde{K}_r) + i(z\widetilde{H}_r - \widetilde{H}_r^-),\end{aligned} \tag{4.42}$$

using (4.11) and (4.12). Next, one employs (3.20) to rewrite

$$(\alpha\phi^- + z) + (\alpha\phi - 1)S^- = \frac{1}{\phi}(z\beta + \phi^-) + \frac{z}{\phi^-}(\beta - \phi)S^-. \tag{4.43}$$

This then allows one to calculate the right-hand side of (4.30) using (4.13) and (4.14)

$$\begin{aligned}&\big((\alpha\phi^- + z) + (\alpha\phi - 1)S^-\big)\big(\widetilde{H}_r + \widetilde{F}_r\phi^2 - (\widetilde{G}_r + \widetilde{K}_r)\phi\big)\\ &= (\alpha\phi^- + z)\widetilde{H}_r + (\alpha\phi - 1)\widetilde{H}_r^- + \phi(z\beta + \phi^-)\widetilde{F}_r + z\phi^-(\beta - \phi)\widetilde{F}_r^-\\ &\quad - (z\beta + \phi^-)(\widetilde{G}_r + \widetilde{K}_r) - z(\beta - \phi)(\widetilde{G}_r^- + \widetilde{K}_r^-)\\ &= \phi\phi^-(\widetilde{F}_r - z\widetilde{F}_r^-) + z\widetilde{H}_r - \widetilde{H}_r^- + \phi^-(\alpha\widetilde{H}_r + z\beta\widetilde{F}_r^-) + \phi(\alpha\widetilde{H}_r^- + z\beta\widetilde{F}_r)\\ &\quad - z\beta(\widetilde{G}_r + \widetilde{K}_r + \widetilde{G}_r^- + \widetilde{K}_r^-) - z\phi(\widetilde{G}_r^- + \widetilde{K}_r^-) - \phi^-(\widetilde{G}_r + \widetilde{K}_r)\end{aligned}$$

$$= \phi\phi^-(\widetilde F_r - z\widetilde F_r^-) + z\widetilde H_r - \widetilde H_r^- - z\beta(\widetilde G_r^- + \widetilde K_r) + (z\phi - \phi^- - z\beta)(\widetilde G_r + \widetilde K_r^-)$$
$$= \phi\phi^-(\widetilde F_r - z\widetilde F_r^-) + z\widetilde H_r - \widetilde H_r^- - z\beta(\widetilde G_r^- + \widetilde K_r) - \alpha\phi\phi^-(\widetilde G_r + \widetilde K_r^-). \tag{4.44}$$

Hence,

$$\Big(\frac{1}{\phi}(z\beta + \phi^-) + \frac{z}{\phi^-}(\beta - \phi)S^-\Big)\big(\phi_{t_r} - i\widetilde H_r - i\widetilde F_r\phi^2 + i(\widetilde G_r + \widetilde K_r)\phi\big) = 0. \tag{4.45}$$

Solving the first-order difference equation (4.45) then yields

$$\begin{aligned}
&\phi_{t_r}(P,n,t_r) - i\widetilde F_r(z,n,t_r)\phi(P,n,t_r)^2 \\
&\quad + i(\widetilde G_r(z,n,t_r) + \widetilde K_r(z,n,t_r))\phi(P,n,t_r) - i\widetilde H_r(z,n,t_r) \\
&= C(P,t_r)\begin{cases} \prod_{j=1}^n B(P,j,t_r)/A(P,j,t_r), & n \geq 1, \\ 1, & n = 0, \\ \prod_{j=n+1}^0 A(P,j,t_r)/B(P,j,t_r), & n \leq -1 \end{cases}
\end{aligned} \tag{4.46}$$

for some n-independent function $C(\,\cdot\,,t_r)$ meromorphic on $\mathcal{K}_{2p-1}$, where

$$A = \phi^{-1}(z\beta + \phi^-), \quad B = -z(\phi^-)^{-1}(\beta - \phi). \tag{4.47}$$

The asymptotic behavior of $\phi(P,n,t_r)$ in (3.43) then yields (for $t_r \in \mathbb{R}$ fixed)

$$\frac{B(P)}{A(P)} \underset{P\to P_{\infty_+}}{=} -(1-\alpha\beta)(\beta^-)^{-1}z^{-1} + O(z^{-2}). \tag{4.48}$$

Since the left-hand side of (4.46) is of order $O(z^r)$ as $P \to P_{\infty_+}$, and C is meromorphic, insertion of (4.48) into (4.46), taking $n \geq 1$ sufficiently large, then yields a contradiction unless $C = 0$. This proves (4.30).

Proving (4.37) is equivalent to showing

$$\psi_{1,t_r} = i(\widetilde G_r - \phi\widetilde F_r)\psi_1, \tag{4.49}$$
$$\psi_1\phi_{t_r} + \phi\psi_{1,t_r} = i(\widetilde H_r - \phi\widetilde K_r)\psi_1, \tag{4.50}$$

using (4.34). Equation (4.50) follows directly from (4.49) and from (4.30),

$$\begin{aligned}
\psi_1\phi_{t_r} + \phi\psi_{1,t_r} &= \psi_1\big(i\widetilde H_r + i\widetilde F_r\phi^2 - i(\widetilde G_r + \widetilde K_r)\phi + i(\widetilde G_r - \phi\widetilde F_r)\phi\big) \\
&= i(\widetilde H_r - \phi\widetilde K_r)\psi_1.
\end{aligned} \tag{4.51}$$

To prove (4.49) we start from

$$\begin{aligned}
&(z + \alpha\phi^-)_{t_r} = \alpha_{t_r}\phi^- + \alpha\phi^-_{t_r} \\
&\quad = \phi^- i\big(z\widetilde F_r^- + \alpha(\widetilde G_r + \widetilde K_r^-) - \widetilde F_r\big) + \alpha i\big(\widetilde H_r^- + \widetilde F_r^-(\phi^-)^2 - (\widetilde G_r^- + \widetilde K_r^-)\phi^-\big) \\
&\quad = i\alpha\phi^-(\widetilde G_r - \widetilde G_r^-) + i(z + \alpha\phi^-)\phi^-\widetilde F_r^- - i\phi^-\widetilde F_r + i\alpha\widetilde H_r^- \\
&\quad = i(z + \alpha\phi^-)\big(\widetilde G_r - \phi\widetilde F_r - (\widetilde G_r^- - \phi^-\widetilde F_r^-)\big),
\end{aligned} \tag{4.52}$$

where we used (4.13) and (3.20) to rewrite

$$i\alpha\widetilde H_r^- - i\phi^-\widetilde F_r = iz(\widetilde G_r - \widetilde G_r^-) - \alpha\phi\phi^-\widetilde F_r - z\phi\widetilde F_r. \tag{4.53}$$

Abbreviating

$$\sigma(P,n_0,t_r) = i\int_0^{t_r} ds\big(\widetilde G_r(z,n_0,s) - \widetilde F_r(z,n_0,s)\phi(P,n_0,s)\big), \tag{4.54}$$

one computes for $n \geq n_0 + 1$,

$$\begin{aligned}
\psi_{1,t_r} &= \bigg(\exp(\sigma) \prod_{m=n_0+1}^{n} (z+\alpha\phi^-)(m) \bigg)_{t_r} \\
&= \sigma_{t_r}\psi_1 + \exp(\sigma) \sum_{m=n_0+1}^{n} (z+\alpha\phi^-)_{t_r}(m) \prod_{\substack{k=1\\ k\neq m}}^{n} (z+\alpha\phi^-)(k) \\
&= \psi_1 \bigg(\sigma_{t_r} + i \sum_{m=n_0+1}^{n} \big((\widetilde{G}_r - \widetilde{F}_r\phi)(m) - (\widetilde{G}_r - \widetilde{F}_r\phi)(m-1) \big) \bigg) \\
&= i(\widetilde{G}_r - \widetilde{F}_r\phi)\psi_1. \qquad (4.55)
\end{aligned}$$

The case $n \leq n_0$ is handled analogously establishing (4.49).

That $\Psi(\,\cdot\,, n, n_0, t_r, t_{0,r})$ is meromorphic on $\mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ if $F_p(\,\cdot\,, n_0, t_r)$ has only simple zeros distinct from zero is a consequence of (4.26) and (4.27) and of

$$-i\widetilde{F}_r(z,n_0,s)\phi(P,n_0,s) \underset{P\to\hat{\mu}_j(n_0,s)}{=} \partial_s \ln\big(F_p(z,n_0,s)\big) + O(1), \qquad (4.56)$$

using (4.20), (4.24), and (4.57). ((4.57) in Lemma 4.3 follows from (4.30), (4.32) and (4.33) which have already been proven.) ☐

Next we consider the t_r-dependence of F_p, G_p, and H_p.

Lemma 4.3. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. In addition, let* $(z,n,t_r) \in \mathbb{C}\times\mathbb{Z}\times\mathbb{R}$. *Then,*

$$F_{p,t_r} = -2iG_p\widetilde{F}_r + i(\widetilde{G}_r + \widetilde{K}_r)F_p, \qquad (4.57)$$

$$G_{p,t_r} = iF_p\widetilde{H}_r - iH_p\widetilde{F}_r, \qquad (4.58)$$

$$H_{p,t_r} = 2iG_p\widetilde{H}_r - i(\widetilde{G}_r + \widetilde{K}_r)H_p. \qquad (4.59)$$

In particular, (4.57)–(4.59) *are equivalent to*

$$V_{p,t_r} = \big[\widetilde{V}_r, V_p\big]. \qquad (4.60)$$

Proof. To prove (4.57) one first differentiates equation (4.33)

$$\phi_{t_r}(P) - \phi_{t_r}(P^*) = -c_{0,+}z^{-p}yF_p^{-2}F_{p,t_r}. \qquad (4.61)$$

The time derivative of ϕ given in (4.30) and (4.32) yield

$$\begin{aligned}
\phi_{t_r}(P) - \phi_{t_r}(P^*) &= i\big(\widetilde{H}_r + \widetilde{F}_r\phi(P)^2 - (\widetilde{G}_r + \widetilde{K}_r)\phi(P)\big) \\
&\quad - i\big(\widetilde{H}_r + \widetilde{F}_r\phi(P^*)^2 - (\widetilde{G}_r + \widetilde{K}_r)\phi(P^*)\big) \\
&= i\widetilde{F}_r(\phi(P) + \phi(P^*))(\phi(P) - \phi(P^*)) \\
&\quad - i(\widetilde{G}_r + \widetilde{K}_r)(\phi(P) - \phi(P^*)) \\
&= 2ic_{0,+}z^{-p}\widetilde{F}_r yG_pF_p^{-2} - ic_{0,+}z^{-p}(\widetilde{G}_r + \widetilde{K}_r)yF_p^{-1}, \qquad (4.62)
\end{aligned}$$

and hence

$$F_{p,t_r} = -2iG_p\widetilde{F}_r + i(\widetilde{G}_r + \widetilde{K}_r)F_p. \qquad (4.63)$$

Similarly, starting from (4.32)

$$\phi_{t_r}(P) + \phi_{t_r}(P^*) = 2F_p^{-2}(F_pG_{p,t_r} - F_{p,t_r}G_p) \qquad (4.64)$$

yields (4.58) and

$$0 = R_{2p+2,t_r} = 2G_p G_{p,t_r} - F_{p,t_r} H_p - F_p H_{p,t_r} \tag{4.65}$$

proves (4.59). □

Next we turn to the Dubrovin equations for the time variation of the zeros μ_j of F_p and ν_j of H_p governed by the AL_r flow (indexed by the parameter t_r).

Lemma 4.4. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold on* $\mathbb{Z} \times I_\mu$ *with* $I_\mu \subseteq \mathbb{R}$ *an open interval. In addition, assume that the zeros* μ_j, $j = 1, \dots, 2p-1$, *of* $F_p(\,\cdot\,)$ *remain distinct and nonzero on* $\mathbb{Z} \times I_\mu$. *Then* $\{\hat{\mu}_j\}_{j=1,\dots,2p-1}$, *defined in* (4.20), *satisfies the following first-order system of differential equations on* $\mathbb{Z} \times I_\mu$,

$$\mu_{j,t_r} = -iy(\hat{\mu}_j)\widetilde{F}_r(\mu_j)(\alpha^+)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^{2p-1} (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, 2p-1. \tag{4.66}$$

For the zeros ν_j, $j = 1, \dots, 2p-1$, *of* $H_p(\,\cdot\,)$, *identical statements hold with* μ_j *and* I_μ *replaced by* ν_j *and* I_ν, *etc.* (*with* $I_\nu \subseteq \mathbb{R}$ *an open interval*). *In particular,* $\{\hat{\nu}_j\}_{j=1,\dots,2p-1}$, *defined in* (4.21), *satisfies the first-order system on* $\mathbb{Z} \times I_\nu$,

$$\nu_{j,t_r} = iy(\hat{\nu}_j)\widetilde{H}_r(\nu_j)(\beta\nu_j)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^{2p-1} (\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, 2p-1. \tag{4.67}$$

Proof. It suffices to consider (4.66) for μ_{j,t_r}. Using the product representation for F_p in (4.9) and employing (4.20) and (4.57), one computes

$$\begin{aligned} F_{p,t_r}(\mu_j) &= \Bigg(c_{0,+}\alpha^+ \mu_j^{-p} \prod_{\substack{k=1 \\ k \neq j}}^{2p-1} (\mu_j - \mu_k) \Bigg) \mu_{j,t_r} = -2iG_p(\mu_j)\widetilde{F}_r(\mu_j) \\ &= -ic_{0,+}\mu_j^{-p} y(\hat{\mu}_j)\widetilde{F}_r(\mu_j), \quad j = 1, \dots, 2p-1, \end{aligned} \tag{4.68}$$

proving (4.66). The case of (4.67) for ν_{j,t_r} is of course analogous using the product representation for H_p in (4.9) and employing (4.21) and (4.59). □

When attempting to solve the Dubrovin systems (4.66) and (4.67), they must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \mathrm{Sym}^{2p-1} \mathcal{K}_{2p-1}$, $t_{0,r} \in I_\mu$, and $\mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,r})} \in \mathrm{Sym}^{2p-1} \mathcal{K}_{2p-1}$, $t_{0,r} \in I_\nu$, as initial conditions.

Since the stationary trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of F_p and H_p in Lemma 3.2 extend line by line to the corresponding time-dependent setting, we next record their t_r-dependent analogs without proof. For simplicity we again confine ourselves to the simplest cases only.

Lemma 4.5. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. Then,*

$$\frac{\alpha}{\alpha^+} = \prod_{j=1}^{2p-1} \mu_j \Bigg(\prod_{m=0}^{4p-1} E_m \Bigg)^{-1/2}, \tag{4.69}$$

$$\frac{\beta^+}{\beta} = \prod_{j=1}^{2p-1} \nu_j \Bigg(\prod_{m=0}^{4p-1} E_m \Bigg)^{-1/2}, \tag{4.70}$$

$$\sum_{j=1}^{2p-1} \mu_j = \alpha^+\beta - \gamma^+ \frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \tag{4.71}$$

$$\sum_{j=1}^{2p-1} \nu_j = \alpha^+\beta - \gamma \frac{\beta^-}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \tag{4.72}$$

Next, we turn to the asymptotic expansions of ϕ and Ψ in a neighborhood of $P_{\infty_\pm}$ and $P_{0,\pm}$.

Lemma 4.6. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. *Then* ϕ *has the asymptotic behavior*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \beta + \beta^-\gamma\zeta + O(\zeta^2), & P \to P_{\infty_+}, \\ -(\alpha^+)^{-1}\zeta^{-1} + (\alpha^+)^{-2}\alpha^{++}\gamma^+ + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{4.73}$$

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \alpha^{-1} - \alpha^{-2}\alpha^-\gamma\zeta + O(\zeta^2), & P \to P_{0,+}, \\ -\beta^+\zeta - \beta^{++}\gamma^+\zeta^2 + O(\zeta^3), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{4.74}$$

The component ψ_1 *of the Baker–Akhiezer vector* Ψ *has the asymptotic behavior*

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) &\underset{\zeta\to 0}{=} \exp\bigg(\pm \frac{i}{2}(t_r - t_{0,r}) \sum_{s=0}^{r} \tilde c_{r-s,+}\zeta^{-s}\bigg)(1 + O(\zeta)) \\ &\times \begin{cases} \zeta^{n_0-n}, & P \to P_{\infty_+}, \\ \Gamma(n, n_0, t_r)\frac{\alpha^+(n,t_r)}{\alpha^+(n_0,t_{0,r})} \\ \times \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\tilde g_{r,+}(n_0, s) - \tilde g_{r,-}(n_0, s)\big)\Big), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \end{aligned} \tag{4.75}$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) &\underset{\zeta\to 0}{=} \exp\bigg(\pm \frac{i}{2}(t_r - t_{0,r}) \sum_{s=0}^{r} \tilde c_{r-s,-}\zeta^{-s}\bigg)(1 + O(\zeta)) \\ &\times \begin{cases} \frac{\alpha(n,t_r)}{\alpha(n_0,t_{0,r})}, & P \to P_{0,+}, \\ \Gamma(n, n_0, t_r)\zeta^{n-n_0} \\ \times \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\tilde g_{r,+}(n_0, s) - \tilde g_{r,-}(n_0, s)\big)\Big), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \end{aligned} \tag{4.76}$$

The divisors (ψ_j) *of* ψ_j, $j = 1, 2$, *are given by*

$$(\psi_1(\,\cdot\,, n, n_0, t_r, t_{0,r})) = \mathcal{D}_{\underline{\hat\mu}(n,t_r)} - \mathcal{D}_{\underline{\hat\mu}(n_0,t_r)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}), \tag{4.77}$$

$$(\psi_2(\,\cdot\,, n, n_0, t_r, t_{0,r})) = \mathcal{D}_{P_{0,-}\underline{\hat\nu}(n,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat\mu}(n_0,t_{0,r})} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}). \tag{4.78}$$

Proof. Since by the definition of ϕ in (4.22) the time parameter t_r can be viewed as an additional but fixed parameter, the asymptotic behavior of ϕ remains the same as in Lemma 3.3. Similarly, also the asymptotic behavior of $\psi_1(P, n, n_0, t_r, t_r)$ is derived in an identical fashion to that in Lemma 3.3. This proves (4.75) and (4.76) for $t_{0,r} = t_r$, that is,

$$\psi_1(P,n,n_0,t_r,t_r) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}(1+O(\zeta)), & P\to P_{\infty_+}, \\ \Gamma(n,n_0,t_r)\frac{\alpha^+(n,t_r)}{\alpha^+(n_0,t_r)} + O(\zeta), & P\to P_{\infty_-}, \end{cases} \qquad \zeta = 1/z, \tag{4.79}$$

$$\psi_1(P,n,n_0,t_r,t_r) \underset{\zeta\to 0}{=} \begin{cases} \frac{\alpha(n,t_r)}{\alpha(n_0,t_r)} + O(\zeta), & P\to P_{0,+}, \\ \Gamma(n,n_0,t_r)\zeta^{n-n_0}(1+O(\zeta)), & P\to P_{0,-}, \end{cases} \qquad \zeta = z. \tag{4.80}$$

It remains to investigate

$$\psi_1(P,n_0,n_0,t_r,t_{0,r}) = \exp\bigg(i\int_{t_{0,r}}^{t_r} dt\big(\widetilde G_r(z,n_0,t) - \widetilde F_r(z,n_0,t)\phi(P,n_0,t)\big)\bigg). \tag{4.81}$$

The asymptotic expansion of the integrand is derived using Theorem A.2. Focusing on the homogeneous coefficients first, one computes as $P\to P_{\infty_\pm}$,

$$\begin{aligned}
\widehat G_s - \widehat F_s\phi &= \widehat G_s - \widehat F_s \frac{G_p + (c_{0,+}/2)z^{-p}y}{F_p} = \widehat G_s - \widehat F_s\bigg(\frac{2z^p}{c_{0,+}}\frac{G_p}{y} + 1\bigg)\bigg(\frac{2z^p}{c_{0,+}}\frac{F_p}{y}\bigg)^{-1} \\
&\underset{\zeta\to 0}{=} \zeta^{-s}\bigg(\sum_{q=0}^{s}\hat g_{q,+}\zeta^q - \bigg(\sum_{\ell=0}^{\infty}\hat g_{\ell,+}\zeta^\ell \mp \frac12\bigg)\bigg(\zeta\sum_{k=0}^{\infty}\hat f_{k,+}\zeta^k\bigg)^{-1} \\
&\qquad \times\zeta\bigg(\sum_{q=0}^{s-1}\hat f_{q,+}\zeta^q + \hat f_{s-1,-}\zeta^s\bigg) + O(\zeta^{s+1})\bigg) \\
&\underset{\zeta\to 0}{=} \zeta^{-s}\bigg(\sum_{\ell=0}^{\infty}\hat g_{\ell,+}\zeta^\ell - \bigg(\sum_{\ell=0}^{\infty}\hat g_{\ell,+}\zeta^\ell \mp \frac12\bigg)\bigg(\sum_{k=0}^{\infty}\hat f_{k,+}\zeta^k\bigg)^{-1} \\
&\qquad \times\bigg(\sum_{m=0}^{\infty}\hat f_{m,+}\zeta^m - (\hat f_{s,+} - \hat f_{s-1,-})\zeta^s\bigg) + O(\zeta^{s+1})\bigg) \\
&\underset{\zeta\to 0}{=} \pm\frac12\zeta^{-s} + \frac{\hat g_{0,+}\mp\frac12}{\hat f_{0,+}}(\hat f_{s,+} - \hat f_{s-1,-}) + O(\zeta), \quad P\to P_{\infty_\pm},\ \zeta = 1/z.
\end{aligned} \tag{4.82}$$

Since

$$\widetilde F_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r}\tilde c_{r-s,+}\widehat F_s + O(\zeta), \quad P\to P_{\infty_\pm},\ \zeta = 1/z, \tag{4.83}$$

$$\widetilde G_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r}\tilde c_{r-s,+}\widehat G_s + O(\zeta), \quad P\to P_{\infty_\pm},\ \zeta = 1/z, \tag{4.84}$$

one infers from (4.73)

$$\widetilde G_r - \widetilde F_r\phi \underset{\zeta\to 0}{=} \frac12\sum_{s=0}^{r}\tilde c_{r-s,+}\zeta^{-s} + O(\zeta), \quad P\to P_{\infty_+},\ \zeta = 1/z. \tag{4.85}$$

Insertion of (4.85) into (4.81) then proves (4.75) as $P\to P_{\infty_+}$.

As $P\to P_{\infty_-}$, we need one additional term in the asymptotic expansion of $\widetilde F_r$, that is, we will use

$$\widetilde F_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r}\tilde c_{r-s,+}\widehat F_s - \sum_{s=0}^{r}(\tilde c_{r-s,+} - \tilde c_{r-s,-})\hat f_{s-1,-}\zeta + O(\zeta^2). \tag{4.86}$$

This then yields

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\zeta^{-s} - (\alpha^+)^{-1}(\tilde{f}_{r,+} - \tilde{f}_{r-1,-}) + O(\zeta), \quad P \to P_{\infty_-},\ \zeta = 1/z. \tag{4.87}$$

Invoking (2.20) and (4.2) one concludes that

$$\begin{aligned} \tilde{f}_{r-1,-} - \tilde{f}_{r,+} &= \tilde{f}_{r-1,-} - \tilde{f}^+_{r-1,+} + \alpha^+(\tilde{g}^+_{r,+} + \tilde{g}_{r,+}) \\ &= \tilde{f}_{r-1,-} - \tilde{f}^+_{r-1,+} + \alpha^+(\tilde{g}^+_{r,+} + \tilde{g}_{r,-}) - \alpha^+(\tilde{g}_{r,-} - \tilde{g}_{r,+}) \\ &= -i\alpha^+_{t_r} + \alpha^+(\tilde{g}_{r,+} - \tilde{g}_{r,-}) \end{aligned} \tag{4.88}$$

and hence

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\zeta^{-s} - \frac{i\alpha^+_{t_r}}{\alpha^+} + \tilde{g}_{r,+} - \tilde{g}_{r,-} + O(\zeta), \quad P \to P_{\infty_-},\ \zeta = 1/z. \tag{4.89}$$

Insertion of (4.89) into (4.81) then proves (4.75) as $P \to P_{\infty_-}$.

Again using Theorem A.2 one obtains in the same manner as $P \to P_{0,\pm}$,

$$\widehat{G}_s - \widehat{F}_s\phi \underset{\zeta\to 0}{=} \pm\frac{1}{2}\zeta^{-s} + \hat{g}_{s,+} - \hat{g}_{s,-} - \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}}\big(\hat{f}_{s-1,+} - \hat{f}_{s,-}\big) + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z. \tag{4.90}$$

Since

$$\widetilde{F}_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r}\tilde{c}_{r-s,-}\big(\widehat{F}_s - \hat{f}_{s-1,+}\big) + \tilde{f}_{r-1,+} + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.91}$$

$$\widetilde{G}_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r}\tilde{c}_{r-s,-}\big(\widehat{G}_s - \hat{g}_{s,+}\big) + \tilde{g}_{r,+} + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.92}$$

(4.90)–(4.92) yield

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} \pm\frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,-}\zeta^{-s} + \tilde{g}_{r,+} - \tilde{g}_{r,-} - \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}}(\tilde{f}_{r-1,+} - \tilde{f}_{r,-}) + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.93}$$

where we again used (4.74), (2.38), and (4.2). As $P \to P_{0,-}$, one thus obtains

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,-}\zeta^{-s} + \tilde{g}_{r,+} - \tilde{g}_{r,-}, \quad P \to P_{0,-},\ \zeta = z. \tag{4.94}$$

Insertion of (4.94) into (4.81) then proves (4.76) as $P \to P_{0,-}$.

As $P \to P_{0,+}$, one obtains

$$\begin{aligned} \widetilde{G}_r - \widetilde{F}_r\phi &\underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,-}\zeta^{-s} + \tilde{g}_{r,+} - \tilde{g}_{r,-} - \frac{1}{\alpha}(\tilde{f}_{r-1,+} - \tilde{f}_{r,-}) + O(\zeta), \\ &\underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,-}\zeta^{-s} - \frac{i\alpha_{t_r}}{\alpha} + O(\zeta), \quad P \to P_{0,+},\ \zeta = z, \end{aligned} \tag{4.95}$$

using $\tilde{f}_{r,-} = \tilde{f}^-_{r-1,-} + \alpha(\tilde{g}_{r,-} - \tilde{g}^-_{r,-})$ (cf. (2.24)) and (4.2). Insertion of (4.95) into (4.81) then proves (4.76) as $P \to P_{0,+}$. $\square$

Finally, we note that Lemma 3.4 on nonspecial divisors in the stationary context extends to the present time-dependent situation without a change. Indeed, since $t_r \in \mathbb{R}$ just plays the role of a parameter, the proof of Lemma 3.4 extends line by line and is hence omitted.

Lemma 4.7. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. Let* $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_{2p-1}\}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\underline{\hat{\nu}}_1, \dots, \underline{\hat{\nu}}_{2p-1}\}$, *be the pole and zero divisors of degree* $2p-1$, *respectively, associated with* α, β, *and* ϕ *defined according to* (4.20) *and* (4.21), *that is,*

$$\begin{aligned}
\hat{\mu}_j(n,t_r) &= (\mu_j(n,t_r), (2/c_{0,+})\mu_j(n,t_r)^p G_p(\mu_j(n,t_r),n,t_r)), \\
&\qquad\qquad j = 1,\dots,2p-1,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R}, \qquad (4.96)\\
\hat{\nu}_j(n,t_r) &= (\nu_j(n,t_r), -(2/c_{0,+})\nu_j(n,t_r)^p G_p(\nu_j(n,t_r),n,t_r)), \\
&\qquad\qquad j = 1,\dots,2p-1,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R}. \qquad (4.97)
\end{aligned}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}$ *are nonspecial for all* $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$.

Finally, we note that

$$\begin{aligned}
&\Gamma(n,n_0,t_r) = \Gamma(n,n_0,t_{0,r}) \\
&\quad \times \exp\bigg(i\int_{t_{0,r}}^{t_r} ds \big(\tilde{g}_{r,+}(n,s) - \tilde{g}_{r,+}(n_0,s) - \tilde{g}_{r,-}(n,s) + \tilde{g}_{r,-}(n_0,s)\big)\bigg),
\end{aligned} \tag{4.98}$$

which follows from (2.84), (3.31), and from

$$\begin{aligned}
\Gamma(n,n_0,t_r)_{t_r} &= \sum_{j=n_0+1}^{n} \gamma(j,t_r)_{t_r} \prod_{\substack{k=n_0+1\\ k\neq j}}^{n} \gamma(j,t_r) \qquad (4.99)\\
&= i\big(\tilde{g}_{r,+}(n,t_r) - \tilde{g}_{r,+}(n_0,t_r) - \tilde{g}_{r,-}(n,t_r) + \tilde{g}_{r,-}(n_0,t_r)\big)\Gamma(n,n_0,t_r)
\end{aligned}$$

after integration with respect to t_r.

The results of Sections 2–4 have been used extensively in [34] to derive the class of algebro-geometric solutions of the Ablowitz–Ladik hierarchy.

Appendix A. Asymptotic Spectral Parameter Expansions and Nonlinear Recursion Relations

In this appendix we consider asymptotic spectral parameter expansions of F_p/y, G_p/y, and H_p/y in a neighborhood of $P_{\infty\pm}$ and $P_{0,\pm}$, the resulting recursion relations for the homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$, their connection with the nonhomogeneous coefficients f_ℓ, g_ℓ, and h_ℓ, and the connection between $c_{\ell,\pm}$ and $c_\ell(\underline{E}^{\pm 1})$. We will employ the notation

$$\underline{E}^{\pm 1} = \big(E_0^{\pm 1}, \dots, E_{4p-1}^{\pm 1}\big). \tag{A.1}$$

We start with the following elementary results (consequences of the binomial expansion) assuming $\eta \in \mathbb{C}$ such that $|\eta| < \min\{|E_0|^{-1}, \dots, |E_{4p-1}|^{-1}\}$:

$$\bigg(\prod_{m=0}^{4p-1} \big(1 - E_m\eta\big)\bigg)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{E})\eta^k, \tag{A.2}$$

where

$$\hat{c}_0(\underline{E}) = 1,$$

$$\hat{c}_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{4p-1}=0\\ j_0+\cdots+j_{4p-1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{4p-1})!}{2^{2k}(j_0!)^2\cdots(j_{4p-1}!)^2} E_0^{j_0}\cdots E_{4p-1}^{j_{4p-1}}, \quad k\in\mathbb{N}. \tag{A.3}$$

The first few coefficients explicitly read

$$\hat{c}_0(\underline{E}) = 1,\ \hat{c}_1(\underline{E}) = \frac{1}{2}\sum_{m=0}^{4p-1} E_m,\ \hat{c}_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{4p-1} E_{m_1}E_{m_2} + \frac{3}{8}\sum_{m=0}^{4p-1} E_m^2, \quad \text{etc.} \tag{A.4}$$

Similarly,

$$\bigg(\prod_{m=0}^{4p-1}\big(1-E_m\eta\big)\bigg)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E})\eta^k, \tag{A.5}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1,\\ &c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{4p-1}=0\\ j_0+\cdots+j_{4p-1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{4p-1})!\, E_0^{j_0}\cdots E_{4p-1}^{j_{4p-1}}}{2^{2k}(j_0!)^2\cdots(j_{4p-1}!)^2(2j_0-1)\cdots(2j_{4p-1}-1)}, \quad k\in\mathbb{N}. \end{aligned} \tag{A.6}$$

The first few coefficients explicitly are given by

$$c_0(\underline{E}) = 1,\ c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{4p-1} E_m,\ c_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{4p-1} E_{m_1}E_{m_2} - \frac{1}{8}\sum_{m=0}^{4p-1} E_m^2, \quad \text{etc.} \tag{A.7}$$

Multiplying (A.2) and (A.5) and comparing coefficients we find

$$\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(\underline{E})c_\ell(\underline{E}) = \delta_{k,0}, \quad k\in\mathbb{N}_0. \tag{A.8}$$

Lemma A.1. *The homogeneous quantities $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, $\hat{h}_{\ell,\pm}$ are uniquely defined by the following recursions:*

$$\begin{aligned} &\hat{g}_{0,+} = \frac{1}{2}, \quad \hat{f}_{0,+} = -\alpha^+, \quad \hat{h}_{0,+} = \beta,\\ &\hat{g}_{l+1,+} = \sum_{k=0}^{l} \hat{f}_{l-k,+}\hat{h}_{k,+} - \sum_{k=1}^{l} \hat{g}_{l+1-k,+}\hat{g}_{k,+},\\ &\hat{f}^-_{l+1,+} = \hat{f}_{l,+} - \alpha(\hat{g}_{l+1,+} + \hat{g}^-_{l+1,+}),\\ &\hat{h}_{l+1,+} = \hat{h}^-_{l,+} + \beta(\hat{g}_{l+1,+} + \hat{g}^-_{l+1,+}), \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} &\hat{g}_{0,-} = \frac{1}{2}, \quad \hat{f}_{0,-} = \alpha, \quad \hat{h}_{0,-} = -\beta^+,\\ &\hat{g}_{l+1,-} = \sum_{k=0}^{l} \hat{f}_{l-k,-}\hat{h}_{k,-} - \sum_{k=1}^{l} \hat{g}_{l+1-k,-}\hat{g}_{k,-},\\ &\hat{f}_{l+1,-} = \hat{f}^-_{l,-} + \alpha(\hat{g}_{l+1,-} + \hat{g}^-_{l+1,-}),\\ &\hat{h}^-_{l+1,-} = \hat{h}_{l,-} - \beta(\hat{g}_{l+1,-} + \hat{g}^-_{l+1,-}). \end{aligned} \tag{A.10}$$

Proof. One verifies that the quantities defined via these recursions satisfy (2.18)–(2.21) (respectively, (2.22)–(2.25)). Since they are homogeneous of the required degree this completes the proof. $\square$

Next we turn to asymptotic expansions. We recall the convention $y(P) = \mp\zeta^{-2p} + O(\zeta^{-2p+1})$ near $P_{\infty_\pm}$ (where $\zeta = 1/z$) and $y(P) = \pm(c_{0,-}/c_{0,+}) + O(\zeta)$ near $P_{0,\pm}$ (where $\zeta = z$).

Theorem A.2. *Assume* (3.1), s-AL$_p(\alpha,\beta) = 0$, *and suppose* $P = (z, y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then* $z^p F_p/y$, $z^p G_p/y$, *and* $z^p H_p/y$ *have the following convergent expansions as* $P \to P_{\infty_\pm}$, *respectively,* $P \to P_{0,\pm}$,

$$\frac{2z^p}{c_{0,+}}\frac{F_p(z)}{y} = \begin{cases} \mp 2\sum_{\ell=0}^\infty \hat f_{\ell,+}\zeta^{\ell+1}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm 2\sum_{\ell=0}^\infty \hat f_{\ell,-}\zeta^{\ell}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{A.11}$$

$$\frac{2z^p}{c_{0,+}}\frac{G_p(z)}{y} = \begin{cases} \mp 2\sum_{\ell=0}^\infty \hat g_{\ell,+}\zeta^{\ell}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm 2\sum_{\ell=0}^\infty \hat g_{\ell,-}\zeta^{\ell}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{A.12}$$

$$\frac{2z^p}{c_{0,+}}\frac{H_p(z)}{y} = \begin{cases} \mp 2\sum_{\ell=0}^\infty \hat h_{\ell,+}\zeta^{\ell}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm 2\sum_{\ell=0}^\infty \hat h_{\ell,-}\zeta^{\ell+1}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{A.13}$$

where $\zeta = 1/z$ *(resp.,* $\zeta = z$*) is the local coordinate near* $P_{\infty_\pm}$ *(resp.,* $P_{0,\pm}$*) and* $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, *and* $\hat h_{\ell,\pm}$ *are the homogeneous coefficients introduced in* (2.35)–(2.37). *In particular,* $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, *and* $\hat h_{\ell,\pm}$ *can be computed from the following nonlinear recursion relations:*[2]

$$\begin{aligned}
&\hat f_{0,+} = -\alpha^+, \quad \hat f_{1,+} = (\alpha^+)^2\beta - \gamma^+\alpha^{++}, \\
&\hat f_{2,+} = -(\alpha^+)^3\beta^2 + (\alpha^+)^2\beta^+ - \alpha^{+++}\gamma^+\gamma^{++} \\
&\qquad + (\alpha^+)^2\big((\beta^-\gamma - 2\alpha^{++}\beta\beta^+) + \alpha^+\alpha^{++}(2\beta - \alpha^{++}(\beta^+)^2\big), \\
&\alpha^4\alpha^+\hat f_{\ell,+} = \frac{1}{2}\bigg((\alpha^+)^4\sum_{m=0}^{\ell-4}\hat f_{m,+}\hat f_{\ell-m-4,+} + \alpha^4\sum_{m=1}^{\ell-1}\hat f_{m,+}\hat f_{\ell-m,+} \\
&\quad - 2(\alpha^+)^2\sum_{m=0}^{\ell-3}\hat f_{m,+}\big(-2\alpha\alpha^+\hat f_{\ell-m-3,+} + (\alpha^+)^2\gamma\hat f^-_{\ell-m-3,+}m + \alpha^2\gamma^+\hat f^+_{\ell-m-3,+}\big) \\
&\quad + \sum_{m=0}^{\ell-2}\big(\alpha^4(\gamma^+)^2\hat f^+_{m,+}\hat f^+_{\ell-m-2,+} + (\alpha^+)^2\gamma\hat f^-_{m,+}((\alpha^+)^2\gamma\hat f^-_{\ell-m-2,+} \\
&\qquad\qquad - 2\alpha^2\gamma^+\hat f^+_{\ell-m-2,+}) \\
&\qquad - 2\alpha\alpha^+\hat f_{m,+}(-3\alpha\alpha^+\hat f_{\ell-m-2,+} + 2(\alpha^+)^2\gamma\hat f^-_{\ell-m-2,+} + 2\alpha^2\gamma^+\hat f^+_{\ell-m-2,+})\big) \\
&\quad - 2\alpha^2\sum_{m=0}^{\ell-1}\hat f_{m,+}\big(-2\alpha\alpha^+\hat f_{\ell-m-1,+} + (\alpha^+)^2\gamma\hat f^-_{\ell-m-1,+}m + \alpha^2\gamma^+\hat f^+_{\ell-m-1,+}\big)\bigg), \\
&\hspace{25em} \ell \geq 3,
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
&\hat f_{0,-} = -\alpha, \quad \hat f_{1,-} = \gamma\alpha^- - \alpha^2\beta^+, \\
&\hat f_{2,-} = \alpha^{--}\gamma^-\gamma - (\alpha^-)^2\beta\gamma - 2\alpha^-\alpha\gamma\beta^+ + \alpha^2\big(\alpha(\beta^+)^2 - \gamma^+\big)\beta^{++},
\end{aligned}$$

[2] Here sums over empty index sets are as usual interpreted as zero.

$$\begin{aligned}
\alpha(\alpha^+)^4\hat f_{\ell,-} &= -\frac{1}{2}\bigg(\alpha^4\sum_{m=0}^{\ell-4}\hat f_{m,-}\hat f_{\ell-m-4,-} + (\alpha^+)^4\sum_{m=1}^{\ell-1}\hat f_{m,-}\hat f_{\ell-m,-}\\
&\quad - 2\alpha^2\sum_{m=0}^{\ell-3}\hat f_{m,-}\big(-2\alpha\alpha^+\hat f_{\ell-m-3,-} + (\alpha^+)^2\gamma\hat f^-_{\ell-m-3,-} + \alpha^2\gamma^+\hat f^+_{\ell-m-3,-}\big)\\
&\quad + \sum_{m=0}^{\ell-2}\big(\alpha^4(\gamma^+)^2\hat f^+_{m,-}\hat f^+_{\ell-m-2,-}\\
&\qquad + (\alpha^+)^2\gamma\hat f^-_{m,-}((\alpha^+)^2\gamma\hat f^-_{\ell-m-2,-} - 2\alpha^2\gamma^+\hat f^+_{\ell-m-2,-})\\
&\qquad - 2\alpha\alpha^+\hat f_{m,-}(-3\alpha\alpha^+\hat f_{\ell-m-2,-} + 2(\alpha^+)^2\gamma\hat f^-_{\ell-m-2,-} + 2\alpha^2\gamma^+\hat f^+_{\ell-m-2,-})\big)\\
&\quad - 2(\alpha^+)^2\sum_{m=0}^{\ell-1}\hat f_{m,-}\big(-2\alpha\alpha^+\hat f_{\ell-m-1,-} + (\alpha^+)^2\gamma\hat f^-_{\ell-m-1,-} + \alpha^2\gamma^+\hat f^+_{\ell-m-1,-}\big)\bigg),\\
&\hspace{20em} \ell\geq 3, \quad \text{(A.15)}
\end{aligned}$$

$$\begin{aligned}
\hat g_{0,+} &= \frac{1}{2}, \quad \hat g_{1,+} = -\alpha^+\beta,\\
\hat g_{2,+} &= -\alpha^{++}\beta + (\alpha^+)^2\beta^2 + \alpha^+\big(-\beta^-\gamma + \alpha^{++}\beta\beta^+\big),\\
(\alpha\beta^+)^2\hat g_{\ell,+} &= -\bigg((\alpha^+)^2\beta^2\sum_{m=0}^{\ell-4}\hat g_{m,+}\hat g_{\ell-m-4,+} + \alpha^2(\beta^+)^2\sum_{m=1}^{\ell-1}\hat g_{m,+}\hat g_{\ell-m,+}\\
&\quad + \alpha^+\beta\sum_{m=0}^{\ell-3}\big(\gamma\gamma^+\hat g^-_{m,+}\hat g^+_{\ell-m-3,+} + \hat g_{m,+}((1+\alpha\beta)(1+\alpha^+\beta^+)\hat g_{\ell-m-3,+}\\
&\qquad - (\gamma+\alpha^+\beta^+\gamma)\hat g^-_{\ell-m-3,+} + (-2+\gamma)\gamma^+\hat g^+_{\ell-m-3,+})\big)\\
&\quad + \sum_{m=0}^{\ell-2}\big(\alpha^+\beta^+\gamma^2\hat g^-_{m,+}\hat g^-_{\ell-m-2,+} + \alpha\beta(\gamma^+)^2\hat g^+_{m,+}\hat g^+_{\ell-m-2,+}\\
&\qquad + \hat g_{m,+}((\alpha^+\beta^+ + \alpha^2\alpha^+\beta^2\beta^+ + \alpha\beta(1+\alpha^+\beta^+)^2)\hat g_{\ell-m-2,+}\\
&\qquad - 2(\alpha^+(1+\alpha\beta)\beta^+\gamma\hat g^-_{\ell-m-2,+} + \alpha\beta(1+\alpha^+\beta^+)\gamma^+\hat g^+_{\ell-m-2,+})\big)\\
&\quad + \alpha\beta^+\sum_{m=0}^{\ell-1}\big(\gamma\gamma^+\hat g^-_{m,+}\hat g^+_{\ell-m-1,+} + \hat g_{m,+}((1+\alpha\beta)(1+\alpha^+\beta^+)\hat g_{\ell-m-1,+}\\
&\qquad - (\gamma+\alpha^+\beta^+\gamma)\hat g^-_{\ell-m-1,+} + (-2+\gamma)\gamma^+\hat g^+_{\ell-m-1,+})\big)\bigg), \quad \ell\geq 3, \quad \text{(A.16)}
\end{aligned}$$

$$\begin{aligned}
\hat g_{0,-} &= \frac{1}{2}, \quad \hat g_{1,-} = -\alpha\beta^+,\\
\hat g_{2,-} &= -\alpha^-\gamma\beta^+ + \alpha\big(\alpha(\beta^+)^2 - \gamma^+\big)\beta^{++},\\
(\alpha^+)^2\beta^2\hat g_{\ell,-} &= -\bigg(\alpha^2(\beta^+)^2\sum_{m=0}^{\ell-4}\hat g_{m,-}\hat g_{\ell-m-4,-} + (\alpha^+)^2\beta^2\sum_{m=1}^{\ell-1}\hat g_{m,-}\hat g_{\ell-m,-}\\
&\quad + \alpha\beta^+\sum_{m=0}^{\ell-3}\big(\gamma\gamma^+\hat g^-_{m,-}\hat g^+_{\ell-m-3,-} + \hat g_{m,-}((1+\alpha\beta)(1+\alpha^+\beta^+)\hat g_{\ell-m-3,-}\\
&\qquad - (\gamma+\alpha^+\beta^+\gamma)\hat g^-_{\ell-m-3,-} + (-2+\gamma)\gamma^+\hat g^+_{\ell-m-3,-})\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=0}^{\ell-2}\big(\alpha^+\beta^+\gamma^2\hat{g}^-_{m,-}\hat{g}^-_{\ell-m-2,-}+\alpha\beta(\gamma^+)^2\hat{g}^+_{m,-}\hat{g}^+_{\ell-m-2,-}\\
&\qquad+\hat{g}_{m,-}((\alpha^+\beta^++\alpha^2\alpha^+\beta^2\beta^++\alpha\beta(1+\alpha^+\beta^+)^2)\hat{g}_{\ell-m-2,-}\\
&\qquad-2(\alpha^+(1+\alpha\beta)\beta^+\gamma\hat{g}^-_{\ell-m-2,-}+\alpha\beta(1+\alpha^+\beta^+)\gamma^+\hat{g}^+_{\ell-m-2,-}))\big)\\
&+\alpha^+\beta\sum_{m=0}^{\ell-1}\big(\gamma\gamma^+\hat{g}^-_{m,-}\hat{g}^+_{\ell-m-1,-}+\hat{g}_{m,-}((1+\alpha\beta)(1+\alpha^+\beta^+)\hat{g}_{\ell-m-1,-}\\
&\qquad-(\gamma+\alpha^+\beta^+\gamma)\hat{g}^-_{\ell-m-1,-}+(-2+\gamma)\gamma^+\hat{g}^+_{\ell-m-1,-})\big)\bigg),\quad \ell\geq 3, \quad\text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
&\hat{h}_{0,+}=\beta,\quad \hat{h}_{1,+}=\gamma\beta^--\alpha^+\beta^2,\\
&\hat{h}_{2,+}=\alpha^2(\beta^-)^2\beta+\beta^{--}\gamma^-\gamma-\alpha\beta^-(\beta^--2\alpha^+\beta^2)\\
&\qquad+\beta\big(-\alpha^{++}\beta+(\alpha^+)^2\beta^2+\alpha^+(-2\beta^-+\alpha^{++}\beta\beta^+)\big),\\
&\beta(\beta^+)^4\hat{h}_{\ell,+}=-\frac{1}{2}\bigg(\beta^4\sum_{m=0}^{\ell-4}\hat{h}_{m,+}\hat{h}_{\ell-m-4,+}+(\beta^+)^4\sum_{m=1}^{\ell-1}\hat{h}_{m,+}\hat{h}_{\ell-m,+}\\
&\quad-2\beta^2\sum_{m=0}^{\ell-3}\hat{h}_{m,+}\big(-2\beta\beta^+\hat{h}_{\ell-m-3,+}+(\beta^+)^2\gamma\hat{h}^-_{\ell-m-3,+}+\beta^2\gamma^+\hat{h}^+_{\ell-m-3,+}\big)\\
&\quad+\sum_{m=0}^{\ell-2}\big(\beta^4(\gamma^+)^2\hat{h}^+_{m,+}\hat{h}^+_{\ell-m-2,+}\\
&\qquad+(\beta^+)^2\gamma\hat{h}^-_{m,+}((\beta^+)^2\gamma\hat{h}^-_{\ell-m-2,+}-2\beta^2\gamma^+\hat{h}^+_{\ell-m-2,+})\\
&\quad-2\beta\beta^+\hat{h}_{m,+}(-3\beta\beta^+\hat{h}_{\ell-m-2,+}+2(\beta^+)^2\gamma\hat{h}^-_{\ell-m-2,+}+2\beta^2\gamma^+\hat{h}^+_{\ell-m-2,+})\big)\\
&\quad-2(\beta^+)^2\sum_{m=0}^{\ell-1}\hat{h}_{m,+}\big(-2\beta\beta^+\hat{h}_{\ell-m-1,+}+(\beta^+)^2\gamma\hat{h}^-_{\ell-m-1,+}+\beta^2\gamma^+\hat{h}^+_{\ell-m-1,+}\big)\bigg),\\
&\hspace{20em}\ell\geq 3,\quad\text{(A.18)}
\end{aligned}
$$

$$
\begin{aligned}
&\hat{h}_{0,-}=-\beta^+,\quad \hat{h}_{1,-}=-\gamma^+\beta^{++}+\alpha(\beta^+)^2,\\
&\hat{h}_{2,-}=\alpha^-\gamma(\beta^+)^2-\alpha^2(\beta^+)^3+2\alpha\beta^+\gamma^+\beta^{++}+\gamma^+\big(\alpha^+(\beta^{++})^2-\gamma^{++}\big)\beta^{+++},\\
&\beta^+\beta^4\hat{h}_{\ell,-}=\frac{1}{2}\bigg((\beta^+)^4\sum_{m=0}^{\ell-4}\hat{h}_{m,-}\hat{h}_{\ell-m-4,-}+\beta^4\sum_{m=1}^{\ell-1}\hat{h}_{m,-}\hat{h}_{\ell-m,-}\\
&\quad-2(\beta^+)^2\sum_{m=0}^{\ell-3}\hat{h}_{m,-}\big(-2\beta\beta^+\hat{h}_{\ell-m-3,-}+(\beta^+)^2\gamma\hat{h}^-_{\ell-m-3,-}+\beta^2\gamma^+\ \hat{h}^+_{\ell-m-3,-}\big)\\
&\quad+\sum_{m=0}^{\ell-2}\big(\beta^4(\gamma^+)^2\hat{h}^+_{m,-}\hat{h}^+_{\ell-m-2,-}\\
&\qquad+(\beta^+)^2\gamma\hat{h}^-_{m,-}((\beta^+)^2\gamma\hat{h}^-_{\ell-m-2,-}-2\beta^2\gamma^+\hat{h}^+_{\ell-m-2,-})\\
&\quad-2\beta\beta^+\hat{h}_{m,-}(-3\beta\beta^+\ \hat{h}_{\ell-m-2,-}+2(\beta^+)^2\gamma\hat{h}^-_{\ell-m-2,-}+2\beta^2\gamma^+\hat{h}^+_{\ell-m-2,-})\big)\\
&\quad-2\beta^2\sum_{m=0}^{\ell-1}\hat{h}_{m,-}\big(-2\beta\beta^+\hat{h}_{\ell-m-1,-}+(\beta^+)^2\gamma\hat{h}^-_{\ell-m-1,-}+\beta^2\gamma^+\hat{h}^+_{\ell-m-1,-}\big)\bigg),
\end{aligned}
$$

$$\ell \geq 3. \quad \text{(A.19)}$$

Proof. Consider first the case (A.11) when $P \to P_{\infty_\pm}$. By introducing $\zeta = 1/z$ and using (A.2), (2.14) we find the convergent expansion

$$\begin{aligned}
\frac{F_p(z)}{R_{2p}(z)^{1/2}} &\underset{P\to P_{\infty_\pm}}{=} \frac{2z^p}{c_{0,+}}\frac{F_p(z)}{y} \\
&\underset{\zeta\to 0}{=} \mp\frac{2\zeta^{-p}}{c_{0,+}\zeta^{-2p}}\bigg(\sum_{k=0}^{\infty}\hat{c}_k(\underline{E})\zeta^k\bigg)\bigg(\sum_{\ell=1}^{p} f_{p-\ell,-}\zeta^{\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}\zeta^{-\ell}\bigg) \\
&\underset{\zeta\to 0}{=} \mp\frac{2}{c_{0,+}}\bigg(\sum_{k=0}^{\infty}\hat{c}_k(\underline{E})\zeta^k\bigg)\bigg(\sum_{\ell=1}^{p} f_{p-\ell,-}\zeta^{p+\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}\zeta^{p-\ell}\bigg)
\end{aligned} \quad \text{(A.20)}$$

$$\underset{\zeta\to 0}{=} \mp 2\sum_{\ell=0}^{\infty} \check{f}_{\ell,+}\zeta^{\ell+1} \quad \text{(A.21)}$$

for some coefficients $\check{f}_{\ell,+}$ to be determined next. From the leading asymptotic behavior one finds that $\check{f}_{0,+} = -\alpha^+$. Similarly, one expands the corresponding quantities for G_p and H_p to obtain

$$\begin{aligned}
\frac{G_p(z)}{R_{2p}(z)^{1/2}} &\underset{P\to P_{\infty_\pm}}{=} \frac{2z^p}{c_{0,+}}\frac{G_p(z)}{y} \\
&\underset{\zeta\to 0}{=} \mp\frac{2}{c_{0,+}}\bigg(\sum_{k=0}^{\infty}\hat{c}_k(\underline{E})\zeta^k\bigg)\bigg(\sum_{\ell=1}^{p} g_{p-\ell,-}\zeta^{p+\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+}\zeta^{p-\ell}\bigg) \\
&\underset{\zeta\to 0}{=} \mp 2\sum_{\ell=0}^{\infty} \check{g}_{\ell,+}\zeta^{\ell},
\end{aligned} \quad \text{(A.22)}$$

using (2.15), and

$$\begin{aligned}
\frac{H_p(z)}{R_{2p}(z)^{1/2}} &\underset{P\to P_{\infty_\pm}}{=} \frac{2z^p}{c_{0,+}}\frac{H_p(z)}{y} \\
&\underset{\zeta\to 0}{=} \mp\frac{2}{c_{0,+}}\bigg(\sum_{k=0}^{\infty}\hat{c}_k(\underline{E})\zeta^k\bigg)\bigg(\sum_{\ell=0}^{p-1} h_{p-1-\ell,-}\zeta^{p+\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+}\zeta^{p-\ell}\bigg) \\
&\underset{\zeta\to 0}{=} \mp 2\sum_{\ell=0}^{\infty} \check{h}_{\ell,+}\zeta^{\ell},
\end{aligned} \quad \text{(A.23)}$$

using (2.16). The coefficients $\check{g}_{\ell,+}, \check{h}_{\ell,+}$ will be determined subsequently. The leading asymptotic behavior yields $\check{g}_{0,+} = \frac{1}{2}$ and $\check{h}_{0,+} = -\beta^+$. Dividing (2.10) by $R_{2p}(z)^{1/2}$ and using the expansions (A.21)–(A.23) one finds that $\check{f}_{\ell,+}, \check{g}_{\ell,+}$, and $\check{h}_{\ell,+}$ satisfy the recursion relations (2.19) with $f_{\ell,+}$ replaced by $\check{f}_{\ell,+}$, $g_{\ell,+}$ replaced by $\check{g}_{\ell,+}$, $h_{\ell,+}$ replaced by $\check{h}_{\ell,+}$, and $\check{g}_{0,+}$ equal to a constant. In the same way one obtains the recursion formulas (2.20) and (2.21) by dividing (2.12) and (2.13), respectively, by $R_{2p}(z)^{1/2}$, using the expansions (A.21)–(A.23). Here we have used (2.61). The coefficients $\check{f}_{0,+}$, $\check{g}_{0,+}$, and $\check{h}_{0,+}$ are consistent with (2.18) for $c_{0,+} = 1$. To conclude that actually,

$$\check{f}_{\ell,+} = \hat{f}_{\ell,+}, \quad \check{g}_{\ell,+} = \hat{g}_{\ell,+}, \quad \check{h}_{\ell,+} = \hat{h}_{\ell,+}, \quad \ell \in \mathbb{N}_0, \quad \text{(A.24)}$$

it remains to show that $c_{\ell,+} = 0$, $\ell \in \mathbb{N}$. To that end we first divide (2.69) by $R_{2p}(z)^{1/2}$ and use the expansion (A.21). One then finds the nonlinear recursion relation (A.14) with $\hat{f}_{\ell,+}$ replaced by $\check{f}_{\ell,+}$. Similarly, dividing both (2.70) and (2.71) by $R_{2p}(z)^{1/2}$ and using the expansions (A.22) and (A.23), respectively, one finds the following nonlinear recursions (A.16) and (A.18) with $\hat{g}_{\ell,+}$ replaced by $\check{g}_{\ell,+}$ (resp., $\hat{h}_{\ell,+}$ replaced by $\check{h}_{\ell,+}$).

At this point we recall the notion of degree (cf. Remark 2.6). By induction one verifies that

$$\deg\big(\check{f}^{(r)}_{\ell,+}\big) = \ell + r + 1, \quad \deg\big(\hat{g}^{(r)}_{\ell,+}\big) = \ell, \quad \deg\big(\check{h}^{(r)}_{\ell,+}\big) = \ell - r, \quad \ell \in \mathbb{N}_0,\ r \in \mathbb{Z} \tag{A.25}$$

(cf. (2.41)). This proves (A.24).

Next we consider the expansions as $P \to P_{0,\pm}$, following the same strategy as in the case $P \to P_{\infty_\pm}$. Thus, one deduces

$$\begin{aligned}
\frac{F_p(z)}{R_{2p}(z)^{1/2}} &\underset{P \to P_{0,\pm}}{=} \frac{2z^p}{c_{0,+}} \frac{F_p(z)}{y} \\
&\underset{\zeta \to 0}{=} \frac{2\zeta^p}{c_{0,+}\big(\prod_{m=0}^{4p-1} E_m\big)^{1/2}} \bigg(\prod_{m=0}^{4p-1}(1-\zeta E_m^{-1})\bigg)^{-1/2} F_p(\zeta) \\
&\underset{\zeta \to 0}{=} \pm\frac{2\zeta^p}{c_{0,-}} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}^{-1})\zeta^k\bigg)\bigg(\sum_{\ell=1}^{p} f_{p-\ell,-}\zeta^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}\zeta^{\ell}\bigg) \\
&\underset{\zeta \to 0}{=} \pm\frac{2}{c_{0,-}} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}^{-1})\zeta^k\bigg)\bigg(\sum_{\ell=1}^{p} f_{p-\ell,-}\zeta^{p-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}\zeta^{p+\ell}\bigg) \\
&\underset{\zeta \to 0}{=} \pm 2\sum_{\ell=0}^{\infty} \check{f}_{\ell,-}\zeta^{\ell},
\end{aligned} \tag{A.26}$$

using (2.67). Similarly,

$$\begin{aligned}
\frac{G_p(z)}{R_{2p}(z)^{1/2}} &\underset{P \to P_{0,\pm}}{=} \frac{2z^p}{c_{0,+}} \frac{G_p(z)}{y} \\
&\underset{\zeta \to 0}{=} \pm\frac{2}{c_{0,-}} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}^{-1})\zeta^k\bigg)\bigg(\sum_{\ell=1}^{p} g_{p-\ell,-}\zeta^{p-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+}\zeta^{p+\ell}\bigg) \\
&\underset{\zeta \to 0}{=} \pm 2\sum_{\ell=0}^{\infty} \check{g}_{\ell,-}\zeta^{\ell},
\end{aligned} \tag{A.27}$$

and

$$\begin{aligned}
\frac{H_p(\zeta)}{R_{2p}(z)^{1/2}} &\underset{P \to P_{0,\pm}}{=} \frac{2z^p}{c_{0,+}} \frac{H_p(z)}{y} \\
&\underset{\zeta \to 0}{=} \pm\frac{2}{c_{0,-}} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}^{-1})\zeta^k\bigg)\bigg(\sum_{\ell=0}^{p-1} h_{p-1-\ell,-}\zeta^{p-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+}\zeta^{p+\ell}\bigg) \\
&\underset{\zeta \to 0}{=} \pm 2\sum_{\ell=0}^{\infty} \check{h}_{\ell,-}\zeta^{\ell+1},
\end{aligned} \tag{A.28}$$

for some coefficients $\check{f}_{\ell,-}, \check{g}_{\ell,-}, \check{h}_{\ell,-}$ to be determined next. The leading order asymptotics yields $\check{f}_{0,-} = \alpha$, $\check{g}_{0,-} = \frac{1}{2}$, and $\check{h}_{0,-} = -\beta^+$. Dividing each of the equations (2.10)–(2.13) by $R_{2p}(z)^{1/2}$ and using the expansions (A.26)–(A.28), one finds that $\check{f}_{\ell,-}, \check{g}_{\ell,-}$, and $\check{h}_{\ell,-}$ satisfy the recursion relations (2.22)–(2.25) with $f_{\ell,-}$ replaced by $\check{f}_{\ell,-}$, $g_{\ell,-}$ replaced by $\check{g}_{\ell,-}$, and $h_{\ell,-}$ replaced by $\check{h}_{\ell,-}$. To conclude that actually,

$$\check{f}_{\ell,-} = \hat{f}_{\ell,-}, \quad \check{g}_{\ell,-} = \hat{g}_{\ell,-}, \quad \check{h}_{\ell,-} = \hat{h}_{\ell,-}, \quad \ell \in \mathbb{N}_0, \tag{A.29}$$

it remains to show that $c_{\ell,-} = 0$, $\ell \in \mathbb{N}$. Dividing (2.69) by $R_{2p}(z)^{1/2}$ and using the expansion (A.26) one finds the nonlinear recursion relation (A.15) with $\hat{f}_{\ell,-}$ replaced by $\check{f}_{\ell,-}$. Similarly, using (2.70), (2.71) one finds the nonlinear recursion formulas (A.17) and (A.19) with $\hat{g}_{\ell,-}$ replaced by $\check{g}_{\ell,-}$ (resp., $\hat{h}_{\ell,-}$ replaced by $\check{h}_{\ell,-}$).

Inductively one verifies that

$$\deg\big(\check{f}^{(r)}_{\ell,-}\big) = -\ell + r, \quad \deg\big(\hat{g}^{(r)}_{\ell,-}\big) = -\ell, \quad \deg\big(\check{h}^{(r)}_{\ell,-}\big) = -\ell - 1 - r, \quad \ell \in \mathbb{N}_0,\ r \in \mathbb{Z} \tag{A.30}$$

(cf. (2.41)). This proves (A.29). □

Moreover, one can prove the following result.

Theorem A.3. *Assume* (3.1) *and* s-AL$_p(\alpha, \beta) = 0$. *Then the E_m-dependent summation constants $c_{\ell,\pm}$, $\ell = 0, \dots, p$, in F_p, G_p, and H_p are given by*

$$c_{\ell,\pm} = c_{0,\pm} c_\ell\big(\underline{E}^{\pm 1}\big), \quad \ell = 0, \dots, p. \tag{A.31}$$

Moreover, one has the following relations between the homogeneous and nonhomogeneous recursion coefficients:

$$\begin{aligned} c_{0,\pm}\hat{f}_{\ell,\pm} &= \sum_{k=0}^{\ell} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) f_{k,\pm}, \quad \ell = 0, \dots, p-1, \\ c_{0,\pm}\hat{f}_{p,\pm} &= \sum_{k=0}^{p-1} \hat{c}_{p-k}\big(\underline{E}^{\pm 1}\big) f_{k,\pm} + \hat{c}_0(\underline{E}^{\pm 1}) f_{p-1,\mp}, \end{aligned} \tag{A.32}$$

$$\begin{aligned} c_{0,\pm}\hat{g}_{\ell,\pm} &= \sum_{k=0}^{\ell} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) g_{k,\pm}, \quad \ell = 0, \dots, p-1, \\ c_{0,\pm}\hat{g}_{p,\pm} &= \sum_{k=0}^{p-1} \hat{c}_{p-k}\big(\underline{E}^{\pm 1}\big) g_{k,\pm} + \hat{c}_0(\underline{E}^{\pm 1}) g_{p,\mp}, \end{aligned} \tag{A.33}$$

$$\begin{aligned} c_{0,\pm}\hat{h}_{\ell,\pm} &= \sum_{k=0}^{\ell} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) h_{k,\pm}, \quad \ell = 0, \dots, p-1, \\ c_{0,\pm}\hat{h}_{p,\pm} &= \sum_{k=0}^{p-1} \hat{c}_{p-k}\big(\underline{E}^{\pm 1}\big) h_{k,\pm} + \hat{c}_0(\underline{E}^{\pm 1}) h_{p-1,\mp}. \end{aligned} \tag{A.34}$$

Furthermore, one has

$$f_{\ell,\pm} = c_{0,\pm} \sum_{k=0}^{\ell} c_{\ell-k}\big(\underline{E}^{\pm 1}\big) \hat{f}_{k,\pm}, \quad \ell = 0, \dots, p, \tag{A.35}$$

$$g_{\ell,\pm} = c_{0,\pm} \sum_{k=0}^{\ell} c_{\ell-k}\big(\underline{E}^{\pm 1}\big)\hat{g}_{k,\pm}, \quad \ell = 0, \dots, p, \tag{A.36}$$

$$h_{\ell,\pm} = c_{0,\pm} \sum_{k=0}^{\ell} c_{\ell-k}\big(\underline{E}^{\pm 1}\big)h_{k,\pm}, \quad \ell = 0, \dots, p. \tag{A.37}$$

For general ℓ (not restricted to $\ell \leq p$) one has[3]

$$c_{0,\pm}\hat{f}_{\ell,\pm} = \begin{cases} \sum_{k=0}^{\ell} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) f_{k,\pm}, & \ell = 0, \dots, p-1, \\ \sum_{k=0}^{p-1} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) f_{k,\pm} & \\ \quad + \sum_{k=(2p-1-\ell)\vee 0}^{p-1} \hat{c}_{\ell+k+1-2p}\big(\underline{E}^{\pm 1}\big) f_{k,\mp}, & \ell \geq p, \end{cases} \tag{A.38}$$

$$c_{0,\pm}\hat{g}_{\ell,\pm} = \begin{cases} \sum_{k=0}^{\ell} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) g_{k,\pm}, & \ell = 0, \dots, p-\delta_\pm, \\ \sum_{k=0}^{p-\delta_\pm} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) g_{k,\pm} & \\ \quad + \sum_{k=(2p-\ell)\vee 0}^{p-\varepsilon_\mp} \hat{c}_{\ell+k-2p}\big(\underline{E}^{\pm 1}\big) g_{k,\mp}, & \ell \geq p-\delta_\pm+1, \end{cases} \tag{A.39}$$

$$c_{0,\pm}\hat{h}_{\ell,\pm} = \begin{cases} \sum_{k=0}^{\ell} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) h_{k,\pm}, & \ell = 0, \dots, p-1, \\ \sum_{k=0}^{p-1} \hat{c}_{\ell-k}\big(\underline{E}^{\pm 1}\big) h_{k,\pm} & \\ \quad + \sum_{k=(2p-1-\ell)\vee 0}^{p-1} \hat{c}_{\ell+k+1-2p}\big(\underline{E}^{\pm 1}\big) h_{k,\mp}, & \ell \geq p. \end{cases} \tag{A.40}$$

Here we used the conventions

$$\delta_\pm = \begin{cases} 0, & +, \\ 1, & -, \end{cases} \qquad \varepsilon_\mp = \begin{cases} 0, & -, \\ 1, & +. \end{cases} \tag{A.41}$$

Proof. Consider first the case (A.32). By comparing the same powers of ζ in (A.21) and (A.20) the formula (A.32) follows (and (A.38) as well). Formulas (A.33) and (A.34) follow by using (A.22) and (A.23), respectively.

To prove (A.35) one uses (A.8) and finds

$$c_{0,\pm} \sum_{m=0}^{\ell} c_{\ell-m}\big(\underline{E}^{\pm 1}\big)\hat{f}_{m,\pm} = \sum_{m=0}^{\ell} c_{\ell-m}\big(\underline{E}\big) \sum_{k=0}^{m} \hat{c}_{m-k}\big(\underline{E}^{\pm 1}\big) f_{k,\pm} = f_{\ell,\pm}. \tag{A.42}$$

The proofs of (A.36) and (A.37) and those of (A.39) and (A.40) are analogous. □

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[3] $m \vee n = \max\{m, n\}$.

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