A proof of the abstract limiting absorption principle by energy estimates

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Abstract

We give a proof in an abstract setting of various resolvent estimates including the limiting absorption principle starting from a Mourre positive commutator estimate, using standard energy estimates arguments.

1 Introduction

Let H, A be two selfadjoint operators on a Hilbert space $\mathcal{H}$ and I an bounded open interval included in the spectrum of H. The *positive commutator method* of Mourre [M1] allows to deduce from a *positive commutator estimate*:

$$\mathbb{1}_I(H)[H, \mathrm{i}A]\mathbb{1}_I(H) \geq c_0 \mathbb{1}_I(H) \tag{1.1}$$

for some $c_0 > 0$, several *resolvent estimates* for H, the most famous of them being the *limiting absorption principle*:

$$\sup_{z \in J^{\pm}} \|\langle A\rangle^{-s}(H-z)^{-1}\langle A\rangle^{-s}\| < \infty,$$

for all closed intervals $J \subset I$ and $s > \frac{1}{2}$, where $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$. It is a far reaching generalization of an argument by Putnam [P].

Mourre's paper had a very deep impact in spectral and scattering theory and drastically changed these two fields. Mourre's result was later improved and generalized in various directions, (we mention among many others the papers [PSS], [Ya], [T], [JMP], [JP]), the most general framework being probably the one exposed in the book [ABG].

The proofs of the abstract limiting absorption principle all rely on a clever differential inequality in ϵ on a family of operators $G_\epsilon(z)$ converging when $\epsilon \to 0$ to $\langle A\rangle^{-s}(H-z)^{-1}\langle A\rangle^{-s}$.

On the other hand, in the field of partial differential equations, more precisely in microlocal analysis, positive commutator methods are very common, under the name of the so called *energy estimates*: typically they rely on following identity on the Hilbert space $L^2(\mathbb{R}^n)$:

$$2\mathrm{Im}(Cu, Pu) = (u, [P_1, \mathrm{i}C]u) + (u, (CP_2 + P_2C)u), \tag{1.2}$$

where

$$P = P_1(x, D) + \mathrm{i}P_2(x, D),\ P_i(x, D) = P_i^*(x, D),\ C(x, D) = C^*(x, D)$$

are pseudodifferential operators. Usually one assumes that $P_2(x, D) \leq 0$ modulo lower order terms and one tries to construct $C \leq 0$ such that

$$[P_1(x, D), \mathrm{i}C(x, D)] \geq B^*(x, D)B(x, D),$$

again modulo lower order terms.

A famous example of the use of (1.2) is the proof by Hörmander [H] of the theorem of propagation of wave front set for operators of real principal type.

Note also that in the study of an abstract unitary group $\mathrm{e}^{-\mathrm{i}tH}$, the idea of looking for a negative observable $C(t)$ with a positive Heisenberg derivative $\partial_t C(t) + [H, \mathrm{i}C(t)]$ was used by Sigal and Soffer [SS], [HSS] to derive *propagation estimates* on $\mathrm{e}^{-\mathrm{i}tH}$ for large t, by the exact time-dependent analog of (1.2).

In [B] Burq proved semiclassical resolvent estimates for Schrödinger operators $-h^2\Delta + V(x)$ on $L^2(\mathbb{R}^n)$, where $V \in C_0^\infty(\mathbb{R}^n)$ near a energy level λ which is *non-trapping* for the classical flow of $p(x, \xi) = \xi^2 + V(x)$ by a contradiction argument. His proof used a propagation theorem for semiclassical measures, which itself is proved by energy estimates. The proof of Burq was later extended by Jecko [J].

Recently Golénia and Jecko [GJ] gave a new proof of the limiting absorption principle in an abstract framework, again by a contradiction argument. Their proof relies on the consideration of what they call *special sequences* (sequences of vectors in $\mathcal{H}$ which contradict the limiting absorption principle), commutator expansions and a virial theorem.

Our purpose in this paper is to give a proof of the limiting absorption principle using only *energy estimates*. We believe that our proof is more transparent than the previous ones and has the advantage of clearly showing a connection with well-known PDE arguments.

The abstract version of (1.2) that we will use is the following:
let C, H be two selfadjoint operators such that H is bounded. Then:

$$2\mathrm{Im}(Cu, (H - z)u) = (u, [H, \mathrm{i}C]u) - 2\mathrm{Im}z(u, Cu), \quad u \in \mathcal{D}(C), \tag{1.3}$$

where the commutator is understood as a quadratic form on $\mathcal{D}(C)$.

Let us now describe the results of this paper.

Let H, A be two selfadjoint operators on a Hilbert space $\mathcal{H}$ such that $H \in C^1(A)$. (The definition of the classes $C^k(A)$ will be recalled in Sect. 2).

We assume that I is a bounded open interval included in $\sigma(H)$, such that the Mourre estimate holds on I, i.e.:

$$\mathbb{1}_I(H)[H, \mathrm{i}A]\mathbb{1}_I(H) \geq c_0 \mathbb{1}_I(H) \tag{1.4}$$

for some $c_0 > 0$. If J is an interval, we set

$$J^\pm = \{z \in \mathbb{C} | \ \mathrm{Re}z \in J, \ \pm\mathrm{Im}z > 0\}.$$

Theorem 1 *Assume that $H \in C^2(A)$ and that (1.4) holds. Then:*

$$\sup_{z \in J^\pm} \|\langle A\rangle^{-s}(H - z)^{-1}\langle A\rangle^{-s}\| < \infty,$$

for all closed intervals $J \subset I$ and $s > \frac{1}{2}$.

Thm. 1 is well known (see [M1], [PSS]) under similar hypotheses. It was later improved in [Ya], [T], [JP], the best results being the ones in [ABG].

We will also give a proof by energy estimates of the following two resolvent estimates, where:

$$P_{\pm}(A) = \mathbb{1}_{\mathbb{R}^{\pm}}(A).$$

(It is customary to interpret the projections $P_{\pm}(A)$ as projections on *outgoing / incoming subspaces*).

Theorem 2 *Assume that $H \in C^3(A)$ and that (1.4) holds. Then:*

$$\sup_{z \in J^{\pm}} \|\langle A\rangle^{-1}(H-z)^{-1}P_{\pm}(A)\| < \infty.$$

for all closed intervals $J \subset I$.

Theorem 3 *Assume that $H \in C^4(A)$ and that (1.4) holds. Then:*

$$\sup_{z \in J^{\pm}} \|P_{\mp}(A)(H-z)^{-1}P_{\pm}(A)\| < \infty,$$

for all closed intervals $J \subset I$.

Thms. 2, 3 were proved before in [M2]. The proofs of these theorems will be given in Sects. 3, 4, 5 respectively. We will only prove the J^+ case, the J^- case being similar. Some preparatory estimates will be given in Sect. 2.

2 Functional calculus and commutator expansions

2.1 The classes $C^k(A)$

We recall in this subsection some definitions from [ABG].

Let A be a selfadjoint operator on $\mathcal{H}$. One says that a bounded operator B is of class $C^k(A)$ for some $k \in \mathbb{N}$ if for all $u \in \mathcal{H}$ the function

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}tA} B \mathrm{e}^{\mathrm{i}tA} u$$

is of class C^k. It $B \in C^1(A)$, then the commutator $[B, A]$ considered as a quadratic form on $\mathcal{D}(A)$ extends as a bounded operator on $\mathcal{H}$ and $B \in C^k(A)$ implies $[A, B] \in C^{k-1}(A)$ for $k \geq 2$. If $B \in C^k(A)$, we will use the standard notation:

$$\mathrm{ad}_A^l B := [A, \mathrm{ad}_A^{l-1} B], \quad ad_A^0 B := B, \text{ for } l \leq k.$$

If H is a selfadjoint operator, one says that $H \in C^k(A)$ if for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$ the operator $(H-z)^{-1}$ is in $C^k(A)$.

The following facts are well known:

Lemma 2.1 *Let H, A be two selfadjoint operators such that $H \in C^1(A)$. Then for all $z \in \mathbb{C}\backslash\mathbb{R}$ and for all $\chi \in C_0^\infty(\mathbb{R})$, the operators $(z-H)^{-1}$ and $\chi(H)$ are bounded on $\mathcal{D}(\langle A\rangle^s)$ for $0 \leq s \leq 1$.*

Lemma 2.2 *Let H, A be selfadjoint operators such that $H \in C^k(A)$. Then for all $\chi \in C_0^\infty(\mathbb{R})$ $\chi(H) \in C^k(A)$.*

Lemma 2.2 can be found in [GJ, Prop. 2.4]. Lemma 2.1 is easy to prove using the identity in $B(\mathcal{D}(A), \mathcal{H})$:

$$A(H-z)^{-1} - (H-z)^{-1}A = (H-z)^{-1}[H, A](H-z)^{-1},$$

and the functional calculus recalled in Subsect. 2.2.

2.2 Almost analytic extensions

Let S^ρ for $\rho \in \mathbb{R}$ be the class of functions f such that:

$$|f^{(m)}(\lambda)| \leq C_m \langle \lambda \rangle^{\rho - m}, \quad m \in \mathbb{N},$$

equipped with the seminorms:

$$\|f\|_m := \sup_{\lambda \in \mathbb{R}, \alpha \leq m} |\langle \lambda \rangle^{-\rho-\alpha} f^{(\alpha)}(\lambda)|.$$

The construction of almost analytic extensions of functions in S^ρ can be found in [DG]. Actually as observed by Ivrii-Sigal [IS] (see also Davies [D]), it is for most purposes sufficient to use *finite order* almost analytic extensions which can be trivially constructed as follows:

let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(s) \equiv 1$ in $|s| \leq 1$, $\chi(s) \equiv 0$ in $|s| \geq 2$. Set:

$$\tilde{f}(x + \mathrm{i}y) := (\sum_{r=0}^{N} f^{(r)}(x) \frac{(\mathrm{i}y)^r}{r!}) \chi(\frac{y}{\langle x \rangle}),$$

for N fixed large enough. Then if $f \in S^\rho$:

$$\begin{aligned}
&\tilde{f}_{|\mathbb{R}} = f, \\
&\operatorname{supp} \tilde{f} \subset \{(x + \mathrm{i}y) | |y| \leq 2\langle x \rangle, \quad x \in \operatorname{supp} f\}, \\
&|\tfrac{\partial \tilde{f}(z)}{\partial \bar{z}}| \leq C \langle x \rangle^{\rho - N - 1} |y|^N,
\end{aligned}$$

If A is selfadjoint and $f \in S^\rho$, then for $u \in \mathcal{D}(\langle A \rangle^\rho)$, one has (see e.g. [GJ]):

$$f(A)u = \lim_{R \to +\infty} \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C} \cap \{|\mathrm{Re} z| \leq R\}} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} (z - A)^{-1} u \, dz \wedge d\bar{z}.$$

2.3 Commutator expansions

We first recall in this subsection a lemma due to Golenia-Jecko [GJ].

Lemma 2.3 *Let $k \in \mathbb{N}^*$, B a bounded selfadjoint operator in $C^k(A)$. Let $\rho < k$ and $f \in S^\rho$. Then as forms on $\mathcal{D}(\langle A \rangle^k)$:*

$$[f(A), B] = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(A) \mathrm{ad}_A^j B + R_k(f, A, B),$$

where:

$$\|\langle A \rangle^s R_k(f, A, B) \langle A \rangle^{s'}\| \leq C(f) \|\mathrm{ad}_A^k B\|,$$

for $s, s' \geq 0$, $s' \leq 1$, $s \leq k$ and $\rho + s + s' < k$, where $C(f)$ is a seminorm of f in S^ρ.

In the rest of this subsection, we will use Lemma 2.3 to obtain three commutator expansions which will be useful later.

Let $0 < \epsilon < 1$ and

$$g(\lambda) = \langle \lambda \rangle^{-(1+\epsilon)/2}, \quad F(\lambda) = -\int_\lambda^{+\infty} g^2(s) ds. \tag{2.5}$$

Note that $g \in S^{-(1+\epsilon)/2}$, $F \in S^0$.

Let us fix $\tau, \chi_1, \chi_2 \in C_0^\infty(\mathbb{R})$ three cutoff functions such that $\chi_1\chi_2 = \chi_1$.

We set as in [GJ]:

$$H_\tau := H\tau(H). \tag{2.6}$$

Note that if $H \in C^k(A)$ then $H_\tau \in C^k(A)$ by Lemma 2.2.

Proposition 2.4 *There exists a constant C such that for all selfadjoint operators H, A such that $H \in C^2(A)$, one has:*

$$\begin{array}{rcl} & & \chi_1(H)[H_\tau, \mathrm{i}F(A)]\chi_1(H) \\ i) & = & \chi_1(H)g(A)\chi_2(H)[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\ & + & \chi_1(H)\langle A\rangle^{-(1+\epsilon)/2}R_1\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H), \end{array}$$

where:

$$\|R_1\| \leq C\Big(\sum_{k+l=2,\, l\leq 1} \|\mathrm{ad}_A^k H_\tau\|\|\mathrm{ad}_A^l \chi_2(H)\|\Big).$$

$$\begin{array}{rcl} ii) & & \chi_1(H)g(A)\chi_2^2(H)g(A)\chi_1(H) \\ & = & \chi_1(H)g^2(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-(1+\epsilon)/2}R_2\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H), \end{array}$$

where:

$$\|R_2\| \leq C\|\mathrm{ad}_A\chi_2(H)\|.$$

Prop. 2.4 will be used in the proof of Thm. 1.

Proof. Applying Lemma 2.3 with $\rho = 0$, $k = 2$, we get:

$$[F(A), H_\tau] = g^2(A)[A, H_\tau] + R_2(F, A, H_\tau),$$

where

$$\|\langle A\rangle^s R_2(F, A, H_\tau)\langle A\rangle^s\| \leq C\|\mathrm{ad}_A^2 H_\tau\|, \tag{2.7}$$

for all $0 \leq s < 1$. Next

$$g^2(A)[A, H_\tau] = g(A)[A, H_\tau]g(A) + g(A)R_1(g, A, \mathrm{ad}_A H_\tau),$$

and applying Lemma 2.3 with $\rho = -(1+\epsilon)/2$ and $k = 1$, we get:

$$\|R_1(g, A, \mathrm{ad}_A H_\tau)\langle A\rangle^s\| \leq C(g)\|\mathrm{ad}_A^2 H_\tau\|, \tag{2.8}$$

for all $0 \leq s < 1$. Collecting (2.7) and (2.8) we obtain since $(1+\epsilon)/2 < 1$:

$$[H_\tau, \mathrm{i}F(A)] = g(A)[H_\tau, \mathrm{i}A]g(A) + \langle A\rangle^{-(1+\epsilon)/2}R_1\langle A\rangle^{-(1+\epsilon)/2}, \tag{2.9}$$

for $\|R_1\| \leq C\|\mathrm{ad}_A^2 H_\tau\|$.

Since $\chi_1\chi_2 = \chi_1$, we have

$$\chi_1(H)g(A) = \chi_1(H)\chi_2(H)g(A) = \chi_1(H)g(A)\chi_2(H) - \chi_1(H)R_1(g, A, \chi_2(H)), \tag{2.10}$$

where as above

$$\|\langle A\rangle^s R_1(g, A, \chi_2(H))\langle A\rangle^{s'}\| \leq C(g)\|\mathrm{ad}_A\chi_2(H)\|,$$

for $0 \leq s, s' < 1$, $s + s' < 1 + (1+\epsilon)/2$. Using this estimate with $(s, s') = (1,0)$ or $(0,1)$, we get:

$$
\begin{aligned}
\chi_1(H)g(A)[H_\tau, \mathrm{i}A]g(A)\chi_1(H) = \;& \chi_1(H)g(A)\chi_2(H)[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\
+\;& \chi_1(H)\langle A\rangle^{-1}R_1[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\
+\;& \chi_1(H)g(A)[H_\tau, \mathrm{i}A]R_2\langle A\rangle^{-1}\chi_1(H),
\end{aligned} \tag{2.11}
$$

Since $(1+\epsilon)/2 < 1$, we can write the sum of the last two terms in the r.h.s. of (2.11) as

$$
\chi_1(H)\langle A\rangle^{-(1+\epsilon)/2}R_3\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H),
$$

with

$$
\|R_3\| \leq C\|\mathrm{ad}_A H_\tau\|\|\mathrm{ad}_A \chi_2(H)\|,
$$

which completes the proof of *i)*. The proof of *ii)* is similar, using again (2.10). $\square$

We now prove two similar commutator expansions for a different F.

Let $g \in C^\infty(\mathbb{R})$ with $0 \leq g \leq 1$, $g(\lambda) \equiv 0$ for $\lambda \geq 2$, $g(\lambda) \equiv 1$ for $\lambda \leq 1$.

Set:

$$
F(\lambda) := -\int_\lambda^{+\infty} g^2(s)ds. \tag{2.12}
$$

Note that $g \in S^0$, $F \in S^1$.

Props. 2.5, 2.6 will be used in the proof of Thms. 2, 3 respectively.

Proposition 2.5 *There exists a constant C such that for all selfadjoint operators H, A such that $H \in C^3(A)$ one has:*

$$
\begin{aligned}
& \chi_1(H)[H_\tau, \mathrm{i}F(A)]\chi_1(H) \\
i) \quad =\;& \chi_1(H)g(A)\chi_2(H)[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\
+\;& \chi_1(H)\langle A\rangle^{-s}R_1 g(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-s}R_2\langle A\rangle^{-s}\chi_1(H) + \text{h.c.},
\end{aligned}
$$

for all $0 \leq s < 1$ where:

$$
\|R_1\| + \|R_2\| \leq C\Big(\sum_{2 \leq k+l \leq 3,\, l \leq 1} \|\mathrm{ad}_A^k H_\tau\|\|\mathrm{ad}_A^l \chi_2(H)\|\Big).
$$

$$
\begin{aligned}
ii) \quad & \chi_1(H)g(A)\chi_2^2(H)g(A)\chi_1(H) \\
=\;& \chi_1(H)g^2(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-s}R_3 g(A)\chi_1(H) + \text{h.c.},
\end{aligned}
$$

for all $0 \leq s < 1$ where:

$$
\|R_3\| \leq C\|\mathrm{ad}_A \chi_2(H)\|.
$$

The next proposition is similar, with slightly better remainder terms.

Proposition 2.6 *There exists a constant C such that for all selfadjoint operators H, A such that $H \in C^4(A)$ one has:*

$$
\begin{aligned}
& \chi_1(H)[H_\tau, \mathrm{i}F(A)]\chi_1(H) \\
i) \quad =\;& \chi_1(H)g(A)\chi_2(H)[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\
+\;& \chi_1(H)\langle A\rangle^{-1}R_1 g(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-1}R_2\langle A\rangle^{-1}\chi_1(H) + \text{h.c.},
\end{aligned}
$$

where:

$$\|R_1\| + \|R_2\| \leq C(\sum_{2\leq k+l\leq 4,\, l\leq 2} \|\mathrm{ad}_A^k H_\tau\| \|\mathrm{ad}_A^l \chi_2(H)\|).$$

$$\begin{aligned} ii) \quad & \chi_1(H)g(A)\chi_2^2(H)g(A)\chi_1(H) \\ = \quad & \chi_1(H)g^2(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-1} R_3 g(A)\chi_1(H) + \text{h.c.}, \end{aligned}$$

where:

$$\|R_3\| \leq C(\sum_{l=1}^{2} \|\mathrm{ad}_A^l \chi_2(H)\|).$$

Proof of Prop. 2.5

Applying Lemma 2.3 for $\rho = 1$, $k = 3$, we get:

$$[F(A), H_\tau] = g^2(A)[A, H_\tau] + gg'(A)\mathrm{ad}_A^2 H_\tau + R_3(F, A, H_\tau),$$

where

(2.13) $$\|\langle A\rangle^s R_3(F, A, H_\tau)\langle A\rangle^s\| \leq C\|\mathrm{ad}_A^3 H_\tau\|, \text{ for } 0 \leq s < 1.$$

Next

$$g^2(A)[A, H_\tau] = g(A)[A, H_\tau]g(A) + g(A)R_1(g, A, \mathrm{ad}_A H_\tau),$$

where

(2.14) $$\|R_1(g, A, \mathrm{ad}_A H_\tau)\langle A\rangle^s\| \leq C\|\mathrm{ad}_A^2 H_\tau\|, \text{ for } 0 \leq s < 1,$$

and:

$$gg'(A)\mathrm{ad}_A^2 H_\tau = g(A)\mathrm{ad}_A^2 H_\tau g'(A) + g(A)R_1(g', A, \mathrm{ad}_A^2 H_\tau),$$

where

(2.15) $$\|R_1(g', A, \mathrm{ad}_A^2 H_\tau)\langle A\rangle^s\| \leq C\|\mathrm{ad}_A^3 H_\tau\|, \text{ for } 0 \leq s < 1.$$

Collecting (2.13), (2.14) and (2.15), we obtain for all $0 \leq s < 1$:

(2.16) $$[H_\tau, \mathrm{i}F(A)] = g(A)[H_\tau, \mathrm{i}A]g(A) + g(A)R_1\langle A\rangle^{-s} + \langle A\rangle^{-s}R_2\langle A\rangle^{-s} + \text{h.c.},$$

where

(2.17) $$\|R_1\| \leq C(\|\mathrm{ad}_A^2 H_\tau\| + \|\mathrm{ad}_A^3 H_\tau\|), \quad \|R_2\| \leq C\|\mathrm{ad}_A^3 H_\tau\|$$

As in (2.10), we have:

$$\chi_1(H)g(A) = \chi_1(H)g(A)\chi_2(H) - \chi_1(H)R_1(g, A, \chi_2(H)),$$

where now:

(2.18) $$\|\langle A\rangle^s R_1(g, A, \chi_2(H))\langle A\rangle^{s'}\| \leq C\|\mathrm{ad}_A \chi_2(H)\|, \text{ for } s + s' < 1.$$

Applying again this estimate with $(s, s') = (s, 0)$ or $(0, s)$, we get:

(2.19) $$\begin{aligned} \chi_1(H)g(A)[H_\tau, \mathrm{i}A]g(A)\chi_1(H) = \quad & \chi_1(H)g(A)\chi_2(H)[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\ + \quad & \chi_1(H)\langle A\rangle^{-s}R_1[H_\tau, \mathrm{i}A]\chi_2(H)g(A)\chi_1(H) \\ + \quad & \chi_1(H)g(A)[H_\tau, \mathrm{i}A]R_2\langle A\rangle^{-s}\chi_1(H). \end{aligned}$$

We can write the last two terms in the r.h.s. of (2.19) as:

$$\chi_1(H)\langle A\rangle^{-s}R_1 g(A)\chi_1(H)+\chi_1(H)g(A)R_2\langle A\rangle^{-s}\chi_1(H), \quad 0\leq s<1$$

where:

$$\|R_i\|\leq C\|\mathrm{ad}_A H_\tau\|\|\mathrm{ad}_A\chi_2(H)\|. \tag{2.20}$$

Collecting (2.17), (2.20), we obtain *i)*. The proof of *ii)* is similar, using (2.18). □

Proof of Prop. 2.6

The proof is very similar to Prop. 2.5, the only difference is that we need to take one more term in the commutator expansions. Therefore we will only sketch it. Applying Lemma 2.3 for $\rho=1$, $k=4$, we get:

$$[F(A),H_\tau]=g^2(A)[A,H_\tau]+gg'(A)\mathrm{ad}_A^2H_\tau+\frac{1}{3}(g'^2+gg")(A)\mathrm{ad}_A^3H_\tau+R_4(F,A,H_\tau), \tag{2.21}$$

where

$$\|\langle A\rangle R_4(F,A,H_\tau)\langle A\rangle\|\leq C\|\mathrm{ad}_A^4H_\tau\|.$$

Using that $g'\in C_0^\infty(\mathrm{I\!R})$, we can symmetrize the second and third terms in the r.h.s. of (2.21) and obtain that:

$$[F(A),H_\tau]=g^2(A)[A,H_\tau]+\langle A\rangle^{-1}R_1\langle A\rangle^{-1},$$

for

$$\|R_1\|\leq C(\sum_{k=2}^{4}\|\mathrm{ad}_A^kH_\tau\|).$$

Next

$$g^2(A)[A,H_\tau]=g(A)[A,H_\tau]g(A)+g(A)\mathrm{ad}_A^2H_\tau g'(A)+g(A)R_2(g,A,\mathrm{ad}_AH_\tau),$$

and the last two terms can be written as

$$g(A)R_2\langle A\rangle^{-1},\ \text{for } \|R_2\|\leq C(\sum_{k=2}^{3}\|\mathrm{ad}_A^kH_\tau\|).$$

We obtain:

$$[H_\tau,\mathrm{i}F(A)]=g(A)[H_\tau,\mathrm{i}A]g(A)+g(A)R_1\langle A\rangle^{-1}+\langle A\rangle^{-1}R_2\langle A\rangle^{-1}+\text{h.c.}, \tag{2.22}$$

where

$$\|R_1\|+\|R_2\|\leq C(\sum_{k=2}^{4}\|\mathrm{ad}_A^kH_\tau\|). \tag{2.23}$$

To handle the cutoffs χ_1, χ_2, we write:

$$\chi_1(H)g(A)=\chi_1(H)g(A)\chi_2(H)+\chi_1(H)g'(A)\mathrm{ad}_A\chi_2(H)+\chi_1(H)R_2(g,A,\chi_2(H)),$$

and since $g'\in C_0^\infty(\mathrm{I\!R})$, the last two terms can be written as

$$\chi_1(H)\langle A\rangle^{-1}R_3,\ \text{for } \|R_3\|\leq C(\sum_{k=1}^{2}\|\mathrm{ad}_A^k\chi_2(H)\|).$$

This yields *i)* and *ii)* by the same arguments as before. □

3 Proof of Thm 1

Let $H \in C^1(A)$ be a selfadjoint operator and let $J \subset \mathbb{R}$ be a compact interval. We recall that:

$$J^+ := \{z \in \mathbb{C} | \ \ \mathrm{Re} z \in J, \ \ \mathrm{Im} z > 0\}$$

Let $\tau, \chi_1 \in C_0^\infty(\mathbb{R})$ be two cutoff functions such that $\chi_1(x) = \tau(x) \equiv 1$ on J and $\tau(x) = 1$ on $\mathrm{supp}\, \chi_1$. We set:

$$H_\tau := \tau(H)H.$$

(This useful idea of replacing H by its local version H_τ is taken from [S] and was also used in [GJ].)

Lemma 3.1 *Let $0 < s \leq 1$. Consider the following three statements:*

$$(L1) \quad \sup_{z \in J^+} \|\langle A\rangle^{-s}(H-z)^{-1}\langle A\rangle^{-s}\| < +\infty,$$

(L2) there exists $C > 0$ such that for all $z \in J^+$, $u \in (H + \mathrm{i})^{-1}\mathcal{D}(\langle A\rangle^s)$ one has:

$$\|\langle A\rangle^{-s}u\| \leq C\|\langle A\rangle^s(H-z)u\|.$$

(L3) there exists $C > 0$ such that for all $z \in J^+$, $u \in \mathcal{D}(\langle A\rangle^s)$ one has:

$$\|\langle A\rangle^{-s}\chi_1(H)u\| \leq C\|\langle A\rangle^s(H_\tau - z)\chi_1(H)u\|.$$

Then

$$(L3) \ \Rightarrow \ (L2) \Rightarrow \ (L1).$$

Proof. Note first that using Lemma 2.1, we see that the estimates *(L2)* and *(L3)* have a meaning, since the operators $(H - z)(H + \mathrm{i})^{-1}$ and $(H_\tau - z)\chi_1(H)$ preserve $\mathcal{D}(\langle A\rangle^s)$ for all $0 \leq s \leq 1$.

Let us prove that $(L2) \ \Rightarrow \ (L1)$. Let $f \in \mathcal{H}$ and $u = (H - z)^{-1}\langle A\rangle^{-s}f$. Then $u \in (H + \mathrm{i})^{-1}\mathcal{D}(\langle A\rangle^s)$ since:

$$u = (H + \mathrm{i})^{-1}\langle A\rangle^{-s}g, \text{ for } g = f + (z + \mathrm{i})\langle A\rangle^s(H - z)^{-1}\langle A\rangle^{-s}f \in \mathcal{H}.$$

Applying *(L2)* to u we obtain *(L1)*.

Let us now prove that $(L3) \ \Rightarrow \ (L2)$. Set $\tilde{\chi}_1 = 1 - \chi_1$. Then:

$$(3.24) \qquad \|\langle A\rangle^{-s}u\| \leq \|\langle A\rangle^{-s}\chi_1(H)u\| + \|\langle A\rangle^{-s}\tilde{\chi}_1(H)u\|,$$

$$\|\langle A\rangle^{-s}\tilde{\chi}_1(H)u\| = \|\langle A\rangle^{-s}\tilde{\chi}_1(H)(H-z)^{-1}(H-z)u\| \leq C\|(H-z)u\|,$$

uniformly for $z \in J^+$ since $\tilde{\chi}_1 \equiv 0$ on J. Next

$$\langle A\rangle^s(H_\tau - z)\chi_1(H)u = \langle A\rangle^s(H-z)\chi_1(H)u = \langle A\rangle^s\chi_1(H)\langle A\rangle^{-s}\langle A\rangle^s(H-z)u,$$

since $\tau \equiv 1$ on $\mathrm{supp}\, \chi_1$. By Lemma 2.1, $\chi_1(H)$ is bounded on $\mathcal{D}(\langle A\rangle^s)$ hence:

$$(3.25) \qquad \|\langle A\rangle^s(H_\tau - z)\chi_1(H)u\| \leq C\|\langle A\rangle^s(H-z)u\|,$$

uniformly for $z \in J^+$. Combining (3.24) and (3.25), we see that $(L3) \ \Rightarrow \ (L2)$.

□

Proof of Thm. 1.

We will prove the estimate $(L3)$ in Lemma 3.1. Let $I \subset \mathbb{R}$ be a bounded open interval on which the Mourre estimate (1.1) holds, and let $J \subset I$ be a closed interval. We choose $\tau \in C_0^\infty(\mathbb{R})$ such that $\tau \equiv 1$ near J. By [GJ, Lemma 2.5] we know that the Mourre estimate for H_τ holds on I, i.e.:

$$\chi(H)[H_\tau, \mathrm{i}A]\chi(H) \geq c_0\chi^2(H), \tag{3.26}$$

for some $c_0 > 0$, if $\operatorname{supp}\chi \subset I$. Let us pick χ_1, χ_2 as in Prop. 2.4 with $\operatorname{supp}\chi_i \subset J$. From Prop. 2.4 *i)* and (3.26), we get:

$$\begin{aligned} \chi_1(H)[H_\tau, \mathrm{i}F(A)]\chi_1(H) \geq\ & c_0\chi_1(H)g(A)\chi_2^2(H)g(A)\chi_1(H) \\ +\ & \chi_1(H)\langle A\rangle^{-(1+\epsilon)/2}R_1\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H), \end{aligned}$$

and using also Prop. 2.4 *ii)*, we obtain:

$$\chi_1(H)[H_\tau, \mathrm{i}F(A)]\chi_1(H) \geq \mathrm{c}_0\chi_1(H)g^2(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-(1+\epsilon)/2}R_2\langle A\rangle^{-(1+\epsilon)/2}, \tag{3.27}$$

where:

$$\|R_2\| \leq C(\|\mathrm{ad}_A^2 H_\tau\| + \|\mathrm{ad}_A H_\tau\|\|\mathrm{ad}_A\chi_2(H)\| + c_0\|\mathrm{ad}_A\chi_2(H)\|). \tag{3.28}$$

We replace now A by $\frac{A}{R}$ for $R \geq 1$. Noting that by Prop. 2.4 the constant C in the r.h.s. of (3.28) is independent on A and that c_0 is replaced by c_0R^{-1}, we obtain:

$$\chi_1(H)[H_\tau, \mathrm{i}F(\frac{A}{R})]\chi_1(H) \geq \frac{c_0}{R}\chi_1(H)\langle\frac{A}{R}\rangle^{-(1+\epsilon)}\chi_1(H) - \frac{C}{R^2}\chi_1(H)\langle\frac{A}{R}\rangle^{-(1+\epsilon)}\chi_1(H). \tag{3.29}$$

Fixing $R \gg 1$ we obtain:

$$\chi_1(H)[H_\tau, \mathrm{i}F(\frac{A}{R})]\chi_1(H) \geq \frac{c_0}{2R}\chi_1(H)\langle\frac{A}{R}\rangle^{-(1+\epsilon)}\chi_1(H). \tag{3.30}$$

Since R is fixed for the rest of the proof we can denote $\frac{A}{R}$ again by A. We apply now identity (1.3) to $C = F(A)$, $H = H_\tau$. Since $F \leq 0$ and $\mathrm{Im}z > 0$ we get for $u \in \mathcal{H}$:

$$\|\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H)u\|^2 \leq C|(F(A)\chi_1(H)u, (H_\tau - z)\chi_1(H)u)|. \tag{3.31}$$

Using that F is a bounded function, we get for $u \in \mathcal{D}(\langle A\rangle^{(1+\epsilon)/2})$:

$$\|\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H)u\|^2 \leq C\|\langle A\rangle^{-(1+\epsilon)/2}\chi_1(H)u\|\|\langle A\rangle^{(1+\epsilon)/2}(H_\tau - z)\chi_1(H)u\|.$$

This implies that the estimate *(L3)* of Lemma 3.1 holds for $z \in J^+$ and $s = (1+\epsilon)/2$. By Lemma 3.1, this proves Thm. 1. □

4 Proof of Thm. 2

Let $H \in C^1(A)$ and J, τ, χ_1 as in Sect. 3.

Lemma 4.1 *Consider the following three statements:*

$$(M1) \quad \sup_{z\in J^+} \|P_-(A)(H-z)^{-1}\langle A\rangle^{-1}\| < +\infty,$$

(M2) there exists $C>0$ such that for all $z\in J^+$, $u\in (H+\mathrm{i})^{-1}\mathcal{D}(\langle A\rangle)$ one has:

$$\|P_-(A)u\| \leq C\|\langle A\rangle (H-z)u\|.$$

(M3) there exists $C>0$ such that for all $z\in J^+$, $u\in \mathcal{D}(\langle A\rangle)$ one has:

$$\|P_-(A)\chi_1(H)u\| \leq C\|\langle A\rangle (H_\tau - z)\chi_1(H)u\|.$$

Then

$$(M3) \ \Rightarrow\ (M2) \Rightarrow\ (M1).$$

Proof. Again by Lemma 2.1 we see that the estimates *(M2)* and *(M3)* have a meaning. We use the notation in the proof of Lemma 3.1. The proof that $(M2) \Rightarrow\ (M1)$ is as in Lemma 3.1. To prove that $(M3)\ \Rightarrow\ (M2)$ we write:

$$\begin{aligned}
&\|P_-(A)u\| \leq \|P_-(A)\chi_1(H)u\| + \|P_-(A)\tilde{\chi}_1(H)u\|,\\
&\|P_-(A)\tilde{\chi}_1(H)u\| = \|P_-(A)\tilde{\chi}_1(H)(H-z)^{-1}(H-z)u\| \leq C\|(H-z)u\|,
\end{aligned}$$

uniformly for $z\in J^+$, since $\tilde{\chi}_1 \equiv 0$ on J. By (3.25) for $s=1$, we get:

$$\|\langle A\rangle (H_\tau - z)\chi_1(H)u\| \leq C\|\langle A\rangle (H-z)u\|,$$

uniformly for $z\in J^+$, which completes the proof. □

Proof of Thm. 2.

We will prove the estimate $(M3)$ in Lemma 4.1. Arguing as in the proof of Thm. 1, using Prop. 2.5 instead of Prop. 2.4, we get for all $0\leq s<1$:

$$\begin{aligned}
\chi_1(H)[H_\tau, \mathrm{i}F(A)]\chi_1(H) \geq\ & \mathrm{c}_0\chi_1(H)g^2(A)\chi_1(H) + \chi_1(H)\langle A\rangle^{-s}R_1 g(A)\chi_1(H)\\
+\ & \chi_1(H)\langle A\rangle^{-s}R_2\langle A\rangle^{-s}\chi_1(H) + \mathrm{h.c.},
\end{aligned} \tag{4.32}$$

where:

$$\begin{aligned}
\|R_1\| \leq\ & C(\|\mathrm{ad}_A^3 H_\tau\| + \|\mathrm{ad}_A^2 H_\tau\| + \|\mathrm{ad}_A H_\tau\|\|\mathrm{ad}_A\chi_2(H)\| + c_0\|\mathrm{ad}_A\chi_2(H)\|),\\
\|R_2\| \leq\ & C\|\mathrm{ad}_A^3 H_\tau\|.
\end{aligned} \tag{4.33}$$

Replacing A by $\frac{A}{R}$ and using the inequality:

$$A_1^*A_2 + A_2^*A_1 \geq -A_1^*A_1 - A_2^*A_2,$$

we get for all $0\leq s<1$:

$$\begin{aligned}
\chi_1(H)[H_\tau, \mathrm{i}F(\tfrac{A}{R})]\chi_1(H) \geq\ & \tfrac{\mathrm{c}_0}{R}\chi_1(H)g^2(\tfrac{A}{R})\chi_1(H) - \tfrac{C}{R^2}\chi_1(H)g^2(\tfrac{A}{R})\chi_1(H)\\
& -\tfrac{C}{R^2}\chi_1(H)\langle \tfrac{A}{R}\rangle^{-2s}\chi_1(H).
\end{aligned} \tag{4.34}$$

Fixing $R \gg 1$ large enough, we obtain:

$$\chi_1(H)[H_\tau, \mathrm{i}F(\frac{A}{R})]\chi_1(H) \geq \frac{c_0}{2R}\chi_1(H)g^2(\frac{A}{R})\chi_1(H) - \frac{C}{R^2}\chi_1(H)\langle\frac{A}{R}\rangle^{-2s}\chi_1(H). \tag{4.35}$$

We denote again $\frac{A}{R}$ by A (note that $P_-(A) = P_-(\frac{A}{R})$), and apply identity (1.3) to $C = F(A)$, $H = H_\tau$ and get:

$$\|g(A)\chi_1(H)u\|^2 \leq C|(F(A)\chi_1(H)u, (H_\tau - z)\chi_1(H)u)| + C\|\langle A\rangle^{-s}\chi_1(H)u\|^2, \quad u \in \mathcal{D}(\langle A\rangle).$$

Next we note that:

$$\begin{aligned} & |(F(A)\chi_1(H)u, (H_\tau - z)\chi_1(H)u)| \\ = \; & |(\langle A\rangle^{-1}F(A)\chi_1(H)u, \langle A\rangle(H_\tau - z)\chi_1(H)u)| \\ \leq \; & \epsilon\|\langle A\rangle^{-1}F(A)\chi_1(H)u\|^2 + \epsilon^{-1}\|\langle A\rangle(H_\tau - z)\chi_1(H)u\|^2, \end{aligned}$$

for all $\epsilon > 0$. Since

$$\langle\lambda\rangle^{-1}|F|(\lambda) \leq Cg(\lambda) + C\langle\lambda\rangle^{-s},$$

for all $0 < s < 1$, we get:

$$\begin{aligned} & \|g(A)\chi_1(H)u\|^2 \\ \leq \; & C\epsilon\|g(A)\chi_1(H)u\|^2 + C\epsilon\|\langle A\rangle^{-s}\chi_1(H)u\|^2 \\ + \; & C\epsilon^{-1}\|\langle A\rangle(H_\tau - z)\chi_1(H)u\|^2 + C\|\langle A\rangle^{-s}\chi_1(H)u\|^2. \end{aligned}$$

Choosing ϵ small enough this gives:

$$\|g(A)\chi_1(H)u\|^2 \leq C\|\langle A\rangle(H_\tau - z)\chi_1(H)u\|^2 + C\|\langle A\rangle^{-s}\chi_1(H)u\|^2.$$

By Thm. 1, we know that:

$$\|\langle A\rangle^{-s}\chi_1(H)u\| \leq C\|\langle A\rangle(H_\tau - z)\chi_1(H)u\|, \text{ uniformly for } z \in J^+,$$

if $\frac{1}{2} < s$. This finally gives:

$$\|g(A)\chi_1(H)u\|^2 \leq C\|\langle A\rangle(H_\tau - z)\chi_1(H)u\|^2.$$

Since $g(\lambda) \geq P_-(\lambda)$, we obtain the estimate *(M3)*, which by Lemma 4.1 completes the proof of Thm. 2. $\square$

5 Proof of Thm. 3

Let $H \in C^2(A)$ and J, τ, χ_1 as in Sect. 3.

Lemma 5.1 *Consider the following three statements:*

$$(N1) \quad \sup_{z \in J^+} \|P_-(A)(H - z)^{-1}P_+(A)\| < +\infty,$$

(N2) there exists $C, b > 0$ such that for all $z \in J^+$, $u \in (H + \mathrm{i})^{-1}\mathcal{D}(\langle A\rangle)$ one has:

$$\|P_-(A)u\| \leq C\|(H - z)u\| + C\|1\!\!1_{]-\infty,b]}(A)\langle A\rangle(H - z)u\|.$$

(N3) there exists $C, b > 0$ such that for all $z \in J^+$, $u \in \mathcal{D}(\langle A\rangle)$ one has:

$$\|P_-(A)\chi_1(H)u\| \leq C\|(H_\tau - z)\chi_1(H)u\| + C\|1\!\!1_{]-\infty,b]}(A)\langle A\rangle(H_\tau - z)\chi_1(H)u\|.$$

Then

$$(N3) \Rightarrow (N2) \Rightarrow (N1).$$

Proof. Let us prove that $(N2) \Rightarrow (N1)$. Let $f \in \mathcal{D}(\langle A\rangle)$, $u = (H - z)^{-1}P_+(A)f$. Note that $u \in (H + \mathrm{i})^{-1}\mathcal{D}(\langle A\rangle)$ and $1\!\!1_{]-\infty,-1]}(A)(H - z)u = 0$. This implies that

$$\|1\!\!1_{]-\infty,b]}(A)\langle A\rangle(H - z)u\| = \|1\!\!1_{]-1,b]}(A)\langle A\rangle(H - z)u\| \leq C\|(H - z)u\|,$$

and hence by $(N2)$:

$$\|P_-(A)u\| \leq C\|(H - z)u\| = C\|P_+(A)f\| \leq C\|f\|,$$

uniformly for $z \in J^+$, which implies $(N1)$.

Let us now prove that $(N3) \Rightarrow (N2)$. Let $u \in (H + \mathrm{i})^{-1}\mathcal{D}(\langle A\rangle)$. As before we have

$$\|P_-(A)\tilde{\chi}_1(H)u\| = \|P_-(A)(H - z)^{-1}\tilde{\chi}_1(H)(H - z)u\| \leq C\|(H - z)u\|,$$

uniformly for $z \in J^+$, since $\tilde{\chi}_1 \equiv 0$ on J. Next we have:

$$\|(H_\tau - z)\chi_1(H)u\| = \|\chi_1(H)(H - z)u\| \leq C\|(H - z)u\|,$$

and

$$\|1\!\!1_{]-\infty,b]}(A)\langle A\rangle(H_\tau - z)\chi_1(H)u\| = \|1\!\!1_{]-\infty,b]}(A)\langle A\rangle\chi_1(H)(H - z)u\| \leq \|F(A)\chi_1(H)(H - z)u\|,$$

for $F \in C^\infty(\mathbb{R})$, $F \geq 0$, $\operatorname{supp} F \subset]-\infty, 2b]$, $F(\lambda) = \langle\lambda\rangle$ in $\lambda \leq b$.

By Lemma 2.3, we have as an identity on $\mathcal{D}(\langle A\rangle)$:

$$F(A)\chi_1(H) = \chi_1(H)F(A) + \mathrm{ad}_A\chi_1(H)F'(A) + R_2,$$

where R_2 is bounded. Therefore:

$$\begin{aligned}\|F(A)\chi_1(H)(H - z)u\| &\leq C\|F(A)(H - z)u\| + C\|(H - z)u\|\\ &\leq C\|(H - z)u\| + C\|1\!\!1_{]-\infty,2b]}(A)\langle A\rangle(H - z)u\|,\end{aligned}$$

since $F(\lambda) \leq C1\!\!1_{]-\infty,2b]}(\lambda)\langle\lambda\rangle$. This completes the proof that $(N3) \Rightarrow (N2)$. □

Proof of Thm. 3.

We will prove the estimate $(N3)$ in Lemma 5.1. Arguing as in the proof of Thm. 2, using now the sharper estimates of Prop. 2.6, we obtain for $R \gg 1$ large enough:

$$\chi_1(H)[H_\tau, \mathrm{i}F(\frac{A}{R})]\chi_1(H) \geq \frac{c_0}{2R}\|g^2(\frac{A}{R})\chi_1(H)u\|^2 - \frac{C}{R^2}\|\langle\frac{A}{R}\rangle^{-1}\chi_1(H)u\|^2.$$

Again we denote $\frac{A}{R}$ by A (note that this change only amounts to changing constants C, b in $(N3)$). By identity (1.3) this gives:

$$\|g(A)\chi_1(H)u\|^2 \leq C|(F(A)\chi_1(H)u, (H_\tau - z)\chi_1(H)u)| + C\|\langle A\rangle^{-1}\chi_1(H)u\|^2, \quad u \in \mathcal{D}(\langle A\rangle),$$

uniformly for $z \in J^+$. Using that $F(\lambda) = -\lambda g^2(\lambda) + \chi(\lambda)$, for some $\chi \in C_0^\infty(\mathbb{R})$, we get:

$$\begin{array}{rl} & \|g(A)\chi_1(H)u\|^2 \\ \leq & C|(g(A)\chi_1(H)u, Ag(A)(H_\tau - z)\chi_1(H)u)| \\ + & C|(\chi(A)\chi_1(H)u, (H_\tau - z)\chi_1(H)u)| + C\|\langle A\rangle^{-1}\chi_1(H)u\|^2 \\ \leq & C\epsilon\|g(A)\chi_1(H)u\|^2 + C\epsilon^{-1}\|\langle A\rangle g(A)(H_\tau - z)\chi_1(H)u)\|^2 \\ + & C\|(H_\tau - z)\chi_1(H)u)\|^2 + C\|\langle A\rangle^{-1}\chi_1(H)u\|^2, \end{array}$$

which choosing ϵ small enough, gives:

$$\|g(A)\chi_1(H)u\|^2 \leq C\|(H_\tau - z)\chi_1(H)u)\|^2 + C\|\langle A\rangle g(A)(H_\tau - z)\chi_1(H)u)\|^2 + C\|\langle A\rangle^{-1}\chi_1(H)u\|^2.$$

We have:

$$\begin{array}{rl} \langle A\rangle^{-1}\chi_1(H)u = & \langle A\rangle^{-1}(H - z)^{-1}(H_\tau - z)\chi_1(H)u \\ = & \langle A\rangle^{-1}(H - z)^{-1}\mathbb{1}_{[0,+\infty[}(A)(H_\tau - z)\chi_1(H)u \\ + & \langle A\rangle^{-1}(H - z)^{-1}\mathbb{1}_{]-\infty,0[}(A)(H_\tau - z)\chi_1(H)u \\ = & \langle A\rangle^{-1}(H - z)^{-1}\mathbb{1}_{[0,+\infty[}(A)(H_\tau - z)\chi_1(H)u \\ + & \langle A\rangle^{-1}(H - z)^{-1}\langle A\rangle^{-1}\langle A\rangle P_-(A)(H_\tau - z)\chi_1(H)u. \end{array}$$

By Thms. 1 and 2, we know that:

$$\sup_{z\in J^+} \|\langle A\rangle^{-1}(H - z)^{-1}\langle A\rangle^{-1}\| \leq C, \quad \sup_{z\in J^+} \|\langle A\rangle^{-1}(H - z)^{-1}\mathbb{1}_{[0,+\infty[}(A)\| \leq C,$$

(the second estimate follows by taking adjoints in the estimate in Thm. 2 for $z \in J^-$). This yields:

$$\|\langle A\rangle^{-1}\chi_1(H)u\|^2 \leq C\|(H_\tau - z)\chi_1(H)u\|^2 + \|\langle A\rangle P_-(A)(H_\tau - z)\chi_1(H)u\|^2,$$

uniformly for $z \in J^+$. Summing up we have:

$$\begin{array}{rl} & \|g(A)\chi_1(H)u\|^2 \\ \leq & C\|(H_\tau - z)\chi_1(H)u)\|^2 + \|\langle A\rangle g(A)(H_\tau - z)\chi_1(H)u)\|^2 \\ \leq & C\|(H_\tau - z)\chi_1(H)u)\|^2 + \|\langle A\rangle\mathbb{1}_{]-\infty,2]}(A)(H_\tau - z)\chi_1(H)u)\|^2. \end{array}$$

Hence the estimate $(N3)$ holds which completes the proof of Thm. 3. $\square$

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