Hiatus perturbation for a singular Schrödinger operator with an interaction supported by a curve in $\mathbb{R}^3$

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We consider Schrödinger operators in $L^2(\mathbb{R}^3)$ with a singular interaction supported by a finite curve Γ. We present a proper definition of the operators and study their properties, in particular, we show that the discrete spectrum can be empty if Γ is short enough. If it is not the case, we investigate properties of the eigenvalues in the situation when the curve has a hiatus of length 2ϵ. We derive an asymptotic expansion with the leading term which a multiple of $\epsilon \ln \epsilon$.

1 Introduction

Singular Schrödinger operators with interactions supported by manifolds of a lower dimension are not a new topic; their properties were investigated already in the beginning of the nineties [BT] or even earlier in cases of a particular symmetry, see, e.g., [AGS, Sha]. Recently, a new motivation appeared when people realized that such operators with *attractive* interaction provide us with a model of "leaky" quantum graphs which have the nice properties of graph description of various nanostructures — see, e.g., the proceedings volumes [BCFK], [EKST], and references therein — but they are more realistic taking possible tunneling between the involved quantum wires into account.

A series of papers devoted to this problem started by [EI] and we refer to [Ex] for a bibliography; among the questions addressed were geometrically induced spectral properties [EI], scattering [EK05], approximations by point interaction Hamiltonians [EN, BO] or strong-coupling asymptotic behavior [EY02]. Another result concerns perturbations of such Hamiltonians caused by alterations of the interaction support. In [EY03] the asymptotic behavior for the eigenvalue shift was derived in the situation when the support is a manifold of codimension one with a "hole" which shrinks to zero, in particular, a curve in $\mathbb{R}^2$ with a hiatus; it was shown that in the leading order the perturbation acts as a repulsive δ interaction with the coupling strength proportional to the hole measure (in particular, the hiatus length).

The aim of this paper is to analyze the analogous question in the situation where the codimension of the manifold is two, specifically, for an interaction supported by a curve in $\mathbb{R}^3$. The extension is by far not trivial since the codimension of the singular interaction support influences properties of such Schrödinger operators substantially [AGHH]. In our particular case we know that to define such a Hamiltonian for a curve in $\mathbb{R}^3$ one cannot use, in contrast to the codimension one case, the "natural" quadratic form and has to resort to appropriate generalized boundary conditions [EK02].

We are going to demonstrate that the asymptotic behavior of the eigenvalues with respect to the hiatus length ϵ is of the following form,

$$\lambda_j(\epsilon) = \lambda_L - s_j(\lambda_L)\epsilon \ln \epsilon + o(\epsilon \ln \epsilon)\,, \quad j = m, ..., n\,,$$

where λ_L is an unperturbed eigenvalue of the Hamiltonian corresponding to the absence of the hiatus, the indices run through a basis in the corresponding eigenspace so that $n - m + 1$ is the multiplicity of λ_L, and $s_j(\lambda_L)$ are coefficients specified in Theorem 6.6. This shows that the asymptotics is in the case of codimension two is substantially different — recall that for codimension one the second term is linear in ϵ — due to the more singular interaction involved. The dependence on the codimension is manifested also in other ways. For instance, while a nontrivial and attractive interaction supported by *any* manifold of codimension one gives rise to bound states, in the situation discussed here a minimum curve length is needed to produce binding as we will demonstrate in Section 4.

2 Preliminaries

As mentioned in the introduction, we are interested in generalized Schrödinger operators with singular potentials supported by sets of lower dimensions. In our case the support of the singular potential will be a finite C^1 smooth curve in $\mathbb{R}^3$ of length L without self-intersections which may and may not be a loop; the corresponding Schrödinger operator can be formally written as

$$-\Delta - \tilde{\alpha}\delta(x-\Gamma) \quad \text{with} \quad \tilde{\alpha} < 0\,. \tag{2.1}$$

We mark the parameter in this formal expression by a tilde to stress that it is different from the true "coupling constant" which will introduce below. It is a natural requirement that the operator which gives a mathematical meaning to (2.1) should act as the Laplacian on the domain $C_0^\infty(\mathbb{R}^3\backslash\Gamma)$, which motivates us to look for self-adjoint extensions of the symmetric operator $-\dot{\Delta} : C_0^\infty(\mathbb{R}^3\backslash\Gamma) \mapsto L^2 := L^2(\mathbb{R}^3)$ such that $\dot{\Delta}f = \Delta f$. The deficiency indices of $-\dot{\Delta}$ are infinite, of course, and looking for operators giving a meaning to (2.1) we will restrict ourselves to a certain "local" one-parameter family of extensions which will be specified in the next section.

We have to say also something more about Γ and to introduce a family of auxiliary "comparison" curves in its vicinity. Since Γ is C^1 smooth by assumption it admits a parameterization by the arc length. This means that Γ is a graph of a C^1 function $\gamma : [0, L] \mapsto \mathbb{R}^3$ such that $|\dot{\gamma}(s)| = 1$, where $\dot{\gamma}$ stands for the derivative. Moreover, we assume that

(a) there exist $c > 0$ and $\mu > 1$ such that

$$|\gamma(s) - \gamma(t)| \geq |s-t|(1 - c|s-t|^\mu) \quad \text{for} \quad c|s-t|^\mu < 1\,.$$

If Γ is not a closed curve then, of course, one of its endpoints is given by $\gamma(0)$. However, if Γ is a loop then there is no such natural "starting point" and we assume that the above property is valid independently of the way the loop is parametrized.

We will say that a family of curves $\{\Gamma_d\}$ is neighboring with Γ if they are graphs of functions $\gamma_d : [0, L] \mapsto \mathbb{R}^3$ with the following properties for any $s \in [0, L]$ and d small enough

(b1) $|\gamma(s) - \gamma_d(s)| = d\,,$

(b2) $|\dot{\gamma}(s) - \dot{\gamma}_d(s)| = \mathcal{O}(d)$ as $d \to 0$, (b3) $\gamma(s) - \gamma_d(s)$ is perpendicular to $\mathrm{t}_d(s) := \dot{\gamma}_d(s)$;

the error term is assumed to be uniform on $[0, L]$. For instance, if the Frénet frame $(\mathrm{t}, \mathrm{n}, \mathrm{b})$ is defined globally for Γ then any family of "shifted" curve defined as the graphs of

$$\gamma + \eta_1 \mathrm{n} + \eta_2 \mathrm{b} : [0, L] \mapsto \mathbb{R}^3 \,, \quad |\eta| = \sqrt{\eta_1^2 + \eta_2^2} = d \,,$$

is neighboring with Γ.

3 Definition of Hamiltonian and its resolvent

In this section we shall construct an operator corresponding to the formal expression (2.1). As mentioned above, it will be defined as a self-adjoint extension of $-\dot{\Delta}$. To this aim we will follow the scheme proposed by Posilicano [Po01, Po04] which generalizes the standard Krein's theory. The self-adjoint extensions are parametrized in it by Birman-Schwinger-type operators entering into expression of their resolvents. As usual, such a resolvent consists of a 'free' term and a 'perturbative' remainder. Since we are in three dimensions, the 'free' resolvent is given by $R_z = (-\Delta - z)^{-1} : L^2 \mapsto L^2$, $z \in \rho(-\Delta)$, which is an integral operator with the kernel

$$G_z(x, y) = \frac{1}{4\pi} \frac{\mathrm{e}^{-\sqrt{-z}|x-y|}}{|x - y|} \,. \tag{3.1}$$

As there is no risk of confusion we will use the same notation $G_z(\cdot)$ for the function of a scalar argument, i.e. $G_z(\rho) = \mathrm{e}^{-\sqrt{-z}|\rho|}(4\pi|\rho|)^{-1}$, $\rho \in \mathbb{R} \setminus \{0\}$.

To construct the second term of the resolvent we need an embedding to the Hilbert space associated with the support of our singular potential. Such a space is naturally defined by $L^2(\mathbb{R}^3, \mu_\Gamma)$, where μ_Γ denotes the Dirac measure on Γ. It is convenient, however, to use a natural identification $L^2(\mathbb{R}^3, \mu_\Gamma) \simeq L^2(I)$, $I \equiv (0, L)$ which we will do in the following. Is is well known that R_z defines a unitary map between L^2 and $W^{2,2}(\mathbb{R}^3) \equiv W^{2,2}$. Moreover, with reference to the Sobolev theorem we claim that the trace operator $\tau : W^{2,2} \to L^2(I)$ is continuous, and consequently, the following operators,

$$\mathrm{R}_z := \tau R_z : L^2 \to L^2(I) \,, \quad \mathrm{R}_z^* : L^2(I) \to L^2 \,,$$

where R_z^* is the adjoint to R_z, are continuous as well.

3.1 Birman–Schwinger operator

The mentioned Birman-Schwinger operators are defined as symmetric operators Θ_z in $L^2(I)$ parameterized by $z \in \rho(-\Delta)$ and satisfying the pseudo-resolvent equivalence

$$\Theta_w - \Theta_z = (w - z)\mathrm{R}_w\mathrm{R}_z^* \quad \text{for} \quad w\,, z \in \rho(-\Delta)\,. \tag{3.2}$$

Furthermore, if the set $Z := \{z \in \rho(-\Delta) : \Theta_z^{-1} \text{ exists and is bounded}\}$ is nonempty then the following operator

$$R_{z;\alpha} = R_z - \mathrm{R}_z(\Theta_z)^{-1}\mathrm{R}_z^* \quad \text{for} \quad z \in Z \tag{3.3}$$

defines the resolvent of certain self-adjoint extension of $-\dot{\Delta}$, cf. [Po01]. Our aim is now to find such an operator Θ_z satisfying (3.2) and corresponding to the singular potential defined by a certain coupling constant. The explicit form of such operator was discussed by [BT] but for our purpose it is useful to derive the other, albeit equivalent form of Θ_z (recall that in the mentioned paper the potential was defined more generally, as a function on I; in our model it is just a "coupling" constant which we will denote as α.)

The most natural way of determining Θ_z would be to take the embedding of R_z to $L^2(I)$, as it is done on the codimension one case [BEKŠ]. However, the explicit formula for G_z, the kernel of R_z, shows that the expression $\tau\mathrm{R}_z^*$ does not make sense because G_z has a singularity; to make use of the approach sketched above the singularity has to be removed by an appropriate regularization. To put it differently, the operator τ can not be canonically extended onto L^2 which is the range of R_z^*. On the other hand, to preserve the equivalence (3.2) we have to consider a special type of regularization which does not depend on z. Using standard facts from the Sobolev space theory [RS] we claim that $\mathrm{R}_z^* f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma) \cap C^\infty(\mathbb{R}^3 \setminus \Gamma)$ for $f \in L^2(I)$, thus the embedding $\mathrm{R}_z^* f \restriction_{\Gamma_d}$ is a C^∞ function on I; recall that Γ_d was introduced in Section 2 as a neighbooring curve with Γ. With these facts in mind we introduce a logarithmic regularization defined through the pointwise limits

$$\breve{f}(s) = \lim_{d \to 0} \left[\mathrm{R}_z^* f \restriction_{\Gamma_d}(s) + \frac{1}{2\pi} f(s) \ln d \right] \quad \text{for} \quad s \in I\,. \tag{3.4}$$

By virtue of the following lemma, the relation (3.4) defines a function belonging to $L^2(I)$ provided $f \in W^{1,2}(I)$.

Lemma 3.1 *The operator Q_z defined by the relation*

$$(Q_z f)(s) := \frac{1}{4\pi}\left(\int_I \frac{f(t)-f(s)}{|t-s|}\,\mathrm{d}t + f(s)\ln 4s(L-s)\right) + \int_I \mathcal{R}_z(s,t)f(t)\mathrm{d}t\,, \tag{3.5}$$

where

$$\mathcal{R}_z(s,t) := G_z(\gamma(s)-\gamma(t)) - (4\pi|s-t|)^{-1}\,, \tag{3.6}$$

maps $W^{2,1}(I) \mapsto L^2(I)$ and $Q_z f = \breve{f}$.

In the following we will also employ the decomposition of the kernel of the last term in (3.5), namely $\mathcal{R}_z(s,t) = \mathcal{A}_z(|s-t|) + \mathcal{D}_z(s,t)$, where

$$\mathcal{A}_z(\rho) := G_z(\rho) - (4\pi|\rho|)^{-1}\,, \quad \mathcal{D}_z(s,t) := G_z(\gamma(s)-\gamma(t)) - G_z(s-t)\,. \tag{3.7}$$

Before starting the proof of the lemma let us make a couple of comments. The operator Q_z defined on the space $W^{1,2}(I)$ with the topology inherited from $L^2(I)$ is essentially self-adjoint. Taking its closure we obtain its unique self-adjoint extension in $L^2(I)$ for which we will use the same notation.

Furthermore, Q_z satisfies the relation (3.2). Indeed, let us note first that the first resolvent formula for R_z can be extended by the continuity to the equivalence $\mathrm{R}^*_w - \mathrm{R}^*_z = (w-z)R_w\mathrm{R}^*_z$. Since $R_w\mathrm{R}^*_z f$ is a continuous function as an element of $W^{2,2}$ we can take the limit

$$\lim_{d\to 0}(\mathrm{R}^*_w - \mathrm{R}^*_z)f \restriction_{\Gamma_d} = (w-z)R_w\mathrm{R}^*_z f \restriction_{\Gamma_d}\,, \quad f\in W^{1,2}(I)\,,\ w,z\in\rho(-\Delta)\,,$$

which consequently, in view of (3.4), gives

$$(Q_w - Q_z)f = (w-z)\mathrm{R}_w\mathrm{R}^*_z f\,. \tag{3.8}$$

Proof of Lemma 3.1. We break the argument into two parts:
Step 1: Assume first that Γ is a straight line segment, i.e. we have $\mathcal{D}_z = 0$. Let us decompose $\mathrm{R}^*_z f\restriction_{\Gamma_d}$ into the sum of two terms,

$$\mathrm{R}^*_z f\restriction_{\Gamma_d}(s) = \int_I G^d_z(s-t)f(t)\,\mathrm{d}t = \int_I \mathcal{S}^d(s-t)f(t)\,\mathrm{d}t + \int_I \mathcal{A}^d_z(s-t)f(t)\,\mathrm{d}t\,, \tag{3.9}$$

where $G^d_z(\rho) := G_z((d^2+\rho^2)^{1/2})$ and

$$S^d(\rho) := (4\pi(d^2+\rho^2)^{1/2})^{-1}\,, \quad \mathcal{A}^d_z(\rho) := G_z(\rho) - \mathcal{S}^d(\rho)\,.$$

The first term at the r.h.s of (3.9) can be rewritten as follows

$$\int_I \mathcal{S}^d(s-t)f(t)\mathrm{d}t = \int_I (f(t)-f(s))\mathcal{S}^d(s-t)\mathrm{d}t + f(s)\int_I \mathcal{S}^d(s-t)\mathrm{d}t\,. \quad (3.10)$$

The integrated function in the first term at the r.h.s. of the last relation can be bounded by $|f(t)-f(s)|(4\pi|t-s|)^{-1}$ which belongs to $L^2(I)$ in view of the fact that $f\in W^{1,2}(I)$. Hence employing Lebesque's dominated convergence theorem we can conclude that the first term at the r.h.s. of (3.10) tends to $\int_I (f(t)-f(s))(4\pi|t-s|)^{-1}\,\mathrm{d}t$ for $d\to 0$. To handle the second term we decompose it integrating separately along $I_\delta = I_{\delta,s} := \{t\in I : |t-s|<\delta\}$ and $I_\delta^c := I\backslash I_\delta$ for δ small enough. As a result we arrive at

$$\int_{I_\delta} \mathcal{S}^d(s-t)\,\mathrm{d}t = \frac{1}{2\pi}\left[\ln\omega_d(\delta)-\ln d\right], \quad \omega_d(a) := a+(a^2+d^2)^{1/2}\,, \quad (3.11)$$

$$\int_{I_\delta^c} \mathcal{S}^d(s-t)\,\mathrm{d}t = \frac{1}{4\pi}\ln\omega_d(s)\,\omega_d(L-s) - \frac{1}{2\pi}\ln\omega_d(\delta)\,. \quad (3.12)$$

Combining the above results with (3.10) we get

$$\begin{aligned}\lim_{d\to 0}&\left(\int_I \mathcal{S}^d(s-t)f(t)\,\mathrm{d}t + \frac{1}{2\pi}f(s)\ln d\right)\\ &= \frac{1}{4\pi}\left(\int_I \frac{f(t)-f(s)}{|t-s|}\,\mathrm{d}t + f(s)\ln 4s(L-s)\right)\,. \end{aligned} \quad (3.13)$$

To obtain the result we have to handle the limit $\int_I \mathcal{A}_z^d(s,t)f(t)$ as $d\to 0$, cf. (3.9). Using the dominated convergence again we find

$$\lim_{d\to 0}\int_I \mathcal{A}_z^d(s,t)f(t)\mathrm{d}t = \int_I \mathcal{A}_z(s,t)f(t)\mathrm{d}t\,, \quad (3.14)$$

which reproduces the remaining term at the r.h.s. of (3.5). Putting together (3.13) and (3.14) we arrive at the sought result,

$$\begin{aligned}\lim_{d\to 0}&\left(\mathrm{R}_z^* f\restriction_{\Gamma_d}(s) + \frac{1}{2\pi}f(s)\ln d\right)\\ &= \frac{1}{4\pi}\left(\int_I \frac{f(t)-f(s)}{|t-s|}\,\mathrm{d}t + f(s)\ln 4s(L-s)\right) + \int_I \mathcal{A}_z(s,t)f(t)\mathrm{d}t\,.\end{aligned}$$

Step 2: Consider next the general case when Γ is a finite curve which may and may not be closed. Then we employ the decomposition

$$\begin{aligned}\lim_{d\to 0}(\mathrm{R}_z^* f\restriction_{\Gamma_d}(s)) &= \int_I G_z(\gamma_d(s)-\gamma(t))f(t)\,\mathrm{d}t \\ &= \int_I G_z^d(s-t)f(t)\,\mathrm{d}t + \int_I \mathcal{D}_z^d(s,t)f(t)\,\mathrm{d}t\,,\end{aligned}$$

where $\mathcal{D}_z^d(s,t) := G_z(\gamma_d(s)-\gamma(t)) - G_z^d(s-t)$ and γ_d is the function whose graph is the neighbooring curve Γ_d with Γ. Using the result of the first step and the limit

$$\lim_{d\to 0}\int_I \mathcal{D}_z^d(s,t)f(t)\,\mathrm{d}t = \int_I \mathcal{D}_z(s,t)f(t)\,\mathrm{d}t$$

discussed in Remark 8.2 below we get the claim, concluding thus the proof of the lemma. ∎

Now we can express the *resolvent of the Hamiltonian*. As was already mentioned one can parameterize self-adjoint extensions of certain symmetric operators by means of operators satisfying the pseudo-resolvent formula. In our model we are specifically interested in extensions of $-\dot{\Delta} : C_0^\infty(\mathbb{R}^3\setminus\Gamma) \mapsto L^2 := L^2(\mathbb{R}^3)$. The operators $\Theta_z = Q_z - \alpha$ are suitable candidates for the role of Birman–Schwinger operators because they satisfy the relation (3.2). The parameter $\alpha\in\mathbb{R}$ appearing here will be referred to as the coupling constant. It is certainly different from the $\tilde{\alpha}$ appearing in (2.1); it is enough to notice that the absence of the interaction is associated with the value $\alpha = \infty$.

To complete the argument one has to make an *a posteriori* claim that the set Z is nonempty, which will be done in Section 4 below. Summing up the discussion, the operator

$$R_{z;\alpha} = R_z - \mathrm{R}_z(Q_z-\alpha)^{-1}\mathrm{R}_z^* \quad \text{for} \quad z\in Z \tag{3.15}$$

is in view of the mentioned result in [Po01] the resolvent of a self-adjoint extension of $-\dot{\Delta}$. We will regard it as a rigorous counterpart of the formal Hamiltonian (2.1) and denote it in the following as $H_{\alpha,\Gamma}$.

3.2 Alternative forms of Q_z

The need to introduce a renormalization makes the use of Birman–Schwinger approach more complicated than in the codimension one case. In addition, the way we have chosen above, with the limit taken over a family of comparison curves "parallel" to the entire Γ is not a particularly elegant one. It is possible to think of other regularizations defining Q_z by

$$\int_I G_z(\tilde{\gamma}_d(s) - \gamma(t)) f(t)\, \mathrm{d}t + \frac{1}{2\pi} f(s) \ln d\,,$$

where $\tilde{\gamma}_d$ correspond to another curve family. One possibility is to consider curves which coincide with Γ everywhere except in the vicinity of the singularity, the point with $s = t$, where they have a recess the size of which is controlled by the parameter d. To describe this and other possible regularizations we will look at a more general class into which all of them fit.

Given s we consider a family of C^2 curves $\tilde{\Gamma}_{d,s}$ which are graphs of $\tilde{\gamma}_{d,s} \equiv \tilde{\gamma}_d : [0, L] \mapsto \mathbb{R}^3$ with $d := |\tilde{\gamma}_d(s) - \gamma_d(s)| = \|\tilde{\gamma}_d - \gamma\|_\infty$. The assumptions (b) of Section 2 will be then replaced by the following modified ones; for any $t \in [0, L]$ and d small enough,

($\widetilde{\mathrm{b}1}$) $|\gamma(t) - \tilde{\gamma}_d(t)| = \mathcal{O}(d)$ as $d \to 0$,

($\widetilde{\mathrm{b}2}$) $|\dot{\gamma}(t) - \dot{\tilde{\gamma}}_d(t)| = \mathcal{O}(d)$ as $d \to 0$,

($\widetilde{\mathrm{b}3}$) $\gamma(s) - \tilde{\gamma}_d(s)$ is perpendicular to $\tilde{\mathrm{t}}_d(s) := \dot{\tilde{\gamma}}_d(s)$,

where we suppose also that the error terms are uniform on $[0, L]$. Let us stress that, in distinction to Γ_d, the curve $\Gamma_{d,s}$ is in general *not* parameterized by its arc length. Then we have the following theorem the proof of which we postpone to Section 8.

Theorem 3.2 *Under the stated assumptions,*

$$(Q_z f)(s) = \lim_{d\to 0} \left[\int_I G_z(\gamma(s) - \tilde{\gamma}_d(t)) j_d(t) f(t)\, dt + \frac{1}{2\pi} f(s) \ln d \right], \qquad (3.16)$$

where $j_d(s) := \left(\sum_{i=1}^3 (\dot{\tilde{\gamma}}_{d,i}(s)^2) \right)^{1/2}$.

4 Existence of bound states

We have said that in distinction to the codimension one case an attractive interaction supported by a finite curve may not induce bound states. The aim of this section is to make this claim precise and to find conditions under which the Hamiltonian $H_{\alpha,\Gamma}$ has a nonempty discrete spectrum. Since the singular potential in our model is supported by a compact set it is easy to check the stability of the essential spectrum,

$$\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = \sigma_{\mathrm{ess}}(-\Delta) = [0,\infty)\,,$$

see [BT]. This means that the negative halfline can contain only the discrete spectrum $\sigma_{\mathrm{d}}(H_{\alpha,\Gamma})$, and consequently the set Z of (3.3) is nonempty. Looking for negative eigenvalues we put $z=\lambda$ with $\lambda<0$. We employ the Birman–Schwinger philosophy, specifically the following result,

$$\lambda \in \sigma_{\mathrm{d}}(H_{\alpha,\Gamma}) \;\Leftrightarrow\; \ker(Q_\lambda - \alpha) \neq \{0\}\,, \tag{4.1}$$

where the multiplicity of λ is equal to $\dim\ker(Q_\lambda-\alpha)$ – cf. [Po04]. In addition, the eigenfunctions of $H_{\alpha,\Gamma}$ corresponding to an eigenvalue λ are given by

$$\psi_\lambda = \mathrm{R}^*_\lambda \phi_\lambda\,, \quad \text{where} \quad \phi_\lambda \in \ker(Q_\lambda-\alpha)\,. \tag{4.2}$$

It is well-known that a point interaction in $\mathbb{R}^2$ always attractive, i.e. it gives rise for any $\alpha\in\mathbb{R}$ to exactly one bound state with the eigenvalue

$$\xi_0 = \xi_0(\alpha) = -4\mathrm{e}^{2(-2\pi\alpha+\psi(1))}\,, \tag{4.3}$$

where $\psi(1) = -0,577...$ is Euler–Mascheroni constant, cf. [AGHH]. Asking about existence of bound states in our model, one may naively expect the same behavior as the perturbation is again of codimension two. It appears, however, that it is not so due to the presence of the third dimension which makes a finite curve in $\mathbb{R}^3$ "more singular" than a point in $\mathbb{R}^2$. We will show that if the length of curve is small enough then our system has no bound states. We need the following result auxiliary result.

Lemma 4.1 *Suppose that $\sigma_{\mathrm{d}}(H_{\alpha,\Gamma})\neq\emptyset$. Then the ground-state eigenvalue $\lambda_0=\min\{\lambda\in\sigma_{\mathrm{d}}(H_{\alpha,\Gamma})\}$ is simple and the corresponding eigenfunction $\psi_0 := \psi_{\lambda_0}$ is a multiple of a positive function.*

Proof. We will employ the form associated with $-Q_z+\alpha$, cf. [BT],

$$\varsigma_z[f] = -\frac{1}{2}\int_{I\times I} |f(s)-f(t)|^2\, G_z(\gamma(s)-\gamma(t))\,\mathrm{d}t\mathrm{d}s - \int_I |f(s)|^2 (a_z(s)+\alpha)\,\mathrm{d}s\,, \tag{4.4}$$

where, with the notation introduced above,

$$a_z(s) := -\int_{I_\delta} G_z(\gamma(s)-\gamma(t))\,\mathrm{d}t \\ +\int_{I_\delta^c}\left(\frac{1}{4\pi|s-t|}-G_z(\gamma(s)-\gamma(t))\right)\mathrm{d}t-\frac{1}{2\pi}\log 2\delta\,.$$

Let us note that the inequality $||f(s)|-|f(t)||\le|f(s)-f(t)|$ implies

$$\varsigma_z[|f|]\le\varsigma_z[f]\,.$$

For a fixed $z<\inf\sigma(H_{\alpha,\Gamma})$ the form ς_z is strictly positive. Using the Beurling–Deny criterion together with the other results from [RS, vol. II, p. 204] we find that $(-Q_z+\alpha)^{-1}$ is positivity preserving. Moreover, the operators R_z, R_z, and R_z^* are positivity improving because they are defined by means of the kernel which is strictly positive. Hence referring to (3.15) we conclude that the resolvent $R_{z;\alpha}$ of $H_{\alpha,\Gamma}$ is positivity improving, and using [RS] again we get the sought positivity of ψ_0. ∎

We begin the discussion concerning the existence of bound state by analyzing the simplest case, namely the situation when Γ is a line segment.

Lemma 4.2 *Suppose that Γ is a finite line segment of length L. If $L<2\mathrm{e}^{2\pi\alpha}$ then there $H_{\alpha,\Gamma}$ has no bound states. On the other hand, if $L>2\pi\mathrm{e}^{2\pi\alpha-\psi(1)}$ then there exists at least one bound state.*

Proof. In order to prove the absence of bound states under the condition $L<2\mathrm{e}^{2\pi\alpha}$ it suffices in view of (4.1) to show that

$$\sup\sigma(Q_\lambda)<\frac{1}{2\pi}\ln\frac{L}{2}\,. \tag{4.5}$$

It is clear that the value $\sup\sigma(Q_\lambda)$ is achieved by $(Q_\lambda\phi_0,\phi_0)$, where $\phi_0\in\ker(Q_{\lambda_0}-\alpha)$ is the normalized function corresponding to the ground state ψ_0 by the relation $\psi_0=\mathrm{R}_{\lambda_0}^*\phi_0$, cf. (4.2). Using the expression of Q_λ given by Lemma 3.1 we get the following asymptotics,

$$\psi_0\restriction_{\Gamma_d}(s)=\mathrm{R}_{\lambda_0}^*\phi_0\restriction_{\Gamma_d}(s)\approx-\frac{1}{2\pi}\phi_0(s)\ln d-(Q_{\lambda_0}\phi_0)(s)\quad\text{as}\quad d\to 0\,,\; s\in I\,. \tag{4.6}$$

Since ψ_0 can be chosen positive by Lemma 4.1 and $\phi_0 \in W^{1,2}(I)$, as we will demonstrate in Section 5 below, we come to the conclusion that ϕ_0 is positive as well because the leading term of (4.6) is determined by ϕ_0. Thus to estimate $\sup \sigma(Q_\lambda)$ it is sufficient to consider the expression $(Q_\lambda f, f)$ for positive functions f only. Using the relation (3.5) again we find

$$\begin{aligned}(Q_\lambda f, f) = \int_{I\times I} \xi_f(s,t)\,\mathrm{d}s\mathrm{d}t + \int_{I\times I} \mathcal{A}_\lambda(s-t) f(s) f(t)\,\mathrm{d}s\mathrm{d}t \\ + (4\pi)^{-1} \int_I f(s)^2 \ln 4s\,\mathrm{d}s\,, \end{aligned} \tag{4.7}$$

where

$$\xi_f(s,t) := \frac{(f(t)-f(s))f(s)}{4\pi|s-t|}\,.$$

A straightforward calculation yields the estimate

$$\xi_f(s,t) - \xi_f(t,s) = -\frac{(f(t)-f(s))^2}{4\pi|s-t|} \le 0\,,$$

which in turn leads to the following inequality

$$\begin{aligned}\int_{I\times I} \xi_f(s,t)\,\mathrm{d}s\mathrm{d}t &= \int_I \int_{s<t} \xi_f(s,t)\,\mathrm{d}s\mathrm{d}t + \int_I \int_{s>t} \xi_f(s,t)\,\mathrm{d}s\mathrm{d}t \\ &= \int_I \int_{s<t} (\xi_f(s,t) + \xi_f(t,s))\,\mathrm{d}s\mathrm{d}t \le 0\,.\end{aligned}$$

On the other hand, we have $\mathcal{A}_\lambda(\rho) \le 0$ and the only positive contribution to $(Q_\lambda f, f)$ comes from the last term of (4.7). Finally, the inequality (4.5) is a consequence of

$$\sup_{s\in I} (4\pi)^{-1} \ln s(L-s) = (2\pi)^{-1} \ln L/2\,.$$

The second part of the lemma, the condition for the *existence* of bound states can be obtained by the Dirichlet bracketing. To be more precise, assume that $\Gamma := \{(s,0,0)\,, s\in[0,L]\}$ and denote by $H^D_{\alpha,\Gamma}$ the Laplace operator with singular potential on Γ and Dirichlet boundary conditions at the planes $(0,y,z)$ and (L,y,z) with $y,z \in \mathbb{R}$; it is well known [RS, Sec. XIII.15] that

$$\inf \sigma(H_{\alpha,\Gamma}) \le \inf \sigma(H^D_{\alpha,\Gamma})\,. \tag{4.8}$$

Furthermore, using a simple separation of variables we find $\inf \sigma(H^D_{\alpha,\Gamma}) = -4\mathrm{e}^{2(-2\pi\alpha+\psi(1))} + L^{-2}$. The operator $H^D_{\alpha,\Gamma}$ has always a ground state with the eigenvalue which becomes negative for a fixed α and L large enough. Since the essential spectrum of $H_{\alpha,\Gamma}$ is $\mathbb{R}^+$, by (4.8) and the minimax principle the Hamiltonian $H_{\alpha,\Gamma}$ has at least one discrete eigenvalue as well; working out the negativity condition quantitatively we arrive at the conclusion. ∎

Remark 4.3 The method we have used in the proof is not particularly precise which explains the gap of $\pi\, e^{-\psi(1)} \approx 5.56$ in the ratio of the lengths L for which the existence and nonexistence of the discrete spectrum were established above.

Let us discuss next the general situation and consider a nontrivial curve which again may or may not be closed. To be concrete we consider a family of curves which are connected subsets of a fixed Γ corresponding to different subintervals of the arc length parameter. The deviation of each such curve from the corresponding straight segment is measured by the quantity $\mathcal{D}_\lambda \neq 0$ given by (3.7). Since $|\gamma(s)-\gamma(t)| \leq |s-t|$ in view of the used parameterization and the function $\rho \mapsto \mathrm{e}^{-\rho}/\rho$ is decreasing we find that $\mathcal{D}_\lambda > 0$ holds on an open set, and moreover

$$\mathcal{D}_\lambda(s,t) \leq \frac{1}{4\pi}\left(\frac{1}{|\gamma(s)-\gamma(t)|} - \frac{1}{|s-t|}\right). \tag{4.9}$$

Using the assumption (a) and mimicking the argument of [EK02] one can show that the operator with kernel defined by the r.h.s. of (4.9) is bounded (or even Hilbert-Schmidt) and denote its norm as D, (see also Remark 8.3). Proceeding as in the proof of Lemma 4.2 we arrive at the conclusion that the operator $H_{\alpha,\Gamma}$ has no bound states if $L < 2\mathrm{e}^{2\pi\alpha-D}$. On the other hand, using arguments borrowed from [EK02] we can claim that in the case $L > 2\pi\mathrm{e}^{2\pi\alpha-\psi(1)}$ the bound states do not disappear when a segment is replaced by a curve of the same length, since the bending acts as an effective attractive interaction. Summarizing this discussion we have the following result.

Theorem 4.4 *For a fixed $\alpha \in \mathbb{R}$ in the described situation, there exists $L_\alpha > 0$ such that the operator $H_{\alpha,\Gamma}$ has no discrete spectrum for $L < L_\alpha$. On the other hand, if $L > 2\pi\mathrm{e}^{2\pi\alpha-\psi(1)}$ then there is at least one bound state.*

5 Regularity of eigenfunction

Before we proceed to our main result we need as a preliminary to investigate the regularity of $\phi \in \ker(Q_{\lambda_L} - \alpha)$, where λ_L is an eigenvalue of $H_{\alpha,\Gamma}$; specifically we will demonstrate that this function belongs to $W^{1,2}$. The proof of this claim is involved and we divide it into several steps. To simplify the presentation we will show first the regularity of the corresponding eigenfunction in the case when Γ is a loop, and then we will comment on an extension of the result. The idea is to compare the loop with a circle of the same length. Suppose Γ is a closed curve satisfying the assumptions of Section 2 and Γ^c is a circle of the length L; up to Euclidean transformations, Γ^c is thus the graph of the function $\gamma^c(\cdot) = \frac{L}{2\pi}(\cos\frac{2\pi}{L}(\cdot), \sin\frac{2\pi}{L}(\cdot), 0) : [0,L] \mapsto \mathbb{R}^3$. The operator Q_z can be defined in analogy with (3.4), i.e.

$$Q_z = T^c_z + \mathcal{D}^c_z\,, \tag{5.1}$$

where

$$T^c_z f = \lim_{d\to 0}\left[\mathrm{R}^*_z f \restriction_{\Gamma^c_d} + \frac{1}{2\pi} f \ln d\right] \quad \text{for} \quad s \in (0,L) \tag{5.2}$$

and $\mathcal{D}^c_z$ is given by the kernel $\mathcal{D}^c_z(s,t) := G_z(\gamma(s)-\gamma(t)) - G_z(\gamma^c(s)-\gamma^c(t))$; in the above expression Γ^c_d stands for a neighbooring curve with Γ_c and the properties described in Section 2.

Lemma 5.1 *Assume that the assumption* (a) *is satisfied; then for any function $f \in L^2(I)$ we have $\mathcal{D}^c_z f \in W^{1,2}(I)$.*

The proof is quite technical and we postpone it to the appendix.

Lemma 5.2 *For $\phi \in \ker(Q_{\lambda_L} - \alpha)$ we have $(T^c_z - \alpha)\phi \in W^{1,2}(I)$.*

Proof. Using the pseudo-resolvent formula (3.8) for $w = \lambda_L$ and the fact that $\mathrm{R}_w \mathrm{R}^*_z \phi \in W^{1,2}(I)$ we get $(Q_z - \alpha)\phi \in W^{1,2}(I)$. Applying then the result of the previous lemma and the decomposition (5.1) we get the claim. ∎

This allows us finally to formulated the indicated result.

Proposition 5.3 *Any eigenfunction $\phi \in \ker(Q_{\lambda_L} - \alpha)$ belongs to $W^{1,2}(I)$.*

Proof. Using the radial symmetry valid for Γ^c one finds

$$T^c_z f = \sum_{k\in\mathbb{Z}} b_k(z) f_k \,\mathrm{e}^{i2\pi k(\cdot)/L}\,,$$

where f_k are Fourier coefficients of f and $b_k(z) \in \mathbb{C}$. Hence T_z^c commutes with the derivative operator D, which implies for $z \in \mathbb{C}^+$

$$\|D\phi\| \le C\|(T_z^c - \alpha)D\phi\| = C\|D(T_z^c - \alpha)\phi\| < \infty \,, \qquad (5.3)$$

where C is a positive constant; we have used the fact that $T_z^c - \alpha$ is invertible with a bounded inverse in combination with Lemma 5.2. The sought claim follows directly from (5.3). ∎

Remark 5.4 In a similar way one can deal with the situation when the curve Γ is not closed; the idea is to compare it to a circular segment. To be precise we introduce Γ^c which is, as before, a circle defined as the graph of $\gamma^c : [0, L+d] \mapsto \mathbb{R}^3$, $d > 0$ and its segment $\Gamma^{c,r}$ being the graph of $\gamma^{c,r} : [0, L] \mapsto \mathbb{R}^3$ such that $\gamma^{c,r}(s) = \gamma^c(s)$ for any $s \in [0, L]$. In analogy with (5.1) we can decompose the operator Q_z corresponding to Γ as

$$Q_z = T_z^{c,r} + \mathcal{D}_z^{c,r} \,,$$

where $T_z^{c,r}$ and $\mathcal{D}_z^{c,r}$ are defined as in (5.1) but by means of $\gamma^{c,r}$, with the variable appropriately restricted. The proofs of Lemmata 5.1, 5.2 can be mimicked directly for the operators $T_z^{c,r}$ and $\mathcal{D}_z^{c,r}$. On the other hand, the proof of Proposition 5.3 requires some comments. Given $\delta > 0$ let us introduce the natural embeddings $\breve{\mathcal{I}} : L^2(0, L+\delta) \mapsto L^2(0, L)$ and $\breve{\mathcal{I}}^* : L^2(0, L) \mapsto L^2(0, L+\delta)$. Using the explicit form of Q_z given by (3.5) we can easily check that $T_z^{c,r} = \breve{\mathcal{I}} T_z^c \breve{\mathcal{I}}^*$. Now can repeat the reasoning which leads to (5.3) but instead of the norm $\|\cdot\|$ in $L^2(0, L)$ we consider the norm $\|\cdot\|_\delta$ in $L^2(\delta, L-\delta)$, where $\delta > 0$ is a constant which can be made arbitrarily small; we get

$$\begin{aligned} \|D\phi\|_\delta \le \quad & C\|(T_z^{c,r} - \alpha)D\phi\|_\delta = C\|\breve{\mathcal{I}}(T_z^c - \alpha)\breve{\mathcal{I}}^* D\phi\|_\delta = \\ & C\|D\breve{\mathcal{O}}(T_z^c - \alpha)\breve{\mathcal{O}}^*\phi\|_\delta < \infty \,. \end{aligned}$$

This means that Proposition 5.3 extends to the case when Γ is not a loop, by which the eigenfunction regularity is finally established generally.

6 A curve with a hiatus

Now we finally come to our main topic. In this section we consider the eigenvalue problem for a curve with a short hiatus. Suppose that we have the system with the singular interaction supported by a curve Γ of length L and satisfying the assumptions of Section 2. Naturally we have to exclude the trivial case assuming that $H_{\alpha,\Gamma}$ has bound states; we know from Theorem 4.4 a sufficient condition for that is $L > 2\pi e^{2\pi\alpha-\psi(1)}$. For simplicity we will suppose first that there is exactly one bound state with corresponding eigenvalue λ_L; the generalization will be provided at the end of this section.

Consider now a family of curves Γ_ϵ which coincides with Γ everywhere apart a short hiatus placed symmetrically w.r.t $x_0 = \Gamma(s_0)$, in other words, Γ_ϵ is a graph of function $\gamma_\epsilon : [0, s_0 - \epsilon) \cup (s_0 + \epsilon, L] \mapsto \mathbb{R}^3$ and $\gamma_\epsilon(s) = \gamma(s)$ for $s \in [0, s_0 - \epsilon) \cup (s_0 + \epsilon, L]$. In the following we will use the notations $I_\epsilon^c \equiv (0, s_0 - \epsilon) \cup (s_0 + \epsilon, L)$ and I_ϵ for $(s_0 - \epsilon, s_0 + \epsilon)$. Our aim is to derive asymptotics of eigenvalue $\lambda(\epsilon)$ of $H_{\alpha,\Gamma_\epsilon}$ for ϵ small. Of course, we may expect that $\lambda(\epsilon) \to \lambda_L$ for $\epsilon \to 0$. Since, as discussed above, the eigenvalue problem can be reduced in view of 4.1 to analysis of the Birman–Schwinger operator, we will seek the function $\lambda(\epsilon)$ such that $\ker(Q^\epsilon_{\lambda(\epsilon)} - \alpha)$ is nontrivial where Q^ϵ_λ denotes the Birman–Schwinger operator corresponding to Γ_ϵ. The first step towards that is to relate Q_λ and Q^ϵ_λ. It is convenient to introduce the natural embedding maps acting between $L^2(I)$ and $L^2(I^c_\epsilon)$. Let $\mathcal{I}_\epsilon$ stand for the canonical embedding from $L^2(I)$ to $L^2(I^c_\epsilon)$ and $\mathcal{I}^c_\epsilon$ for its adjoint acting from $L^2(I^c_\epsilon)$ to $L^2(I)$. We will also use the abbreviation $Q^{\epsilon c}_\lambda := \mathcal{I}^c_\epsilon Q^\epsilon_\lambda \mathcal{I}_\epsilon$.

Lemma 6.1 *The asymptotic expansion*

$$(Q^\epsilon_\lambda \mathcal{I}_\epsilon f, \mathcal{I}_\epsilon f) = (Q_\lambda f, f) + \frac{2}{\pi}|f(s_0)|^2 \epsilon \ln \epsilon + o(\epsilon \ln \epsilon) \tag{6.1}$$

holds for $\epsilon \to \infty$ and any $f \in D(Q_\lambda) \cap W^{1,2}(I)$.

Proof. Let us first note that for any $f \in L^2(I)$ such that $\mathcal{I}_\epsilon f \in D(Q^\epsilon_\lambda)$ we have $(Q^\epsilon_\lambda \mathcal{I}_\epsilon f, \mathcal{I}_\epsilon f) = (Q^{\epsilon c}_\lambda f, f)$ and $Q^{\epsilon c}_\lambda f$ can be decomposed as,

$$Q^{\epsilon c}_\lambda f = \lim_{d\to 0} \left[\int_{I^c_\epsilon} G_\lambda(\gamma_d(\cdot) - \gamma(t)) f(t) \mathrm{d}t + \frac{1}{2\pi} \ln d\, f \right] \chi^c_\epsilon = Q_\lambda f - Jf - J'f - Tf\,, \tag{6.2}$$

where

$$\begin{aligned} Jf &:= \left[\lim_{d\to 0} \int_{I_\epsilon} G_\lambda(\gamma_d(\cdot) - \gamma_d(t)) f(t) \mathrm{d}t \right] \chi^c_\epsilon\,, \\ J'f &:= \left[\lim_{d\to 0} \int_{I^c_\epsilon} G_\lambda(\gamma_d(\cdot) - \gamma(t)) f(t) \mathrm{d}t \right] \chi_\epsilon \end{aligned} \tag{6.3}$$

and

$$Tf = \lim_{d\to 0} \left[\int_{I_\epsilon} G_\lambda(\gamma^d(\cdot) - \gamma(t)) f(t)\,\mathrm{d}t + \frac{1}{2\pi} \ln d\, f \right] \chi_\epsilon\,.$$

The symbols χ_ϵ, χ_ϵ^c stand for the characteristic functions of I_ϵ and I_ϵ^c, respectively. Let us show how the last term of (6.2) emerges. In analogy with the proof of Lemma 3.1, see eq. (3.5), one shows that

$$\begin{aligned}(Tf)(s) &= \frac{1}{4\pi} f(s) \ln 4(s-s_0+\epsilon)(s_0-s+\epsilon)\chi_\epsilon(s) \\ &+ \left(\int_{I_\epsilon} \frac{f(t)-f(s)}{4\pi|s-t|}\,\mathrm{d}t + \int_{I_\epsilon} \mathcal{R}_\lambda(s,t) f(t)\,\mathrm{d}t \right) \chi_\epsilon(s)\,; \qquad (6.4)\end{aligned}$$

recall that $\mathcal{R}_\lambda(s,t) = \lim_{d\to 0} \mathcal{R}_\lambda{}^d(s,t) = \lim_{d\to 0}(G_\lambda(\gamma_d(s)-\gamma(t)) - S^d(s-t))$ and $S^d(s-t) = (4\pi(d^2+(s-t)^2)^{1/2})^{-1}$. Using the identity

$$\int_{I_\epsilon} \ln 4(s-s_0+\epsilon)(s_0-s+\epsilon)\mathrm{d}s = 8\epsilon \ln 2\epsilon$$

together with the expansion $f(s) = f(s_0) + o(1)$ for $s \sim s_0$, which can be performed in view of the fact that $f \in W^{1,2}(I)$ we obtain

$$(Tf, f) = \frac{2}{\pi} |f(s_0)|^2 \epsilon \ln \epsilon + \mathcal{O}(\epsilon)\,; \qquad (6.5)$$

note that the second and the third term of (6.4) can be uniformly bounded w.r.t. s, cf. Remark 8.3 below, and consequently, they contribute in (6.5) to the error term only. The latter depends on λ, however, it is important for us that it can be uniformly bounded together with its derivative being $\mathcal{O}(\epsilon)$.

Let us now consider the term Jf appearing in (6.2). Applying to (6.3) the decomposition $G_\lambda(\gamma_d(s) - \gamma(t)) = S^d(s-t) + \mathcal{R}_\lambda{}^d(s,t)$ we get by a straightforward computation

$$(Jf)(s) = \left((f(s_0) + o_\epsilon(1)) j_\epsilon(s) + \int_{I_\epsilon} \mathcal{R}_\lambda(s,t) f(t)\,\mathrm{d}t \right) \chi_\epsilon^c(s)\,, \qquad (6.6)$$

where the error term $o_\epsilon(1)$ means the asymptotics for $\epsilon \to 0$, and

$$j_\epsilon(s) := \lim_{d\to 0} \int_{I_\epsilon} S^d(s-t) f(s)\,\mathrm{d}s = \frac{1}{4\pi} \ln \frac{|s-s_0|+\epsilon}{|s-s_0|-\epsilon} \quad \text{for} \quad |s-s_0| > \epsilon\,.$$

Our aim is to estimate

$$(Jf,f) = (f(s_0)+o_\epsilon(1))\int_{I_\epsilon^c} j_\epsilon(s)\overline{f(s)}\,\mathrm{d}s + \int_{I_\epsilon^c}\int_{I_\epsilon} \mathcal{R}_\lambda(s,t)f(t)\overline{f(s)}\,\mathrm{d}t\mathrm{d}s\,. \tag{6.7}$$

By an analogous argument as in the first step of proof we can check that the last term of (6.7) contributes to $\mathcal{O}(\epsilon)$. To handle the first term at the r.h.s. of (6.7) we integrate by parts

$$\int_{I_\epsilon^c} j_\epsilon(s)\overline{f(s)}\mathrm{d}s = \hat{j}_\epsilon(s)\overline{f(s)}\mid_{I_\epsilon^c} - \int_{I_\epsilon^c}\hat{j}_\epsilon(s)\overline{f'(s)}\mathrm{d}s\,, \tag{6.8}$$

$$\hat{j}_\epsilon(s) := \frac{1}{4\pi}\sum_{k=\{-1,1\}} k(|s-s_0|-k\epsilon)\Big[\ln(|s-s_0|-k\epsilon)-1\Big]\frac{|s_0-s|}{s_0-s}\,.$$

Consequently, the first term of (6.8) takes the following form

$$\hat{j}_\epsilon(s)\overline{f(s)}\mid_{I_\epsilon^c} = -\frac{2}{\pi}\epsilon\,\ln\epsilon\overline{f(s_0)}+o(\epsilon\ln\epsilon) \quad \text{for} \quad s\in I_\epsilon^c\,.$$

Furthermore, the second term can be estimated as

$$\Big|\int_{I_\epsilon^c}\hat{j}_\epsilon(s)\overline{f'(s)}\,\mathrm{d}s\Big| \le \|\hat{j}_\epsilon\|_{L^2(I_\epsilon^c)}\|f\|_{W^{1,2}(I)}\,.$$

One can check directly that $\|\hat{j}_\epsilon\|_{L^2(I_\epsilon^c)} = o(\epsilon\ln\epsilon)$. Summarizing, we get

$$\int_{I_\epsilon^c} j_\epsilon(s)\overline{f(s)}\,\mathrm{d}s = -\frac{2}{\pi}\epsilon\ln\epsilon\overline{f(s_0)}+o(\epsilon\ln\epsilon)\,,$$

and consequently, $(Jf,f) = -\frac{2}{\pi}|f(s_0)|^2\epsilon\ln\epsilon + o(\epsilon\ln\epsilon)$. Using the fact that $(Jf,f)=(J'f,f)$ in combination with (6.5) we get the claim. ∎

With the above lemma we are ready to demonstrate the following result.

Lemma 6.2 *The eigenvalues of Q_λ^ϵ tend to the eigenvalues of Q_λ for $\epsilon\to 0$. Moreover, if ϵ and $\lambda-\lambda_L$ are small enough the operator Q_λ^ϵ has an eigenvalue $\eta(\lambda,\epsilon)$ which tends to α as $\epsilon\to 0$ and $\lambda\to\lambda_L$.*

Proof. Since $Q_\lambda^{\epsilon c}$ is the natural embedding of Q_λ^ϵ to space $L^2(I)$ it suffices to show the claim for $Q_\lambda^{\epsilon c}$. Let us make the following decomposition

$$(Q_\lambda^{\epsilon c} f, f) = ((Q_\lambda^{\epsilon c} f, f) - (Q_\lambda f, f)) + ((Q_\lambda f, f) - (Q_{\lambda_L} f, f)) + (Q_{\lambda_L} f, f)\,. \tag{6.9}$$

The convergence of the first term at the r.h.s. of (6.9) is proved in the previous lemma, precisely we have $0 < (Q_\lambda f, f) - (Q_\lambda^{\epsilon c} f, f) \to 0$ for $\epsilon \to 0$; combining this with the results of [Ka, Chap. XIII] we arrive at the first statement of the lemma. Moreover, using pseudo-resolvent identity (3.8) we get that $Q_\lambda - Q_{\lambda_L} \to 0$ for $\lambda \to \lambda_L$ and the convergence is understood in the norm sense. Since α is an eigenvalue of Q_{λ_L} we get the final claim. ∎

Relying on the last lemma and 4.1 we state that the eigenvalue of $H_{\alpha,\Gamma_\epsilon}$ approaches the eigenvalue of $H_{\alpha,\Gamma}$. Furthermore, for ϵ and $\lambda - \lambda_L$ small enough we can introduce the eigenprojector P_λ^ϵ onto the spaces spanned by the eigenvectors of Q_λ^ϵ corresponding to $\eta(\lambda, \epsilon)$. In the following we will use the representation of P_λ^ϵ by means of the resolvent of Q_λ^ϵ, i.e.

$$P_\lambda^\epsilon = \frac{1}{2\pi i} \oint_C R_\lambda^\epsilon(z)\, \mathrm{d}z \quad \text{with} \quad R_\lambda^\epsilon(z) := (Q_\lambda^\epsilon - z)^{-1} \tag{6.10}$$

and $C := \{\alpha + r\mathrm{e}^{i\varphi} :\ \varphi \in [0, 2\pi)\,, 0 < r < |\alpha|\}$. Furthermore, $R_\lambda^\epsilon(z)$ satisfies a first-resolvent-type identity of the following form

$$R_\lambda^\epsilon(z) = \mathcal{I}_\epsilon R_\lambda(z) \mathcal{I}_\epsilon^c + R_\lambda^\epsilon(z)(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) R_\lambda(z) \mathcal{I}_\epsilon^c\,. \tag{6.11}$$

According to the previous discussion the eigenvalue $\lambda(\epsilon)$ is a zero of the function $\eta(\lambda, \epsilon) - \alpha$ by (4.1), i.e. we have $\eta(\lambda(\epsilon), \epsilon) - \alpha = 0$. Thus to derive the asymptotics of $\lambda(\epsilon)$ the most natural way is employ the implicit function theorem which requires to know the asymptotics of $\eta(\lambda, \epsilon)$. Let us note that

$$\eta(\lambda, \epsilon) = (Q_\lambda^\epsilon P_\lambda^\epsilon \mathcal{I}_\epsilon \phi, P_\lambda^\epsilon \mathcal{I}_\epsilon \phi) \|P_\lambda^\epsilon \mathcal{I}_\epsilon \phi\|^{-2}\,, \tag{6.12}$$

where $\phi \in \ker(Q_{\lambda_L} - \alpha)$. To recover the asymptotics of $\eta(\lambda, \epsilon)$ we write it as

$$\eta(\lambda, \epsilon) = A(\lambda, \epsilon) + B(\lambda, \epsilon) + C(\lambda, \epsilon) - \alpha\,, \tag{6.13}$$

where $A(\lambda, \epsilon) := \eta(\lambda, \epsilon) - (Q_\lambda^{\epsilon c} \phi, \phi)$, $B(\lambda, \epsilon) := (Q_\lambda^{\epsilon c} \phi, \phi) - (Q_\lambda \phi, \phi)$, and $C(\lambda, \epsilon) := (Q_\lambda \phi, \phi) - (Q_{\lambda_L} \phi, \phi)$. The asymptotics of $B(\lambda, \epsilon)$ was already derived in Lemma 6.1, now we want to find the asymptotics of $A(\lambda, \epsilon)$. To this aim we first prove the following lemma.

Lemma 6.3 *As $\epsilon \to 0$ and $\lambda - \lambda_L \to 0$, we have the relation*

$$\|(P_\lambda^\epsilon - I)\mathcal{I}_\epsilon \phi\| = \mathcal{O}(\epsilon \ln \epsilon) + \mathcal{O}(\lambda - \lambda_L)\,. \tag{6.14}$$

Proof. Applying (6.10), (6.11) and using the fact that $\mathcal{I}_\epsilon^c \mathcal{I}_\epsilon \phi = \chi_\epsilon^c \phi$ we get by a straightforward calculation

$$\begin{aligned} &\|(P_\lambda^\epsilon - I)\mathcal{I}_\epsilon \phi\| \\ &\le \|\mathcal{I}_\epsilon (P_\lambda \chi_\epsilon^c - I)\phi\| + \frac{1}{2\pi} \oint_C \|R_\lambda^\epsilon(z)(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) R_\lambda(z) \chi_\epsilon^c \phi\| |\mathrm{d}z|\,. \end{aligned} \tag{6.15}$$

To handle the first r.h.s. term in (6.15) we employ the triangle inequality,

$$\|\mathcal{I}_\epsilon (P_\lambda \chi_\epsilon^c - I)\phi\| \le \|\mathcal{I}_\epsilon (P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| + \|\mathcal{I}_\epsilon (P_{\lambda_L} \chi_\epsilon^c - I)\phi\|\,. \tag{6.16}$$

Using the pseudo-resolvent formula (3.2) and the representation of the projector by means of the resolvent we get $\|\mathcal{I}_\epsilon (P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| = \mathcal{O}(\lambda - \lambda_L)$. Moreover, since P_{λ_L} is the eigenprojector onto the space spanned by ϕ we have $\|\mathcal{I}_\epsilon (P_{\lambda_L} \chi_\epsilon^c - I)\phi\| = \mathcal{O}(\epsilon)$. To estimate the second term of (6.15) we consider $\|(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) f\|$ where $f \in D(Q_\lambda) \cap W^{2,1}(I)$. Using (6.2), (6.3) and the results of Lemma 6.1 we obtain

$$\|(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) f\| = \|Jf\| = |f(s_0)| \mathcal{O}(\epsilon \ln \epsilon)\,. \tag{6.17}$$

Moreover, let us note that the function $g = R_\lambda(z)\chi_\epsilon^c \phi$ belongs to $W^{2,1}(I)$. Indeed, to see this consider $(Q_\lambda - z)g$ which is a function from $W^{1,2}(I)$ because $\chi_\epsilon^c \phi \in W^{1,2}(I)$ by Lemma 5.3. Now we can repeat the arguments from Lemmata 5.1 and 5.3, i.e. we have $\mathcal{D}_\lambda^c g \in W^{1,2}(I)$, and therefore $(T_\lambda^c - z)g \in W^{1,2}(I)$, so finally

$$\|Dg\| \le C \|D(T_\lambda^c - z)g\| < \infty\,;$$

see (5.3). Since $g \in W^{2,1}(I)$ it makes sense to consider $g(s_0)$ and to employ (6.17). Consequently, the second term in (6.15) can be estimated as

$$\|R_\lambda^\epsilon(z)(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) g\| \le \frac{1}{r} \|(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) g\| = \mathcal{O}(\epsilon \ln \epsilon)\,, \tag{6.18}$$

where $r = |z - \alpha|$. Combining these estimates we get the sought claim. ∎

The asymptotics for $A(\lambda, \epsilon)$ is given in the following lemma.

Lemma 6.4 *In the limits $\epsilon \to 0$ and $\lambda - \lambda_L \to 0$ we have*

$$|A(\lambda,\epsilon)| = |\eta(\lambda,\epsilon) - (Q_\lambda^\epsilon \mathcal{I}_\epsilon \phi, \mathcal{I}_\epsilon \phi)| = o(\epsilon \ln \epsilon) + \mathcal{O}((\lambda-\lambda_L)^2) + \mathcal{O}(\epsilon \ln \epsilon)\mathcal{O}(\lambda - \lambda_L)\,.$$

Proof. Let us note that using the properties of the eigenprojector and the asymptotics $\|P_\lambda^\epsilon \mathcal{I}_\epsilon \phi\| = 1 + \mathcal{O}(\epsilon \ln \epsilon) + \mathcal{O}(\lambda - \lambda_L)$ which is a consequence of the previous lemma we can estimate

$$\begin{aligned} |A(\lambda,\epsilon)| &= |(Q_\lambda^\epsilon P_\lambda^\epsilon \mathcal{I}_\epsilon \phi, P_\lambda^\epsilon \mathcal{I}_\epsilon \phi)\|P_\lambda^\epsilon \mathcal{I}_\epsilon \phi\|^{-2} - (Q_\lambda^\epsilon \mathcal{I}_\epsilon \phi, \mathcal{I}_\epsilon \phi)| \\ &\le \|Q_\lambda^\epsilon (P_\lambda^\epsilon - I)\mathcal{I}_\epsilon \phi\| \|(P_\lambda^\epsilon - I)\mathcal{I}_\epsilon \phi\| (1 + \mathcal{O}(\epsilon \ln \epsilon) + \mathcal{O}(\lambda - \lambda_L))\,. \end{aligned} \tag{6.19}$$

The asymptotics for $\|(P_\lambda^\epsilon - I)\mathcal{I}_\epsilon \phi\|$ was explicitly derived in Lemma 6.3. Furthermore, proceeding in analogy with (6.15) we find

$$\begin{aligned} \|Q_\lambda^\epsilon (P_\lambda^\epsilon - I)\mathcal{I}_\epsilon \phi\| &\le \|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_\lambda \chi_\epsilon^c - I)\phi\| \\ &+ \frac{1}{2\pi} \oint_C \|Q_\lambda^\epsilon R_\lambda^\epsilon(z)(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) R_\lambda(z) \chi_\epsilon^c \phi\| |\mathrm{d}z|\,. \end{aligned} \tag{6.20}$$

Mimicking now the argument of (6.16) we estimate the first term on the r.h.s. of (6.20) obtaining

$$\|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_\lambda \chi_\epsilon^c - I)\phi\| \le \|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| + \|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_{\lambda_L} \chi_\epsilon^c - I)\phi\|\,. \tag{6.21}$$

Furthermore

$$\begin{aligned} \|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| &\le \|(Q_\lambda^\epsilon \mathcal{I}_\epsilon - \mathcal{I}_\epsilon Q_\lambda)(P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| \\ &+ \|\mathcal{I}_\epsilon Q_\lambda (P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\|\,, \end{aligned} \tag{6.22}$$

where $\|(Q_\lambda^\epsilon \mathcal{I}_\epsilon - \mathcal{I}_\epsilon Q_\lambda)(P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| = \mathcal{O}(\epsilon \ln \epsilon)\mathcal{O}(\lambda - \lambda_L)$ and $\|\mathcal{I}_\epsilon Q_\lambda (P_\lambda - P_{\lambda_L})\chi_\epsilon^c \phi\| = \mathcal{O}(\lambda - \lambda_L) + \mathcal{O}(\epsilon)$. Proceeding analogously as with the second term of (6.21) we get $\|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_{\lambda_L} \chi_\epsilon^c - I)\phi\| = \mathcal{O}(\epsilon \ln \epsilon)$, and therefore

$$\|Q_\lambda^\epsilon \mathcal{I}_\epsilon (P_\lambda \chi_\epsilon^c - I)\phi\| = \mathcal{O}(\epsilon \ln \epsilon) + \mathcal{O}(\lambda - \lambda_L) + \mathcal{O}(\epsilon \ln \epsilon)\mathcal{O}(\lambda - \lambda_L)\,. \tag{6.23}$$

To handle the second term in (6.20) let us note that

$$\|Q_\lambda^\epsilon R_\lambda^\epsilon(z)\| \le 1 + \frac{|z|}{r} \le 2 + \frac{|\alpha|}{r}\,,$$

hence using (6.18) we obtain

$$\|Q_\lambda^\epsilon R_\lambda^\epsilon(z)(\mathcal{I}_\epsilon Q_\lambda - Q_\lambda^\epsilon \mathcal{I}_\epsilon) R_\lambda(z) \chi_\epsilon^c \phi\| = \mathcal{O}(\epsilon \ln \epsilon)\,,$$

which finally gives

$$\|Q_\lambda^\epsilon(P_\lambda^\epsilon-1)\mathcal{O}_\epsilon\phi\| = \mathcal{O}(\epsilon\ln\epsilon)+\mathcal{O}(\lambda-\lambda_L)+\mathcal{O}(\epsilon\ln\epsilon)\mathcal{O}(\lambda-\lambda_L)\,. \qquad (6.24)$$

Putting the above results together and applying Lemma 6.3 to (6.19) we get the claim of the lemma. ∎

Putting the results of Lemmata 6.1, 6.3, 6.4 together and applying (6.12) we get

$$\begin{aligned}\eta(\lambda,\epsilon) = {} & \frac{2}{\pi}|\phi(s_0)|^2\epsilon\ln\epsilon+(Q_\lambda\phi,\phi)+ \\ & o(\epsilon\ln\epsilon)+\mathcal{O}((\lambda-\lambda_L)^2)+\mathcal{O}(\epsilon\ln\epsilon)\mathcal{O}(\lambda-\lambda_L)\,,\end{aligned} \qquad (6.25)$$

as the hiatus half-length ϵ and the eigenvalue difference $\lambda-\lambda_L$ tend to zero.

Let us keep the notation λ_L for the eigenvalue of $H_{\alpha,\Gamma}$ which means that $\ker(Q_{\lambda_L}-\alpha)$ is nontrivial and suppose as before that $\phi\in\ker(Q_{\lambda_L}-\alpha)$ is the normalized function in $L^2(I)$. Our goal is to find an asymptotic expression for the eigenvalue of $H_{\Gamma_\epsilon,\alpha}$ by means of λ_L and ϕ.

Theorem 6.5 *The eigenvalue of $H_{\alpha,\Gamma}$ admits the following asymptotic expansion as $\epsilon\to 0$,*

$$\lambda(\epsilon)=\lambda_L-\omega(\kappa_L)|\phi(s_0)|^2\epsilon\ln\epsilon+o(\epsilon\ln\epsilon)\,, \qquad (6.26)$$

where

$$\omega(\lambda_L)=16\kappa_L\left(\int_{I\times I}\mathrm{e}^{-\kappa_L|\gamma(s)-\gamma(t)|}\phi(s)\overline{\phi(t)}\,\mathrm{d}s\mathrm{d}t\right)^{-1},\quad \kappa_L:=\sqrt{-\lambda_L}\,.$$

Proof. Due to (4.1) the eigenvalue $\lambda(\epsilon)$ is determined by the condition $\ker(Q^\epsilon_{\lambda(\epsilon)}-\alpha)\neq\{0\}$. It is convenient to put

$$\hat\eta(\lambda,\delta)\equiv\eta(\lambda,\epsilon)-\alpha : U_0\times\mathbb{C}\mapsto\mathbb{C} \quad \text{where} \quad \delta:=\epsilon\ln\epsilon$$

and U_0 is a neighborhood of zero. Our aim is to find where the function $\hat\eta$ vanishes. Using the fact that $\hat\eta(\lambda_L,0)=0$ and $\hat\eta\in C^1\times C^\infty$ and relying on the implicit function theorem we can evaluate

$$\lambda(\epsilon)=\lambda_L-(\partial_\delta\hat\eta)\mid_{\theta_L}(\partial_\lambda\hat\eta)^{-1}\mid_{\theta_L}\delta+o(\delta)\,,\quad \theta_L\equiv(\lambda_L,0)\,.$$

To find $\partial_\delta \hat\eta \mid_{\theta_L}$ we use the asymptotics (6.25)

$$\frac{1}{\delta}\Big(\hat\eta(\lambda_L,\delta)-\hat\eta(\lambda_L,0)\Big) \to \frac{2}{\pi}|\phi(s_0)|^2 \quad \text{as} \quad \delta\to 0\,.$$

To find the other derivative we use (6.25) to state

$$(\partial_\lambda \hat\eta)\mid_{\theta_L} = (\partial_\lambda (Q_\lambda\phi,\phi))\mid_{\theta_L}\ .$$

On the other hand the derivative of Q_λ w.r.t. λ coincides with the derivative of G_λ because the regularization we made was independent of the spectral parameter λ; therefore we have

$$(\partial_\lambda Q_\lambda(s,t))\mid_{\theta_L} = \frac{1}{8\pi\kappa_L}e^{-\kappa_L|\gamma(s)-\gamma(s)|}\,. \tag{6.27}$$

Putting together (6.27), (6.27) we get the sought result. ∎

As the final step of is this section we return to the general question and extend the above theorem to the case when $H_{\alpha,\Gamma}$ have more than one eigenvalue; recall that since Γ is finite by assumption we have $\sharp\sigma_{\mathrm{d}}(H_{\alpha,\Gamma})<\infty$. Suppose that $\lambda_L^1<\lambda_L^2\le ...\le\lambda_L^N$, $N\in\mathbb{N}$ are the eigenvalues of $H_{\alpha,\Gamma}$ and $\{\phi_i\}_{i=1}^N$ is the corresponding eigenfunction system which is assumed to be normalized. Given $\lambda_L\in\sigma_{\mathrm{d}}(H_{\alpha,\Gamma})$ define

$$m(\lambda_L):=\min\{j=1,...,N\,:\,\lambda_j=\lambda_L\}\,,\quad n(\lambda_L):=\max\{j=1,...,N\,:\,\lambda_j=\lambda_L\}$$

and the matrix $C(\lambda_L)$ given by

$$[C(\lambda_L)]_{ij}:=\phi_i(s_0)\overline{\phi_j(s_0)}\omega_{ij}\,,\quad i,j=m(\lambda_L),...,n(\lambda_L)\,,$$

where

$$\omega_{ij}(\lambda_L):=\left(\int_{I\times I}\mathrm{e}^{-\kappa_L|\gamma(s)-\gamma(t)|}\phi_i(s)\overline{\phi_j(t)}\,\mathrm{d}s\mathrm{d}t\right)^{-1}\,.$$

Using this notation we can state our main result:

Theorem 6.6 *Let $\lambda_L\in\sigma_{\mathrm{d}}(H_{\alpha,\Gamma})$. Then the corresponding eigenvalues of $H_{\alpha,\Gamma_\epsilon}$ have the following asymptotic expansion,*

$$\lambda_j(\epsilon)=\lambda_L-s_j(\lambda_L)\epsilon\ln\epsilon+o(\epsilon\ln\epsilon)\,,\quad m(\lambda_L)\le j\le n(\lambda_L)\,,$$

as $\epsilon\to 0$, where $s_j(\lambda_L)$ are the eigenvalues of matrix $C(\lambda_L)$.

The proof of essentially repeats the reasoning used above; the only new element is that different eigenfunctions corresponding to the same eigenvalue λ_L correspond to the appropriate scalar products. This consequently leads to the appearance of the matrix $C(\lambda_L)$ which reduces to $|\phi(s_0)|^2\omega(\lambda_L)$ if λ_L is a simple eigenvalue.

7 Concluding remarks

First we note that the hiatus perturbation of a curve in $\mathbb{R}^3$ can be regarded as an effective repulsive interaction. The presence of a hiatus pushes the eigenvalues up which can be easily seen from (6.26) since $\omega(\lambda_L) > 0$ and $\epsilon \ln \epsilon < 0$ for small enough ϵ. This might be expected, of course, because the interaction supported by a curve in $\mathbb{R}^3$ is attractive as manifested by the fact that it produces bound states, at least if the curve is sufficiently long, cf. Theorem 4.4.

Comparing the eigenvalue asymptotics (6.26) with the analogous result for a curve in $\mathbb{R}^2$ derived in [EY03] we can see the difference in the first asymptotic term, which in the codimension one case behaves as $\mathcal{O}(\epsilon)$ in contrast to $\mathcal{O}(\epsilon \ln \epsilon)$ obtained here. The former result is a natural consequence of the additive character of the singular potential manifested by the sum-type quadratic form representation of the corresponding Hamiltonian. Such a representation does not exists if the potential is supported by a set of codimension two. To find a self-adjoint realization of the δ interaction in this case we have to perform, for instance, a logarithmic regularization of the appropriate quantities, and consequently, the eigenvalue asymptotics w.r.t. the length of the hiatus, as well as its derivation, are more involved.

8 Appendix: the remaining proofs

To prove Theorem 3.2 we need the following lemma.

Lemma 8.1 *Given $s \in [0, L]$ corresponding to $\tilde{\gamma}_{d,s} = \tilde{\gamma}_d$ and $d > 0$, and making $|s - t|$ small we have*

$$|\gamma(s) - \tilde{\gamma}_d(t)|^2 = (s-t)^2(1 + \mathcal{O}(d)) + d^2 + \mathcal{O}((s-t)^3)\,.$$

Proof. An elementary cosine formula gives

$$|\gamma(s) - \tilde{\gamma}_d(t)|^2 = |\tilde{\gamma}_d(s) - \tilde{\gamma}_d(t)|^2 + d^2 - 2\iota(s,t)\,,$$

where $\iota(s,t) := (\gamma(s) - \tilde{\gamma}_d(s)\,, \tilde{\gamma}_d(s) - \tilde{\gamma}_d(t))$. Note that $\iota(s,t)|_{t=s} = 0$ and $\partial_t \iota(s,t)|_{t=s} = 0$ holds in view of the assumption $(\widetilde{\text{b3}})$. Furthermore, the Taylor expansion in the corresponding "shifted" points of the coordinate projections of the curve $\tilde{\gamma}_d$ yields

$$|\tilde{\gamma}_d(s) - \tilde{\gamma}_d(t)|^2 = \sum_{i=1}^{3} \dot{\tilde{\gamma}}_{d,i}(\theta_i)^2 (s-t)^2 + \mathcal{O}((s-t)^3\,.$$

Using the Taylor expansion again and combining it with the asymptotics given by $(\widetilde{\mathrm{b}1})$, $(\widetilde{\mathrm{b}2})$ and the fact that $\sum_{i=1}^{3} \dot{\gamma}_i(s)^2 = 1$ we get the claim. ∎

Proof of Theorem 3.2. The first step is show that in the following limits

$$\lim_{d\to 0} \int_I \left[G_z(\gamma(s) - \tilde{\gamma}_d(t)) - \frac{1}{4\pi|\gamma(s) - \tilde{\gamma}_d(t)|} \right] f(t)\,\mathrm{d}t \tag{8.1}$$

and

$$\lim_{d\to 0} \int_I \left[\frac{1}{|\gamma(s) - \tilde{\gamma}_d(t)|} - \frac{1}{((s-t)^2 + d^2)^{1/2}} \right] f(t)\,\mathrm{d}t \tag{8.2}$$

we can interchange the limit with the integration. Using the inequality $|(\mathrm{e}^{-\kappa x} - 1)x^{-1}| \le \kappa$ for κ and x positive we can use the dominated convergence to prove claim concerning (8.1). To handle the second limit we can use Lemma 8.1 and show that

$$\begin{aligned} &\frac{1}{|\gamma(s) - \tilde{\gamma}_d(t)|} - \frac{1}{((s-t)^2 + d^2)^{1/2}} \\ &\quad = \Big((s-t)^2 \mathcal{O}(d) + \mathcal{O}((s-t)^3) \Big) \Big((s-t)^2 + d^2 \Big)^{-1} \le \text{const}\,. \end{aligned} \tag{8.3}$$

Therefore using the dominated convergence again we can perform the interchange in (8.2). The resulting limit of the sum of both the expressions (8.1) and (8.2) is given by

$$\begin{aligned} \lim_{d\to 0} \int_I &\left[G_z(\gamma(s) - \tilde{\gamma}_d(t)) - \frac{1}{4\pi((s-t)^2 + d^2)^{1/2}} \right] f(t)\,\mathrm{d}t \\ &= \int_I \mathcal{A}_z(s-t) f(t)\mathrm{d}t + \int_I \mathcal{D}_z(s,t) f(t)\,\mathrm{d}t\,, \end{aligned} \tag{8.4}$$

where $\mathcal{A}_z$ and $\mathcal{D}_z$ are defined in Lemma 3.5. Repeating the argument from the proof of this lemma, see (3.11) and (3.12), we get

$$\begin{aligned} \lim_{d\to 0} &\left[\int_{I_\delta} \frac{1}{4\pi((s-t)^2 + d^2)^{1/2}} f(t)\,\mathrm{d}t + \frac{1}{2\pi} f(s) \ln d \right] \\ &= \frac{1}{4\pi} \Big(\int_I \frac{f(t) - f(s)}{|t-s|}\,\mathrm{d}t + \ln 4s(L-s) f(s) \Big)\,. \end{aligned} \tag{8.5}$$

The final step is to note that

$$\int_I G_z(\gamma(s) - \tilde{\gamma}_d(t))(1 - j_d(t))f(t)\,\mathrm{d}t = o(1)$$

as $d \to 0$, because $j_d = 1 + \mathcal{O}(d)$ and $\int_I G_z(\gamma(s) - \tilde{\gamma}_d(t))f(t)\,\mathrm{d}t$ has a singularity of the type $f(s)\ln d$. Combing this with (8.4) and (8.5) we conclude the proof of Theorem 3.2

Remark 8.2 Using the same arguments as in (8.3) we can estimate

$$\begin{aligned} &\frac{1}{|\gamma_d(s) - \gamma(t)|} - \frac{1}{((s-t)^2 + d^2)^{1/2}} \\ &\quad = \Big((s-t)^2\mathcal{O}(d) + \mathcal{O}((s-t)^3)\Big)\Big((s-t)^2 + d^2\Big)^{-1} \le \text{const}\,, \end{aligned} \tag{8.6}$$

which directly implies

$$|\mathcal{D}_z^d(s,t)| = |G_z(\gamma_d(s) - \gamma(t)) - G_z^d(s-t)| \le \text{const}\,,$$

for any $s \in [0, L]$.

Proof of Lemma 5.1. Recall that our goal is to show that

$$\int_I \mathcal{D}_z^c(s,t)f(t)\,\mathrm{d}t \in W^{1,2} \ \text{ for } \ f \in L^2(I)\,, \tag{8.7}$$

where

$$\mathcal{D}_z^c(s,t) := G_z(\gamma(s) - \gamma(t)) - G_z(\gamma^c(s) - \gamma^c(t))$$

and $G_z(\rho) = \mathrm{e}^{-\sqrt{-z}|\rho|}(4\pi|\rho|)^{-1}$. We proceed in three steps:
Step 1: We show that the following inequality holds

$$||\gamma(s) - \gamma(t)| - |\gamma^c(s) - \gamma^c(t)|| \le c_1|s-t|^{\mu+1} \tag{8.8}$$

for $c_1|s-t|^\mu < 1$. By a straightforward calculation one can find that

$$|\gamma^c(s) - \gamma^c(t)|^2 = \frac{L^2}{2\pi^2}\left(1 - \cos\frac{2\pi(s-t)}{L}\right)\,.$$

Consequently, there exists a positive constant $\tilde{c}$ such that

$$|\gamma^c(s) - \gamma^c(t)| \ge |s-t|(1 - \tilde{c}|s-t|^2) \tag{8.9}$$

for $\tilde{c}|s-t|^2<1$. Using the above inequality we have

$$|\gamma(s)-\gamma(t)| \le |s-t| \le |\gamma^c(s)-\gamma^c(t)| + \tilde{c}|s-t|^3 \,. \tag{8.10}$$

On the other hand, using the assumption (a) we obtain

$$|\gamma(s)-\gamma(t)| \ge |s-t| - c|s-t|^{\mu+1} \ge |\gamma^c(s)-\gamma^c(t)| - c|s-t|^{\mu+1} \,. \tag{8.11}$$

Combining (8.10) and (8.11) we arrive at (8.8). *Step 2:* The aim of this part of the proof is to show the following asymptotics,

$$\mathcal{D}_z^c(s,t) = \mathcal{O}(|s-t|^{\mu-1}) \,. \tag{8.12}$$

Using (8.8), (8.9) and the assumption (a) we get

$$\begin{aligned} |T(s,t)| &:= \left| \frac{1}{|\gamma(s)-\gamma(t)|} - \frac{1}{|\gamma^c(s)-\gamma^c(t)|} \right| \\ &\le \frac{c_1 |s-t|^{\mu+1}}{|s-t|^2(1-c|s-t|^{\mu})(1-\tilde{c}|s-t|^2)} \le c_2 |s-t|^{\mu-1} \,, \end{aligned} \tag{8.13}$$

where c_2 is a positive constant. Furthermore, using the the exponential function expansion and (8.8) we find

$$\mathcal{D}_z^c(s,t) = T(s,t) + \mathcal{O}(|s-t|^{\mu+1}) \,,$$

which in view of (8.13) implies (8.12). *Step 3:* Let us note that for $f \in L^2(I)$ we have

$$\Big| \int_I \mathcal{D}_z^c(s,t) f(t)\, \mathrm{d}t \Big| \le a(s) \|f\|_{L^2(I)} \,, \quad \text{where} \quad a(s) := \int_I |\mathcal{D}_z^c(s,t)|^2 \mathrm{d}t \,.$$

Using (8.12) we claim that $a'(s)$ is an integrable function, and therefore we can use the dominated convergence to show that

$$\begin{aligned} \int_I \Big| D_s \int_I \mathcal{D}_z^c(s,t) f(t)\, \mathrm{d}t \Big|^2 \mathrm{d}s &= \int_I \Big| \int_I D_s \mathcal{D}_z^c(s,t) f(t)\, \mathrm{d}t \Big|^2 \mathrm{d}s \\ &\le \int_I \int_I |D_s \mathcal{D}_z^c(s,t)|^2 \, \mathrm{d}t \, \mathrm{d}s \, \|f\|_{L^2(I)} < \infty \,, \end{aligned} \tag{8.14}$$

where D_s stands for the derivative; in the above estimates we have again used (8.12) to check that the last term in the cahin (8.14) is finite. This finally proves (8.7), and by that Lemma 5.1.

Remark 8.3 Let us note that in analogy with (8.13) we can estimate

$$\left|\tfrac{1}{|\gamma(s)-\gamma(t)|}-\tfrac{1}{|s-t|}\right| \leq \tfrac{c|s-t|^{\mu+1}}{|s-t|^2(1-c|s-t|^\mu)} \leq c_3|s-t|^{\mu-1}\,, \tag{8.15}$$

where we have again used the assumption (a). This implies

$$|\mathcal{R}_\lambda(s,t)| = |G_\lambda(\gamma(s)-\gamma(t)) - (4\pi|s-t|)^{-1}| \leq c_4|s-t|^{\mu-1}\,.$$

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