On the optimization of the principal eigenvalue for single-centre point-interaction operators in a bounded region

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Abstract

We investigate relations between spectral properties of a single-centre point-interaction Hamiltonian describing a particle confined to a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with Dirichlet boundary, and the geometry of Ω. For this class of operators Krein's formula yields an explicit representation of the resolvent in terms of the integral kernel of the unperturbed one, $\left(-\Delta_\Omega^D + z\right)^{-1}$. We use a moving plane analysis to characterize the behaviour of the ground-state energy of the Hamiltonian with respect to the point-interaction position and the shape of Ω, in particular, we establish some conditions showing how to place the interaction to optimize the principal eigenvalue.

1 Introduction

Relations between geometry of a domain and spectral properties of corresponding operators belong to the most traditional question in mathematical physics; one can recall, e.g., the Faber-Krahn inequality [Fa23, Kr25] or the Payne-Pólya-Weinberger conjecture [PPW55] proved by Ashbaugh and Benguria [AB92a, AB92b]. A more recent example concerns the situation where the domain in question is not simply connected and one asks, in particular, how to place a circular hard-wall obstacle within a circular planar cavity to minimize the ground-state eigenvalue; it appears that the minimum is reached when the obstacle touches the boundary [HKK01].

In connection with the last mentioned problem one can also ask what happens if such a "hard" obstacle is replaced by another object, say, by a potential barrier or well. In this paper we are going to address this question in the particular case when such a potential is singular, in other words, a point interaction. Recalling basic results about these interactions [AGHH05] we see that the problem makes sense in dimension $d \leq 3$. Furthermore, a simple perturbative argument shows that in the one dimensional situation the answer may depend on the sign of the δ potential; we restrict here our attention to the more singular case of dimension $d = 2, 3$.

Problems of this type were to our knowledge solved so far only in cases where the domain has a simple geometry such as a straight strip in $\mathbb{R}^2$, see [EGŠT96], or a planar layer in $\mathbb{R}^3$, see [EN02], where the eigenvalue problem can be solved more or less explicitly. Here we consider that the domain Ω, to which the particle is confined, is bounded and otherwise quite general, see below. The operator of interest will be the corresponding Dirichlet Laplacian perturbed by a single point interaction of a fixed coupling constant; we will ask about the dependence of its principal eigenvalue on the perturbation position. Using a figurative expression, to be made

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precise below, we are going to show that the ground-state energy *increases* as the point moves towards the boundary of Ω, in contrast to the case of a Dirichlet obstacle mentioned above.

Our method is based on the fact that one is able to express the resolvent of the operator in question by means of the Krein's formula. The resulting spectral condition allow us to characterize the principal eigenvalue as a function of the interaction position in a non perturbative setting. Then we exploit the maximum principle and a domain reflection technique, analogous to the one used in [HKK01], to demonstrate our main result, Theorem 4.3, expressing strict monotonicity of the principal eigenvalue with respect to certain directions. Using this conclusion, we formulate then some conditions under which the eigenvalue reaches its minimum value for a fixed Ω.

2 Confined point interactions in two and three dimensions

The definition domain and the spectral properties of point interaction Hamiltonians in dimensions two and three are usually expressed in terms of the free Green function, i.e. the integral kernel of the operator $(-\Delta+z)^{-1}$, cf. [AGHH05]. This is true both if the configuration space is the whole $\mathbb{R}^d$ or if the particle is confined to a subset Ω of it by a hard wall corresponding to Dirichlet boundary condition. In the latter case Green's function is defined by the equation

$$\begin{cases} (-\Delta+z)\,\mathcal{G}_0^z(\underline{x},\underline{x}')=\delta(\underline{x}-\underline{x}') \\ \mathcal{G}_0^z(\underline{x},\underline{x}')|_{\underline{x}\in\partial\Omega}=0 \end{cases} \quad ; \quad \underline{x}'\in\Omega \tag{2.1}$$

which admits a solution in $L^2(\Omega)$ whenever $-z$ does not belong to the spectrum of the Dirichlet Laplacian $-\Delta_\Omega^D$ defined in the standard way through the associated quadratic form [RS78]. Throughout this paper, Ω is supposed to be an open regular set in $\mathbb{R}^d$, $d=2,3$, bounded and connected, not necessarily simply; we assume that $\partial\Omega$ is piecewise C^1.

Under these hypotheses $-\Delta_\Omega^D$ has a purely discrete spectrum; we denote by $\{\lambda_n\}_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, its eigenvalues, and by $\{\psi_{n,k}\}_{\substack{k=1\\ n\in\mathbb{N}_0}}^{N_n}$ the corresponding system of eigenfunctions, N_n being the multiplicity of the n-th eigenvalue. Projecting (2.1) on the vectors $\psi_{n,k}$, it is easy to check the validity of the following standard Fourier expansion of $\mathcal{G}_0^z$,

$$\mathcal{G}_0^z(\underline{x},\underline{x}')=\sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}}\frac{\psi_{n,k}(\underline{x}')\,\psi_{n,k}(\underline{x})}{\lambda_n+z}\,. \tag{2.2}$$

An alternative representation of the kernel, which will be extensively used in this paper, can be given in terms of the 'free' Green's function $\mathcal{G}^z(\underline{x},\underline{x}')$, in other words, the integral kernel of $(-\Delta+z)^{-1}$ in the whole space[1],

$$\mathcal{G}^z(\underline{x},\underline{x}')=\frac{1}{2\pi}K_0(\sqrt{z}\,|\underline{x}-\underline{x}'|)\quad \text{in } \mathbb{R}^2 \tag{2.3}$$

$$\mathcal{G}^z(\underline{x},\underline{x}')=\frac{e^{-\sqrt{z}|\underline{x}-\underline{x}'|}}{4\pi\,|\underline{x}-\underline{x}'|}\qquad \text{in } \mathbb{R}^3 \tag{2.4}$$

Here K_0 denotes the Macdonald (or modified Hankel) function. Using the boundary conditions in (2.1), we can express the kernel of interest as

$$\mathcal{G}_0^z(\underline{x},\underline{x}')=\mathcal{G}^z(\underline{x},\underline{x}')-h(\underline{x},\underline{x}',\sqrt{z}) \tag{2.5}$$

[1]Here and in the following we use the convention in which the negative real axis represents the cut of the square root $\sqrt{z}$ in the complex plane.

where $\mathcal{G}^z$ is defined by (2.3)–(2.4) and $h(\underline{x}, \underline{x}', \sqrt{z})$ solves the boundary value problem

$$\begin{cases} (-\Delta + z)\, h(\underline{x}, \underline{x}', \sqrt{z}) = 0 \\ h(\underline{x}, \underline{x}', \sqrt{z})|_{\underline{x}\in\partial\Omega} = \mathcal{G}^z(\underline{x}, \underline{x}')|_{\underline{x}\in\partial\Omega} \end{cases} \quad \text{for any } \underline{x}' \in \Omega \tag{2.6}$$

In the next step we construct the operator which will be our main object perturbing $-\Delta_\Omega^D$ by a single point interaction with the support at a point $\underline{x}_0 \in \Omega$. Such Hamiltonians are defined by the self-adjoint extensions of the symmetric operator

$$\begin{cases} D(H_0) = \left\{ \psi \in H^2 \cap H_0^1(\Omega) \middle|\, \psi(\underline{x}_0) = 0 \right\} \\ H_0\psi = -\Delta\psi \end{cases} \tag{2.7}$$

Following the von Neumann theory [AG63, RS75], we observe that the restriction (2.7) has deficiency indices $(1,1)$; consequently, we arrive at a one-parameter family of self-adjoint operators H_α. For fixed $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and a domain $\Omega \subset \mathbb{R}^2$ we have

$$\begin{aligned} D(H_\alpha) = \Big\{ \psi \in L^2(\Omega) \Big|\, & \psi = \phi^\lambda + q\mathcal{G}_0^\lambda(\cdot, \underline{x}_0),\ \phi^\lambda \in H^2 \cap H_0^1(\Omega), \\ & \phi^\lambda(\underline{x}_0) = \frac{q}{2\pi}\left(\alpha - \ln\sqrt{\lambda} - 2\pi\, h(\underline{x}_0, \underline{x}_0, \sqrt{\lambda})\right) \Big\} \end{aligned} \tag{2.8}$$

while for $\Omega \subset \mathbb{R}^3$ the operator domain is

$$\begin{aligned} D(H_\alpha) = \Big\{ \psi \in L^2(\Omega) \Big| : & \psi = \phi^\lambda + q\mathcal{G}_0^\lambda(\cdot, \underline{x}_0),\ \phi^\lambda \in H^2 \cap H_0^1(\Omega), \\ & \phi^\lambda(\underline{x}_0) = q\left(\alpha + \frac{\sqrt{\lambda}}{4\pi} + h(\underline{x}_0, \underline{x}_0, \sqrt{\lambda})\right) \Big\} \end{aligned} \tag{2.9}$$

In both cases the parameter λ determines a representation of the operator domain, roughly speaking, a split between the regular and singular part; for a fixed choice of λ the action of H_α is the following

$$H_\alpha\psi = -\Delta\phi^\lambda - \lambda q\, \mathcal{G}_0^\lambda(\cdot, \underline{x}_0)\,. \tag{2.10}$$

It is convenient to include into this description also infinite values of the parameter α, in which case the coefficient q of the singular part — sometimes referred to as the *charge* of the state — vanishes. It is equivalent to the absence of the point interaction: the domain reduces in this case to $H^2 \cap H_0^1(\Omega)$ and the self-adjoint extensions corresponding to $\alpha = \pm\infty$ is identified with the unperturbed operator $-\Delta_\Omega^D$.

Making use of Krein's formula [AG63] the action of the resolvent, $R_z^\alpha = (H_\alpha + z)^{-1}$ on $L^2(\Omega)$, can be expressed as a rank-one perturbation of its 'free' counterpart $(-\Delta + z)^{-1}$, specifically

$$R_z^\alpha\varphi = (-\Delta + z)^{-1}\varphi + q_z\left((-\Delta + z)^{-1}\varphi\right)(\underline{x}_0)\, \mathcal{G}_0^z(\cdot, \underline{x}_0)\,. \tag{2.11}$$

In this formula, of course, the first term is the regular part of the function $R_z^\alpha\varphi \in D(H_\alpha)$ while the value $q_z\left((-\Delta + z)^{-1}\varphi\right)(\underline{x}_0)$ denotes the corresponding charge. Using the boundary conditions in (2.8) and (2.9) we can identify the coefficient q_z with

$$q_z = \left(\alpha - \ln\sqrt{z} - 2\pi\, h(\underline{x}_0, \underline{x}_0, \sqrt{z})\right)^{-1}, \quad \Omega \subset \mathbb{R}^2 \tag{2.12}$$

$$q_z = \left(\alpha + \frac{\sqrt{z}}{4\pi} + h(\underline{x}_0, \underline{x}_0, \sqrt{z})\right)^{-1}, \quad \Omega \subset \mathbb{R}^3 \tag{2.13}$$

Furthermore, the Fourier expansion

$$(-\Delta + z)^{-1}\varphi = \sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}} \frac{(\varphi,\psi_{n,k})}{\lambda_n+z}\psi_{n,k} \tag{2.14}$$

yields the following explicit expression

$$R_z^\alpha\varphi = \sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}} \frac{(\varphi,\psi_{n,k})}{\lambda_n+z}\psi_{n,k} + 2\pi \sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}} \frac{(\varphi,\psi_{n,k})}{\lambda_n+z}\,\frac{\psi_{n,k}(\underline{x}_0)}{\alpha-\ln\sqrt{z}-2\pi\, h(\underline{x}_0,\underline{x}_0,\sqrt{z})}\mathcal{G}_0^z(\cdot,\underline{x}_0) \tag{2.15}$$

for $\Omega\subset\mathbb{R}^2$, and its counterpart

$$R_z^\alpha\varphi = \sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}} \frac{(\varphi,\psi_{n,k})}{\lambda_n+z}\psi_{n,k} + \sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}} \frac{(\varphi,\psi_{n,k})}{\lambda_n+z}\,\frac{\psi_{n,k}(\underline{x}_0)}{\alpha+\frac{\sqrt{z}}{4\pi}+h(\underline{x}_0,\underline{x}_0,\sqrt{z})}\mathcal{G}_0^z(\cdot,\underline{x}_0) \tag{2.16}$$

for $\Omega\subset\mathbb{R}^3$.

3 The principal eigenvalue of H_α

As usual, the Krein formula allows us to determine the spectrum through the denominator of the perturbation term. In particular, it follows from the resolvent equations (2.15)–(2.16) that the spectrum of H_α is formed by the solutions of the equations

$$\begin{aligned} &\alpha-\ln\sqrt{-\xi}-2\pi\, h(\underline{x}_0,\underline{x}_0,\sqrt{-\xi})=0\,, \qquad \Omega\subset\mathbb{R}^2\\ &\alpha+\tfrac{\sqrt{-\xi}}{4\pi}+h(\underline{x}_0,\underline{x}_0,\sqrt{-\xi})=0\,, \qquad \Omega\subset\mathbb{R}^3 \end{aligned} \tag{3.1}$$

to which one has to add eigenvalues of $-\Delta_\Omega^D$, the degenerate ones in any case and the non-degenerate ones, $\lambda_{\bar n}$, provided that the corresponding eigenfunction $\psi_{\bar n}$ satisfies the condition (see e.g. in [BFM07])

$$\psi_{\bar n}(\underline{x}_0)=0 \tag{3.2}$$

This does not concern, however, the bottom of the spectrum which we are interested in here, because the ground state of $-\Delta_D^\Omega$ is non-degenerate and can be represented by a positive function.

The subject of this section is the principal eigenvalue of the point-interaction operator H_α. We are going to show, in particular, that for any $\alpha\in\mathbb{R}$ there exists a unique simple eigenvalue of H_α below the spectral threshold of $-\Delta_\Omega^D$. As a preliminary, we need to characterize the derivatives of $h(\underline{x}_0,\underline{x}_0,\sqrt{z})$ w.r.t. the variable $\sqrt{z}$.

Lemma 3.1 *Let z be a positive real number; the function $y\mapsto h(\underline{x},\underline{x}',y)$, defined by (2.6) with $y:=\sqrt{z}$, satisfies the conditions*

$$\begin{aligned} &\tfrac{1}{y}\left(\tfrac{1}{y}+2\pi\,\partial_y h(\underline{x}',\underline{x}',y)\right)>0\,, \qquad \Omega\subset\mathbb{R}^2\\ &\tfrac{1}{y}\left(\tfrac{1}{4\pi}+\partial_y h(\underline{x}',\underline{x}',y)\right)>0\,, \qquad \Omega\subset\mathbb{R}^3 \end{aligned} \tag{3.3}$$

Furthermore, for any $z<0$ and $h(\underline{x},\underline{x}',iy)$, defined by (2.6) with $y:=\sqrt{|z|}$, we have

$$\begin{aligned} &\tfrac{\pi}{2}+2\pi\,\operatorname{Im} h(\underline{x}',\underline{x}',i\,y)=0\,, \qquad \Omega\subset\mathbb{R}^2\\ &\tfrac{y}{4\pi}+\operatorname{Im} h(\underline{x}',\underline{x}',i\,y)=0\,, \qquad \Omega\subset\mathbb{R}^3 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\tfrac{1}{y} + 2\pi\, \partial_y \operatorname{Re} h(\underline{x}', \underline{x}', i\, y) < 0\,, \qquad \Omega \subset \mathbb{R}^2 \\ &\partial_y \operatorname{Re} h(\underline{x}', \underline{x}', i\, y) < 0\,, \qquad \Omega \subset \mathbb{R}^3 \end{aligned} \tag{3.5}$$

Proof. We start with the 3D case. Let z_j, $j = 1, 2$, be a pair of positive values; setting $y_j := \sqrt{z_j}$ we get from equation (2.5) in combination with the first resolvent formula the relation

$$\left(y_1^2 - y_2^2\right) \left(\mathcal{G}_0^{y_1^2}, \mathcal{G}_0^{y_2^2}\right)_{L^2(\Omega)} = \lim_{\underline{x} \to \underline{x}'} \left[\mathcal{G}^{y_2^2}(\underline{x}, \underline{x}') - \mathcal{G}^{y_1^2}(\underline{x}, \underline{x}')\right] + h(\underline{x}', \underline{x}', y_1) - h(\underline{x}', \underline{x}', y_2)\,, \tag{3.6}$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the scalar product in $L^2(\Omega)$. The limit at the r.h.s. is easily seen to be

$$\lim_{\underline{x} \to \underline{x}'} \left[\mathcal{G}^{y_2^2}(\underline{x}, \underline{x}') - \mathcal{G}^{y_1^2}(\underline{x}, \underline{x}')\right] = \frac{1}{4\pi}\,(y_1 - y_2) \tag{3.7}$$

Substituting this expression into (3.6) we get

$$(y_1 + y_2) \left(\mathcal{G}_0^{y_1^2}, \mathcal{G}_0^{y_2^2}\right)_{L^2(\Omega)} = \frac{1}{4\pi} + \frac{h(\underline{x}', \underline{x}', y_1) - h(\underline{x}', \underline{x}', y_2)}{y_1 - y_2}\,, \tag{3.8}$$

and consequently, in the limit $y_1 \to y_2$ we arrive at

$$\frac{1}{4\pi} + \partial_y h(\underline{x}', \underline{x}', y_2) = 2y_2 \left\|\mathcal{G}_0^{y_2^2}\right\|^2_{L^2(\Omega)} > 0\,. \tag{3.9}$$

On the other hand, for negative values of z a similar argument yields

$$\left(y_1^2 - y_2^2\right) \left(\mathcal{G}_0^{-y_1^2}, \mathcal{G}_0^{-y_2^2}\right)_{L^2(\Omega)} = \lim_{\underline{x} \to \underline{x}'} \left[\mathcal{G}^{-y_1^2}(\underline{x}, \underline{x}') - \left(\mathcal{G}^{-y_2^2}\right)^* (\underline{x}, \underline{x}')\right] - h(\underline{x}', \underline{x}', i\, y_1) + h^*(\underline{x}', \underline{x}', i\, y_2) \tag{3.10}$$

with the asterisk denoting complex conjugation and $y_j := \sqrt{|z_j|}$. The limit at the r.h.s. is

$$\lim_{\underline{x} \to \underline{x}'} \left[\mathcal{G}^{-y_1^2}(\underline{x}, \underline{x}') - \left(\mathcal{G}^{-y_2^2}\right)^* (\underline{x}, \underline{x}')\right] = -\frac{i}{4\pi}\,(y_1 + y_2) \tag{3.11}$$

from which we get a relation replacing (3.8), namely

$$\left(y_1^2 - y_2^2\right) \left(\mathcal{G}_0^{-y_1^2}, \mathcal{G}_0^{-y_2^2}\right)_{L^2(\Omega)} = -\frac{i}{4\pi}\,(y_1 + y_2) - h(\underline{x}', \underline{x}', i\, y_1) + h^*(\underline{x}', \underline{x}', i\, y_2)\,. \tag{3.12}$$

The second one of the relations (3.4) then follows from here by setting $y_1 = y_2$. As for the real part of (3.10) given by

$$\begin{aligned} &\frac{1}{2}\left(y_1^2 - y_2^2\right) \left[\left(\mathcal{G}_0^{-y_1^2}, \mathcal{G}_0^{-y_2^2}\right)_{L^2(\Omega)} + \left(\mathcal{G}_0^{-y_2^2}, \mathcal{G}_0^{-y_1^2}\right)_{L^2(\Omega)}\right] = \\ &\quad = \lim_{\underline{x} \to \underline{x}'} \left[\operatorname{Re} \mathcal{G}^{-y_1^2}(\underline{x}, \underline{x}') - \operatorname{Re} \mathcal{G}^{-y_2^2}(\underline{x}, \underline{x}')\right] - \operatorname{Re} h(\underline{x}', \underline{x}', i\, y_1) + \operatorname{Re} h(\underline{x}', \underline{x}', i\, y_2) \end{aligned} \tag{3.13}$$

we notice that the first term at the r.h.s. is in fact zero; dividing the remaining ones by $y_1 - y_2$ and taking the limit as $y_1 \to y_2$, we arrive at

$$-\partial_y \operatorname{Re} h(\underline{x}', \underline{x}', i\, y_2) = 2y_2 \left\|\mathcal{G}_0^{-y_2^2}\right\|^2_{L^2(\Omega)} > 0\,. \tag{3.14}$$

In the 2D case the validity of relations (3.3), (3.4) and (3.5) can be checked following the same idea. Taking into account the logarithmic singularity of $\mathcal{G}^z(\underline{x}, \underline{x}')$ as $\underline{x} \to \underline{x}'$ — see, e.g., [AS72] — we find for $z > 0$

$$\lim_{\underline{x}\to\underline{x}'} \left[\mathcal{G}^{y_2^2}(\underline{x}, \underline{x}') - \mathcal{G}^{y_1^2}(\underline{x}, \underline{x}')\right] = \frac{1}{2\pi}\left(\ln y_1 - \ln y_2\right) , \tag{3.15}$$

while for $z < 0$ we have

$$\lim_{\underline{x}\to\underline{x}'} \left[\mathcal{G}^{-y_1^2}(\underline{x}, \underline{x}') - \left(\mathcal{G}^{-y_2^2}\right)^*(\underline{x}, \underline{x}')\right] = \frac{1}{2\pi}\left(-i\pi + \ln y_1 - \ln y_2\right) ; \tag{3.16}$$

this concludes the proof. ■

In the next two lemmata, we deal with the solutions of equations (3.1) below the spectrum of $-\Delta_\Omega^D$. We are going to show that for a fixed real α there is a unique such solution. It is convenient to treat the 2D and 3D cases separately.

Lemma 3.2 *Let λ_0 denote the first eigenvalue of $-\Delta_\Omega^D$ corresponding to the domain $\Omega \subset \mathbb{R}^3$. For any $\alpha \in \mathbb{R}$, the equation*

$$\alpha + \frac{\sqrt{-\xi}}{4\pi} + h(\underline{x}_0, \underline{x}_0, \sqrt{-\xi}) = 0 , \quad \xi \in (-\infty, \lambda_0) \tag{3.17}$$

admits a unique solution, denoted $\xi(\alpha)$, such that

$$\lim_{\alpha\to-\infty} \xi(\alpha) = -\infty , \qquad \xi(-h(\underline{x}_0, \underline{x}_0, 0)) = 0 , \tag{3.18}$$

and

$$\lim_{\alpha\to+\infty} \xi(\alpha) = \lambda_0 . \tag{3.19}$$

Proof. In order to study solutions of (3.17), we need to find the dependence of $h(\underline{x}_0, \underline{x}_0, \sqrt{-\xi})$ on the variable $\sqrt{-\xi}$. We start by considering the case $\xi \le 0$. Setting $y = \sqrt{|\xi|}$, the equation (3.17) assumes the form

$$\frac{y}{4\pi} = -\alpha - h(\underline{x}_0, \underline{x}_0, y) = 0 , \quad y \ge 0 \tag{3.20}$$

and its solutions can be geometrically interpreted as the abscissas of the intersection points between the curves at the left and the right-hand side of (3.20). We will show that for any fixed choice of $\underline{x}_0 \in \Omega$, $h(\underline{x}_0, \underline{x}_0, y)$ is a positive and strictly decreasing function of the variable y, such that

$$\lim_{y\to+\infty} h(\underline{x}_0, \underline{x}_0, y) = 0 . \tag{3.21}$$

Let us consider the boundary value problem

$$\left\{ \begin{array}{l} \left(-\Delta + y^2\right) h(\underline{x}, \underline{x}_0, y) = 0 \\ h(\underline{x}, \underline{x}_0, y)|_{\underline{x}\in\partial\Omega} = \left.\frac{e^{-y|\underline{x}-\underline{x}_0|}}{4\pi|\underline{x}-\underline{x}_0|}\right|_{\underline{x}\in\partial\Omega} \end{array} \right. ; \quad \underline{x}_0 \in \Omega \tag{3.22}$$

The solution of (3.22) is infinitely smooth in the open set Ω, continuous and positive on its boundary. The strong maximum principle — cf. [Ev98] — in this case allows us to claim that h is strictly positive in Ω reaching its maximum on the boundary,

$$0 < h(\underline{x}, \underline{x}_0, y) < \sup_{\underline{x}'\in\partial\Omega} \frac{e^{-y|\underline{x}'-\underline{x}_0|}}{4\pi\,|\underline{x}' - \underline{x}_0|} \quad \text{for } \forall \underline{x} \in \Omega,\, y \ge 0 . \tag{3.23}$$

Furthermore. the derivative $\partial_y h$ satisfies the equation

$$\begin{cases} \left(-\Delta + y^2\right)\, \partial_y h(\underline{x}, \underline{x}_0, y) = -2y h(\underline{x}, \underline{x}_0, y) \\ \partial_y h(\underline{x}, \underline{x}_0, y)|_{\underline{x}\in\partial\Omega} = -\left.\frac{e^{-y|\underline{x}-\underline{x}_0|}}{4\pi}\right|_{\underline{x}\in\partial\Omega} \end{cases} ; \quad \underline{x}_0 \in \Omega \tag{3.24}$$

the solution of which belongs to $C^2(\Omega) \cap C(\bar{\Omega})$ in view of the regularity of the source term and the boundary value. The maximum principle — see the version given in [Br83, Thm IX.27] — in this case implies

$$\partial_y h(\underline{x}, \underline{x}_0, y) < 0 \quad \text{for } \forall \underline{x} \in \Omega,\ y > 0\,. \tag{3.25}$$

In particular, the solution of (3.24) for $y = 0$ is $\partial_y h(\underline{x}, \underline{x}_0, 0) = -\frac{1}{4\pi}$. This characterization of $h(\underline{x}_0, \underline{x}_0, y)$ allows us to claim that the equation (3.20) admits at least one solution for any $\alpha \in (-\infty, -h(\underline{x}_0, \underline{x}_0, 0)]$ and that the conditions (3.18) hold. Moreover, the second one of the relations (3.3) in Lemma 3.1 implies the monotonicity of the function $\alpha \mapsto y(\alpha)$ implicitly defined by (3.20); this grants the uniqueness of the solution.

Next we turn to (3.17) for $\xi \in (0, \lambda_0)$. In this case, setting $y := \sqrt{\xi}$, the equation reads as

$$\alpha + \frac{i\,y}{4\pi} + h(\underline{x}_0, \underline{x}_0, i\,y) = 0\,. \tag{3.26}$$

According to the second one of the relations (3.4), this is equivalent to

$$\alpha + \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\,y) = 0\,, \tag{3.27}$$

where $\operatorname{Re} h$ satisfies the boundary value problem

$$\begin{cases} \left(-\Delta - y^2\right) \operatorname{Re} h(\underline{x}, \underline{x}_0, i\,y) = 0 \\ \operatorname{Re} h(\underline{x}, \underline{x}_0, y)|_{\underline{x}\in\partial\Omega} = \left.\frac{\cos y\,|\underline{x}-\underline{x}_0|}{4\pi\,|\underline{x}-\underline{x}_0|}\right|_{\underline{x}\in\partial\Omega} \end{cases} ; \qquad \underline{x}_0 \in \Omega \tag{3.28}$$

It is worthwhile to notice that $h(\underline{x}, \underline{x}_0, y)$ is not defined for $y = \sqrt{\lambda_0}$. In particular, one can show that[2]

$$\lim_{|\varepsilon|\to 0} \left\| \varepsilon h(\cdot, \underline{x}_0, i\sqrt{\lambda_0 - \varepsilon}) + \psi_0(\underline{x}_0)\, \psi_0(\cdot) \right\|_{L^2(U)} = 0\,, \tag{3.29}$$

where ψ_0 is the principal eigenstate of the Dirichlet Laplacian and U is any subset of Ω. In view of the boundedness of ψ_0 and the arbitrariness of U, this relation also implies

$$\lim_{y\to\sqrt{\lambda_0}^-} |h(\underline{x}_0, \underline{x}_0, i\,y)| = +\infty \tag{3.30}$$

Using this result together with the conditions (3.4) and (3.5) of Lemma 3.1, we conclude that $\operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\,y)$ is a strictly decreasing function of $y \in \left(0, \sqrt{\lambda_0}\right)$ whose behavior for $y \to \sqrt{\lambda_0}$ is given by

$$\lim_{y\to\sqrt{\lambda_0}^-} \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\,y) = -\infty \tag{3.31}$$

Summing up this discussion, the equation (3.27) has a unique positive solution $y = y(\alpha)$ for any $\alpha \in (-h(\underline{x}_0, \underline{x}_0, 0), +\infty)$, and this solution asymptotically approaches the value λ_0 as $\alpha \to +\infty$; in combination with the first part this concludes the proof of the lemma. ■

[2]Relation (3.29) easily follows, e.g., from Lemma 2 in [BFM07].

Next we deal with the eigenvalue equation in the two-dimensional case. Recall that the free Green's function related to this problem is the modified Bessel function K_0, which is strictly positive and convex in $\mathbb{R}^+$ and admits the following representation [AS72]

$$K_0(x) = -\left\{\ln\frac{x}{2} + \gamma\right\} I_0(x) + \sum_{n=1}^{+\infty} c_n \, \frac{x^{2n}}{(2\,n!)^{2n}}\,, \tag{3.32}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, $c_n = \sum_{k=1}^{n} \frac{1}{k}$, and $I_0(x)$ is the other modified Bessel function given by the series

$$I_0(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n!)^{2n}}\,. \tag{3.33}$$

In the following proof we will make use of the asymptotic properties of K_0,

$$\lim_{x\to 0^+} K_0(x) = +\infty\,, \qquad \lim_{x\to+\infty} K_0(x) = 0\,. \tag{3.34}$$

Lemma 3.3 *Let λ_0 denote the first eigenvalue of $-\Delta_\Omega^D$ corresponding to the domain $\Omega \subset \mathbb{R}^2$. For any $\alpha \in \mathbb{R}$, the equation*

$$\alpha - \ln\sqrt{-\xi} - 2\pi\, h(\underline{x}_0, \underline{x}_0, \sqrt{-\xi}) = 0\,, \quad \xi \in (-\infty, \lambda_0)\,, \tag{3.35}$$

admits an unique solution, denoted $\xi(\alpha)$, such that

$$\lim_{\alpha\to+\infty} \xi(\alpha) = -\infty\,, \qquad \xi(f(\underline{x}_0, \underline{x}_0, 0)) = 0\,, \tag{3.36}$$

and

$$\lim_{\alpha\to-\infty} \xi(\alpha) = \lambda_0\,, \tag{3.37}$$

where $f(\underline{x}, \underline{x}_0, \sqrt{-\xi})$ in the second one of the relations (3.36) is a $C^{+\infty}(\Omega) \cap C(\bar{\Omega})$–regular function of both the spatial variables defined by

$$f(\underline{x}, \underline{x}_0, \sqrt{-\xi}) = 2\pi\, h(\underline{x}, \underline{x}_0, \sqrt{-\xi}) + \ln\sqrt{-\xi}\, I_0(\sqrt{-\xi}\ |\underline{x} - \underline{x}_0|)\,, \quad \xi < \lambda_0\,. \tag{3.38}$$

Proof. The argument follows the same line as in Lemma 3.2, the main difference coming from the specific form of the Green functions in two dimensions. For $\xi \le 0$ and $y := \sqrt{|\xi|}$, equation (3.35) reads

$$\alpha = \ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y)\,, \tag{3.39}$$

where $h(\underline{x}, \underline{x}_0, \sqrt{y})$, the solution of the boundary value problem

$$\begin{cases} \left(-\Delta + y^2\right)\, h(\underline{x}, \underline{x}_0, y) = 0 \\ h(\underline{x}, \underline{x}_0, y)\big|_{\underline{x}\in\partial\Omega} = \frac{1}{2\pi} K_0(y\ |\underline{x} - \underline{x}_0|)\big|_{\underline{x}\in\partial\Omega} \end{cases} ; \quad \underline{x}_0 \in \Omega \tag{3.40}$$

is strictly positive in Ω, as it follows from the maximum principle and the positivity of the boundary values. Consequently, the r.h.s. of (3.39) diverges as $y \to +\infty$. Moreover, from (3.3) we know that $\ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y)$ is strictly increasing as a function of y in the whole $\mathbb{R}^+$. In order to study the behaviour for $y \to 0$, we introduce the auxiliary function $f(\underline{x}, \underline{x}_0, y)$, defined in (3.38) which solves the boundary value problem

$$\begin{cases} \left(-\Delta + y^2\right) f(\underline{x}, \underline{x}_0, y) = 0 \\ h(\underline{x}, \underline{x}_0, y)\big|_{\underline{x}\in\partial\Omega} = \frac{1}{2\pi} K_0(y\, |\underline{x} - \underline{x}_0|) + \ln y\, I_0(\sqrt{-\xi}\, |\underline{x} - \underline{x}_0|)\big|_{\underline{x}\in\partial\Omega} \end{cases} ; \quad \underline{x}_0 \in \Omega \tag{3.41}$$

Using the definition (3.33), it is easy to verify that the quantity $\ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y)$ coincides with the value of f in $\underline{x}_0$, thus for $y \to 0^+$ we have

$$\lim_{y\to 0^+} (\ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y)) = \lim_{y^+\to 0} f(\underline{x}_0, \underline{x}_0, y)\,.$$

In this limit, the problem (3.41) simplifies to

$$\begin{cases} -\Delta\, f(\underline{x}, \underline{x}_0, 0) = 0 \\ h(\underline{x}, \underline{x}_0, 0)\big|_{\underline{x}\in\partial\Omega} = -\frac{1}{2\pi} \left\{ \ln \frac{|\underline{x}-\underline{x}_0|}{2} + \gamma \right\}\Big|_{\underline{x}\in\partial\Omega} \end{cases} ; \quad \underline{x}_0 \in \Omega \tag{3.42}$$

where the representation (3.32) has been considered; the regularity of the boundary condition in (3.42) implies $f(\underline{x}, \underline{x}_0, 0) \in C^{+\infty}(\Omega) \cap C(\bar{\Omega})$. So far we have shown that $(\ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y))$ is a strictly increasing function of $y \in \mathbb{R}^+$ such that

$$\lim_{y\to +\infty} (\ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y)) = +\infty \tag{3.43}$$

and

$$\lim_{y\to 0^+} (\ln y + 2\pi\, h(\underline{x}_0, \underline{x}_0, y)) = f(\underline{x}_0, \underline{x}_0, 0)\,. \tag{3.44}$$

From here we can conclude that the equation (3.35) admits an unique solution, $\xi(\alpha) \le 0$, for any $\alpha \in [f(\underline{x}_0, \underline{x}_0, 0), +\infty)$, which satisfies the conditions (3.36).

In the case of a positive eigenvalue $\xi \in (0, \lambda_0)$, setting $y = \sqrt{\xi}$ we rewrite equation (3.35) as

$$\alpha = \ln i\, y + 2\pi\, h(\underline{x}_0, \underline{x}_0, i\, y) \tag{3.45}$$

which due to (3.4) is equivalent to

$$\alpha = \ln y + 2\pi\, \mathrm{Re}\, h(\underline{x}_0, \underline{x}_0, i\, y)\,. \tag{3.46}$$

As in the 3D case, we notice that the r.h.s of (3.46) is a strictly decreasing function, cf. (3.5), diverging as $y \to \sqrt{\lambda_0}$; the sought conclusion easily follows. ■

Since the spectrum of H_α is determined by the solutions of the equations (3.1), the above lemmata have the following implication which means that in a sense point interactions in dimension two and three can be always regarded as "attractive".

Corollary 3.4 *For any real α, the operator H_α defined by (2.8)–(2.10) has a unique spectral point below the spectral threshold of $-\Delta_\Omega^D$.*

4 Dependence of the principal eigenvalue on the position of the interaction

Now we pass to our main topic. We will characterize the behaviour of the principal eigenvalue of the point-interaction Hamiltonians H_α for a fixed bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, as the interaction site moves towards the boundary of Ω. We will restrict our attention to domains having an interior reflection property w.r.t. a suitable hyperplane, in the following sense:

Definition 4.1 *Consider a hyperplane P of dimension $d-1$ in $\mathbb{R}^d$ and denote by S^P the mirror image of a set $S \subset \mathbb{R}^d$ w.r.t. P provided $S \cap P = \varnothing$. The domain Ω is said to have the* interior reflection *property w.r.t. P if $P \cap \Omega \neq \varnothing$ and there is an open connected component $\Omega_s \subset \Omega \backslash P$ such that Ω_s^P is a proper subset of $\Omega \backslash \bar{\Omega}_s$. We call Ω_s the* smaller side *of Ω and P an* interior reflection *hyperplane.*

To prove our main result, we need following auxiliary statement.

Lemma 4.2 *Let $\mathcal{G}_0^z(\underline{x}, \underline{x}')$ be defined by (2.5)-(2.6) and $z \in \mathbb{R}$. For values of z above $-\lambda_0$, the following implications hold,*

$$z \in [0, +\infty) \implies \mathcal{G}_0^z(\underline{x}, \underline{x}') > 0 \quad \text{in } \Omega \tag{4.1}$$

and

$$z \in (-\lambda_0, 0) :\Longrightarrow: \operatorname{Re} \mathcal{G}_0^z(\underline{x}, \underline{x}') > 0 \quad \text{in } \Omega\,. \tag{4.2}$$

Proof. For $\Omega \subset \mathbb{R}^3$ and $z \in [0, +\infty)$, the Green function

$$\mathcal{G}_0^z(\underline{x}, \underline{x}') = \frac{e^{-\sqrt{z}|\underline{x}-\underline{x}'|}}{4\pi\,|\underline{x}-\underline{x}'|} - h(\underline{x}, \underline{x}', \sqrt{z}) \tag{4.3}$$

is certainly positive in a small enough open neighbourhood $B_{\underline{x}'}$ of the point $\underline{x}'$ due to the boundedness of $h(\underline{x}, \underline{x}', \sqrt{z})$. Moreover, it solves the boundary value problem

$$\left\{ \begin{array}{l} (-\Delta + z)\,\mathcal{G}_0^z(\underline{x}, \underline{x}') = 0 \quad in\ \Omega \backslash B_{\underline{x}'} \\ \mathcal{G}_0^z|_{\partial\Omega} = 0\,, \quad \mathcal{G}_0|_{\partial B_{\underline{x}'}} > 0 \end{array} \right. \quad ; \qquad \underline{x}' \in \Omega \tag{4.4}$$

It follows from the maximum principle that $\mathcal{G}_0^z$ is strictly positive in the whole Ω.

To prove the other implication in the 3D case, notice that for $z \in (-\lambda_0, 0)$ we have

$$\operatorname{Re} \mathcal{G}_0^z(\underline{x}, \underline{x}') = \frac{\cos\sqrt{|z|}\,|\underline{x}-\underline{x}'|}{4\pi\,|\underline{x}-\underline{x}'|} - \operatorname{Re} h(\underline{x}, \underline{x}', i\sqrt{|z|})\,. \tag{4.5}$$

Once more we can find a suitable open neighbourhood of the point $\underline{x}'$, which we call $B_{\underline{x}'}$, where this function is positive. In $\Omega \backslash B_{\underline{x}'}$ $\operatorname{Re} \mathcal{G}_0^z(\underline{x}, \underline{x}')$ solves the boundary value problem

$$\left\{ \begin{array}{l} (-\Delta + z) \operatorname{Re} \mathcal{G}_0^z(\underline{x}, \underline{x}') = 0 \quad in\ \Omega \backslash B_{\underline{x}'} \\ \mathcal{G}_0|_{\partial\Omega} = 0\,, \quad \mathcal{G}_0|_{\partial B_{\underline{x}'}} > 0 \end{array} \right. \quad ; \qquad \underline{x}' \in \Omega \tag{4.6}$$

Under the condition $z > -\lambda_0$ we can still apply the maximum principle obtaining in this way $\operatorname{Re} \mathcal{G}_0^z(\underline{x}, \underline{x}') > 0$ in Ω. Finally, in the 2D case the proof follows the same line with the replacement (4.3) and (4.5) by the corresponding 2D Green's function

$$\mathcal{G}_0^z(\underline{x}, \underline{x}') = K_0(\sqrt{z}\,|\underline{x}-\underline{x}'|) - h(\underline{x}, \underline{x}', \sqrt{z}) \tag{4.7}$$

and taking into account the asymptotic properties of $K_0(x)$ as $x \to 0$. ■

Now we are in position to prove our main result. The next theorem shows that, under the interior reflection conditions imposed on the domain Ω, the principal eigenvalue of H_α increases as the interaction site moves towards the boundary of the smaller side Ω_s of Ω.

Theorem 4.3 *Let P be an interior reflection hyperplane for the domain Ω and denote by $\underline{n}$ the normal vector to P pointing towards Ω_s. Assume that $\underline{x}_0 \in \Omega \cap (\partial\Omega_s \cap P)$; then the principal eigenvalue ξ of the point-interaction H_α with the perturbation placed at $\underline{x}_0$ satisfies the condition*

$$\underline{n} \cdot \nabla_{\underline{x}_0} \xi > 0\,. \tag{4.8}$$

Proof. Consider first the 3D case. To analyze the dependence of ξ on the interaction position $\underline{x}_0$, we have to distinguish between the negative and positive spectral points. If $\alpha \in (-\infty, -h(\underline{x}_0, \underline{x}_0, 0)]$, then $\xi = \xi(\underline{x}_0)$ is by Lemma 3.2 a negative solution of (3.20). Replacing $\sqrt{|\xi|}$ with $y(\underline{x}_0)$ and taking the gradient w.r.t. $\underline{x}_0$ in (3.20) we find

$$\nabla_{\underline{x}_0} y(\underline{x}_0) \left(\frac{1}{4\pi} + \partial_y h \right) = -\nabla_{\underline{x}_0} h(\underline{x}_0, \underline{x}_0, y) \,. \tag{4.9}$$

Next we consider the term $\nabla_{\underline{x}_0} h(\underline{x}_0, \underline{x}_0, y)$ at the r.h.s. of the last equation; under our interior reflection assumptions we will show that this vector is oriented towards the smaller side of Ω. To this aim we notice that, in view of the relations (2.2)–(2.4), $h(\underline{x}, \underline{x}', y)$ can be written as

$$h(\underline{x}, \underline{x}', y) = \frac{e^{-y|\underline{x}-\underline{x}'|}}{4\pi\,|\underline{x}-\underline{x}'|} - \sum_{\substack{n\in\mathbb{N}_0\\ k\le N_n}} \frac{\psi_{n,k}(\underline{x}')\,\psi_{n,k}(\underline{x})}{\lambda_n + y^2} \tag{4.10}$$

for any $\underline{x} \neq \underline{x}'$. From the symmetry of this expression and the regularity $h(\underline{x}, \underline{x}', y)$ it follows that

$$\nabla_{\underline{x}'} h(\underline{x}, \underline{x}', y)\big|_{\underline{x}=\underline{x}'=\underline{x}_0} = \nabla_{\underline{x}} h(\underline{x}, \underline{x}', y)\big|_{\underline{x}=\underline{x}'=\underline{x}_0} \tag{4.11}$$

and

$$\nabla_{\underline{x}_0} h(\underline{x}_0, \underline{x}_0, y) = 2\,\nabla_{\underline{x}} h(\underline{x}, \underline{x}', y)\big|_{\underline{x}=\underline{x}'=\underline{x}_0} \,. \tag{4.12}$$

To analyze the orientation of this vector, we introduce the function u defined on the smaller part of Ω by

$$u(\underline{x}, \underline{x}_0, y) = h(\underline{x}, \underline{x}_0, y) - h(\underline{x}^P, \underline{x}_0, y), \quad \underline{x} \in \Omega_s \,, \tag{4.13}$$

where $\underline{x}^P$ denotes the mirror image of $\underline{x} \in \Omega_s$ through the plane P. The following equation holds

$$\begin{cases} \left(-\Delta + y^2\right) u = 0 & \text{in } \Omega_s \\ u|_{P\cap\Omega} = 0\,, \quad u|_{\partial\Omega_s\cap\partial\Omega} = \frac{e^{-y|\underline{x}-\underline{x}_0|}}{4\pi|\underline{x}-\underline{x}_0|} - h(\underline{x}^P, \underline{x}_0, y)\Big|_{\underline{x}\in\partial\Omega s\cap\partial\Omega} \end{cases} ; \quad \underline{x}_0 \in \Omega \cap P \tag{4.14}$$

It is worthwhile to notice that the boundary value on $\partial\Omega_s \cap \partial\Omega$ can be identified with the value of $\mathcal{G}_0^{y^2}(\underline{x}, \underline{x}_0)$ on the set $(\partial\Omega s \cap \partial\Omega)^P$, indeed we have

$$\mathcal{G}_0^{y^2}(\underline{x}, \underline{x}_0)\Big|_{(\partial\Omega s\cap\partial\Omega)^P} = \frac{e^{-y|\underline{x}-\underline{x}_0|}}{4\pi\,|\underline{x}-\underline{x}_0|} - h(\underline{x}, \underline{x}_0, y)\Bigg|_{(\partial\Omega s\cap\partial\Omega)^P} = \frac{e^{-y|\underline{x}-\underline{x}_0|}}{4\pi\,|\underline{x}-\underline{x}_0|} - h(\underline{x}^P, \underline{x}_0, y)\Bigg|_{\partial\Omega s\cap\partial\Omega} \tag{4.15}$$

Then it follows from (4.1) that u is positive on $\partial\Omega s \cap \partial\Omega$ and by the maximum principle, $u > 0$ holds in Ω_s. In particular, u reaches its minimum on the points of the open surface $P \cap \Omega$; the Hopf boundary-point lemma in this case implies

$$\underline{n} \cdot \nabla_{\underline{x}} u > 0 \quad \text{for } \forall \underline{x} \in P \cap \Omega \,. \tag{4.16}$$

Due to the definition (4.13), in combination with the relation

$$\underline{n} \cdot \nabla_{\underline{x}} h(\underline{x}^P, \underline{x}_0, y) = -\underline{n} \cdot \nabla_{\underline{x}^P} h(\underline{x}^P, \underline{x}_0, y) \quad \textit{in } \Omega_s \cup (P \cap \Omega) \,, \tag{4.17}$$

the last inequality also implies

$$2\,\underline{n} \cdot \nabla_{\underline{x}} h(\underline{x}, \underline{x}_0, y) > 0 \quad \text{for } \forall \underline{x} \in P \cap \Omega \,. \tag{4.18}$$

Substituting (4.12) and (4.18) into the r.h.s. of (4.9) and taking the projection in the direction of the vector $\underline{n}$ we get

$$\underline{n} \cdot \nabla_{\underline{x}_0} y(\underline{x}_0) \left(\frac{1}{4\pi} + \partial_y h \right) = -\underline{n} \cdot \nabla_{\underline{x}_0} h(\underline{x}_0, \underline{x}_0, y) < 0 \,. \tag{4.19}$$

The term $\nabla_{\underline{x}_0} y(\underline{x}_0)$ at the l.h.s. of (4.19) is related to $\nabla_{\underline{x}_0} \xi$ by

$$\nabla_{\underline{x}_0} y = -\frac{1}{y} \nabla_{\underline{x}_0} \xi \tag{4.20}$$

from which it follows that

$$\frac{1}{y} \left(\frac{1}{4\pi} + \partial_y h \right) \nabla_{\underline{x}_0} \xi > 0 \tag{4.21}$$

The sought inequality (4.8) follows easily from (4.21) taking into account the condition (3.3).

In the opposite case, $\alpha > -h(\underline{x}_0, \underline{x}_0, 0)$, the first spectral point of H_α is a strictly positive solution of the equation

$$\alpha + \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\sqrt{\xi}) = 0 \tag{4.22}$$

with $\xi < \lambda_0$ — cf. (3.27) in Lemma 3.2. Replacing $\sqrt{\xi}$ with $y(\underline{x}_0)$ and taking the gradient w.r.t. $\underline{x}_0$ in (4.22), we get

$$\nabla_{\underline{x}_0} y \, \partial_y \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\, y) = -\nabla_{\underline{x}_0} \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\, y) \,. \tag{4.23}$$

In order to check the orientation of the vector at the r.h.s of this expression, we notice again that due to the symmetry of the function

$$\operatorname{Re} h(\underline{x}, \underline{x}', i\, y) = \frac{\cos y \, |\underline{x} - \underline{x}'|}{4\pi \, |\underline{x} - \underline{x}'|} - \sum_{\substack{n \in \mathbb{N}_0 \\ k \le N_n}} \frac{\psi_{n,k}(\underline{x}') \, \psi_{n,k}(\underline{x})}{\lambda_n - y^2} \tag{4.24}$$

the gradient $\nabla_{\underline{x}_0} \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\, y)$ can be expressed as

$$\nabla_{\underline{x}_0} \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\, y) = 2 \, \nabla_{\underline{x}} \operatorname{Re} h(\underline{x}, \underline{x}', y) \big|_{\underline{x} = \underline{x}' = \underline{x}_0} \,. \tag{4.25}$$

Then we follow the line of the first part of the proof introducing the function u,

$$u(\underline{x}, \underline{x}_0, y) = \operatorname{Re} h(\underline{x}, \underline{x}_0, i\, y) - \operatorname{Re} h(\underline{x}^P, \underline{x}_0, i\, y) \,, \quad \underline{x} \in \Omega_s \,. \tag{4.26}$$

Proceeding as before and taking into account the implication (4.2) we find easily

$$\underline{n} \cdot \nabla_{\underline{x}} u > 0 \tag{4.27}$$

and

$$2 \, \underline{n} \cdot \nabla_{\underline{x}} \operatorname{Re} h(\underline{x}, \underline{x}_0, y) > 0 \quad \text{for } \forall \underline{x} \in P \cap \Omega \,; \tag{4.28}$$

substituting (4.25) and (4.28) into the r.h.s. of (4.23), we conclude that

$$\underline{n} \cdot \nabla_{\underline{x}_0} y \, \partial_y \operatorname{Re} h(\underline{x}_0, \underline{x}_0, i\, y) < 0 \tag{4.29}$$

The claim (4.8) is then obtained from (4.29) by taking into account the relation

$$\nabla_{\underline{x}_0} y = \frac{1}{y} \nabla_{\underline{x}_0} \xi \tag{4.30}$$

and the inequality (3.5) from Lemma 3.1. This concludes the argument in the three-dimensional case; the two-dimensional one can be dealt with in the same way, step by step. ■

5 Optimization of $\xi(\underline{x}_0)$

By Theorem 4.3 the spectral threshold of the operator H_α increases as the interaction position $\underline{x}_0$ moves towards the boundary of the domain Ω. This result provides us with some insights on how to place the point-interaction centre to minimize the principal eigenvalue of the Hamiltonian H_α. For the sake of simplicity, we begin with the case of a convex Ω. Let Π be the set of all the hyperplanes P of interior reflection for Ω; we denote by $\Omega_{s,P}$ the smaller part related to $P \in \Pi$, provided it exists, and by Σ the union

$$\Sigma = \bigcup_{P\in\Pi} \Omega_{s,P} \tag{5.1}$$

The following claim is a straightforward consequence of the Theorem 4.3.

Corollary 5.1 *Let Ω be an open convex domain in $\mathbb{R}^d$, $d = 2, 3$, and let H_α be a point-interaction operator in Ω with the perturbation placed at $\underline{x}_0$. The principal eigenvalue of H_α, considered as a function of the interaction centre, takes its minimum value when $\underline{x}_0$ belongs to the open set $\Omega\backslash\Sigma$.*

Proof. Notice first that the continuity of $h(\underline{x}, \underline{x}_0, y)$ implies the continuity of solutions of the eigenvalue equations (3.1), thus the principal eigenvalue $\xi(\underline{x}_0)$ has at least one minimum point $\underline{x}_0^m \in \bar{\Omega}$. We use reduction ad absurdum: assume $\underline{x}_0^m \in \Omega_{s,P}$ for a suitable hyperplane $P \in \Pi$. Due to the convexity of the domain, it exists another hyperplane $P' \in \Pi$ parallel to P and such that

$$\underline{x}_0^m \in \partial\Omega_{s,P'} \cap P' \,, \tag{5.2}$$

however, under this assumption Theorem 4.3 implies the inequality

$$\underline{n} \cdot \nabla_{\underline{x}_0} \xi > 0 \tag{5.3}$$

from which a contradiction follows easily. ■

In the case of highly symmetric domains such as the interior of a circle or an ellipse in the plane, and similarly a ball and an interior of an ellipsoid in three dimensions, it is easy to identify the set Σ with the center of such a domain. More generally, the convexity of Ω ensures the validity of the interior reflection property needed in Theorem 4.3 with respect to some hyperplane passing through a point sufficiently close to the boundary. In this situation the above result can be used to localize the optimal position of $\underline{x}_0$ in a 'central' subset of the domain Ω.

A slight generalization of the above argument lead us to an analogous rule to localize the minimum points of the principal eigenvalue for a point interaction within non-convex domains. Let $P \in \Pi$ and consider the one parameter family of hyperplanes P_t,

$$\begin{cases} P_t = \{\underline{x} + \underline{n}\,t : \underline{x} \in P,\ t \in [0, T]\} \\ \\ T = \max\{t \in \mathbb{R}^+ : P_t \cap \Omega \neq \varnothing\} \end{cases} \tag{5.4}$$

where $\underline{n}$ denotes the unit normal to P directed towards the smaller part Ω_s of Ω. We denote as Π' the subset formed by all those hyperplanes $P \in \Pi$ such that

$$P_t \in \Pi \quad \text{for } \forall\, t \in [0, T] \,. \tag{5.5}$$

It is important to notice that in the non-convex case, to any hyperplane of interior reflection there may correspond more than one smaller part. Next we denote by $\Theta_{s,P}$ the union of all the smaller parts related to P, and by Σ' the set

$$\Sigma' = \bigcup_{P\in\Pi'} \Theta_{s,P} \,. \tag{5.6}$$

Corollary 5.2 *Assume that Ω is an open domain in $\mathbb{R}^d$, $d = 2, 3$, and H_α is a point-interaction operator in Ω with the perturbation placed at $\underline{x}_0$. The principal eigenvalue of H_α, regarded as a function of $\underline{x}_0$, takes its minimum value when $\underline{x}_0$ belongs to the open set $\Omega\backslash\Sigma'$.*

Proof. The argument is an easy modification of the proof of Corollary 5.1. ■

Non-convex domains with a reasonably simple boundary such as, for instance, the union of two intersecting disks or a dog-bone profile in two dimensions, can be easily analyzed using Corollary 5.2. It is also worthwhile to stress that the results of this section do not depend on the parameter α, hence the optimal placement of the point interaction with respect to the minimum of the principal eigenvalues can be the same irrespective of the interaction "strength".

Let us finally comment on he relation to the work [HKK01] mentioned in the introduction. We have said that for a hard-wall obstacle the principal eigenvalue *decreases* as it moves towards the boundary. The difference of the two effects can be traced back to the different boundary conditions which characterize the operator domains in the two cases. While the hard obstacle is characterized by Dirichlet boundary condition, the point-interaction operator H_α considered here can be obtained as the norm-resolvent limit of a family of sphere interactions Hamiltonians $H_\alpha(r)$ with the boundary condition of a *mixed type* as the radius $r \to 0$. In the three-dimensional case, for instance, the operator $H_\alpha(r)$ is explicitly given by

$$\begin{cases} H_\alpha(r) = -\Delta \qquad \text{on } \Omega\backslash S_r \\ D(H_\alpha(r)) = \left\{ \psi \in H^1(\mathbb{R}^3) \middle| \, (\partial_n\psi)_+ - (\partial_n\psi)_- = \frac{1}{4\pi\alpha\, r^2 + r}\psi \right\} \end{cases}$$

where S_r denotes the sphere of radius r centered at $\underline{x}_0$ and $(\partial_n\psi)_+ - (\partial_n\psi)_-$ is the jump of the normal derivative of ψ on the interaction surface [FT93, Sh03]. Another insight into the difference of the two situations can be obtained from [EŠ96].

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