A UNIFORM QUANTUM VERSION OF THE CHERRY THEOREM

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Abstract

Consider in $L^2(\mathbb{R}^2)$ the operator family $H(\epsilon) := P_0(\hbar,\omega) + \epsilon F_0$. P_0 is the quantum harmonic oscillator with diophantine frequency vector ω, F_0 a bounded pseudodifferential operator with symbol decreasing to zero at infinity in phase space, and $\epsilon \in \mathbb{C}$. Then there exist $\epsilon^* > 0$ independent of $\hbar$ and an open set $\Omega \subset \mathbb{C}^2 \setminus \mathbb{R}^2$ such that if $|\epsilon| < \epsilon^*$ and $\omega \in \Omega$ the quantum normal form near P_0 converges uniformly with respect to $\hbar$. This yields an exact quantization formula for the eigenvalues, and for $\hbar = 0$ the classical Cherry theorem on convergence of Birkhoff's normal form for complex frequencies is recovered.

1 Introduction and statement of the results

Consider in the phase space $\mathbb{R}^{2l}$ with canonical coordinates denoted (x,ξ) the Hamiltonian system defined by the principal function

$$p_\epsilon(x,\xi;\omega) := p_0(x,\xi) + \epsilon f_0(x,\xi) \tag{1.1}$$

$$p_0(x,\xi;\omega) := \frac{1}{2}(|\xi|^2 + |\omega x|^2) = \sum_{k=1}^{l} \omega_k I_k(x,\xi), \tag{1.2}$$

$$I_k(x,\xi) := \frac{1}{2\omega_k}[\xi_k^2 + \omega_k^2 x_k^2], \quad k = 1,\dots,l. \tag{1.3}$$

Here $f_0 : \mathbb{R}^{2l} \to \mathbb{R}$ is analytic; $f_0 = O([|\xi|^2 + |\omega x|^2]^{s/2})$, $s \geq 3$, as $|x| + |\xi| \to 0$, and $\epsilon \in \mathbb{R}$. Any analytic Hamiltonian near a non-degenerate elliptic equilibrium point can be written in the form (1.1). Let the *frequencies* $\omega := (\omega_1,\dots,\omega_l)$ fulfill a diophantine condition, i.e

$$\langle\omega,k\rangle \geq \gamma|k|^{-\tau}, \quad \forall k \in \mathbb{Z}^l \setminus \{0\},\ |k| := |k_1| + \ldots + |k_l|,\ ,\ \gamma > 0,\ \tau > l-1. \tag{1.4}$$

Under these circumstances the Birkhoff theorem holds, namely (see e.g.[SM], §30): $\forall N \in \mathbb{N}$, $\forall p \in \mathbb{N}$, $\forall \epsilon \in \mathbb{R}$ *one can construct an analytic, canonical bijection* $(y,\eta) = \chi_{\epsilon,N}(x,\xi) : \mathbb{R}^{2l} \leftrightarrow \mathbb{R}^{2l}$ *and a sequence of analytic functions* $Y_p(I;\omega) : \mathbb{R}_+^l \to \mathbb{R}$ *such that:*

$$p_\epsilon \circ \chi_{\epsilon,N}^{-1}(y,\eta) = \sum_{k=1}^{l} \omega_k I_k(y,\eta) + \sum_{p=1}^{N-1} Y_p(I(y,\eta);\omega)\epsilon^p + \epsilon^N R_N(y,\eta;\epsilon). \tag{1.5}$$

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The l functions $I := (I_k(y,\eta) : k = 1, \dots, l)$, the mechanical actions, are thus first integrals of the transformed Hamiltonian up to an error of order ϵ^N. Hence the system is integrable if the remainder in (1.4) vanishes as $N \to \infty$, namely if the *Birkhoff normal form*

$$B(I;\omega,\epsilon) := \langle \omega, I\rangle + \sum_{p=1}^{\infty} Y_p(I;\omega)\epsilon^p, \quad \langle \omega, I\rangle := \sum_{k=1}^{l} \omega_k I_k \tag{1.6}$$

converges when the actions belong to some ball $|I| < R$ of $\mathbb{R}^l_+$. However, as proved by C.L.Siegel[Si] in 1941, (1.6) is generically divergent (a particular convergence criterion has been later isolated by Rüssmann[Ru]; see also [Ga]. It states that (1.6) converges if $Y_p(I,\omega) = Y_p(\langle \omega, I\rangle))$. Already in 1928, on the other hand, T.M.Cherry[Ch] (see also [SM], §30; a more recent proof can be found in [Ot]) remarked that, when $l = 2$, the normal form is convergent provided the frequencies ω are complex with non vanishing imaginary part. Under this assumption the small denominator mechanism which generates the divergence becomes instead a large denominator one entailing the convergence.

We prove here that under the same assumptions on the frequencies, but much more restrictive conditions on the perturbation, the Cherry theorem holds in quantum mechanics as well, with estimates uniform with respect to the Planck constant $\hbar$. Namely, the quantum Birhoff normal form (see [Sj]) converges uniformly with respect to $\hbar$, and this yields an *exact* quantization formula for the quantum spectrum.
Consider indeed in $L^2(\mathbb{R}^2)$ the operator $H(\epsilon) = P_0(\hbar,\omega) + \epsilon F_0$ under the assumptions:

(A1) $P_0(\hbar,\omega)$ is the harmonic-oscillator Schrödinger operator with frequencies ω:

$$P_0(\hbar,\omega)\psi = -\frac{1}{2}\hbar^2\Delta\psi + \frac{1}{2}[\omega_1^2 x_1^2 + \omega_2^2 x_2^2]\psi, \;\; D(P_0) = H^2(\mathbb{R}^2) \cap L^2_2(\mathbb{R}^2). \tag{1.7}$$

(A2) Let $\omega_1 = a + ib$, $\omega_2 = c + id$, $a \neq 0, c \neq 0$, $\langle \omega_1, \omega_2\rangle := ac + bd$. Then $\omega \in \Gamma \subset \mathbb{C}^2$, where:

$$\Gamma := \left\{ \omega \in \mathbb{C}^2 \,|\, 0 < \delta_1 \le |\omega| \le \delta_2 \,|\, \frac{|\langle \omega_1, \omega_2\rangle|}{|\omega_1\omega_2|} = \frac{|ac + bd|}{\sqrt{(a^2+b^2)(c^2+d^2)}} \le \delta < 1 \right\} \tag{1.8}$$

To state the assumption on the perturbation F_0, define an analytic action $\Psi : \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$, $\quad (x,\xi) \mapsto (x',\xi') = \Psi_{\phi,\omega}(x,\xi)$ of $\mathbb{T}^2$ into $\mathbb{C}^2 \times \mathbb{C}^2$ through the flow of $p_0(\cdot,\omega)$ of *real* initial data $u := (x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2$ but *complex* frequencies $\omega \in \Gamma$:

$$\begin{cases} x'_k := \dfrac{\xi_k}{\omega_k}\sin\phi_k + x_k\cos\phi_k, \\ \xi'_k := \xi_k\cos\phi_k - \omega_k x_k \sin\phi_k, \end{cases} \qquad k = 1,2 \tag{1.9}$$

Let $f(z) \in C(\mathbb{C}^2 \times \mathbb{C}^2; \mathbb{C})$. Then $f \circ \Psi_{\phi,\omega}(x,\xi) = f \circ \Psi_{\phi,\omega}(u)$, denoted $f_{\phi,\omega}(u)$, is a $\phi - 2\pi$-periodic function $\forall\, (u,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \Gamma$ fixed. We further denote: $f(u) := f_{\phi,\omega}|_{\phi=0}$ and

1. $f_{\nu,\omega}(u)$ the Fourier coefficients of $f_{\phi,\omega}(u)$:

$$f_{\nu,\omega}(u) := \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(\Psi_{\phi,\omega}(u))e^{-i\langle\nu,\phi\rangle}\,d\phi, \quad \nu\in\mathbb{Z}^2.$$

2.

$$\widehat{f}_{\nu,\omega}(s) := \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} f_{\nu,\omega}(u)e^{-i\langle s,u\rangle}\,du\,. \tag{1.10}$$

their space Fourier transform. Here $\widehat{g}(s)$ is the Fourier transform of g:

$$\widehat{g}(s) = \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} g(u)e^{-i\langle s,u\rangle}\,du\,, \quad g(u)\in L^1(\mathbb{R}^2\times\mathbb{R}^2).$$

3. $\mathcal{F}_\sigma := \{f\in L^1(\mathbb{R}^2\times\mathbb{R}^2)\,|\,\|f\|_\sigma < +\infty\}$, $\sigma>0$. Here:

$$\|f\|_\sigma := \int_{R^2\times\mathbb{R}^2} |\widehat{f}(s)|e^{\sigma|s|}\,ds < +\infty \tag{1.11}$$

4. $\mathcal{A}_{\Gamma,\rho,\sigma} := \{f\in L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap C(\mathbb{C}^2\times\mathbb{C}^2)\,|\,\|f\|_{\Gamma,\rho,\sigma} < +\infty\}$, $\rho>0$, $\sigma>0$. Here:

$$\|f\|_{\Gamma,\rho,\sigma} := \sup_{\omega\in\Gamma}\sum_{\nu\in\mathbb{Z}^2} e^{\rho|\nu|}\|f_{\nu,\omega}\|_\sigma; \tag{1.12}$$

We can now state our assumption on the perturbation.

(A3) F_0 is a semiclassical pseudodifferential operator of order ≤ 0 with (Weyl) symbol $f_0\in\mathcal{A}_{\Gamma,\rho,\sigma}$ for some $\rho>0$, $\sigma>0$. Explicitly: (notation as in [Ro]) $F_0 = Op_h^W(f_0)$,

$$(F_0\psi)(x) = \frac{1}{h^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} e^{i\langle(x-y),\xi\rangle/\hbar} f_0((x+y)/2,\xi)\psi(y)\,dyd\xi, \quad \psi\in\mathcal{S}(\mathbb{R}^2). \tag{1.13}$$

Remarks

1. Since ([Ro], §II.4) $\|F\|_{L^2\to L^2}\le\|\widehat{f}\|_{L^1}$, F_0 extends to a continuous operator in $L^2(\mathbb{R}^2)$ because:

$$\|F_0\|_{L^2\to L^2}\le\|\widehat{f_0}\|_{L^1}\le\|f_0\|_\sigma\le\|f_0\|_{\Gamma,\rho,\sigma}. \tag{1.14}$$

2. Any $f\in\mathcal{A}_{\Gamma,\rho,\sigma}$ admits a holomorphic continuation from $u=(x,\xi)\in\mathbb{R}^2\times\mathbb{R}^2$ to the strip $\{z=(z_1,z_2)\in\mathbb{C}^2\times\mathbb{C}^2\,|\,|\mathrm{Im}\,z|<\sigma\}$. Obviously this holomorphic continuation can be different from the function $f\circ\Psi_{\phi,\omega}(z_1,z_2):\mathbb{C}^2\times\mathbb{C}^2\to\mathbb{R}$, as in the example $f=e^{-|z|^2}P(z):\mathbb{C}^2\times\mathbb{C}^2\to\mathbb{C}$, P any polynomial, discussed in Appendix.

Since F_0 is bounded, $H(\epsilon)$ defined on $D(P_0)$ is closed with pure-point spectrum $\forall\,\epsilon\in\mathbb{C}$, and is self-adjoint for $\epsilon\in\mathbb{R}$ if $\omega\in\mathbb{R}^2_+$. Moreover, P_0 can be considered a semiclassical pseudodifferential operator of order 2 with symbol $p_0(x,\xi;\omega)$.

Theorem 1.1 *Let (A1-A3) be verified and let $h^* > 0$. Then there exists $\epsilon^* > 0$ independent of $\hbar \in [0, \hbar^*]$ such that if $|\epsilon| < \epsilon^*$ the spectrum of $H(\epsilon)$ is given by the quantization formula*

$$E_n(\hbar, \epsilon) = \langle \omega, n \rangle \hbar + \frac{1}{2}(\omega_1 + \omega_2)\hbar + \mathcal{N}(n\hbar, \hbar; \epsilon). \tag{1.15}$$

$$\mathcal{N}(n\hbar, \hbar; \epsilon) = \sum_{p=1}^{\infty} \mathcal{N}_p(n\hbar, \hbar)\epsilon^p \tag{1.16}$$

Here $n = (n_1, n_2)$, $n_i = 0, 1, \ldots,$ and:
1. $\mathcal{N}_p(I, \hbar) : \mathbb{R}^2_+ \times [0, h^*] \to \mathbb{C}$ is analytic in I and continuous in $\hbar$;
2. The series (1.15) has convergence radius ϵ^* uniformy with respect to $(I, \hbar) \in \Omega \times [0, h^*]$. Here Ω is any compact of $\mathbb{R}^2_+$;
3. $\mathcal{N}_p(I, \hbar) : p = 1, 2, \ldots$ admits an asymptotic expansion to all orders in $\hbar$; the order 0 *term is the coefficient $Y_p(I)$ of the Birkhoff normal form.*

Remarks

1. The conditions of the Cherry theorem are much less restrictive than the present ones. In particular, the standard Schrödinger operator in which f_0 depends only on x is excluded. On the other hand, in the classical case $\hbar = 0$ we obtain an improved version of the theorem: indeed, in our conditions the Birkhoff normal form converges, for ϵ small enough, in *any* compact of $\mathbb{R}^2$. To our knowledge this result is new.

2. Taking $\hbar = 0$ in $\mathcal{N}_p(I, \hbar)$ (1.15) becomes $E^{BS}_\nu(\hbar, \epsilon) := \langle \omega, n \rangle \hbar + \frac{1}{2}(\omega_1 + \omega_2)\hbar + \sum_{p=1}^{\infty} Y_p(n\hbar)\epsilon^p$, namely the Bohr-Sommerfeld quantization of the Birkhoff normal form. Formula (1.15) yields all corrections needed to recover the eigenvalues $E_n(\hbar, \epsilon)$.

3. For any fixed n and $\hbar$ the series (1.15) coincides with the Rayleigh-Schrödinger perturbation expansion near the simple eigenvalue $\langle \omega, n \rangle \hbar + \frac{1}{2}(\omega_1 + \omega_2)\hbar$ of P_0 [GP].

4. The eigenvalues $E_n(\hbar, \epsilon)$ admit the interpretation of quantum resonances of a self-adjoint Schrödinger operator. For this matter the reader is referred to [MS], where under much more general conditions on f_0 the eigenvalues are obtained by an exact quantization of the KAM iteration scheme.

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2 Proof of the results

The proof is to be obtained in four steps.

1. Perturbation theory: the formal construction

Look for a unitary transformation $U(\omega,\epsilon,\hbar)=e^{iW(\epsilon)/\hbar}: L^2\leftrightarrow L^2$, $W(\epsilon)=W^*(\epsilon)$, $\epsilon\in\mathbb{R}$, such that:

$$S(\epsilon):=UH(\epsilon)U^{-1}=P_0(\hbar,\omega)+\epsilon Z_1+\epsilon^2 Z_2+\ldots+\epsilon^k R_k(\epsilon) \tag{2.1}$$

where $[Z_p,P_0]=0$, $p=1,\ldots,k-1$. Recall the formal commutator expansion:

$$e^{itW(\epsilon)/\hbar}He^{-itW(\epsilon)/\hbar}=\sum_{l=0}^{\infty}t^lH_l,\quad H_0:=H,\quad H_l:=\frac{[W,H_{l-1}]}{i\hbar l},\ l\geq 1 \tag{2.2}$$

Looking for $W(\epsilon)$ under the form of a power series, $W(\epsilon)=\epsilon W_1+\epsilon^2W_2+\ldots$, (2.2) becomes:

$$S=\sum_{s=0}^{k}\epsilon^sP_s+\epsilon^{k+1}R^{(k+1)} \tag{2.3}$$

where

$$P_s=\frac{[W_s,P_0]}{i\hbar}+F_s,\quad s\geq 1,\ F_1\equiv F_0 \tag{2.4}$$

$$\begin{aligned}F_s &= \sum_{r=2}^{s}\frac{1}{r!}\sum_{\substack{j_1+\ldots+j_r=s\\ j_l\geq 1}}\frac{[W_{j_1},[W_{j_2},\ldots,[W_{j_r},P_0]\ldots]}{(i\hbar)^r}\\ &+\sum_{r=2}^{s-1}\frac{1}{r!}\sum_{\substack{j_1+\ldots+j_r=s-1\\ j_l\geq 1}}\frac{[W_{j_1},[W_{j_2},\ldots,[W_{j_r},F_0]\ldots]}{(i\hbar)^r}\end{aligned}$$

Since F_s depends on $W_1,\ldots,W_{s-1}$, (2.1) yields the recursive homological equations:

$$\frac{[W_s,P_0]}{i\hbar}+F_s=Z_s,\qquad [P_0,Z_s]=0 \tag{2.5}$$

To solve for S, W_s, Z_s, we can equivalently look for their symbols; from now on, we denote by the same letter, but in small case, the symbol $\sigma(A)$ of an operator A, except for the symbol of S, denoted Σ. Let us now recall the following relevant results (see e.g. [Fo],§3.4):

1. $\sigma([A,B]/i\hbar)=\{a,b\}_M$, where $\{a,b\}_M$ is the Moyal bracket of a and b.
2. Given $(g,g')\in\mathcal{A}_{\omega,\sigma}$, their Moyal bracket $\{g,g'\}_M$ is defined as

$$\{g,g'\}_M=g\#g'-g'\#g,$$

   where $\#$ is the composition of g,g' considered as Weyl symbols.

3. In the Fourier transform representation, used throughout the paper, the Moyal bracket has the expression

$$(\{g,g'\}_M)^\wedge(s)=\frac{2}{\hbar'}\int_{\mathbb{R}^{2n}}\widehat{g}(s^1)\widehat{g'}(s-s^1)\sin\left[\hbar(s-s^1)\wedge s^1/2\right]ds^1, \tag{2.6}$$

where, given two vectors $s=(v,w)$ and $s^1=(v^1,w^1)$, $s\wedge s^1:=\langle w,v_1\rangle-\langle v,w_1\rangle$.

4. $\{g,g'\}_M=\{g,g'\}$ if either g or g' is quadratic in (x,ξ).

The equations (2.2,2.3,2.4) then become, once written for the symbols:

$$\sigma(e^{iW(\epsilon)/\hbar}He^{-iW(\epsilon)/\hbar}) = \sum_{l=0}^{\infty}\mathcal{H}_l,\ \mathcal{H}_0:=p_0+\epsilon f_0,\ \mathcal{H}_l:=\frac{\{w,\mathcal{H}_{l-1}\}_M}{l},\ l\geq 1 \tag{2.7}$$

$$\Sigma(\epsilon) = \sum_{s=0}^{k}\epsilon^s p_s+\epsilon^{k+1}r^{(k+1)} \tag{2.8}$$

where

$$p_s:=\{w_s,p_0\}_M+f_s,\quad s=1, f_1\equiv f_0 \tag{2.9}$$

$$\begin{aligned} f_s &:= \sum_{r=2}^{s}\frac{1}{r!}\sum_{\substack{j_1+\ldots+j_r=s\\ j_l\geq 1}}\{w_{j_1},\{w_{j_2},\ldots,\{w_{j_r},p_0\}_M\ldots\}_M \\ &+ \sum_{r=2}^{s-1}\frac{1}{r!}\sum_{\substack{j_1+\ldots+j_r=s-1\\ j_l\geq 1}}\{w_{j_1},\{w_{j_2},\ldots,\{w_{j_r},f_0\}_M\ldots\}_M,\quad s>1 \end{aligned} \tag{2.10}$$

In turn, the recursive homological equations become:

$$\{w_s,p_0\}_M+f_s=\zeta_s,\qquad \{p_0,\zeta_s\}_M=0 \tag{2.11}$$

2. Solution of the homological equation and estimates of the solution

$f\in\mathcal{A}_{\omega,\rho,\sigma}$ clearly entails the existence of the Fourier expansion of $f_{\phi,\omega}(u)$, and its uniform convergence with respect to $\phi\in\mathbb{T}^2$, u on compacts of $\mathbb{R}^2\times\mathbb{R}^2$, and $\omega\in\Gamma$, namely:

$$f_{\phi,\omega}(u)=\sum_{\nu\in\mathbb{Z}^l}f_{\nu,\omega}(u)e^{i\langle\nu,\phi\rangle}\Longrightarrow f(u)=\sum_{\nu\in\mathbb{Z}^l}f_{\nu,\omega}(u). \tag{2.12}$$

We further denote, for $\omega\in\Gamma$, and $\rho>0$:

$$\|f\|_{\omega,\sigma} := \sum_{\nu\in\mathbb{Z}^2}\|f_{\nu,\omega}\|_\sigma;\quad \mathcal{A}_{\omega,\sigma}:=\{f(u)\in\mathcal{F}_\sigma\,|\,\|f(u)\|_{\omega,\sigma}<+\infty\} \tag{2.13}$$

$$\|f\|_{\omega,\rho,\sigma} := \sum_{\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\|f_{\nu,\omega}\|_\sigma;\quad \mathcal{A}_{\omega,\rho,\sigma}:=\{f(u)\in\mathcal{A}_{\omega,\sigma}\,|\,\|f(u)\|_{\omega,\rho,\sigma}<+\infty\} \tag{2.14}$$

$$\|f\|_{\Gamma,\sigma} := \sup_{\omega\in\Gamma}\|f\|_{\omega,\sigma};\quad \mathcal{A}_{\Gamma,\sigma}:=\{f(u)\in\mathcal{F}_\sigma\,|\,\|f(u)\|_{\Gamma,\sigma}<+\infty\} \tag{2.15}$$

$$\|f\|_{\Gamma,\rho,\sigma} := \sup_{\omega\in\Gamma}\|f\|_{\omega,\rho,\sigma}; \tag{2.16}$$

Hence $\mathcal{A}_{\Gamma,\rho,\sigma} = \{f(u) \in \mathcal{F}_\sigma \mid \|f(u)\|_{\Gamma,\rho,\sigma} < +\infty\}$ and clearly $\mathcal{A}_{\Gamma,\rho,\sigma} \subset \mathcal{A}_{\Gamma,\sigma} \subset \mathcal{F}_\sigma$. Moreover the following inequalities obviously hold:

$$\begin{aligned} \sup_{u\in\mathbb{R}^2\times\mathbb{R}^2} |f_{\nu,\omega}(u)| &\leq \|\hat{f}_{\nu,\omega}(s)\|_{L^1} \leq \|f_{\nu,\omega}\|_\sigma \leq \|f\|_{\Gamma,\sigma} \leq \|f\|_{\Gamma,\rho,\sigma} && (2.17)\\ \|\hat{f}\|_{L^1} \leq \|f\|_\sigma &\leq \|f\|_\sigma \leq \|f\|_{\Gamma,\sigma} \leq \|f\|_{\Gamma,\rho,\sigma} && (2.18)\end{aligned}$$

Now the key remark is that $\{a, p_0\}_M = \{a, p_0\}$ for any symbol a because p_0 is quadratic in (x, ξ). The homological equation (2.11) becomes therefore

$$\{w_s, p_0\} + f_s = \zeta_s, \qquad \{p_0, \zeta_s\} = 0 \tag{2.19}$$

We then have:

Proposition 2.1 *Let $f \in \mathcal{A}_{\Gamma,\rho,\sigma}$. Then the equation*

$$\{w, p_0\} + f = \zeta, \qquad \{p_0, \zeta\} = 0 \tag{2.20}$$

admits the solutions $\zeta \in \mathcal{A}_{\Gamma,\sigma}$, $w \in \mathcal{A}_{\Gamma,\rho,\sigma}$

$$\zeta := f_{0,\omega}; \qquad w := \sum_{\nu\neq 0} \frac{f_{\nu,\omega}}{i\langle\omega,\nu\rangle}, \tag{2.21}$$

with the property $\zeta \circ \Psi_\phi = \zeta$; i.e., ζ depends only on I_1, I_2. Moreover:

$$\|\zeta\|_{\Gamma,\sigma} \leq \|f\|_{\Gamma,\sigma}; \quad \|w\|_{\Gamma,\rho,\sigma} \leq \|f\|_{\Gamma,\rho,\sigma}, \quad \|\nabla w\|_{\Gamma,\rho,\sigma} \leq \frac{4C}{\sigma}\|f\|_{\Gamma,\rho,\sigma} \tag{2.22}$$

for some $C(\Gamma, \delta) > 0$.

To prove the Proposition we need a preliminary result.

Lemma 2.1 *Let w be defined by (2.21), and $\Psi_{\phi,\omega}(x, \xi)$ by (1.9). Set:*

$$\Xi_{\phi,\omega}(x, \xi) := \Phi_{i\phi,i\omega}(x, \xi), \tag{2.23}$$

that is: $\Xi_{\phi,\omega}(x, \xi) := (x'_k, \xi'_k)$, where:

$$\begin{cases} x'_k = x_k \cosh\phi_k + \dfrac{\xi_k}{\omega_k}\sinh\phi_k \\ \xi'_k = \xi_k\cosh\phi_k + \omega_k x_k \sinh\phi_k \end{cases} \qquad k = 1, 2 \tag{2.24}$$

Then one has, uniformly with respect to (x, ξ) on compacts of $\mathbb{R}^4$:

$$\begin{aligned} w \circ \Psi_{\phi,\omega}(x, \xi) &= \sum_{\nu\neq 0} \frac{f_{\nu,\omega}(x, \xi)}{i\langle\omega,\nu\rangle} e^{i\langle\nu,\phi\rangle}, \quad \phi \in \mathbb{T}^2 && (2.25)\\ w \circ \Xi_{\phi,\omega}(x, \xi) &= \sum_{\nu\neq 0} \frac{f_{\nu,i\omega}(x, \xi)}{\langle\omega,\nu\rangle} e^{-\langle\nu,\phi\rangle}, \quad |\phi| \leq \rho - \eta, \ \forall\, 0 < \eta < \rho && (2.26)\end{aligned}$$

Moreover there is $C(\delta) > 0$ such that:

$$\|w\|_{\omega,\rho,\sigma} \leq C\|f\|_{\omega,\rho,\sigma}; \quad \|w\|_{i\omega,\rho,\sigma} \leq C\|f\|_{i\omega,\rho,\sigma} \tag{2.27}$$

Proof

Let us first prove that (2.21), whose convergence is proved below, solves (2.20), and that $w\circ\Psi_{\phi,\omega}(x,\xi)$ admits the representation (2.25). Following the argument of ([BGP]), Lemma 3.6, let us write:

$$\{p_0,pw\}(x,\xi)=\frac{d}{dt}\Big|_{t=0} w\circ\Psi_{\omega t,\omega}(x,\xi)=\frac{d}{dt}\Big|_{t=0}\sum_{0\neq\nu\in\mathbb{Z}^2}\frac{f_{\nu,\omega}\circ\Psi_{\omega t,\omega}(u)}{i\langle\omega,\nu\rangle}$$

$$=\frac{d}{dt}\Big|_{t=0}\sum_{0\neq\nu\in\mathbb{Z}^2}\frac{f_{\nu,\omega}\circ\Psi_{\omega t,\omega}(u)}{i\langle\omega,\nu\rangle}=\frac{d}{dt}\Big|_{t=0}\sum_{0\neq\nu\in\mathbb{Z}^2}\frac{f_{\nu,\omega}(u)e^{i\langle\nu,\omega t\rangle}}{i\langle\omega,\nu\rangle}=\sum_{0\neq\nu\in\mathbb{Z}^2}f_{\nu,\omega}(u)$$

Clearly, this equality also entails $\zeta=f_{0,\omega}$. Consider now the expansions (2.25, 2.26). First, it is easy to check that $\omega\in\Gamma$ if and only if $i\omega\in\Gamma$. Now we have:

$$w_{\nu,\omega}=\frac{f_{\nu,\omega}(x,\xi)}{i\langle\omega,\nu\rangle}$$

and therefore, by a straightforward application of Lemma 2.5:

$$\|w_{\nu,\omega}\|_\sigma\leq C\|f_{\nu,\omega}\|_\sigma.$$

Hence:

$$\|w\|_{\omega,\rho,\sigma}=\sum_{\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\|w_{\nu,\omega}\|_\sigma\leq C\sum_{\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\|f_{\nu,\omega}\|_\sigma=\|f\|_{\omega,\rho,\sigma}\quad\forall\,\omega\in\Gamma.$$

Therefore $q\in\mathcal{A}_{\Gamma,\rho,\sigma}$ entails $w\circ\Psi_{\omega,\phi}\in\mathcal{A}_{\Gamma,\rho,\sigma}$, whence the uniform convergence of the series (2.25). Now $i\omega\in\Gamma$ if $\omega\in\Gamma$; hence $w\circ\Psi_{i\omega,\phi}\in\mathcal{A}_{\Gamma,\rho,\sigma}$. On the other hand, the replacement $\phi\to i\phi$ maps $\Psi_{\phi,i\omega}(x,\xi)$ into $\Xi_{\phi,\omega}(x,\xi)$, and the series (2.26) is uniformly convergent if $|\mathrm{Im}\,\phi|<\rho-\eta$, $0<\eta<\rho$. Formula (2.26) is therefore proved. This concludes the proof of the Lemma.

Proof of Proposition 2.1

Let us first prove that ζ depends only on I_1,I_2. Consider for the sake of simplicity $u=(x,\xi)\in\mathbb{R}^2$. Since $f\in\mathcal{A}_{\Gamma,\rho,\sigma}$, we can write:

$$f_{\phi,\omega}(x,\xi)=\sum_{m.n=0}^{\infty}\frac{a_{mn}}{2^{m+n}}\left[(x+\frac{\xi}{i\omega})e^{i\phi}+(x-\frac{\xi}{i\omega})e^{-i\phi}\right]^m\left[(-i\omega x+\xi)e^{i\phi}+(i\omega x+\xi)e^{-i\phi}\right]^n$$

The average over ϕ eliminates all terms but those proportional to

$$[(x+\frac{\xi}{i\omega})(x-\frac{\xi}{i\omega})]^k[(-i\omega x+\xi)(i\omega x+\xi)]^l$$

i.e. to I^kI^l. The estimate $\|\zeta\|_{\omega,\sigma}\leq\|f\|_{\omega,\sigma}$ is obvious, and entails $\|\zeta\|_{\Gamma,\sigma}\leq\|f\|_{\Gamma,\sigma}$. The second estimate in (2.22) has been proved in Lemma 2.1 above. To prove the third one, consider the function $f\circ\Psi_{\phi,\omega}(z)$ and compute, for $j=1,2$:

$$
\begin{aligned}
\frac{d}{d\phi_j}w\circ\Psi_{\phi,\omega}(z)|_{\phi=0} &= \left.\frac{\partial w}{\partial x_j}\frac{\partial x_j'}{\partial\phi_j}+\frac{\partial w}{\partial\xi_j}\frac{\partial\xi_j'}{\partial\phi_j}\right|_{\phi=0}=\\
\frac{\partial w}{\partial x_j}\frac{\xi_j}{\omega_j}-\frac{\partial w}{\partial\xi_j}\omega_jx_j &= \sum_{0\neq\nu\in\mathbb{Z}^2}\frac{\nu_jf_{\nu,\omega}}{i\langle\omega,\nu\rangle}
\end{aligned}
$$

Therefore, once more by Lemma 2.5,

$$
\begin{aligned}
\left\|\frac{\partial w}{\partial x_j}\frac{\xi_j}{\omega_j}-\frac{\partial w}{\partial\xi_j}\omega_jx_j\right\|_{\omega,\rho,\sigma} &\leq \sum_{0\neq\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\frac{|\nu_j|}{|\langle\omega,\nu\rangle|}\|f_{\nu,\omega}\|_{\omega,\sigma}\\
&\leq C\sum_{0\neq\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\|f_{\nu,\omega}\|_{\omega,\sigma}=C\|f\|_{\omega,\rho,\sigma}.
\end{aligned}
$$

This yields:

$$
\left\|\frac{\partial w}{\partial x_j}\frac{\xi_j}{\omega_j}-\frac{\partial w}{\partial\xi_j}\omega_jx_j\right\|_{\Gamma,\rho,\sigma}\leq C\|f\|_{\Gamma,\rho,\sigma}. \tag{2.28}
$$

In the same way:

$$
\begin{aligned}
\frac{d}{d\phi_j}w\circ\Xi_{\phi,\omega}(z)|_{\phi=0} &= \left.\frac{\partial w}{\partial x_j}\frac{\partial x_j'}{\partial\phi_j}+\frac{\partial w}{\partial\xi_j}\frac{\partial\xi_j'}{\partial\phi_j}\right|_{\phi=0}=\\
\frac{\partial w}{\partial x_j}\frac{\xi_j}{\omega_j}+\frac{\partial w}{\partial\xi_j}\omega_jx_j &= \sum_{0\neq\nu\in\mathbb{Z}^2}\frac{\nu_jf_{\nu,i\omega}}{\langle\omega,\nu\rangle}
\end{aligned}
$$

whence, by Lemma 2.5,

$$
\begin{aligned}
\left\|\frac{\partial w}{\partial x_j}\frac{\xi_j}{\omega_j}+\frac{\partial w}{\partial\xi_j}\omega_jx_j\right\|_{i\omega,\rho,\sigma} &\leq \sum_{0\neq\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\frac{|\nu_j|}{|\langle\omega,\nu\rangle|}\|f_{\nu,i\omega}\|_{i\omega,\sigma}\\
&\leq C\sum_{0\neq\nu\in\mathbb{Z}^2}e^{\rho|\nu|}\|f_{\nu,i\omega}\|_{i\omega,\sigma}=C\|f\|_{i\omega,\rho,\sigma}.
\end{aligned}
$$

Recalling that $\omega\in\Gamma$ if and only if $i\omega\in\Gamma$ we get:

$$
\left\|\frac{\partial w}{\partial x_j}\frac{\xi_j}{\omega_j}+\frac{\partial w}{\partial\xi_j}\omega_jx_j\right\|_{\Gamma,\rho,\sigma}\leq C\|f\|_{\Gamma,\rho,\sigma}. \tag{2.29}
$$

Denote now s_j, t_j the Fourier dual variables of (x_j,ξ_j), $j=1,2$. Then, by definition (we drop for the sake of simplicity the dependence of ω):

$$
\left\|\frac{\partial w}{\partial x_j}\xi_j\right\|_{\sigma}=\int_{\mathbb{R}^4}\left|s_j\frac{\partial\widehat{w}(s_j,t_j)}{\partial t_j}\right|e^{\sigma(|s|+|t|)}\,dsdt
$$

Applying Lemma 2.3 to the integration over t_j we get:

$$
\begin{aligned}
\left\|\frac{\partial w}{\partial x_j}\right\|_{\omega,\sigma} &= \sum_{\nu\in\mathbb{Z}^2}\int_{\mathbb{R}^4}|s_j\widehat{w}_{\nu,\omega}(s_j,t_j)|\,e^{\sigma(|s|+|t|)}\,dsdt\leq\\
&\leq \frac{2}{\sigma}\sum_{\nu\in\mathbb{Z}^2}\int_{\mathbb{R}^4}\left|s_j\frac{\partial\widehat{w}_{\nu,\omega}(s_j,t_j)}{\partial t_j}\right|e^{\sigma(|s|+|t|)}\,dsdt\\
&= \frac{2}{\sigma}\sum_{\nu\in\mathbb{Z}^2}\left\|\frac{\partial w_{\nu,\omega}}{\partial x_j}\xi_j\right\|_{\sigma}=\frac{2}{\sigma}\left\|\frac{\partial w}{\partial x_j}\xi_j\right\|_{\omega,\sigma}
\end{aligned}
$$

Therefore, by (2.28,2.29)

$$\left\|\frac{\partial w}{\partial x_j}\right\|_{\Gamma,\rho,\sigma} \leq \frac{2C|\omega_j|}{\sigma}\|f\|_{\Gamma,\omega,\sigma}$$

Analogously, applying this time Lemma 2.3 to the integration over s_j:

$$\left\|\frac{\partial w}{\partial \xi_j}\right\|_{\Gamma,\rho,\sigma} \leq \frac{2C}{\sigma|\omega_j|}\|f\|_{\Gamma,\omega,\sigma}.$$

This is enough to prove the Proposition.

3. Iterative Lemma

Proposition 2.2 *Set:*

$$\mu := \frac{4\epsilon\|f_0\|_{\Gamma,\rho,\sigma}}{\sigma}.$$

Let $\mu < 1/4$ and consider for $k = 1, 2, \ldots$ the function

$$\Sigma_k := p_0 + \epsilon\mathcal{Z}_k + v_k \tag{2.30}$$

with $\mathcal{Z}_k, v_k \in \mathcal{A}_{\Gamma,\rho,\sigma}$, and let $\mathcal{Z}_k$ depend on (I_1, I_2) only.
Assume moreover:

$$\begin{aligned} \|\mathcal{Z}_k\|_{\Gamma,\sigma} &\leq \begin{cases} 0 & \text{if } k=0 \\ \displaystyle\sum_{s=0}^{k-1}(2\mu)^s & \text{if } k \geq 1 \end{cases} && (2.31) \\ \|v_k\|_{\Gamma,\rho,\sigma} &\leq \epsilon(2\mu)^k\|f_0\|_{\Gamma,\rho,\sigma} && (2.32) \end{aligned}$$

Let S_k be the Weyl quantization of Σ_k. Then there exists a unitary map $T_k : L^2 \to L^2$, $T_k := e^{i\epsilon W/\hbar}$ such that the Weyl symbol of the transformed operator $T_k S_k T_k^* := S_{k+1}$ is given by (2.30) with $k+1$ in place of k and satisfies (2.31,2.32) with $k+1$ in place of k.

Proof As in [BGP], Proposition 3.2, the homological equation:

$$\{p_0, w\} + v_k = \epsilon\mathcal{V}_k \tag{2.33}$$

determines the symbol w of W. Here the second unknown $\mathcal{V}_k$ has to depend on (x, ξ) only through I_1, I_2. Applying Proposition 1 we find that w and $\mathcal{V}_k$ exist and fulfill the estimates

$$\|w\|_{\Gamma,\rho,\sigma} \leq \epsilon\|f_0\|_{\Gamma,\rho,\sigma}(2\mu)^k; \quad \|\nabla w\|_{\Gamma,\rho,\sigma} \leq (2\mu)^{k+1}; \quad \|\mathcal{V}_k\|_{\Gamma,\rho,\sigma} \leq \|f_0\|_{\Gamma,\rho,\sigma}(2\mu)^k.$$

Define now:

$$\begin{aligned} \mathcal{Z}_{k+1} &:= \mathcal{Z}_k + \mathcal{V}_k; \quad v_{k+1} := \epsilon\sum_{l\geq 1}\mathcal{Z}_k^l + \sum_{l\geq 1}v_k^l + \sum_{l\geq 1}p_{l0} \\ \mathcal{Z}_k^0 &:= \mathcal{Z}_k; \quad \mathcal{Z}_k^l := \frac{1}{l}\{w, \mathcal{Z}_k^{l-1}\}_M \end{aligned}$$

and analogous definitions for v_k^l and p_{l0}. Clearly $v_{k+1}\in\mathcal{A}_{\Gamma,\rho,\sigma}$ by Lemma 2.4 below. Then the symbol of the transformed operator has the form (2.30) with $k+1$ in place of k. To get the estimates, for $k\geq 1$ we can write, by Proposition 1 and Lemmas 2.2. 2.3. 2.4:

$$\begin{aligned}\sum_{l\geq 1}\|(v_k^l)\|_{\Gamma,\rho,\sigma} &\leq \epsilon(2\mu)^k\sum_{l\geq 1}(2\mu)^l=\frac{\epsilon(2\mu)^{k+1}}{1-2\mu}\leq\epsilon(2\mu)^{k+1}\\ \sum_{l\geq 1}\|\mathcal{Z}_k^l\|_{\Gamma,\sigma} &\leq \|\mathcal{Z}_k^l\|_{\Gamma,\sigma}\cdot\frac{\mu}{1-\mu}\leq 2\mu,\quad \sum_{l\geq 2}\|p_{l0}\|_{\Gamma,\rho,\sigma}\leq\epsilon(2\mu)^{k+1}\end{aligned}$$

whence the assertion in a straightforward way.

Proof of Theorem 1

By Proposition 2 there is $\epsilon^*>0$ such that

$$\lim_{k\to\infty}p_0+\epsilon\mathcal{Z}_k:=\Sigma(\epsilon)$$

exists in the

$|\cdot\|_{\Gamma,\rho,\sigma}$ norm if $|\epsilon|<\epsilon^*$. Then $S(\epsilon):=Op_h^W(\Sigma(\epsilon))$ is unitarily equivalent to $H(\epsilon)$. Since $\mathcal{Z}_k$ is a polynomial of order $k-1$ in ϵ, we can write $\Sigma_k=p_0+\sum_{l=1}^{k}\zeta^{(l)}\epsilon^l+v_k$, where $\zeta^{(l)}(I_1,I_2)$ are solutions of the homological equations (2.11); therefore $S(\epsilon)$ has the form (2.1). Note that $\lim_{k\to\infty}\|v_k\|_{\Gamma,\rho,\sigma}=0$ entails $\lim_{k\to\infty}\|R_k\|_{L^2\to L^2}=0$. To sum up, the Weyl symbol $\Sigma(\epsilon,\hbar)$ has the convergent (uniform with respect to $\hbar$) normal form

$$\Sigma(\epsilon,\hbar)=p_0(I)+\sum_{n=1}^{\infty}\mathcal{Z}_n(I,\hbar)\epsilon^n$$

Then the assertions of Theorem 1 follow exactly as in [Sj] (see also [BGP]). This concludes the proof.

4. Auxiliary results

Lemma 2.2 *Let $(g,g',\nabla g,\nabla g')\in\mathcal{F}_\sigma$. Then:*

$$\|\{g,g'\}_M\|_\sigma\leq\|\nabla g\|_\sigma\|\nabla g'\|_\sigma. \tag{2.34}$$

If $(g,g',\nabla g,\nabla g')\in\mathcal{A}_{\omega,\rho,\sigma}$ then

$$\|\{g,g'\}_M\|_{\omega,\rho,\sigma}\leq\|\nabla g\|_{\omega,\rho,\sigma}\|\nabla g'\|_{\omega,\rho,\sigma}. \tag{2.35}$$

and if $(g,g',\nabla g,\nabla g')\in\mathcal{A}_{\Gamma,\rho,\sigma}$:

$$\|\{g,g'\}_M\|_{\Gamma,\rho,\sigma}\leq\|\nabla g\|_{\Gamma,\rho,\sigma}\|\nabla g'\|_{\Gamma,\rho,\sigma}. \tag{2.36}$$

Proof

We repeat the argument of [BGP], Lemma 3.1. We have $|s\wedge s^1|\leq |s|\cdot|s^1|$. Hence by (2.6) and the definition of the $\sigma-$ norm we get:

$$\begin{aligned}\|\{g,g'\}_M\|_\sigma &= \frac{2}{\hbar}\int_{\mathbb{R}^{2l}} e^{\sigma|s|}\,ds\int_{\mathbb{R}^{2l}}|\hat{g}(s)\hat{g'}(s-s^1)|\cdot|\sinh(\hbar(s-s^1)\wedge s^1)/2|\,ds^1\\ &\leq \frac{2}{\hbar}\int_{\mathbb{R}^{2l}} ds\int_{\mathbb{R}^{2l}} e^{\sigma(|s|+|s^1|)}|\hat{g}(s)\hat{g'}(s^1)|\cdot|\sinh(\hbar s\wedge s^1)/2|\,ds^1\\ &\leq \int_{\mathbb{R}^{2l}} e^{\sigma|s|}|\hat{g}(s)|\,ds\int_{\mathbb{R}^{2l}} e^{\sigma|s^1|}|\hat{g'}(s^1)|\cdot|s\wedge s^1|\,ds^1=\\ &\leq \int_{\mathbb{R}^{2l}} e^{\sigma|s|}|\hat{g}(s)||s|\,ds\int_{\mathbb{R}^{2l}} e^{\sigma|s^1|}|\hat{g'}(s^1)|\cdot|s^1|\,ds^1=\|\nabla g\|_\sigma\|\nabla g'\|_\sigma.\end{aligned}$$

The remaining two inequalities follow from the first one by exactly the same argument of [BGP], Lemma 3.4. This concludes the proof of the Lemma.

Lemma 2.3 *Let $g\in\mathcal{F}_\sigma$, $u=(x,\xi)\in\mathbb{R}^{2l}$. Then:*

$$\|g\|_\sigma\leq\frac{1}{\sigma}\|ug\|_\sigma \tag{2.37}$$

Proof

Setting $f(s):=\hat{g}(s)$ (2.37) is clearly equivalent to

$$\int_{\mathbb{R}^{2l}} e^{\sigma|s|}|f(s)|\,ds\leq\frac{1}{\sigma}\int_{\mathbb{R}^{2l}} e^{\sigma|s|}|\nabla f(s)|\,ds \tag{2.38}$$

We may limit ourselves to prove this inequality in the one-dimensional case, namely to show that:

$$\int_{\mathbb{R}} e^{\sigma|s|}|f(s)|\,ds\leq\frac{1}{\sigma}\int_{\mathbb{R}} e^{\sigma|s|}|f'(s)|\,ds \tag{2.39}$$

To see this, first write, for $s>0$:

$$e^{\sigma s}f(s)=-\int_s^\infty e^{\sigma t}f'(t)e^{\sigma(s-t)}\,dt$$

whence, for $A>0$:

$$\begin{aligned}\int_A^\infty|e^{\sigma s}f(s)|\,ds &\leq \iint_{A\leq s\leq t\leq\infty}|f'(t)|e^{\sigma s}\,dsdt=\int_A^\infty|f'(t)|\int_A^t e^{\sigma s}\,dsdt=\\ &= \sigma^{-1}\int_A^\infty|f'(t)|(e^{\sigma t}-e^{\sigma A})\,dt\leq\sigma^{-1}\int_A^\infty|f'(t)|e^{\sigma t}\,dt\end{aligned}$$

Likewise, for $s<0$, $A<0$:

$$e^{-\sigma s}f(s)=\int_{-\infty}^s e^{-\sigma t}f'(t)e^{-\sigma(s-t)}\,dt$$

$$\begin{aligned}\int_{-\infty}^{A}|e^{-\sigma s}f(s)|\,ds &= \iint_{-\infty\le t\le s\le A}|f'(t)|e^{-\sigma s}\,dsdt = \int_{-\infty}^{A}|f'(t)|\int_{t}^{A}e^{-\sigma s}\,dsdt = \\ &= \sigma^{-1}\int_{-\infty}^{A}|f'(t)|(e^{-\sigma t}-e^{-\sigma A})\,dt \le \sigma^{-1}\int_{-\infty}^{A}|f'(t)|e^{-\sigma t}\,dt\end{aligned}$$

Performing the limit $A\to 0$ in both inequalities we get (2.39). This concludes the proof of the Lemma.

Lemma 2.4 *Let $g\in\mathcal{A}_{\Gamma,\rho,\sigma}$, $w\in\mathcal{A}_{\Gamma,\rho,\sigma}$.*

1. Define

$$g_r := \frac{1}{r}\{w, g_{r-1}\}_M, \qquad r\ge 1;\quad g_0 := g.$$

Then $g_r\in\mathcal{A}_{\Gamma,\rho,\sigma}$ and the following estimate holds

$$\|g_r\|_{\Gamma,\rho,\sigma} \le \left(4\frac{\nabla w_{\Gamma,\rho,\sigma}}{\sigma}\right)^r \|g\|_{\Gamma,\rho,\sigma}. \tag{2.40}$$

2. Let w solve the homological equation (2.11). Define the sequence $p_{r0}: r=0,1,\ldots$:

$$p_{00} := p_0; \qquad p_{r0} := \frac{1}{r}\{w, p_{r-10}\}_M,\ r\ge 1.$$

Then $p_{r0}\in\mathcal{A}_{\omega,\sigma}$ and fulfills the following estimate

$$\|p_{r0}\|_{\Gamma,\rho,\sigma} \le \left(4\sigma^{-1}\|\nabla w\|_{\Gamma,\rho,\sigma}\right)^{r-1}\|f_0\|_{\Gamma,\rho,\sigma}, \quad r\ge 1. \tag{2.41}$$

Proof

Both estimates (2.40,2.41) are straightforward consequences of Lemmas 2.2 and 2.3: as far as (2.41) is concerned, it is indeed enough to note that $\{w,p_0\}=\zeta-q$ whence

$$\|p_{10}\|_{\Gamma,\rho,\sigma} + \|\nabla p_{10}\|_{\Gamma,\rho,\sigma} \le \frac{4\|f_0\|_{\Gamma,\rho,\sigma}}{\sigma}.$$

Lemma 2.5 *If (A3) holds there is $C_\delta>0$ independent of $\omega\in\Gamma$ such that*

$$|\omega_1\nu_1+\omega_2\nu_2| \ge C_\delta\sqrt{v_1^2+\nu_2^2} \tag{2.42}$$

Proof

We have to show the existence of $C_\delta>0$ such that

$$f(\nu_1,\nu_2) := \frac{|\omega_1\nu_1+\omega_2\nu_2|^2}{v_1^2+\nu_2^2} \ge C_\delta, \quad \forall\,(\nu_1,\nu_2)\in\mathbb{Z}^2, (\nu_1,\nu_2)\ne(0,0) \tag{2.43}$$

Notice that f is homogeneous of degree 0, namely $f(\mu\nu_1,\mu\nu_2) = f(\nu_1,\nu_2)$ $\forall\,(\nu_1,\nu_2)\in\mathbb{Z}^2, (\nu_1,\nu_2)\ne(0,0)$, $\forall\,\mu\in\mathbb{R}$, $\mu\ne 0$. Hence it is enough to show that

$$F(x,y) := |\omega_1 x+\omega_2 y|^2 \ge C_\delta, \quad \forall\,(x,y)\in S^1 \tag{2.44}$$

or, writing $x = \cos\theta, y = \sin\theta$:

$$F(\theta) := \frac{1}{2}[|\omega_1|^2 + |\omega_2|^2] + \frac{1}{2}[|\omega_1|^2 - |\omega_2|^2]\cos 2\theta + \langle\omega_1, \omega_2\rangle \sin 2\theta \geq C$$

Note that $F(0) = F(2\pi) = |\omega_1|^2$. A simple study of the function $F(\theta) : S^1 \to \mathbb{R}$ under the assumption (A2) shows the existence of $C_\delta \downarrow 0$ as $\delta \uparrow 1$ such that $|F(\theta)| \geq C_\delta \ \forall\, \theta \in S^1$. We omit the elementary details.

Appendix

Consider the function $f : \mathbb{C}^4 \to \mathbb{R}$

$$f(z) := e^{-|z|^2} P_n(z), \quad z \in \mathbb{C}^4,\ |z| = \sum |z_k|^2.$$

Here $P_n(z)$ is a polynomial of degree n.

Let us verify that f belongs to $\mathcal{A}_{\Gamma,\rho,\sigma}$; namely, there are $\rho > 0$, $\sigma > 0$ such that:

$$\sup_{\omega\in\Gamma} \sum_{\nu\in\mathbb{Z}^2} e^{\rho|\nu|} \|f_{\nu,\omega}(u)\|_\sigma < +\infty.$$

It is clearly enough to consider the case $u = (x, \xi) \in \mathbb{R}^2$, $n = 0$.

Set: $\omega := \gamma e^{i\theta}, 0 \leq \theta \leq 2\pi$, $\delta_1 \leq \gamma \leq \delta_2$. Then:

$$|\Psi_{\phi,\omega}(u)|^2 = \left| x\mathrm{cos}\phi + \frac{\xi}{\omega}\mathrm{sin}\phi \right|^2 + |\xi\mathrm{cos}\phi - \omega x\mathrm{sin}\phi|^2 = Ax^2 + Bx\xi + C\xi^2$$

$$A := \cos^2\phi + \gamma^2 \sin^2\phi; \quad B := \cos\theta(\gamma^{-1} - \gamma)\sin 2\phi, \quad C := \cos^2\phi + \gamma^{-2}\sin^2\phi$$

Therefore we can write:

$$\begin{aligned}
f_{\phi,\omega}(u) &:= f \circ \Psi_{\omega,\phi}(u) = e^{-\langle Q(\gamma,\theta,\phi)u,u\rangle}, \qquad Q(\gamma,\theta,\phi) := \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} \\
\det Q &= \cos^4\phi + \sin^4\phi + [(\gamma^{-2} + \gamma^2) - \cos^2\theta(\gamma^{-1} - \gamma)^2]\sin^2\phi\cos^2\phi \\
&= 1 + \kappa(1 - \cos^2\theta)\mathrm{sin}^2\phi\mathrm{cos}^2\phi \\
\mathrm{Tr}\, Q &= 2 + \kappa\mathrm{sin}^2\phi \\
\kappa &:= \gamma^{-2} + \gamma^2 - 2 \geq 0
\end{aligned}$$

whence, $\forall\, (\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$

$$1 \leq \lambda_1\lambda_2 \leq 1 + \kappa, \quad 2 \leq \lambda_1 + \lambda_2 \leq 2 + \kappa$$

where $0 < \lambda_1(\gamma, \theta, \phi) \leq \lambda_2(\gamma, \theta, \phi)$ denote the eigenvalues of $Q(\gamma, \theta, \phi) > 0$. This easily yields the uniform estimate:

$$\frac{1}{D} \leq \lambda_1(\gamma, \theta, \phi) \leq \lambda_2(\gamma, \theta, \phi) \leq D, \quad D := \frac{1}{2}[2 + \kappa + \sqrt{(2 + \kappa)^2 - 4}].$$

Consider now the Fourier coefficients $f_{\nu,\omega}(u) = f_{\nu,\gamma,\theta}(u)$:

$$f_{\nu,\gamma,\theta}(u) := \frac{1}{2\pi}\int_0^{2\pi} f\circ\Psi_{\omega,\phi}(u)e^{-i\nu\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{-\langle Q(\gamma,\theta,\phi)u,u\rangle}e^{-i\nu\phi}\,d\phi$$

and compute their Fourier transform:

$$\begin{aligned}
\hat f_{\nu,\gamma,\theta}(s) &= \frac{1}{2(\pi)^2}\int_{\mathbb{R}^2}\int_0^{2\pi} e^{-\langle Q(\gamma,\theta,\phi)u,u\rangle}e^{-i\nu\phi}e^{-i\langle u,s\rangle}\,d\phi\,du \\
&= \frac{2}{(2\pi)^2\sqrt{\det Q}}\int_0^{2\pi} e^{-\langle Q^{-1}(\gamma,\theta,\phi)s,s\rangle/2}e^{-i\nu\phi}\,d\phi, \quad s\in\mathbb{R}^2 \\
Q^{-1}(\gamma,\theta,\phi) &= \frac{1}{\det Q}\begin{pmatrix} C & -\frac{1}{2}B \\ -\frac{1}{2}B & A\end{pmatrix}.
\end{aligned}$$

Since

$$\langle s, Q^{-1}(\gamma,\theta,\phi)s\rangle \ge \lambda_2^{-1}s^2 \ge \frac{s^2}{D}$$

$\forall\,(\theta,\phi)\in[0,2\pi]\times[0,2\pi]$ we get the (ν,θ,ϕ)-independent estimate

$$|\hat f_{\nu,\gamma,\theta}(s)| \le \frac{2}{(2\pi)^2}e^{-|s|^2/D}\int_0^{2\pi} d\phi = \frac{1}{\pi}e^{-|s|^2/D}$$

Therefore $\|f_{\nu,\omega}\|_\sigma < +\infty\ \forall\,\sigma>0,\ \forall\,\nu\in\mathbb{Z}^2$.

Let now $\phi\in\mathbb{C}$. Writing:

$$\det Q(\gamma,\theta,\phi) = 1+\frac{A(\gamma,\theta)}{4}\sin^2(2\phi), \quad A(\gamma,\theta) := \kappa(1-\cos^2\theta)\ge 0$$

we get (omitting the elementary details):

$$\det Q(\gamma,\theta,\phi)\neq 0, \qquad |\mathrm{Im}\,\phi| < \frac{1}{4}\mathrm{arccosh}(1+8/\kappa)\,.$$

Therefore the function

$$\phi\mapsto\frac{e^{-\langle Q^{-1}(\gamma,\theta,\phi)s,s\rangle}}{\sqrt{\det Q(\gamma,\theta,\phi)}} := G_{\gamma,\theta,s}(\phi)$$

is analytic with respect to ϕ in the strip $|\mathrm{Im}\,\phi| < \frac{1}{4}\mathrm{arccosh}(1+8/\kappa) := m(\kappa)$ uniformly with respect to $(\gamma,\theta,s)\in[\delta_1,\delta_2]\times[0,2\pi]\times\mathbb{R}^2$.

In turn the analyticity entails, as is well known, that for any $0<\eta<m(\kappa)$ there exists $\rho_1 > m(\kappa)-\eta$ independent of $(\gamma,\theta,s)\in[\delta_1,\delta_2]\times[0,2\pi]\times\mathbb{R}^2$ such that

$$|\hat f_{\nu,\gamma,\theta}(s)| \le \sup_{|\mathrm{Im}\,\phi|\le\eta}|G_{\gamma,\theta,s}(\phi)|e^{-\rho_1|\nu|}.$$

Since $\det Q(\gamma,\theta,\phi)\neq 0$ for $|\mathrm{Im}\,\phi|\le\eta$, there exist $K_1(\eta) >, K_2(\eta) > 0$ independent of (γ,θ) such that:

$$|\langle Q^{-1}(\gamma,\theta,\phi)s,s\rangle| \ge K_1|s|^2, \quad \frac{1}{|\sqrt{\det Q(\gamma,\theta,\phi)}|} < K_2(\eta)$$

and therefore

$$|\hat{f}_{\nu,\gamma,\theta}(s)| \le \frac{K_2(\eta)}{2\pi} e^{-K_1|s|^2} e^{-\rho_1|\nu|}.$$

This in turn entails the existence of $K_3(\eta) > 0$ independent of ν such that, $\forall\, \sigma > 0$:

$$\|f_{\nu,\omega}\|_\sigma = \int_{\mathbb{R}^2} e^{\sigma|s|} |\hat{f}_{\nu,\gamma,\theta}(s)|\, ds \le K_3 e^{-\rho_1|\nu|}.$$

Hence, $\forall\, 0 < \rho < \rho_1$:

$$\|f\|_{\omega,\rho,\sigma} = \sum_{\nu\in\mathbb{Z}^2} e^{\rho|\nu|} \|f_{\nu,\omega}\|_\sigma < K(\eta)$$

for some $K(\eta) > 0$ independent of $\omega \in \Gamma$. We can thus conclude that

$$\|f\|_{\Gamma,\rho,\sigma} = \sup_{\omega\in\Gamma} \sum_{\nu\in\mathbb{Z}^2} e^{\rho|\nu|} \|f_{\nu,\omega}\|_\sigma < K$$

i.e., $f \in \mathcal{A}_{\Gamma,\rho,\sigma}$.

Remark

We have checked that $f \in \mathcal{A}_{\Gamma,\rho,\sigma}$. This entails $f \in \mathcal{F}_\sigma$. By the Paley-Wiener theorem, $f_{\phi,\omega}(u) = e^{-(Ax^2+Bx\xi+C\xi^2)}$ must have, $\forall\,(\phi,\omega)$, a holomorphic continuation $g_{\phi,\omega}(z_1,z_2)$ from $u = (x,\xi) \in \mathbb{R}\times\mathbb{R}$ to $z = (z_1,z_2) = (x+iy, \xi+i\eta) \in \mathbb{C}\times\mathbb{C}$. This holomorphic continuation is clearly

$$g_{\phi,\omega}(z_1,z_2) := e^{-Az_1^2+Bz_1z_2+Cz_2^2}.$$

$g_{\phi,\omega}(z_1,z_2)$ of course does not coincide with

$$f\circ\Psi_{\phi,\omega}((z_1,z_2)) = \exp\{-[|z_1\cos\phi + \frac{z_2}{\omega}\sin\phi|^2 + |z_2\cos\phi - \omega z_1\sin\phi|^2]\}$$

when $(y,\eta) \neq (0,0)$.

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