Quantum Fluctuations and Rate of Convergence towards Mean Field Dynamics

Igor Rodnianski[1*] and Benjamin Schlein[2†]

Department of Mathematics, Princeton University
Princeton, NJ, USA[1]

Institute of Mathematics, University of Munich,
Theresienstr. 39, D-80333 Munich, Germany[2]

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Abstract

The nonlinear Hartree equation describes the macroscopic dynamics of initially factorized N-boson states, in the limit of large N. In this paper we provide estimates on the rate of convergence of the microscopic quantum mechanical evolution towards the limiting Hartree dynamics. More precisely, we prove bounds on the difference between the one-particle density associated with the solution of the N-body Schrödinger equation and the orthogonal projection onto the solution of the Hartree equation.

1 Introduction

We consider an N boson system described on the Hilbert space $L^2_s(\mathbb{R}^{3N})$ (the subspace of $L^2(\mathbb{R}^{3N})$ consisting of all functions symmetric with respect to arbitrary permutations of the N particles) by a mean field Hamiltonian of the form

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N V(x_i - x_j)\,. \tag{1.1}$$

We will specify later assumptions on the interaction potential V. Note the coupling constant $1/N$ in front of the potential energy which characterizes mean-field models; it makes sure that in the limit of large N the potential and the kinetic energy are typically of the same order, and thus can compete to generate nontrivial effective equation for the macroscopic dynamics of the system.

We consider a factorized initial wave function

$$L^2_s(\mathbb{R}^{3N}) \ni \;\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j) \quad \text{for some } \varphi \in H^1(\mathbb{R}^3) \tag{1.2}$$

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with normalization $\|\varphi\|_{L^2(\mathbb{R}^3)} = 1$ (so that $\|\psi_N\|_{L^2(\mathbb{R}^{3N})} = 1$) and we study its time-evolution $\psi_{N,t}$, given by the solution of the N body Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t} \qquad \text{with initial data } \psi_{N,0} = \psi_N. \tag{1.3}$$

In (1.2) and in what follows we use the notation $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$.

Clearly, because of the interaction among the particles, the factorization of the wave function is not preserved by the time evolution. However, due to the presence of the small constant $1/N$ in front of the potential energy in (1.1), we may expect the total potential experienced by each particle to be approximated, for large N, by an effective mean field potential, and thus that, in the limit $N \to \infty$, the solution $\psi_{N,t}$ of (1.3) is still approximately (and in an appropriate sense) factorized. We may expect, in other words, that in an appropriate sense

$$\psi_{N,t}(\mathbf{x}) \simeq \prod_{j=1}^{N} \varphi_t(x_j) \qquad \text{for large } N. \tag{1.4}$$

If (1.4) is indeed correct, it is easy to derive a self-consistent equation for the evolution of the one-particle wave function φ_t. In fact, it follows from (1.4) that the total potential experienced by a particle at x can be approximated by the convolution $(V * |\varphi_t|^2)(x)$, and thus that the evolution of the one-particle wave function φ_t is described by the nonlinear Hartree equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + (V * |\varphi_t|^2)\varphi_t \,. \tag{1.5}$$

To understand in which sense (1.4) holds true, we need to introduce marginal densities. The density matrix $\gamma_{N,t} = |\psi_{N,t}\rangle\langle\psi_{N,t}|$ associated with $\psi_{N,t}$ is defined as the orthogonal projection onto $\psi_{N,t}$ (we use here Dirac's bracket notation; for $f, g, h \in L^2(\mathbb{R}^d)$, $|f\rangle\langle g| : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the operator defined by $|f\rangle\langle g|(h) = \langle g, h\rangle_{L^2}\, f$). The kernel of $\gamma_{N,t}$ is thus given by

$$\gamma_{N,t}(\mathbf{x}; \mathbf{x}') = \psi_{N,t}(\mathbf{x})\overline{\psi}_{N,t}(\mathbf{x}').$$

For $k = 1, \dots, N$, we define then the k-particle marginal density $\gamma^{(k)}_{N,t}$ associated with $\psi_{N,t}$ by taking the partial trace of $\gamma_{N,t}$ over the last $N-k$ particles. In other words, we define $\gamma^{(k)}_{N,t}$ as a positive trace class operator on $L^2(\mathbb{R}^{3k})$ with kernel

$$\gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k}\; \gamma_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k}; \mathbf{x}'_k, \mathbf{x}_{N-k}) \,. \tag{1.6}$$

Since $\|\psi_{N,t}\|_{L^2(\mathbb{R}^{3N})} = 1$, we immediately obtain $\mathrm{Tr}\, \gamma^{(k)}_{N,t} = 1$ for all $N \geq 1$, $k = 1, \dots, N$, and $t \in \mathbb{R}$.

By the choice of the initial wave function (1.2), at time $t = 0$ we have $\gamma^{(k)}_{N,0} = |\varphi\rangle\langle\varphi|^{\otimes k}$. It turns out that (1.4) should be understood in terms of convergence of marginal densities. For a large class of interaction potentials V, for every fixed $k \geq 1$, and $t \in \mathbb{R}$, one can in fact show that

$$\gamma^{(k)}_{N,t} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \qquad \text{as } N \to \infty \tag{1.7}$$

where φ_t is a solution of the nonlinear Hartree equation (1.5). The convergence (1.7) holds in the trace norm topology. In particular, (1.7) implies that for arbitrary k and for an arbitrary bounded operator $J^{(k)}$ on $L^2(\mathbb{R}^{3k})$,

$$\left\langle \psi_{N,t}, \left(J^{(k)} \otimes 1^{(N-k)}\right) \psi_{N,t} \right\rangle \to \langle \varphi_t^{\otimes k}, J^{(k)} \varphi_t^{\otimes k}\rangle$$

as $N \to \infty$. The approximate identity (1.4) can thus be be interpreted as follows: as long as we are interested in the expectation of observables depending non-trivially only on a fixed number of particles, the N-body wave function $\psi_{N,t}$ can be approximated by the N-fold tensor product of the solution ϕ_t to the nonlinear Hartree equation (1.5).

The first rigorous proof of (1.7) was obtained by Spohn in [11], under the assumption of a bounded interaction potential V. The problem of proving (1.7) becomes substantially more involved for singular potentials. In [7], Erdős and Yau extended Spohn's approach to obtain a rigorous derivation of the Hartree equation (1.5) for a Coulomb interaction $V(x) = \text{const}/|x|$ (partial results for the Coulomb interaction were also obtained by Bardos, Golse, and Mauser in [2]). In [4], the Hartree equation with Coulomb interaction was derived for semirelativistic bosons; in the semirelativistic setting, the dispersion of the bosons only grows linearly in the momentum (for large momenta), and thus the control of the Coulomb singularity is more delicate. In [3, 5, 6], models described by the Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^{N} N^{3\beta} V(N^\beta(x_i - x_j)) \qquad \text{with } \beta \in (0,1]$$

with an N-dependent potential were considered (in the one-dimensional case, N-dependent potentials were considered by Adami, Golse and Teta in [1]). These models are used to describe systems of physical interest, such as Bose-Einstein condensates. Assuming the interaction to be positive ($V(x) \geq 0$ for all $x \in \mathbb{R}^3$) and sufficiently small, the main result was again a proof of the convergence (1.7); this time, however, φ_t is a solution of the cubic nonlinear Schrödinger equation (with local nonlinearity)

$$i\partial_t \varphi_t = -\Delta\varphi_t + \sigma|\varphi_t|^2\varphi_t \quad \text{with } \sigma = \begin{cases} b_0 & \text{if } 0 < \beta < 1 \\ 8\pi a_0 & \text{if } \beta = 1 \end{cases} . \tag{1.8}$$

Here $b_0 = \int \mathrm{d}x V(x)$ and a_0 is the scattering length of V. The emergence of the scattering length a_0 for $\beta = 1$ (for all other choices of $0 < \beta < 1$ the coupling constant is given by b_0, which is the first Born approximation to $8\pi a_0$) is a consequence of the short scale correlation structure developed in solutions of the Schrödinger equation, which, in the case $\beta = 1$, is characterized by the same length scale $O(1/N)$ as the scale of the interaction potential.

The results described above have been obtained by extensions of the approach introduced by Spohn in [11], which was based on the study of the BBGKY hierarchy

$$\begin{aligned} i\partial_t \gamma^{(k)}_{N,t} = &\sum_{j=1}^{k} [-\Delta_{x_j}, \gamma^{(k)}_{N,t}] + \frac{1}{N}\sum_{i<j}^{k} [V(x_i - x_j), \gamma^{(k)}_{N,t}] \\ &+ \left(\frac{N-k}{N}\right) \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\, [V(x_j - x_{k+1}), \gamma^{(k+1)}_{N,t}] \end{aligned} \tag{1.9}$$

for the evolution of the marginal densities $\gamma^{(k)}_{N,t}$, $k = 1, \ldots, N$ (here Tr_{k+1} denotes the partial trace over the $(k+1)$-th particle; this hierarchy is equivalent to the Schrödinger equation (1.3) for $\psi_{N,t}$). Because of the compactness of the sequence $\gamma^{(k)}_{N,t}$, $N \geq k$, the proof of (1.7) reduces to two main steps. The first step consists in proving that an arbitrary family of limit points $\{\gamma^{(k)}_{\infty,t}\}_{k\geq 1}$ satisfies the infinite hierarchy

$$i\partial_t \gamma^{(k)}_{\infty,t} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma^{(k)}_{\infty,t}] + \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\, [V(x_j - x_{k+1}), \gamma^{(k+1)}_{\infty,t}] . \tag{1.10}$$

The second step is a proof of the uniqueness of the solution of (1.10). Since the factorized family $\gamma^{(k)}_{\infty,t} = |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$, with φ_t determined by (1.5), is a solution of the infinite hierarchy (1.10), these two steps are sufficient to obtain (1.7).

Despite its many successes, this method has some limitations. The main one, from our point of view, is that, because of the use of abstract arguments related to the compactness of the sequence $\gamma^{(k)}_{N,t}$, this technique does not provide any information on the rate of convergence of $\gamma^{(k)}_{N,t}$ to $|\varphi_t\rangle\langle\varphi_t|^{\otimes k}$.

In some cases, instead of comparing the solution of (1.9) with the solution of the infinite hierarchy (1.10), it is also possible to expand it in a Duhamel series and to compare it directly with the corresponding expansion for the factorized densities $|\varphi_t\rangle\langle\varphi_t|^{\otimes k}$. This approach (see [11]) leads to bounds of the form

$$\mathrm{Tr}\left|\gamma^{(k)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right| \leq \frac{C^k}{N} \tag{1.11}$$

for all sufficiently small times $|t| \leq t_0$. The restriction to small times is needed to guarantee the convergence of the Duhamel expansion of the solution to (1.9). Iterating the arguments used to obtain (1.11), one can derive bounds of the form

$$\mathrm{Tr}\left|\gamma^{(k)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right| \leq \frac{C^k}{N^{\frac{1}{2^t}}}$$

which hold for all $t \in \mathbb{R}$, but deteriorate very fast in time and are therefore not effective and not very useful. Next theorem, which is the main result of this paper, provides much stronger bounds on the difference between the true quantum mechanical evolution of the marginal densities and their Hartree evolution; in particular it shows that for every fixed time $t \in \mathbb{R}$, the error is at most of the order $O(N^{-1/2})$.

Theorem 1.1. *Suppose that there exists $D > 0$ such that the operator inequality*

$$V^2(x) \leq D\,(1 - \Delta_x) \tag{1.12}$$

holds true. Let

$$\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j), \tag{1.13}$$

for some $\varphi \in H^1(\mathbb{R}^3)$ with[1] $\|\varphi\| = 1$. Denote by $\psi_{N,t} = e^{-iH_N t}\psi_N$ the solution to the Schrödinger equation (1.3) with initial data $\psi_{N,0} = \psi_N$, and let $\gamma^{(1)}_{N,t}$ be the one-particle density associated with $\psi_{N,t}$. Then there exist constants C, K, depending only on the H^1 norm of φ and on the constant D on the r.h.s. of (1.12) such that

$$Tr\left|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right| \leq \frac{C}{N^{1/2}}\, e^{Kt}\,. \tag{1.14}$$

Here φ_t is the solution to the nonlinear Hartree equation

$$i\partial_t\varphi_t = -\Delta\varphi_t + (V * |\varphi_t|^2)\varphi_t \tag{1.15}$$

with initial data $\varphi_{t=0} = \varphi$.

[1] In what follows, for a function f we will always denote by $\|f\|$ its L^2 norm, while, for an operator A, $\|A\|$ will mean its L^2 operator norm.

Remark 1.2. *The assumption on the potential V means that the most singular potential we can handle is the Coulomb potential $V(x) = \kappa/|x|$. Note that our theorem applies both to the attractive ($\kappa < 0$) and the repulsive case ($\kappa > 0$). In particular Theorem 1.1 implies the result obtained by Erdös and Yau in [7].*

Remark 1.3. *Note that under the assumption (1.12) on the interaction potential V, the nonlinear equation (1.15) is known to be globally well-posed in $H^1(\mathbb{R}^3)$. This follows from the conservation of the mass $\|\varphi\|$ and of the energy*

$$\mathcal{E}(\varphi) = \int \mathrm{d}x \, |\nabla\varphi(x)|^2 + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, V(x-y)|\varphi(x)|^2|\varphi(y)|^2$$

and from the observation that there exist constants c_1, c_2 such that

$$\mathcal{E}(\varphi) \leq c_1\|\varphi\|^2_{H^1}(1+\|\varphi\|^2) \quad \text{and} \quad \|\varphi\|^2_{H^1} \leq c_2\left(\mathcal{E}(\varphi) + \|\varphi\|^4 + \|\varphi\|^2\right). \tag{1.16}$$

Both bounds can be proven using that, by (1.12),

$$\int \mathrm{d}y \, V(x-y)|\varphi(y)|^2 \leq \varepsilon\|\nabla\varphi\|^2 + \varepsilon^{-1}\|\varphi\|^2$$

for all $\varepsilon > 0$, uniformly in $x \in \mathbb{R}^3$.

Remark 1.4. *Instead of (1.14) we will prove that*

$$\|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\|_{HS} \leq \frac{C}{N^{1/2}}\, e^{Kt} \tag{1.17}$$

where $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm. Although in general the trace norm is bigger than the Hilbert-Schmidt norm, in this case they differ at most by a factor of two[2]*. In fact, since $|\varphi_t\rangle\langle\varphi_t|$ is a rank one projection, the operator $A = \gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|$ can only have one negative eigenvalue $\lambda_{neg} < 0$. Since moreover*

$$Tr\left(\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right) = 0$$

it follows that the negative eigenvalue of A is equal, in absolute value, to the sum of all positive eigenvalues. The trace norm of A is equal, therefore, to $2|\lambda_{neg}| = 2\|A\|$, where $\|A\|$ denotes the operator norm of A. Since $\|A\| \leq \|A\|_{HS}$, we immediately obtain that $Tr\,|A| \leq 2\|A\|_{HS}$.

Remark 1.5. *The bound (1.14) is not optimal. As mentioned above, for short times and bounded potentials, the quantity on the l.h.s. of (1.14) is known to be of the order $1/N$. Nevertheless Theorem 1.1 is the first estimate on the rate of convergence towards the mean-field limit which holds for all times and remains of the same order $N^{-1/2}$ for all fixed times.*

Remark 1.6. *Although, in order to simplify the analysis, we only consider the rate of convergence of the one-particle density $\gamma^{(1)}_{N,t}$ to $|\varphi_t\rangle\langle\varphi_t|$, our method can also be used to prove bounds of the form*

$$Tr\left|\gamma^{(j)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|^{\otimes j}\right| \leq \frac{C(j)}{N^{1/2}}\, e^{K(j)\,t}$$

for all j, t, N and for j-dependent constants $C(j), K(j)$.

[2]We would like to thank Robert Seiringer for pointing out this argument to us.

In this paper we avoid the use of the BBGKY hierarchy and instead revive an approach, introduced by Hepp in [9] and extended by Ginibre and Velo in [8], to the study of a semiclassical limit of quantum many-boson systems[3]. This approach is based on embedding the N-body Schrödinger system into the second quantized Fock-space representation and on the use of coherent states as initial data. The use of the Fock-space representation is in particular dictated by the fact that coherent states do not have a fixed number of particles.

The Hartree dynamics emerges as the main component of the evolution of coherent states in the mean field limit (or, in the language of [9, 8], in the semiclassical limit). The problem then reduces to the study of quantum fluctuations, described by an N-dependent two-parameter unitary group $\mathcal{U}_N(t;s)$, around the Hartree dynamics. In [9, 8], Hepp (for smooth interaction potentials) and Ginibre and Velo (for singular potentials) proved that, in the limit $N \to \infty$, the fluctuation dynamics $\mathcal{U}_N(t;s)$ approaches a limiting evolution $\mathcal{U}(t;s)$. This important result shows the relevance of the Hartree dynamics in the mean field limit (at least in the case of coherent initial states). It does not prove, however, the convergence (1.7) of the one-particle marginal density to the orthogonal projection onto the solution of the Hartree equation, nor does it imply convergence results for the evolution of factorized initial sates. The problem of convergence of marginals requires additional control on the growth of the number[4] of fluctuations generated by the evolution $\mathcal{U}_N(t;s)$. This analysis, which, technically, is the most difficult part of the present paper (see Proposition 3.3), is new[5]. Another novel part of our work is the derivation of convergence towards Hartree dynamics for factorized initial sates from the corresponding statements for the evolution of coherent states.

Although we are mainly concerned with the dynamics of factorized initial data, the result we obtain for coherent states (see Theorem 3.1) is of independent interest, especially because, in this case, our bound is optimal in its N-dependence (for coherent states, we show that the error is at most of the order $1/N$ for every fixed time).

The paper is organized as follows. In Section 2, we define the Fock space representation of the mean field system, introduce coherent states and review their main properties. In Section 3, we consider the evolution of a coherent state and we prove that, in this case, the rate of convergence to the mean field solution remains of the order $1/N$ for all fixed times. Finally, in Section 4, we show how to use coherent states to obtain information on the dynamics of factorized states, and we prove Theorem 1.1.

2 Fock space representation

We define the bosonic Fock space over $L^2(\mathbb{R}^3, \mathrm{d}x)$ as the Hilbert space

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2(\mathbb{R}^3, \mathrm{d}x)^{\otimes_s n} = \mathbb{C} \oplus \bigoplus_{n\geq 1} L^2_s(\mathbb{R}^{3n}, \mathrm{d}x_1 \dots \mathrm{d}x_n)\,,$$

with the convention $L^2(\mathbb{R}^3)^{\otimes_s 0} = \mathbb{C}$. Vectors in $\mathcal{F}$ are sequences $\psi = \{\psi^{(n)}\}_{n\geq 0}$ of n-particle wave functions $\psi^{(n)} \in L^2_s(\mathbb{R}^{3n})$. The scalar product on $\mathcal{F}$ is defined by

$$\langle \psi_1, \psi_2 \rangle = \sum_{n\geq 0} \langle \psi_1^{(n)}, \psi_2^{(n)} \rangle_{L^2(\mathbb{R}^{3n})} = \overline{\psi_1^{(0)}} \psi_2^{(0)} + \sum_{n\geq 1} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \, \overline{\psi_1^{(n)}}(x_1, \dots, x_n) \psi_2^{(n)}(x_1, \dots, x_n)\,.$$

[3]Mathematically, the semiclassical limit considered in [9, 8] is equivalent to the mean field limit considered in the present manuscript.

[4]Fluctuations around the Hartree dynamics will be considered as particle excitations and thus it will be possible to compute their number.

[5]A more precise discussion of the results of [9, 8], and of their relation with our work can be found at the end of Section 3.

An N particle state with wave function ψ_N is described on $\mathcal{F}$ by the sequence $\{\psi^{(n)}\}_{n\geq 0}$ where $\psi^{(n)} = 0$ for all $n \neq N$ and $\psi^{(N)} = \psi_N$. The vector $\{1, 0, 0, \dots\} \in \mathcal{F}$ is called the vacuum, and will be denoted by Ω.

On $\mathcal{F}$, we define the number of particles operator $\mathcal{N}$, by $(\mathcal{N}\psi)^{(n)} = n\psi^{(n)}$. Eigenvectors of $\mathcal{N}$ are vectors of the form $\{0, \dots, 0, \psi^{(m)}, 0, \dots\}$ with a fixed number of particles. For $f \in L^2(\mathbb{R}^3)$ we also define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $\mathcal{F}$ by

$$\begin{aligned} \left(a^*(f)\psi\right)^{(n)}(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^n f(x_j)\psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ \left(a(f)\psi\right)^{(n)}(x_1, \dots, x_n) &= \sqrt{n+1} \int \mathrm{d}x \, \overline{f(x)} \, \psi^{(n+1)}(x, x_1, \dots, x_n)\,. \end{aligned} \tag{2.1}$$

The operators $a^*(f)$ and $a(f)$ are unbounded, densely defined, closed operators. The creation operator $a^*(f)$ is the adjoint of the annihilation operator $a(f)$ (note that by definition $a(f)$ is anti-linear in f), and they satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \langle f, g\rangle_{L^2(\mathbb{R}^3)}, \qquad [a(f), a(g)] = [a^*(f), a^*(g)] = 0\,. \tag{2.2}$$

For every $f \in L^2(\mathbb{R}^3)$, we introduce the self adjoint operator

$$\phi(f) = a^*(f) + a(f)\,.$$

We will also make use of operator valued distributions a^*_x and a_x ($x \in \mathbb{R}^3$), defined so that

$$\begin{aligned} a^*(f) &= \int \mathrm{d}x \, f(x) \, a^*_x \\ a(f) &= \int \mathrm{d}x \, \overline{f(x)} \, a_x \end{aligned} \tag{2.3}$$

for every $f \in L^2(\mathbb{R}^3)$. The canonical commutation relations assume the form

$$[a_x, a^*_y] = \delta(x-y) \qquad [a_x, a_y] = [a^*_x, a^*_y] = 0\,.$$

The number of particle operator, expressed through the distributions a_x, a^*_x, is given by

$$\mathcal{N} = \int \mathrm{d}x \, a^*_x a_x\,.$$

The following lemma provides some useful bounds to control creation and annihilation operators in terms of the number of particle operator $\mathcal{N}$.

Lemma 2.1. *Let $f \in L^2(\mathbb{R}^3)$. Then*

$$\begin{aligned} \|a(f)\psi\| &\leq \|f\| \, \|\mathcal{N}^{1/2}\psi\| \\ \|a^*(f)\psi\| &\leq \|f\| \, \| \left(\mathcal{N}+1\right)^{1/2} \psi\| \\ \|\phi(f)\psi\| &\leq 2\|f\| \| \left(\mathcal{N}+1\right)^{1/2} \psi\| \end{aligned} \tag{2.4}$$

Proof. The last inequality clearly follows from the first two. To prove the first bound we note that

$$\begin{aligned} \|a(f)\psi\| &\leq \int \mathrm{d}x \, |f(x)| \, \|a_x\psi\| \leq \left(\int \mathrm{d}x \, |f(x)|^2\right)^{1/2} \left(\int \mathrm{d}x \, \|a_x\psi\|^2\right)^{1/2} \\ &= \|f\| \, \|\mathcal{N}^{1/2}\psi\|\,. \end{aligned} \tag{2.5}$$

The second estimate follows by the canonical commutation relations (2.2) because

$$\begin{aligned}\|a^*(f)\psi\|^2 &= \langle \psi, a(f)a^*(f)\psi\rangle = \langle \psi, a^*(f)a(f)\psi\rangle + \|f\|^2\|\psi\|^2 \\ &= \|a(f)\psi\|^2 + \|f\|^2\|\psi\|^2 \leq \|f\|^2 \left(\|\mathcal{N}^{1/2}\psi\| + \|\psi\|^2\right) = \|f\|^2\| \left(\mathcal{N}+1\right)^{1/2} \psi\|^2 .\end{aligned} \tag{2.6}$$

□

Given $\psi \in \mathcal{F}$, we define the one-particle density $\gamma_\psi^{(1)}$ associated with ψ as the positive trace class operator on $L^2(\mathbb{R}^3)$ with kernel given by

$$\gamma_\psi^{(1)}(x;y) = \frac{1}{\langle \psi, \mathcal{N}\psi\rangle} \langle \psi, a_y^* a_x \psi\rangle . \tag{2.7}$$

By definition, $\gamma_\psi^{(1)}$ is a positive trace class operator on $L^2(\mathbb{R}^3)$ with $\mathrm{Tr}\,\gamma_\psi^{(1)} = 1$. For every N-particle state with wave function $\psi_N \in L^2_s(\mathbb{R}^{3N})$ (described on $\mathcal{F}$ by the sequence $\{0,0,\dots,\psi_N,0,0,\dots\}$) it is simple to see that this definition is equivalent to the definition (1.6).

We define the Hamiltonian $\mathcal{H}_N$ on $\mathcal{F}$ by $(\mathcal{H}_N\psi)^{(n)} = \mathcal{H}_N^{(n)}\psi^{(n)}$, with

$$\mathcal{H}_N^{(n)} = -\sum_{j=1}^n \Delta_j + \frac{1}{N}\sum_{i<j}^n V(x_i - x_j) .$$

Using the distributions a_x, a_x^*, $\mathcal{H}_N$ can be rewritten as

$$\mathcal{H}_N = \int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N}\int \mathrm{d}x\mathrm{d}y\, V(x-y) a_x^* a_y^* a_y a_x . \tag{2.8}$$

By definition the Hamiltonian $\mathcal{H}_N$ leaves sectors of $\mathcal{F}$ with a fixed number of particles invariant. Moreover, it is clear that on the N-particle sector, $\mathcal{H}_N$ agrees with the Hamiltonian H_N (the subscript N in $\mathcal{H}_N$ is a reminder of the scaling factor $1/N$ in front of the potential energy). We will study the dynamics generated by the operator $\mathcal{H}_N$. In particular we will consider the time evolution of coherent states, which we introduce next.

For $f \in L^2(\mathbb{R}^3)$, we define the Weyl-operator

$$W(f) = \exp\left(a^*(f) - a(f)\right) = \exp\left(\int \mathrm{d}x\, \left(f(x)a_x^* - \overline{f}(x)a_x\right)\right) . \tag{2.9}$$

Then the coherent state $\psi(f) \in \mathcal{F}$ with one-particle wave function f is defined by

$$\psi(f) = W(f)\Omega .$$

Notice that

$$\psi(f) = W(f)\Omega = e^{-\|f\|^2/2}\sum_{n\geq 0}\frac{(a^*(f))^n}{n!}\Omega = e^{-\|f\|^2/2}\sum_{n\geq 0}\frac{1}{\sqrt{n!}}\, f^{\otimes n} , \tag{2.10}$$

where $f^{\otimes n}$ indicates the Fock-vector $\{0,\dots,0,f^{\otimes n},0,\dots\}$. This follows from

$$\exp(a^*(f) - a(f)) = e^{-\|f\|^2/2}\exp(a^*(f))\exp(-a(f))$$

which is a consequence of the fact that the commutator $[a(f), a^*(f)] = \|f\|^2$ commutes with $a(f)$ and $a^*(f)$. From Eq. (2.10) we see that coherent states are superpositions of states with different number of particles (the probability of having n particles in $\psi(f)$ is given by $e^{-\|f\|^2}\|f\|^{2n}/n!$).

In the following lemma we collect some important and well known properties of Weyl operators and coherent states.

Lemma 2.2. *Let $f, g \in L^2(\mathbb{R}^3)$.*

i) The Weyl operator satisfy the relations

$$W(f)W(g) = W(g)W(f)e^{-2i\, Im\,\langle f,g\rangle} = W(f+g)e^{-i\, Im\,\langle f,g\rangle}\,.$$

ii) $W(f)$ is a unitary operator and

$$W(f)^* = W(f)^{-1} = W(-f).$$

iii) We have

$$W^*(f)a_x W(f) = a_x + f(x), \qquad \text{and} \quad W^*(f)a_x^* W(f) = a_x^* + \overline{f}(x)\,.$$

iv) From iii) we see that coherent states are eigenvectors of annihilation operators

$$a_x \psi(f) = f(x)\psi(f) \qquad \Rightarrow \qquad a(g)\psi(f) = \langle g, f\rangle_{L^2}\psi(f)\,.$$

v) The expectation of the number of particles in the coherent state $\psi(f)$ is given by $\|f\|^2$, that is

$$\langle \psi(f), \mathcal{N}\psi(f)\rangle = \|f\|^2\,.$$

Also the variance of the number of particles in $\psi(f)$ is given by $\|f\|^2$ (the distribution of $\mathcal{N}$ is Poisson), that is

$$\langle \psi(f), \mathcal{N}^2\psi(f)\rangle - \langle \psi(f), \mathcal{N}\psi(f)\rangle^2 = \|f\|^2\,.$$

vi) Coherent states are normalized but not orthogonal to each other. In fact

$$\langle \psi(f), \psi(g)\rangle = e^{-\frac{1}{2}\left(\|f\|^2 + \|g\|^2 - 2(f,g)\right)} \quad \Rightarrow \quad |\langle \psi(f), \psi(g)\rangle| = e^{-\frac{1}{2}\|f-g\|^2}\,.$$

3 Time evolution of coherent states

Next we study the dynamics of coherent states with expected number of particles N in the limit $N \to \infty$. We choose the initial data

$$\psi(\sqrt{N}\varphi) = W(\sqrt{N}\varphi)\Omega \qquad \text{for } \varphi \in H^1(\mathbb{R}^3) \text{ with } \|\varphi\| = 1 \tag{3.1}$$

and we study its time evolution $\psi(N,t) = e^{-i\mathcal{H}_N t}\psi(\sqrt{N}\varphi)$ with the Hamiltonian $\mathcal{H}_N$ defined in (2.8).

Theorem 3.1. *Suppose that there exists $D > 0$ such that the operator inequality*

$$V^2(x) \leq D(1 - \Delta_x) \tag{3.2}$$

holds true. Let $\Gamma^{(1)}_{N,t}$ be the one-particle marginal associated with $\psi(N,t) = e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega$ (as defined in (2.7)). Then there exist constants $C, K > 0$ (only depending on the H^1-norm of φ and on the constant D appearing in (3.2)) such that

$$Tr\left|\Gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right| \leq \frac{C}{N}\, e^{Kt} \tag{3.3}$$

for all $t \in \mathbb{R}$.

Remark 3.2. *The use of coherent states as initial data allows us to obtain the optimal rate of convergence $1/N$ for all fixed times (while for the evolution of factorized N-particle states we only get the rate $1/\sqrt{N}$; see (1.14)).*

Proof. The proof of Theorem 3.1 will occupy the remaining subsections of section 3.

3.1 Dynamics $\mathcal{U}_N$ of quantum fluctuations

By (2.7), the kernel of $\Gamma^{(1)}_{N,t}$ is given by

$$\begin{aligned}
\Gamma^{(1)}_{N,t}(x;y) =& \frac{1}{N}\left\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}a^*_y a_x e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\right\rangle \\
=& \varphi_t(x)\overline{\varphi}_t(y) + \frac{\overline{\varphi}_t(y)}{\sqrt{N}}\left\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}(a_x - \sqrt{N}\varphi_t(x))e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\right\rangle \\
&+ \frac{\varphi_t(x)}{\sqrt{N}}\left\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}(a^*_y - \sqrt{N}\overline{\varphi}_t(y))e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\right\rangle \\
&+ \frac{1}{N}\left\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}(a^*_y - \sqrt{N}\overline{\varphi}_t(y))(a_x - \sqrt{N}\varphi_t(x))e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\right\rangle .
\end{aligned} \tag{3.4}$$

It was observed by Hepp in [9] (see also Eqs. (1.17)-(1.28) in [8]) that

$$\begin{aligned}
W^*(\sqrt{N}\varphi_s)\, e^{i\mathcal{H}_N(t-s)}(a_x - \sqrt{N}\varphi_t(x))e^{-i\mathcal{H}_N(t-s)}W(\sqrt{N}\varphi_s) &= \mathcal{U}_N(t;s)^*\, a_x\, \mathcal{U}_N(t;s) \\
&= \mathcal{U}_N(s;t)\, a_x\, \mathcal{U}_N(t;s)
\end{aligned} \tag{3.5}$$

where the unitary evolution $\mathcal{U}_N(t;s)$ is determined by the equation[6]

$$i\partial_t \mathcal{U}_N(t;s) = \mathcal{L}_N(t)\mathcal{U}_N(t;s) \qquad \text{and} \quad \mathcal{U}_N(s;s) = 1 \tag{3.6}$$

with the generator

$$\begin{aligned}
\mathcal{L}_N(t) =& \int \mathrm{d}x\, \nabla_x a^*_x \nabla_x a_x + \int \mathrm{d}x\, \left(V * |\varphi_t|^2\right)(x)\, a^*_x a_x + \int \mathrm{d}x\mathrm{d}y\, V(x-y)\, \overline{\varphi_t}(x)\varphi_t(y) a^*_y a_x \\
&+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\left(\varphi_t(x)\varphi_t(y)a^*_x a^*_y + \overline{\varphi_t}(x)\overline{\varphi_t}(y)a_x a_y\right) \\
&+ \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\, a^*_x\left(\varphi_t(y)a^*_y + \overline{\varphi_t}(y)a_y\right) a_x \\
&+ \frac{1}{2N}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\, a^*_x a^*_y a_y a_x .
\end{aligned} \tag{3.7}$$

It follows from (3.4) that

$$\begin{aligned}
\Gamma^{(1)}_{N,t}(x,y) - \varphi_t(x)\overline{\varphi}_t(y) =& \frac{1}{N}\left\langle \Omega, \mathcal{U}_N(t;0)^* a^*_y a_x \mathcal{U}_N(t;0)\Omega\right\rangle \\
&+ \frac{\varphi_t(x)}{\sqrt{N}}\left\langle \Omega, \mathcal{U}_N(t;0)^* a^*_y \mathcal{U}_N(t;0)\Omega\right\rangle \\
&+ \frac{\overline{\varphi}_t(y)}{\sqrt{N}}\left\langle \Omega, \mathcal{U}_N(t;0)^* a_x \mathcal{U}_N(t;0)\Omega\right\rangle .
\end{aligned} \tag{3.8}$$

In order to produce another decaying factor $1/\sqrt{N}$ in the last two term on the r.h.s. of the last equation, we compare the evolution $\mathcal{U}_N(t;0)$ with another evolution $\widetilde{\mathcal{U}}_N(t;0)$ defined through the equation

$$i\partial_t \widetilde{\mathcal{U}}_N(t;s) = \widetilde{\mathcal{L}}_N(t)\,\widetilde{\mathcal{U}}_N(t;s) \qquad \text{with} \quad \widetilde{\mathcal{U}}_N(s;s) = 1 \tag{3.9}$$

[6] Note that, explicitly, $\mathcal{U}_N(t,s) = W^*(\sqrt{N}\phi_t)e^{-i\mathcal{H}_N(t-s)}W(\sqrt{N}\phi_s)$.

with the time-dependent generator

$$\begin{aligned}\widetilde{\mathcal{L}}_N(t) = &\int \mathrm{d}x\, \nabla_x a_x^* \nabla_x a_x + \int \mathrm{d}x\, \left(V * |\varphi_t|^2\right)(x)\, a_x^* a_x + \int \mathrm{d}x\mathrm{d}y\, V(x-y)\overline{\varphi_t}(x)\varphi_t(y) a_y^* a_x \\ &+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\left(\varphi_t(x)\varphi_t(y) a_x^* a_y^* + \overline{\varphi_t}(x)\overline{\varphi_t}(y) a_x a_y\right) \\ &+ \frac{1}{2N}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\, a_x^* a_y^* a_y a_x\,. \end{aligned} \tag{3.10}$$

From (3.8) we find

$$\begin{aligned} &\Gamma^{(1)}_{N,t}(x;y) - \varphi_t(x)\overline{\varphi}_t(y) \\ &= \frac{1}{N}\langle \Omega, \mathcal{U}_N(t;0)^* a_y^* a_x \mathcal{U}_N(t;0)\Omega\rangle \\ &+ \frac{\varphi_t(x)}{\sqrt{N}}\left(\left\langle \Omega, \mathcal{U}_N(t;0)^* a_y^* \left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\rangle + \left\langle \Omega, \left(\mathcal{U}_N(t;0)^* - \widetilde{\mathcal{U}}_N(t;0)^*\right) a_y^* \widetilde{\mathcal{U}}_N(t;0)\Omega\right\rangle\right) \\ &+ \frac{\overline{\varphi}_t(y)}{\sqrt{N}}\left(\left\langle \Omega, \mathcal{U}_N(t;0)^* a_x \left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\rangle + \left\langle \Omega, \left(\mathcal{U}_N(t;0)^* - \widetilde{\mathcal{U}}_N(t;0)^*\right) a_x \widetilde{\mathcal{U}}_N(t;0)\Omega\right\rangle\right). \end{aligned} \tag{3.11}$$

Here we used the fact that

$$\left\langle \Omega, \widetilde{\mathcal{U}}_N(t;0)^* a_y \widetilde{\mathcal{U}}_N(t;0)\Omega\right\rangle = \left\langle \Omega, \widetilde{\mathcal{U}}_N(t;0)^* a_x^* \widetilde{\mathcal{U}}_N(t;0)\Omega\right\rangle = 0\,.$$

This follows from the observation that, although the evolution $\widetilde{\mathcal{U}}_N(t)$ does not preserve the number of particles, it preserves the parity (it commutes with $(-1)^{\mathcal{N}}$). Multiplying (3.11) with the kernel $J(x,y)$ of a Hilbert-Schmidt operator J over $L^2(\mathbb{R}^3)$ and taking the trace, we obtain

$$\begin{aligned} &\mathrm{Tr}\, J\left(\Gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right) \\ &= \frac{1}{N}\int \mathrm{d}x\mathrm{d}y\, J(x,y)\langle a_y \mathcal{U}_N(t;0)\Omega, a_x \mathcal{U}_N(t;0)\Omega\rangle \\ &\quad + \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, J(x,y)\varphi_t(x)\langle a_y \mathcal{U}_N(t;0)\Omega, \left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\rangle \\ &\quad + \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, J(x,y)\varphi_t(x)\langle \left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega, a_y^* \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\ &\quad + \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, J(x,y)\,\overline{\varphi}_t(y)\langle a_x^* \mathcal{U}_N(t;0)\Omega, \left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\rangle \\ &\quad + \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y J(x,y)\,\overline{\varphi}_t(y)\langle \left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega, a_x \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle\,. \end{aligned}$$

Hence

$$
\begin{aligned}
&\left|\operatorname{Tr} J\left(\Gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right)\right| \\
&\quad \leq \frac{1}{N}\left(\int \mathrm{d}x\mathrm{d}y\, |J(x,y)|^2\right)^{1/2} \int \mathrm{d}x \|a_x \mathcal{U}_N(t;0)\Omega\|^2 \\
&\qquad + \frac{1}{\sqrt{N}}\left\|\left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\| \int \mathrm{d}x\, |\varphi_t(x)| \|a(J(x,.))\mathcal{U}_N(t;0)\Omega\| \\
&\qquad + \frac{1}{\sqrt{N}}\left\|\left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\| \int \mathrm{d}x |\varphi_t(x)| \|a^*(J(x,.))\widetilde{\mathcal{U}}_N(t;0)\Omega\| \\
&\qquad + \frac{1}{\sqrt{N}}\left\|\left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\| \int \mathrm{d}y\, |\varphi_t(y)| \|a^*(J(.,y))\mathcal{U}_N(t;0)\Omega\| \\
&\qquad + \frac{1}{\sqrt{N}}\left\|\left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\| \int \mathrm{d}y\, |\varphi_t(y)| \|a(J(.,y))\widetilde{\mathcal{U}}_N(t;0)\Omega\|
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\left|\operatorname{Tr} J\left(\Gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right)\right| \leq\; & \frac{\|J\|_{\mathrm{HS}}}{N}\, \langle \mathcal{U}_N(t;0)\Omega, \mathcal{N}\mathcal{U}_N(t;0)\Omega\rangle \\
& + \frac{2\|J\|_{\mathrm{HS}}}{\sqrt{N}} \|(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0))\Omega\| \, \|(\mathcal{N}+1)^{1/2}\mathcal{U}_N(t;0)\Omega\| \\
& + \frac{2\|J\|_{\mathrm{HS}}}{\sqrt{N}} \|(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0))\Omega\| \, \|(\mathcal{N}+1)^{1/2}\widetilde{\mathcal{U}}_N(t;0)\Omega\| \,.
\end{aligned}
$$

The proof of Theorem 3.1 now follows from Proposition 3.3, Lemma 3.8, Lemma 3.9, and from the remark that the trace norm can be controlled, in this case, by twice the Hilbert-Schmidt norm (see Remark 3 after Theorem 1.1). ☐

Proposition 3.3. *Let $\mathcal{U}_N(t;s)$ be the unitary evolution defined in (3.6). Then there exists a constant K, and, for every $j \in \mathbb{N}$, constants $C(j), K(j)$ (depending only on $\|\varphi\|_{H^1}$ and on the constant D appearing in (3.2)) such that*

$$
\langle \mathcal{U}_N(t;s)\psi, \mathcal{N}^j \mathcal{U}_N(t;s)\psi\rangle \leq C(j)\langle \psi, (\mathcal{N}+1)^{2j+2}\psi\rangle\, e^{K(j)|t-s|}\,. \tag{3.12}
$$

for all $\psi \in \mathcal{F}$, and for all $t, s \in \mathbb{R}$.

Remark 3.4. *Proposition 3.3 states that the number of particles produced by the dynamics $\mathcal{U}_N$ of quantum fluctuations is independent of N and grows in time with at most exponential rate. This N-independence plays an important role in our analysis. Its proof requires the introduction of yet another dynamics $\mathcal{U}_N^{(M)}$, whose generator looks very similar to $\mathcal{L}_N(t)$ but contains a cutoff, in the cubic term, guaranteeing that the number of particles is smaller than a given M.*

Proof. We start by introducing a new unitary dynamics with time-dependent generator $\mathcal{L}_N^{(M)}(t)$ similar to $\mathcal{L}_N(t)$ but with a cutoff in the number of particles in the cubic term.

3.2 Truncated dynamics $\mathcal{U}_N^{(M)}$

For a fixed $M > 0$ (at the end we will choose $M = N$), we consider the time-dependent generator

$$\begin{aligned}\mathcal{L}_N^{(M)}(t) = & \int \mathrm{d}x\, \nabla_x a_x^* \nabla_x a_x + \int \mathrm{d}x\, \left(V * |\varphi_t|^2\right)(x)\, a_x^* a_x + \int \mathrm{d}x\mathrm{d}y\, V(x-y)\, \overline{\varphi_t}(x)\varphi_t(y) a_y^* a_x \\ & + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\left(\varphi_t(x)\varphi_t(y) a_x^* a_y^* + \overline{\varphi_t}(x)\overline{\varphi_t}(y) a_x a_y\right) \\ & + \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\, a_x^*\left(\overline{\varphi}_t(y) a_y \chi(\mathcal{N}\le M) + \varphi_t(y)\chi(\mathcal{N}\le M) a_y^*\right) a_x \\ & + \frac{1}{2N}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\, a_x^* a_y^* a_y a_x \end{aligned} \tag{3.13}$$

and the corresponding time-evolution $\mathcal{U}_N^{(M)}(t;s)$, defined by

$$i\partial_t \mathcal{U}_N^{(M)}(t;s) = \mathcal{L}_N^{(M)}(t)\mathcal{U}_N^{(M)}(t;s) \qquad \text{with} \qquad \mathcal{U}_N^{(M)}(s;s) = 1\,.$$

Step 1. in the proof of Proposition 3.3

Lemma 3.5. *There exists a constant K (only depending on $\|\varphi\|_{H^1}$ and on the constant D in (3.2)), such that, for all $N, M \in \mathbb{N}$, $\psi \in \mathcal{F}$, and $t, s \in \mathbb{R}$*

$$\langle \mathcal{U}_N^{(M)}(t;s)\psi, \mathcal{N}^j \mathcal{U}_N^{(M)}(t;s)\psi\rangle \le \langle \psi, (\mathcal{N}+1)^j \psi\rangle\, \exp\left(4^j\, K|t-s|(1+\sqrt{M/N})\right)\,. \tag{3.14}$$

Proof of Lemma 3.5. To prove (3.14) we compute the time-derivative of the expectation of $(\mathcal{N}+1)^j$. It suffices to consider the case $s = 0$. We find

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{U}_N^{(M)}(t;0)\psi, (\mathcal{N}+1)^j \mathcal{U}_N^{(M)}(t;0)\psi\rangle \\ & \quad = \langle \mathcal{U}_N^{(M)}(t;0)\psi, [i\mathcal{L}_N^{(M)}(t), (\mathcal{N}+1)^j]\mathcal{U}_N^{(M)}(t;0)\psi\rangle \\ & \quad = \mathrm{Im} \int \mathrm{d}x\mathrm{d}y V(x-y)\, \varphi_t(x)\varphi_t(y) \langle \mathcal{U}_N^{(M)}(t;0)\psi, [a_x^* a_y^*, (\mathcal{N}+1)^j]\mathcal{U}_N^{(M)}(t;0)\psi\rangle \\ & \quad\; + \frac{2}{\sqrt{N}}\,\mathrm{Im}\int \mathrm{d}x\mathrm{d}y V(x-y)\, \overline{\varphi}_t(y)\langle \mathcal{U}_N^{(M)}(t;0)\psi, [a_x^* a_y \chi(\mathcal{N}\le M) a_x, (\mathcal{N}+1)^j]\, \mathcal{U}_N^{(M)}(t;0)\psi\rangle \end{aligned}$$

Using the pull-through formulae $a_x\mathcal{N} = (\mathcal{N}+1)a_x$, $a_x^*\mathcal{N} = (\mathcal{N}-1)a_x^*$, we find

$$[a_x^*, (\mathcal{N}+1)^j] = \sum_{k=0}^{j-1}\binom{j}{k}(-1)^k(\mathcal{N}+1)^k a_x^*, \qquad [a_x, (\mathcal{N}+1)^j] = \sum_{k=0}^{j-1}\binom{j}{k}(\mathcal{N}+1)^k a_x.$$

As a consequence,

$$\begin{aligned} [a_x^* a_y^*, (\mathcal{N}+1)^j] &= \sum_{k=0}^{j-1}\binom{j}{k}(-1)^k\left(a_x^*(\mathcal{N}+1)^k a_y^* + (\mathcal{N}+1)^k a_x^* a_y^*\right) \\ &= \sum_{k=0}^{j-1}\binom{j}{k}(-1)^k\left(\mathcal{N}^{\frac{k}{2}} a_x^* a_y^* (\mathcal{N}+2)^{\frac{k}{2}} + (\mathcal{N}+1)^{\frac{k}{2}} a_x^* a_y^* (\mathcal{N}+3)^{\frac{k}{2}}\right), \\ [a_x, (\mathcal{N}+1)^j] &= \sum_{k=0}^{j-1}\binom{j}{k}(\mathcal{N}+1)^k a_x = \sum_{k=0}^{j-1}\binom{j}{k}\,(\mathcal{N}+1)^{\frac{k}{2}} a_x \mathcal{N}^{\frac{k}{2}}. \end{aligned}$$

Therefore

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} &\langle \mathcal{U}_N^{(M)}(t;0)\psi, (\mathcal{N}+1)^j \mathcal{U}_N^{(M)}(t;0)\psi\rangle \\
&= \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \, \mathrm{Im} \int \mathrm{d}x\mathrm{d}y \, V(x-y)\, \varphi_t(x)\varphi_t(y) \\
&\qquad \times \langle \mathcal{U}_N^{(M)}(t;0)\psi, \left(\mathcal{N}^{\frac{k}{2}} a_x^* a_y^* (\mathcal{N}+2)^{\frac{k}{2}} + (\mathcal{N}+1)^{\frac{k}{2}} a_x^* a_y^* (\mathcal{N}+3)^{\frac{k}{2}}\right) \mathcal{U}_N^{(M)}(t;0)\psi\rangle \\
&\quad + \frac{2}{\sqrt{N}} \sum_{k=0}^{j-1} \binom{j}{k} \mathrm{Im} \int \mathrm{d}x \\
&\qquad \times \langle \mathcal{U}_N^{(M)}(t;0)\psi, a_x^* a(V(x-.)\varphi_t) \chi(\mathcal{N} \le M)(\mathcal{N}+1)^{\frac{k}{2}} a_x \mathcal{N}^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\rangle.
\end{aligned} \tag{3.15}
$$

To control contributions from the first term we use bounds of the form

$$
\begin{aligned}
\Big| \int & \mathrm{d}x\mathrm{d}y V(x-y)\, \varphi_t(x)\varphi_t(y)\, \langle \mathcal{U}_N^{(M)}(t;0)\psi, (\mathcal{N}+1)^{\frac{k}{2}} a_x^* a_y^* (\mathcal{N}+3)^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\rangle \Big| \\
&\le \int \mathrm{d}x |\varphi_t(x)| \|a_x (\mathcal{N}+1)^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\| \, \|a^*(V(x-.)\varphi_t)(\mathcal{N}+3)^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\| \\
&\le \text{const} \sup_x \left(\int V(x-y)^2 |\varphi_t(y)|^2 \right)^{1/2} \|(\mathcal{N}+3)^{\frac{k+1}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\|^2 \\
&\le K \, \|(\mathcal{N}+3)^{\frac{k+1}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\|^2 \, .
\end{aligned}
$$

Here we used that, by (3.2),

$$
\sup_x \int \mathrm{d}y V^2(x-y) |\varphi_t(y)|^2 \le D \|\varphi_t\|^2_{H^1} \le \text{const} D \|\varphi\|^2_{H^1} \le K \tag{3.16}
$$

is bounded uniformly in t (as follows from (1.16)). Similar estimates are applied to the term containing $\mathcal{N}^{\frac{k}{2}} a_x^* a_y^* (\mathcal{N}+2)^{\frac{k}{2}}$.

On the other hand, to control contributions arising from the second integral on the r.h.s. of (3.15), we use estimates of the form

$$
\begin{aligned}
\Big| \int & \mathrm{d}x \, \langle \mathcal{U}_N^{(M)}(t;0)\psi, a_x^* a(V(x-.)\varphi_t) \chi(\mathcal{N} \le M)(\mathcal{N}+1)^{\frac{k}{2}} a_x \mathcal{N}^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\rangle \Big| \\
&\le \int \mathrm{d}x \, \|a_x (\mathcal{N}+1)^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\| \, \|a(V(x-.)\varphi_t)\chi(\mathcal{N} \le M)\| \, \|a_x \mathcal{N}^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\| \\
&\le M^{1/2} \sup_x \|V(x-.)\varphi_t\| \, \|\mathcal{N}^{\frac{k+1}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\| \, \|\mathcal{N}^{1/2}(\mathcal{N}+1)^{\frac{k}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\| \\
&\le K M^{1/2} \|(\mathcal{N}+1)^{\frac{k+1}{2}} \mathcal{U}_N^{(M)}(t;0)\psi\|^2 \, .
\end{aligned}
$$

This implies

$$
\begin{aligned}
\Big| \frac{\mathrm{d}}{\mathrm{d}t} & \langle \mathcal{U}_N^{(M)}(t;0)\psi, (\mathcal{N}+1)^j \mathcal{U}_N^{(M)}(t;0)\psi\rangle \Big| \\
&\le \; K(1+\sqrt{M/N}) \sum_{k=0}^{j} \binom{j}{k} \langle \mathcal{U}_N^{(M)}(t;0)\psi, (\mathcal{N}+3)^k \mathcal{U}_N^{(M)}(t;0)\psi\rangle \\
&\le \; 4^j \, K(1+\sqrt{M/N}) \, \langle \mathcal{U}_N^{(M)}(t;0)\psi, (\mathcal{N}+1)^j \, \mathcal{U}_N^{(M)}(t;0)\psi\rangle \, .
\end{aligned}
$$

From Gronwall Lemma, we find (3.14). □

Step 2. of the proof of Proposition 3.3

3.3 Weak bounds on the $\mathcal{U}_N$ dynamics

To compare the evolution $\mathcal{U}_N(t;s)$ with the cutoff evolution $\mathcal{U}_N^{(M)}(t;s)$, we first need some (very weak) a-priori bound on the growth of the number of particle with respect to $\mathcal{U}_N(t;s)$.

Lemma 3.6. *For arbitrary $t,s \in \mathbb{R}$ and $\psi \in \mathcal{F}$, we have*

$$\langle \psi, \mathcal{U}_N(t;s)\mathcal{N}\mathcal{U}_N(t;s)^*\psi\rangle \leq 6\langle \psi, (\mathcal{N}+N+1)\psi\rangle\,. \tag{3.17}$$

Moreover, for every $\ell \in \mathbb{N}$, there exists a constant $C(\ell)$ such that

$$\langle \psi, \mathcal{U}_N(t;s)\mathcal{N}^{2\ell}\mathcal{U}_N(t;s)^*\psi\rangle \leq C(\ell)\,\langle \psi, (\mathcal{N}+N)^{2\ell}\psi\rangle \tag{3.18}$$

$$\langle \psi, \mathcal{U}_N(t;s)\mathcal{N}^{2\ell+1}\mathcal{U}_N(t;s)^*\psi\rangle \leq C(\ell)\,\langle \psi, (\mathcal{N}+N)^{2\ell+1}(\mathcal{N}+1)\psi\rangle \tag{3.19}$$

for all $t,s \in \mathbb{R}$ and $\psi \in \mathcal{F}$.

Proof of Lemma 3.6. Eq. (3.19) follows from (3.18). In fact, assuming (3.18) to hold true, we have

$$\begin{aligned}
&\langle \psi, \mathcal{U}_N(t;s)\mathcal{N}^{2\ell+1}\mathcal{U}_N(t;s)^*\psi\rangle \\
&\qquad\qquad \leq \frac{1}{2N}\langle \psi, \mathcal{U}_N(t;s)\mathcal{N}^{2\ell+2}\mathcal{U}_N(t;s)^*\psi\rangle + \frac{N}{2}\langle \psi, \mathcal{U}_N(t;s)\mathcal{N}^{2\ell}\mathcal{U}_N(t;s)^*\psi\rangle \\
&\qquad\qquad \leq \frac{C(\ell+1)}{2N}\langle \psi, (\mathcal{N}+N)^{2\ell+2}\psi\rangle + \frac{C(\ell)N}{2}\langle \psi, (\mathcal{N}+N)^{2\ell}\psi\rangle \\
&\qquad\qquad \leq D(\ell)\,\langle \psi, (\mathcal{N}+N)^{2\ell+1}(\mathcal{N}+1)\psi\rangle
\end{aligned} \tag{3.20}$$

for an appropriate constant $D(\ell)$.

To prove (3.17) and (3.18) we observe that, by (3.5),

$$\begin{aligned}
&\mathcal{U}_N^*(t;s)\mathcal{N}\mathcal{U}_N(t;s) \\
&\quad = \int \mathrm{d}x\, \mathcal{U}_N^*(t;s)a_x^*a_x\mathcal{U}_N(t;s) \\
&\quad = \int \mathrm{d}x\, W^*(\sqrt{N}\varphi_s)e^{i\mathcal{H}_N(t-s)}(a_x^* - \sqrt{N}\overline{\varphi}_t(x))(a_x - \sqrt{N}\varphi_t(x))e^{-i\mathcal{H}_N(t-s)}W(\sqrt{N}\varphi_s) \\
&\quad = W^*(\sqrt{N}\varphi_s)\left(\mathcal{N} - \sqrt{N}e^{i\mathcal{H}_N(t-s)}\phi(\varphi_t)e^{-i\mathcal{H}_N(t-s)} + N\right)W(\sqrt{N}\varphi_s)\,.
\end{aligned} \tag{3.21}$$

(Recall that $\phi(\varphi) = a^*(\varphi) + a(\varphi) = \int \mathrm{d}x(\varphi(x)a_x^* + \overline{\varphi}(x)a_x)$). From Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
\langle \psi, \mathcal{U}_N^*(t;s)\mathcal{N}\mathcal{U}_N(t;s)\psi\rangle &\leq 2\langle \psi, W^*(\sqrt{N}\varphi_s)(\mathcal{N}+N+1)W(\sqrt{N}\varphi_s)\psi\rangle \\
&= 2\langle \psi, (\mathcal{N} + \sqrt{N}\phi(\varphi_s) + 2N + 1)\psi\rangle \\
&\leq 6\langle \psi, (\mathcal{N}+N+1)\psi\rangle
\end{aligned} \tag{3.22}$$

which shows (3.17). To complete the proof of (3.18), we define

$$X_{t,s} = (\mathcal{N} - \sqrt{N}e^{i\mathcal{H}_N(t-s)}\phi(\varphi_t)e^{-i\mathcal{H}_N(t-s)} + N)\,.$$

Then, using the notation $\mathrm{ad}_A(B) = [B, A]$, it is simple to prove that there exists a constant $C > 0$ such that

$$X_{t,s}^2 \le C(N + \mathcal{N})^2 \qquad \text{and} \qquad \left|\mathrm{ad}^m_{X_{t,s}}(\mathcal{N})\right| \le C(N + \mathcal{N}) \qquad \text{for all } m \in \mathbb{N}. \tag{3.23}$$

By induction it follows that, for every $\ell \in \mathbb{N}$, there exist constants $D(\ell), C(\ell)$ with

$$X_{t,s}^{\ell-1}(\mathcal{N} + N)^2 X_{t,s}^{\ell-1} \le D(\ell)(\mathcal{N} + N)^{2\ell} \qquad \text{and} \qquad X_{t,s}^{2\ell} \le C(\ell)(\mathcal{N} + N)^{2\ell}\,. \tag{3.24}$$

In fact, for $\ell = 1$ (3.24) reduces to (3.23). Assuming (3.24) to hold for all $\ell < k$, we can prove it for $\ell = k$ by noticing that

$$\begin{aligned} X_{t,s}^{k-1}(\mathcal{N} + N)^2 X_{t,s}^{k-1} &\le 2(\mathcal{N} + N)X_{t,s}^{2k-2}(\mathcal{N} + N) + 2|[X_{t,s}^{k-1}, \mathcal{N}]|^2 \\ &\le 2(\mathcal{N} + N)X_{t,s}^{2k-2}(\mathcal{N} + N) + 4^k \sum_{m=0}^{k-2} X_{t,s}^m \left|\mathrm{ad}^{k-1-m}_{X_{t,s}}(\mathcal{N})\right|^2 X_{t-s}^m \\ &\le 2(\mathcal{N} + N)X_{t,s}^{2k-2}(\mathcal{N} + N) + 4^k C \sum_{m=0}^{k-2} X_{t,s}^m(\mathcal{N} + N)^2 X_{t-s}^m \\ &\le D(k)\,(\mathcal{N} + N)^{2k} \end{aligned} \tag{3.25}$$

for an appropriate constant $D(k)$, and that, by (3.23) and (3.25),

$$X_{t,s}^{2k} \le C X_{t,s}^{k-1}(\mathcal{N} + N)^2 X_{t,s}^{k-1} \le C D(k)(\mathcal{N} + N)^{2k} = C(k)(\mathcal{N} + N)^{2k}\,.$$

In (3.25), we used the commutator expansion

$$[A^n, B] = \sum_{m=0}^{n-1} \binom{n}{m} A^m \mathrm{ad}_A^{n-m}(B)$$

in the second line, the bound (3.23) in the third line, and the induction assumption in the last line.

From (3.21) and (3.24), we obtain that

$$\begin{aligned} \langle \psi, \mathcal{U}_N(t;s)\mathcal{N}^{2\ell}\mathcal{U}_N(t;s)^*\psi\rangle &= \langle W(\sqrt{N}\varphi_s)\psi, X_{t,s}^{2\ell} W(\sqrt{N}\varphi_s)\psi\rangle \\ &\le C(\ell)\langle W(\sqrt{N}\varphi_s)\psi, (\mathcal{N} + N)^{2\ell} W(\sqrt{N}\varphi_s)\psi\rangle \\ &= C(\ell)\langle \psi, (\mathcal{N} + \sqrt{N}\phi(\varphi_s) + 2N)^{2\ell}\psi\rangle\,. \end{aligned} \tag{3.26}$$

Analogously to (3.24), it is possible to prove that, for every $\ell \in \mathbb{N}$, there exists a constant $C(\ell)$ with

$$(\mathcal{N} + \sqrt{N}\phi(\varphi_s) + 2N)^{2\ell} \le C(\ell)(\mathcal{N} + N)^{2\ell}\,.$$

Eq. (3.18) follows therefore from (3.26). $\square$

Step 3. of the proof of Proposition 3.3

3.4 Comparison of the $\mathcal{U}_N$ and $\mathcal{U}_N^{(M)}$ dynamics

Lemma 3.7. *For every $j \in \mathbb{N}$ there exist constants $C(j), K(j)$ (depending only on j, on $\|\varphi\|_{H^1}$ and on the constant D in (3.2)) such that*

$$\begin{aligned} &\left| \langle \mathcal{U}_N(t;s)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;s) - \mathcal{U}_N^{(M)}(t;s) \right) \psi \rangle \right| \\ &\qquad\qquad \leq C(j) \frac{(N/M)^j \, \|(\mathcal{N}+1)^{j+1}\psi\|^2}{(1+\sqrt{M/N})} \exp\left(K(j)(1+\sqrt{M/N})|t-s| \right) \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\left| \langle \mathcal{U}_N^{(M)}(t;s)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;s) - \mathcal{U}_N^{(M)}(t;s) \right) \psi \rangle \right| \\ &\qquad\qquad \leq C \frac{\|(\mathcal{N}+1)^j \psi\|^2}{M^j(1+\sqrt{M/N})} \exp\left(K(j)(1+\sqrt{M/N})|t-s| \right) , \end{aligned} \tag{3.28}$$

for all $\psi \in \mathcal{F}$ and for all $t, s \in \mathbb{R}$.

Proof of Lemma 3.7. To simplify the notation we consider the case $s = 0$ and $t > 0$ (but the other cases can be treated identically). To prove (3.27), we expand the difference of the two evolutions:

$$\begin{aligned} &\langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0) \right) \psi \rangle \\ &\quad = \langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \mathcal{U}_N(t;0) \left(1 - \mathcal{U}_N(t;0)^* \mathcal{U}_N^{(M)}(t;0) \right) \psi \rangle \\ &\quad = -i \int_0^t \mathrm{d}s \, \langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \mathcal{U}_N(t;s) \left(\mathcal{L}_N(s) - \mathcal{L}_N^{(M)}(s) \right) \mathcal{U}_N^{(M)}(s;0)\psi \rangle \\ &\quad = -\frac{i}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x\mathrm{d}y V(x-y) \\ &\qquad\quad \times \langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \mathcal{U}_N(t;s) a_x^* \left(\overline{\varphi}_t(y) a_y \chi(\mathcal{N} > M) + \varphi_t(y) \chi(\mathcal{N} > M) a_y^* \right) a_x \mathcal{U}_N^{(M)}(s;0)\psi \rangle \\ &\quad = -\frac{i}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x \langle a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi, a(V(x-.)\varphi_t) \chi(\mathcal{N} > M) a_x \mathcal{U}_N^{(M)}(s;0)\psi \rangle \\ &\qquad - \frac{i}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x \langle a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi, \chi(\mathcal{N} > M) a^*(V(x-.)\varphi_t) a_x \mathcal{U}_N^{(M)}(s;0)\psi \rangle . \end{aligned} \tag{3.29}$$

Hence

$$\begin{aligned}
&\left|\langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0)\right)\psi\rangle\right| \\
&\quad \leq \frac{1}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x \|a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\| \, \|a(V(x-.)\varphi_t) a_x \chi(\mathcal{N} > M+1)\mathcal{U}_N^{(M)}(s;0)\psi\| \\
&\qquad + \frac{1}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x \|a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\| \|a^*(V(x-.)\varphi_t) a_x \chi(\mathcal{N} > M)\mathcal{U}_N^{(M)}(s;0)\psi\| \\
&\quad \leq \frac{1}{\sqrt{N}} \sup_x \|V(x-.)\varphi_t\| \int_0^t \mathrm{d}s \int \mathrm{d}x \|a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\| \\
&\qquad\qquad \times \|a_x \mathcal{N}^{1/2} \chi(\mathcal{N} > M+1) \mathcal{U}_N^{(M)}(s;0)\psi\| \\
&\qquad + \frac{1}{\sqrt{N}} \sup_x \|V(x-.)\varphi_t\| \int_0^t \mathrm{d}s \int \mathrm{d}x \|a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\| \\
&\qquad\qquad \times \|a_x \mathcal{N}^{1/2} \chi(\mathcal{N} > M) \mathcal{U}_N^{(M)}(s;0)\psi\| \\
&\quad \leq \frac{C}{\sqrt{N}} \int_0^t \mathrm{d}s \|\mathcal{N}^{1/2} \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\| \, \|\mathcal{N} \chi(\mathcal{N} > M) \mathcal{U}_N^{(M)}(s;0)\psi\|
\end{aligned}$$

where we used (3.16) once again. From Lemma 3.6, we obtain

$$\begin{aligned}
\|\mathcal{N}^{1/2} \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\|^2 &= \langle \mathcal{N}^j \mathcal{U}_N(t;0)\psi, \mathcal{U}(t;s) \mathcal{N} \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N(t;0)\psi\rangle \\
&\leq 6 \langle \mathcal{N}^j \mathcal{U}_N(t;0)\psi, (\mathcal{N}+N+1) \mathcal{N}^j \mathcal{U}_N(t;0)\psi\rangle \\
&\leq C(j) \langle \psi, (\mathcal{N}+N)^{2j+1} (\mathcal{N}+1)\psi\rangle \\
&\leq C(j) N^{2j+1} \langle \psi, (\mathcal{N}+1)^{2j+2}\psi\rangle \,.
\end{aligned} \tag{3.30}$$

Therefore, using the inequality $\chi(\mathcal{N} > M) \leq (\mathcal{N}/M)^{2j}$, we obtain

$$\begin{aligned}
&\left|\langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0)\right)\psi\rangle\right| \\
&\qquad \leq C(j) N^j \, \|(\mathcal{N}+1)^{j+1}\psi\| \int_0^t \mathrm{d}s \; \langle \mathcal{U}_N^{(M)}(s;0)\psi, \mathcal{N}^2 \chi(\mathcal{N} > M) \mathcal{U}_N^{(M)}(s;0)\psi\rangle^{1/2} \\
&\qquad \leq C(j) N^j \, \|(\mathcal{N}+1)^{j+1}\psi\| \int_0^t \mathrm{d}s \; \langle \mathcal{U}_N^{(M)}(s;0)\psi, \frac{\mathcal{N}^{2j+2}}{M^{2j}} \mathcal{U}_N^{(M)}(s;0)\psi\rangle^{1/2} \,.
\end{aligned}$$

Finally, from (3.14), we conclude that

$$\begin{aligned}
&\left|\langle \mathcal{U}_N(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0)\right)\psi\rangle\right| \\
&\qquad \leq C(j) (N/M)^j \|(\mathcal{N}+1)^{j+1}\psi\|^2 \int_0^t \mathrm{d}s \; \exp\left(K(j)\, s \,(1+\sqrt{M/N})\right) \\
&\qquad \leq C(j) \frac{(N/M)^j \|(\mathcal{N}+1)^{j+1}\psi\|^2}{1+\sqrt{M/N}} \; \exp\left(K(j)\, t\, (1+\sqrt{M/N})\right) .
\end{aligned}$$

To prove (3.28), we proceed similarly; analogously to (3.29) we find

$$\begin{aligned}
&\langle \mathcal{U}_N^{(M)}(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0)\right)\psi\rangle \\
&\quad = -\frac{i}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x \langle a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N^{(M)}(t;0)\psi, a(V(x-.)\varphi_t) \chi(\mathcal{N} > M) a_x \mathcal{U}_N^{(M)}(s;0)\psi\rangle \\
&\qquad - \frac{i}{\sqrt{N}} \int_0^t \mathrm{d}s \int \mathrm{d}x \langle a_x \mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N^{(M)}(t;0)\psi, \chi(\mathcal{N} > M) a^*(V(x-.)\varphi_t) a_x \mathcal{U}_N^{(M)}(s;0)\psi\rangle
\end{aligned}$$

and thus

$$\begin{aligned}
&\left|\langle \mathcal{U}_N^{(M)}(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0)\right)\psi\rangle\right| \\
&\qquad \leq \frac{C}{\sqrt{N}} \int_0^t \mathrm{d}s \|\mathcal{N}^{1/2}\mathcal{U}_N(t;s)^* \mathcal{N}^j \mathcal{U}_N^{(M)}(t;0)\psi\| \, \|\mathcal{N}\chi(\mathcal{N} > M)\mathcal{U}_N^{(M)}(s;0)\psi\| \, .
\end{aligned} \tag{3.31}$$

Again, applying (3.18) and (3.14) we find

$$\left|\langle \mathcal{U}_N^{(M)}(t;0)\psi, \mathcal{N}^j \left(\mathcal{U}_N(t;0) - \mathcal{U}_N^{(M)}(t;0)\right)\psi\rangle\right| \leq C \frac{\|(\mathcal{N}+1)^{j+1}\psi\|^2}{M^j(1+\sqrt{M/N})} \exp\left(K(j)\, t\, (1+\sqrt{M/N})\right).$$

□

Step 4. Conclusion of the proof of Proposition 3.3

From (3.27), (3.28) and (3.14) we obtain, choosing $M = N$,

$$\begin{aligned}
\langle \mathcal{U}_N(t;s)\psi, \mathcal{N}^j \mathcal{U}_N(t;s)\psi\rangle &= \langle \mathcal{U}_N(t;s)\psi, \mathcal{N}^j (\mathcal{U}_N(t;s) - \mathcal{U}_N^{(M)}(t;s))\psi\rangle \\
&\quad + \langle (\mathcal{U}_N(t;s) - \mathcal{U}_N^{(M)}(t;s))\psi, \mathcal{N}^j \, \mathcal{U}_N^{(M)}(t;s)\psi\rangle \\
&\quad + \langle \mathcal{U}_N^{(M)}(t;s)\psi, \mathcal{N}^j \, \mathcal{U}_N^{(M)}(t;s)\psi\rangle \\
&\leq C(j)\|(\mathcal{N}+1)^{j+1}\psi\|^2 e^{K(j)|t-s|} \, .
\end{aligned}$$

□

3.5 Approximate dynamics $\widetilde{\mathcal{U}}_N(t;s)$

We now consider the dynamics $\widetilde{\mathcal{U}}_N(t;s)$, defined in (3.9) by

$$i\partial_t \widetilde{\mathcal{U}}_N(t;s) = \widetilde{\mathcal{L}}_N(t)\,\widetilde{\mathcal{U}}_N(t;s) \qquad \text{with} \quad \widetilde{\mathcal{U}}_N(s;s) = 1$$

with the time-dependent generator

$$\begin{aligned}
\widetilde{\mathcal{L}}_N(t) = &\int \mathrm{d}x \, \nabla_x a_x^* \nabla_x a_x + \int \mathrm{d}x \, \left(V * |\varphi_t|^2\right)(x)\, a_x^* a_x + \int \mathrm{d}x \mathrm{d}y \, V(x-y) \overline{\varphi_t}(x) \varphi_t(y) a_y^* a_x \\
&+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, V(x-y) \left(\varphi_t(x)\varphi_t(y) a_x^* a_y^* + \overline{\varphi_t}(x)\overline{\varphi_t}(y) a_x a_y\right) \\
&+ \frac{1}{2N}\int \mathrm{d}x\mathrm{d}y \, V(x-y)\, a_x^* a_y^* a_y a_x \, .
\end{aligned} \tag{3.32}$$

Lemma 3.8. *There exists a constant $K > 0$, only depending on $\|\varphi\|_{H^1}$ and on the constant D appearing in (3.2), such that*

$$\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, \mathcal{N}^3 \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \leq e^{Kt} \, . \tag{3.33}$$

Proof. We compute the derivative

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, (\mathcal{N}+1)^3 \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\quad = \langle \widetilde{\mathcal{U}}_N(t;0)\Omega, [i\widetilde{\mathcal{L}}_N(t), (\mathcal{N}+1)^3]\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\quad = 2\mathrm{Im}\int \mathrm{d}x\mathrm{d}y V(x-y)\varphi_t(x)\varphi_t(y)\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, [a_x^* a_y^*, (\mathcal{N}+1)^3]\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\quad = 4\mathrm{Im}\int \mathrm{d}x\mathrm{d}y V(x-y)\varphi_t(x)\varphi_t(y)\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, \\
&\qquad\qquad \left(a_x^* a_y^*(\mathcal{N}+1)^2 + (\mathcal{N}+1)a_x^* a_y^*(\mathcal{N}+1) + (\mathcal{N}+1)^2 a_x^* a_y^*\right)\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\quad = 4\mathrm{Im}\int \mathrm{d}x\mathrm{d}y V(x-y)\varphi_t(x)\varphi_t(y)\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, \\
&\qquad\qquad \left((\mathcal{N}-1)a_x^* a_y^*(\mathcal{N}+1) + (\mathcal{N}+1)a_x^* a_y^*(\mathcal{N}+1) + (\mathcal{N}+1)a_x^* a_y^*(\mathcal{N}+3)\right)\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\quad = 4\mathrm{Im}\int \mathrm{d}x\mathrm{d}y V(x-y)\varphi_t(x)\varphi_t(y)\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, \left(3(\mathcal{N}+1)a_x^* a_y^*(\mathcal{N}+1) - 4a_x^* a_y^*\right)\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, (\mathcal{N}+1)^3 \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\qquad = 12\mathrm{Im}\int \mathrm{d}x\varphi_t(x)\langle a_x(\mathcal{N}+1)\widetilde{\mathcal{U}}_N(t;0)\Omega, a^*(V(x-.)\varphi_t)(\mathcal{N}+1)\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle \\
&\qquad\quad - 16\mathrm{Im}\int \mathrm{d}x\varphi_t(x)\langle a_x\widetilde{\mathcal{U}}_N(t;0)\Omega, a^*(V(x-.)\varphi_t)\widetilde{\mathcal{U}}_N(t;0)\Omega\rangle\,.
\end{aligned}
$$

Taking the absolute value, we find

$$
\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, (\mathcal{N}+1)^3 \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle\right| \\
&\qquad \le 12\int \mathrm{d}x|\varphi_t(x)|\|a_x(\mathcal{N}+1)\widetilde{\mathcal{U}}_N(t;0)\Omega\|\,\|a^*(V(x-.)\varphi_t)(\mathcal{N}+1)\widetilde{\mathcal{U}}_N(t;0)\Omega\| \\
&\qquad\quad + 16\int \mathrm{d}x|\varphi_t(x)|\|a_x\widetilde{\mathcal{U}}_N(t;0)\Omega\|\,\|a^*(V(x-.)\varphi_t)\widetilde{\mathcal{U}}_N(t;0)\Omega\| \\
&\qquad \le 28\sup_x\|V(x-.)\varphi_t\|\|(\mathcal{N}+1)^{3/2}\widetilde{\mathcal{U}}_N(t;0)\Omega\|^2 \\
&\qquad \le C\,\langle \widetilde{\mathcal{U}}_N(t;0)\Omega, (\mathcal{N}+1)^3 \widetilde{\mathcal{U}}_N(t;0)\Omega\rangle\,.
\end{aligned}
$$

Applying Gronwall Lemma, we obtain (3.33). □

3.6 Comparison of the $\mathcal{U}_N$ and $\widetilde{\mathcal{U}}_N$ dynamics

The final step in the proof of Theorem 3.1 is the comparison of evolutions generated by $\mathcal{U}_N$ and $\widetilde{\mathcal{U}}_N$.

Lemma 3.9. *Let the evolutions $\mathcal{U}_N(t;s)$ and $\widetilde{\mathcal{U}}_N(t;s)$ be defined as in (3.6) and (3.9), respectively. Then there exist constants $C, K > 0$, only depending on $\|\varphi\|_{H^1}$ and on the constant D in (3.2), such that*

$$\left\|\left(\mathcal{U}_N(t;0) - \widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\| \le \frac{C}{\sqrt{N}}\,e^{Kt}\,. \tag{3.34}$$

Proof. We write

$$
\begin{aligned}
\Big(\mathcal{U}_N(t;0)-&\widetilde{\mathcal{U}}_N(t;0)\Big)\Omega \\
&= \mathcal{U}_N(t;0)\left(1-\mathcal{U}_N(t;0)^*\widetilde{\mathcal{U}}_N(t;0)\right)\Omega \\
&= -i\int_0^t \mathrm{d}s\,\mathcal{U}_N(t;s)\left(\mathcal{L}_N(s)-\widetilde{\mathcal{L}}_N(s)\right)\widetilde{\mathcal{U}}_N(s;0)\Omega \\
&= -\frac{i}{\sqrt{N}}\int_0^t \mathrm{d}s\int \mathrm{d}x\mathrm{d}y\,V(x-y)\,\mathcal{U}_N(t;s)a_x^*\left(\varphi_t(y)a_y^*+\overline{\varphi}_t(y)a_y\right)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega \\
&= -\frac{i}{\sqrt{N}}\int_0^t \mathrm{d}s\int \mathrm{d}x\,\mathcal{U}_N(t;s)a_x^*\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\,.
\end{aligned}
$$

Hence

$$
\left\|\left(\mathcal{U}_N(t;0)-\widetilde{\mathcal{U}}_N(t;0)\right)\Omega\right\| \leq \frac{1}{\sqrt{N}}\int_0^t \mathrm{d}s\,\left\|\int \mathrm{d}x\,a_x^*\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\right\|\,. \tag{3.35}
$$

Next, we observe that

$$
\begin{aligned}
\Big\|\int \mathrm{d}x\,&a_x^*\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\Big\|^2 \\
=& \int \mathrm{d}y\mathrm{d}x\langle a_y\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(y-.)\varphi_t)a_ya_x^*\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
=& \int \mathrm{d}y\mathrm{d}x\langle a_y\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(y-.)\varphi_t)a_x^*a_y\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
&+\int \mathrm{d}x\langle a_x\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(x-.)\varphi_t)\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
=& \int \mathrm{d}y\mathrm{d}x\langle a_y\widetilde{\mathcal{U}}_N(s;0)\Omega,(a_x^*\phi(V(y-.)\varphi_t)+V(y-x)\overline{\varphi}_t(x)) \\
&\qquad\qquad\times\left(\phi(V(x-.)\varphi_t)a_y+V(x-y)\varphi_t(y)\right)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
&+\int \mathrm{d}x\langle a_x\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(x-.)\varphi_t)\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle\,.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\Big\|\int \mathrm{d}x\,&a_x^*\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\Big\|^2 \\
=& \int \mathrm{d}y\mathrm{d}x\langle a_xa_y\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(y-.)\varphi_t)\phi(V(x-.)\varphi_t)a_ya_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
&+\int \mathrm{d}y\mathrm{d}xV(x-y)\overline{\varphi}_t(x)\langle a_y\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(x-.)\varphi_t)a_ya_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
&+\int \mathrm{d}y\mathrm{d}xV(x-y)\varphi_t(y)\langle a_xa_y\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(y-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
&+\int \mathrm{d}y\mathrm{d}xV(x-y)^2\overline{\varphi}_t(x)\varphi_t(y)\langle a_y\widetilde{\mathcal{U}}_N(s;0)\Omega,a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle \\
&+\int \mathrm{d}x\langle a_x\widetilde{\mathcal{U}}_N(s;0)\Omega,\phi(V(x-.)\varphi_t)\phi(V(x-.)\varphi_t)a_x\widetilde{\mathcal{U}}_N(s;0)\Omega\rangle\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\Big\| \int \mathrm{d}x \, a_x^* \phi(V(x-.)\varphi_t) a_x \widetilde{\mathcal{U}}_N(s;0)\Omega \Big\|^2 \\
&\leq \sup_x \|V(x-.)\varphi_t\|^2 \int \mathrm{d}y\mathrm{d}x \|(\mathcal{N}+2)^{1/2} a_x a_y \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \\
&\quad + \sup_x \|V(x-.)\varphi_t\| \int \mathrm{d}y\mathrm{d}x |V(x-y)||\varphi_t(x)| \|(\mathcal{N}+1)^{1/2} a_y \widetilde{\mathcal{U}}_N(s;0)\Omega\| \|a_y a_x \widetilde{\mathcal{U}}_N(s;0)\Omega\| \\
&\quad + \sup_y \|V(y-.)\varphi_t\| \int \mathrm{d}y\mathrm{d}x |V(x-y)||\varphi_t(y)| \|a_x a_y \widetilde{\mathcal{U}}_N(s;0)\Omega\| \|(\mathcal{N}+1)^{1/2} a_x \widetilde{\mathcal{U}}_N(s;0)\Omega\| \\
&\quad + \int \mathrm{d}y\mathrm{d}x V(x-y)^2 |\varphi_t(x)||\varphi_t(y)| \|a_y \widetilde{\mathcal{U}}_N(s;0)\Omega\| \|a_x \widetilde{\mathcal{U}}_N(s;0)\Omega\| \\
&\quad + \sup_x \|V(x-.)\varphi_t\|^2 \int \mathrm{d}x \|(\mathcal{N}+1)^{1/2} a_x \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \, .
\end{aligned}
$$

Using (3.16), we obtain

$$
\begin{aligned}
&\Big\| \int \mathrm{d}x \, a_x^* \phi(V(x-.)\varphi_t) a_x \widetilde{\mathcal{U}}_N(s;0)\Omega \Big\|^2 \\
&\leq C \int \mathrm{d}y\mathrm{d}x \|a_x a_y \mathcal{N}^{1/2} \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \\
&\quad + C \left(\int \mathrm{d}y\mathrm{d}x |V(x-y)|^2 |\varphi_t(x)|^2 \|a_y \mathcal{N}^{1/2} \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \right)^{1/2} \left(\int \mathrm{d}x\mathrm{d}y \|a_y a_x \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \right)^{1/2} \\
&\quad + C \left(\int \mathrm{d}y\mathrm{d}x \|a_x a_y \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \right)^{1/2} \left(\int \mathrm{d}y\mathrm{d}x |V(x-y)|^2 |\varphi_t(y)|^2 \|a_x \mathcal{N}^{1/2} \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \right)^{1/2} \\
&\quad + \int \mathrm{d}y\mathrm{d}x V(x-y)^2 |\varphi_t(x)|^2 \|a_y \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \\
&\quad + C \int \mathrm{d}x \|a_x \mathcal{N}^{1/2} \widetilde{\mathcal{U}}_N(s;0)\Omega\|^2 \, .
\end{aligned}
$$

From

$$
\int \mathrm{d}y\mathrm{d}x |V(x-y)|^2 |\varphi_t(y)|^2 \|a_x \psi\|^2 \leq \left(\sup_x \int \mathrm{d}y V(x-y)^2 |\varphi_t(y)|^2 \right) \|\mathcal{N}^{1/2}\psi\|^2 \leq C \|\mathcal{N}^{1/2}\psi\|^2
$$

we thus find

$$
\Big\| \int \mathrm{d}x \, a_x^* \phi(V(x-.)\varphi_t) a_x \widetilde{\mathcal{U}}_N(s;0)\Omega \Big\|^2 \leq C \|(\mathcal{N}+1)^{3/2} \widetilde{\mathcal{U}}_N(t;0)\Omega\|^2 \, .
$$

Inserting the last bound in (3.35) and using the result of Lemma 3.8 we obtain (3.34). □

This concludes the proof of Theorem 3.1.

3.7 Discussion

As mentioned in the introduction, our approach to the study of the mean field limit of the N-body Schrödinger dynamics mirrors that used by Hepp and Ginibre-Velo in [9, 8] in the study of the semi-classical limit of quantum many-boson systems. In the language of the mean field limit, the main result obtained by Hepp (for smooth potentials) and by Ginibre and Velo (for singular potentials) was the convergence of the fluctuation dynamics $\mathcal{U}_N(t;s)$ (defined in (3.6)) to a limiting N-independent dynamics $\mathcal{U}(t;s)$ in the sense that

$$s-\lim_{N\to\infty}\mathcal{U}_N(t;s)=\mathcal{U}(t;s) \tag{3.36}$$

for all fixed t and s. Here the limiting dynamics $\mathcal{U}(t;s)$ is defined by

$$i\partial_t\mathcal{U}(t;s)=\mathcal{L}(t)\mathcal{U}(t;s) \qquad \text{with } \mathcal{U}(s;s)=1$$

and with generator

$$\begin{aligned}\mathcal{L}(t)=&\int \mathrm{d}x\,\nabla_x a_x^*\nabla_x a_x+\int \mathrm{d}x\,\left(V*|\varphi_t|^2\right)(x)\,a_x^*a_x+\int \mathrm{d}x\mathrm{d}y\,V(x-y)\overline{\varphi_t}(x)\varphi_t(y)a_y^*a_x\\&+\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,V(x-y)\left(\varphi_t(x)\varphi_t(y)a_x^*a_y^*+\overline{\varphi_t}(x)\overline{\varphi_t}(y)a_xa_y\right)\end{aligned} \tag{3.37}$$

The convergence (3.36) does not give any information about the convergence of the one-particle marginal $\Gamma^{(1)}_{N,t}$, associated with the evolution of the coherent initial state, to the orthogonal projection $|\varphi_t\rangle\langle\varphi_t|$. The definition of the marginal density $\Gamma^{(1)}_{N,t}$ involves *unbounded* creation and annihilation operators. This also explains why the derivation of the bound (3.3) in Theorem 3.1 is in general more complicated than the proof of the convergence (3.36). The proof of (3.36) requires control of the growth of the expectation of powers of the number of particle operator $\mathcal{N}$ *only* with respect to the *limiting dynamics*. To prove (3.3), on the other hand, we need to control the growth of the expectation of $\mathcal{N}$ with respect to the N-dependent fluctuation dynamics $\mathcal{U}_N(t;s)$.

4 Time evolution of factorized states

This section is devoted to the proof of Theorem 1.1. The main idea in the proof is that we can write the factorized N-particle state $\psi_N=\varphi^{\otimes N}$ (whose evolution is considered in Theorem 1.1) as a linear combination of coherent states, whose dynamics can be studied using the results of Section 3.

Proof of Theorem 1.1. We start by writing $\psi_N=\varphi^{\otimes N}$ or, more precisely, the sequence

$$\{0,0,\dots,0,\psi_N,0,0,\dots\}=\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega\in\mathcal{F}$$

as a linear combination of coherent states. While it is always possible in principle our goal is to represent ψ_N with the *least* number of coherent states.

Lemma 4.1. *We have the following representation.*

$$\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega=d_N\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\,e^{i\theta N}W(e^{-i\theta}\sqrt{N}\varphi)\Omega \tag{4.1}$$

with the constant

$$d_N=\frac{\sqrt{N!}}{N^{N/2}e^{-N/2}}\simeq N^{1/4}\,. \tag{4.2}$$

Proof. To prove the representation (4.1) observe that, from (2.10) and since $\|\varphi\| = 1$,

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{i\theta N} W(e^{-i\theta}\sqrt{N}\varphi)\Omega &= e^{-N/2}\sum_{j=1}^{\infty} N^{j/2}\left(\int \frac{\mathrm{d}\theta}{2\pi}\, e^{i\theta(N-j)}\right)\frac{(a^*(\varphi))^j}{j!}\Omega \\ &= \frac{e^{-N/2}N^{N/2}}{\sqrt{N!}}\,\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega\,.\end{aligned} \tag{4.3}$$

□

The kernel of the one-particle density $\gamma^{(1)}_{N,t}$ associated with the solution of the Schrödinger equation

$$e^{-it\mathcal{H}_N}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega$$

is given by (see (2.7))

$$\begin{aligned}\gamma^{(1)}_{N,t}(x;y) &= \frac{1}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, e^{i\mathcal{H}_N t}a_y^* a_x e^{-i\mathcal{H}_N t}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega\right\rangle \\ &= \frac{d_N^2}{N}\int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, e^{-i\theta_1 N}e^{i\theta_2 N}\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega, a_y^*(t)a_x(t)W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega\rangle\end{aligned} \tag{4.4}$$

where we introduced the notation $a_x(t) = e^{i\mathcal{H}_N t}a_x e^{-i\mathcal{H}_N t}$. Next, we expand

$$\begin{aligned}\gamma^{(1)}_{N,t}(x;y) = \;&\frac{d_N^2}{N}\int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, e^{-i\theta_1 N}e^{i\theta_2 N}\Big\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega,\, \Big(a_y^*(t) - e^{i\theta_1}\sqrt{N}\overline{\varphi}_t(y)\Big) \\ &\qquad\times \Big(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\Big)\, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega\Big\rangle \\ &+ \frac{d_N^2\,\overline{\varphi}_t(y)}{\sqrt{N}}\int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, e^{-i\theta_1(N-1)}e^{i\theta_2 N} \\ &\qquad\times \Big\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega,\, \Big(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\Big)\, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega\Big\rangle \\ &+ \frac{d_N^2\,\varphi_t(x)}{\sqrt{N}}\int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, e^{-i\theta_1 N}e^{i\theta_2(N-1)} \\ &\qquad\times \Big\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega,\, \Big(a_y^*(t) - e^{i\theta_1}\sqrt{N}\overline{\varphi}_t(y)\Big)\, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega\Big\rangle \\ &+ d_N^2\,\varphi_t(x)\overline{\varphi}_t(y)\int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, e^{-i\theta_1(N-1)}e^{i\theta_2(N-1)} \\ &\qquad\times \Big\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega,\, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega\Big\rangle\,.\end{aligned} \tag{4.5}$$

We introduce the notation

$$\begin{aligned}f_N(x) =\,& d_N^2\int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, e^{-i\theta_1(N-1)}e^{i\theta_2 N} \\ &\times \Big\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega,\, \Big(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\Big)\, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega\Big\rangle\,.\end{aligned} \tag{4.6}$$

Since

$$
\begin{aligned}
d_N \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{i\theta(N-1)} W(e^{-i\theta}\sqrt{N}\varphi)\Omega &= d_N e^{-N/2} \sum_{j=0}^{\infty} \left(\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{i\theta(N-1-j)} \right) N^{j/2} \frac{(a^*(\varphi))^j}{j!} \Omega \\
&= d_N \frac{e^{-N/2} N^{(N-1)/2}}{\sqrt{N-1!}} \frac{(a^*(\varphi))^{N-1}}{N-1!} \Omega \\
&= \varphi^{\otimes N-1} ,
\end{aligned} \tag{4.7}
$$

we obtain, from (4.5), that

$$
\begin{aligned}
\gamma^{(1)}_{N,t}(x;y) = \frac{d_N^2}{N} \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \, & e^{-i\theta_1 N} e^{i\theta_2 N} \Big\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega, \, \Big(a_y^*(t) - e^{i\theta_1}\sqrt{N}\overline{\varphi}_t(y)\Big) \\
& \times \Big(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\Big) \, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega \Big\rangle \\
+ \frac{\overline{\varphi}_t(y) f_N(x)}{\sqrt{N}} + \frac{\varphi_t(x)\overline{f_N}(y)}{\sqrt{N}} & + \varphi_t(x)\overline{\varphi}_t(y) \, .
\end{aligned} \tag{4.8}
$$

Thus

$$
\begin{aligned}
\Big| \gamma^{(1)}_{N,t}(x;y) - \varphi_t(x)\overline{\varphi}_t(y) \Big| \leq \frac{d_N^2}{N} \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \, & \Big\| \Big(a_y(t) - e^{-i\theta_1}\sqrt{N}\varphi_t(y)\Big) W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega \Big\| \\
& \times \Big\| \Big(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\Big) \, W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega \Big\| \\
+ \frac{|\varphi_t(x)||f_N(y)|}{\sqrt{N}} + \frac{|\varphi_t(y)||f_N(x)|}{\sqrt{N}} & \\
\leq \frac{d_N^2}{N} \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \, & \|a_y \mathcal{U}_N^{\theta_1}(t;0)\Omega\| \, \|a_x \mathcal{U}_N^{\theta_2}(t;0)\Omega\| \\
+ \frac{|\varphi_t(x)||f_N(y)|}{\sqrt{N}} + \frac{|\varphi_t(y)||f_N(x)|}{\sqrt{N}} &
\end{aligned} \tag{4.9}
$$

where the unitary evolutions $\mathcal{U}_N^{\theta}(t;s)$ are defined as in (3.6), but with φ_t replaced[7] by $e^{-i\theta}\varphi_t$ in the generator (3.7). Taking the square of (4.9) and integrating over x, y, we obtain

$$
\begin{aligned}
\int \mathrm{d}x\mathrm{d}y \, & \Big| \gamma^{(1)}_{N,t}(x;y) - \varphi_t(x)\overline{\varphi}_t(y) \Big|^2 \\
& \leq 2\frac{d_N^4}{N^2} \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \, \|\mathcal{N}^{1/2}\mathcal{U}_N^{\theta_1}(t;0)\Omega\|^2 \, \|\mathcal{N}^{1/2}\mathcal{U}_N^{\theta_2}(t;0)\Omega\|^2 \\
& \quad + \frac{4}{N} \int \mathrm{d}x |f_N(x)|^2
\end{aligned} \tag{4.10}
$$

Using Proposition 3.3 and the fact that $d_N \simeq N^{1/4}$ to control the first term, and using Lemma 4.2 to control the second term on the r.h.s. of the last equation, we find constants C, K, only depending on $\|\varphi\|_{H^1}$ and on the constant D in (1.12) such that

$$
\|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\|_{\mathrm{HS}} \leq \frac{C}{N^{1/2}} \exp(Kt) \, . \tag{4.11}
$$

This proves (1.17) and thus concludes the proof of Theorem 1.1. □

[7]We are making use here of the important fact that if φ_t solves the nonlinear equation (1.15), then $e^{i\theta}\varphi_t$ is also a solution of the same equation, for any fixed real θ.

Lemma 4.2. *Let φ_t be a solution to the Hartree equation (1.5) with initial data $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\| = 1$. Let*

$$\begin{aligned} f_N(x) =& d_N^2 \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \, e^{-i\theta_1(N-1)} e^{i\theta_2 N} \\ & \times \left\langle W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega, \left(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\right) W(e^{-i\theta_2}\sqrt{N}\varphi)\Omega \right\rangle . \end{aligned}$$

Then there exist constants C, K (only depending on $\|\varphi\|_{H^1}$ and on the constant D in (1.12) such that

$$\int \mathrm{d}x \, |f_N(x)|^2 \leq C e^{Kt} .$$

Proof. Using that

$$\left(a_x(t) - e^{-i\theta_2}\sqrt{N}\varphi_t(x)\right) W(e^{-i\theta_2}\sqrt{N}\varphi) = W(e^{-i\theta_2}\sqrt{N}\varphi)\mathcal{U}_N^{\theta_2}(0;t) a_x \mathcal{U}_N^{\theta_2}(t;0)$$

where the unitary evolution $\mathcal{U}_N^{\theta}(t;s)$ is defined as in (3.6), but with φ_t replaced by $e^{-i\theta}\varphi_t$ in the generator (3.7), we can rewrite $f_N(x)$ as

$$f_N(x) = \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \left\langle \psi(\theta_2), \mathcal{U}_N^{\theta_2}(0;t) a_x \mathcal{U}_N^{\theta_2}(t;0)\Omega \right\rangle \tag{4.12}$$

with

$$\psi(\theta_2) = d_N^2 \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \, e^{i\theta_1(N-1)} e^{-i\theta_2 N} W^*(e^{-i\theta_2}\sqrt{N}\varphi) W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega \, . \tag{4.13}$$

Performing the integration over θ_1, we immediately obtain

$$\psi(\theta_2) = d_N \, e^{-i\theta_2 N} W^*(e^{-i\theta_2}\sqrt{N}\varphi)\varphi^{\otimes(N-1)} \, . \tag{4.14}$$

It is also possible to expand $\psi(\theta_2)$ in a sum of factors living in the different sectors of the Fock space. From Eq. (2.10) and Lemma 2.2, we compute

$$\begin{aligned} W^*(e^{-i\theta_2}\sqrt{N}\varphi) W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega &= W(-e^{-i\theta_2}\sqrt{N}\varphi) W(e^{-i\theta_1}\sqrt{N}\varphi)\Omega \\ &= e^{iN\mathrm{Im} e^{i(\theta_2-\theta_1)}} W((e^{-i\theta_1} - e^{-i\theta_2})\sqrt{N}\varphi)\Omega \\ &= e^{-N} e^{N e^{i(\theta_2-\theta_1)}} \sum_{m\geq 0} \frac{N^{m/2}(e^{-i\theta_1} - e^{-i\theta_2})^m}{\sqrt{m!}} \varphi^{\otimes m} \end{aligned} \tag{4.15}$$

which implies (using the periodicity in the variable θ_1)

$$\psi(\theta_2) = d_N^2 e^{-N} \sum_{m=0}^{\infty} \frac{N^{m/2}}{\sqrt{m!}} \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \, e^{i\theta_1(N-1)} e^{-i\theta_2(m+1)} e^{N e^{-i\theta_1}} (e^{-i\theta_1} - 1)^m \varphi^{\otimes m} \, .$$

Switching to the complex variable $z = e^{-i\theta_1}$ we obtain

$$\psi(\theta_2) = \; - d_N^2 e^{-N} \sum_{m\geq 0} \frac{N^{m/2}}{\sqrt{m!}} e^{-i\theta_2(m+1)} \int \frac{\mathrm{d}z}{2\pi i} \, z^{-N} e^{Nz} (z-1)^m \varphi^{\otimes m}$$

where the z integral is over the circle of radius one around the origin (in clock-wise sense). Changing variables $z \to Nz$, and using that $d_N = e^{N/2}\sqrt{N!}/N^{N/2}$, we obtain

$$\begin{aligned}\psi(\theta_2) =& -(N-1)! \sum_{m=0}^{\infty} \frac{N^{-\frac{m}{2}}}{\sqrt{m!}} e^{-i\theta_2(m+1)} \int \frac{\mathrm{d}z}{2\pi i}\, z^{-N} e^{z}(z-N)^m \varphi^{\otimes m} \\ =& \sum_{m=0}^{\infty} \frac{N^{-\frac{m}{2}}}{\sqrt{m!}} \mathcal{R}_m\, e^{-i\theta_2(m+1)} \varphi^{\otimes m}\end{aligned} \tag{4.16}$$

where we defined

$$\mathcal{R}_m = \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \left(e^z (z-N)^m\right)|_{z=0}\,. \tag{4.17}$$

Comparing (4.16) with (4.14), we obtain the identity

$$\sum_{m=0}^{\infty} \frac{\mathcal{R}_m^2}{N^m m!} = d_N^2\,. \tag{4.18}$$

It is also possible to obtain pointwise bounds on the coefficients $\mathcal{R}_m$. From (4.17) we deduce that for $m \le (N-1)$

$$\mathcal{R}_m = \sum_{k=0}^{m} (-1)^{m-k} \frac{(N-1)!m!N^{m-k}}{k!(N-1-k)!(m-k)!} = \sum_{k=0}^{m} (-1)^{m-k} N^{m-k} (N-1)...(N-k) \frac{m!}{k!(m-k)!}\,. \tag{4.19}$$

The coefficients $\mathcal{R}_m$ turn out to be intimately connected with the classical system of orthogonal Laguerre polynomials. Recall that the associated Laguerre polynomial $L_n^{(\alpha)}(x)$ admits the following representation

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \frac{(n+\alpha)!}{k!(n-k)!(\alpha+k)!} x^k.$$

Therefore

$$\mathcal{R}_m = (-1)^m m!\, L_m^{(N-m-1)}(N),$$

which, for $N > m+1$, involves the value of the Laguerre polynomial $L_n^{(\alpha)}(N)$ with a *positive* index α. Asymptotic expansions and estimates for the Laguerre polynomials is a classical subject, see [12] and references therein. However for the indices $\alpha = N-m-1$, $n = m$ with $N \gg m$ the value of $x = N$ belongs to the oscillatory regime of the behavior of $L_n^{(\alpha)}(x)$ and the sharp estimates for those values of parameters have been only obtained recently in [10], where it is proven that, for $\alpha > -1$, $n \ge 2$ and the values of $x \in (q^2, s^2)$ the function $L_n^{(\alpha)}(x)$ obeys the bound

$$|L_n^{(\alpha)}(x)| < \sqrt{\frac{(n+\alpha)!}{n!}} \sqrt{\frac{x(s^2-q^2)}{r(x)}} e^{\frac{x}{2}} x^{-\frac{\alpha+1}{2}},$$

where

$$s = (n+\alpha+1)^{\frac{1}{2}} + n^{\frac{1}{2}}, \quad q = (n+\alpha+1)^{\frac{1}{2}} - n^{\frac{1}{2}}, \quad r(x) = (x-q^2)(s^2-x).$$

As a consequence, we obtain that

$$|L_m^{(N-m-1)}(N)| < \sqrt{\frac{(N-1)!}{m!}} \sqrt{\frac{4N\sqrt{Nm}}{4Nm-m^2}} e^{\frac{N}{2}} N^{-\frac{N-m}{2}}$$

Assuming that $m \leq N$ and using the asymptotics $(N-1)! \sim N^{N-1/2}e^{-N}$ we obtain

$$|L_m^{(N-m-1)}(N)| \lesssim m^{-\frac{1}{4}}(m!)^{-\frac{1}{2}}N^{\frac{m}{2}}$$

and therefore

$$\frac{\mathcal{R}_m}{(m!)^{\frac{1}{2}}N^{\frac{m}{2}}} \lesssim m^{-\frac{1}{4}}\,.$$

Summarizing, the coefficients $\mathcal{A}_m = \mathcal{R}_m/(m!^{1/2}N^{m/2})$ appearing in the expansion (4.16) of $\psi(\theta_2)$ satisfy the bounds

$$\begin{aligned} |\mathcal{A}_m| &\leq Cm^{-1/4} \qquad \text{for all } m \leq N \quad \text{and} \\ \sum_{m=0}^{\infty} \mathcal{A}_m^2 &= d_N^2 \leq CN^{1/2}\,. \end{aligned} \tag{4.20}$$

Inserting (4.16) into (4.12) we obtain

$$f_N(x) = \sum_{m=0}^{\infty} \mathcal{A}_m \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{i\theta(m+1)} \left\langle \varphi^{\otimes m}, \mathcal{U}_N^\theta(0;t)a_x\mathcal{U}_N^\theta(t;0)\Omega \right\rangle \tag{4.21}$$

and therefore

$$\begin{aligned} |f_N(x)| &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \sum_{m=0}^{\infty} \frac{|\mathcal{A}_m|}{\sqrt{m+1}} \left| \left\langle \varphi^{\otimes m}, (\mathcal{N}+1)^{1/2}\, \mathcal{U}_N^\theta(0;t)a_x\mathcal{U}_N^\theta(t;0)\Omega \right\rangle \right| \\ &\leq \left(\sum_{m=0}^{\infty} \frac{|\mathcal{A}_m|^2}{m+1} \right)^{1/2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \left\| (\mathcal{N}+1)^{1/2}\, \mathcal{U}_N^\theta(0;t)a_x\mathcal{U}_N^\theta(t;0)\Omega \right\|\,. \end{aligned} \tag{4.22}$$

From (4.20), we obtain

$$\sum_{m=0}^{\infty} \frac{|\mathcal{A}_m|^2}{m+1} \leq C \sum_{m=0}^{N-1} \frac{1}{(m+1)^{3/2}} + \frac{1}{N} \sum_{m \geq N} |A_m|^2 \leq \text{const}\,. \tag{4.23}$$

On the other hand, from Proposition 3.3, we have

$$\left\| (\mathcal{N}+1)^{1/2}\, \mathcal{U}_N^\theta(0;t)a_x\mathcal{U}_N^\theta(t;0)\Omega \right\|^2 \leq Ce^{\widetilde{K}t} \left\| (\mathcal{N}+1)^2 a_x\mathcal{U}_N^\theta(t;0)\Omega \right\|^2 \leq Ce^{\widetilde{K}t} \left\| a_x\mathcal{N}^2\mathcal{U}_N^\theta(t;0)\Omega \right\|^2.$$

Thus, applying once more Proposition 3.3, we find

$$\int \mathrm{d}x\, |f_N(x)|^2 \leq Ce^{\widetilde{K}t} \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \langle \mathcal{U}_N^\theta(t;0)\Omega, \mathcal{N}^5\mathcal{U}_N^\theta(t;0)\Omega \rangle \leq Ce^{Kt}\,.$$

$\square$

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