LOCALIZATION ON QUANTUM GRAPHS WITH RANDOM VERTEX COUPLINGS

FRÉDÉRIC KLOPP AND KONSTANTIN PANKRASHKIN

Abstract. We consider Schrödinger operators on a class of periodic quantum graphs with randomly distributed Kirchhoff coupling constants at all vertices. Using the technique of self-adjoint extensions we obtain conditions for localization on quantum graphs in terms of finite volume criteria for some energy-dependent discrete Hamiltonians. These conditions hold in the strong disorder limit and at the spectral edges.

Introduction

In the present work we study spectral properties for a special type of random interactions on quantum graphs, the so-called random Kirchhoff model. We are going to show that such models can be effectively treated using well-established methods for the discrete Anderson model, in particular, with the help of finite volume fractional moment criteria.

The study of random Schrödinger operators on quantum graphs has become especially active during the last years. In [3] weakly disordered tree graphs were studied; it was shown that the absolutely continuous spectrum is stable in the weak disorder limit. Random interaction on radial tree-like graphs were studied in [17]; for the random edge length and random coupling constants it was shown that the corresponding Schrödinger operators exhibit the Anderson localization at all energies. This generalizes previously known results on the random necklace graphs [24]. Schrödinger operators with random potentials on the edges have been studied in [11] using the multiscale method, where the presence of the dense pure point spectrum at the bottom of the spectrum was shown. The authors of [14, 15] have proved the existence of the integrated density of states and Wegner estimates for periodic quantum graphs with random interactions (for both random potentials and random boundary conditions).

Our method consists in a reduction of the spectral problem on quantum graphs to the study of a family of energy dependent discrete operators with a random potential. To perform this reduction we use the theory of self-adjoint extensions, or, more precisely, the machinery of abstract Weyl functions [6]. A reduction of continuous problems to discrete ones within the localization framework was exploited in numerous papers on Schrödinger operators with random or quasiperiodic point interactions, see e.g. [5, 9, 16, 12, 19, 20], but, as we will see below, such a correspondence is particularly explicit and efficient for quantum graphs.

We consider periodic quantum graphs spanned by simple $\mathbb{Z}^d$-lattices with randomly distributed Kirchhoff coupling constants at all vertices (the precise construction is given in section 1). The edges can carry additional scalar potentials and the quantum graph is not assumed to be isotropic. Actually the scheme presented below can be directly extended to graphs with more complicated combinatorial properties, but we do not do this to avoid technicalities. The central points of the paper are theorem 1, where a condition of the Schrödinger operator on a quantum graph to have a pure point spectrum in terms of upper spectral measures is obtained, and proposition 6, where we provide estimates for the spectral measures of quantum graphs in terms of associated discrete operators. These tools reduce the problem to a direct application of finite volume criteria for discrete Hamiltonians. Using these criteria we establish localization in the strong disorder regime (section 4) and localization at the band edges (section 5) using the Lifshitz asymptotics for the density of states.

1. Schrödinger operator on a quantum graph

1.1. **Construction of Hamiltonians.** For general matters concerning the theory and applications of quantum graphs, we refer to [13, 25, 26].

We consider a quantum graph whose set of vertices is identified with $\mathbb{Z}^d$. By h_j, $j = 1, \dots, d$, we denote the standard basis vectors of $\mathbb{Z}^d$.

Two vertices m, m' are connected by an oriented edge $m \to m'$ iff $|m - m'| := \sum_{j=1}^{d} |m_j - m'_j| = 1$ and $m_j \le m'_j$ for all $j = 1, \dots, d$; one says that m is the initial vertex and m' is the terminal vertex. Hence, each edge ϵ has the form $m \to (m + h_j)$ with some $m \in \mathbb{Z}^d$ and $j \in \{1, \dots, d\}$; in this case we will write $\epsilon = (m, j)$.

Fix some $l_j > 0$, $j \in \{1, \dots, d\}$, and replace each edge (m, j) by a copy of the segment $[0, l_j]$ in such a way that 0 is identified with m and l_j is identified with $m+h_j$. In this way we arrive at a certain topological set carrying a natural metric structure. We will parameterize the points of the edges by the distance from the initial vertex. Point x lying on the edge (m, j) on the distance $t \in [0, l_j)$ from m will be denoted as $x = (m, j, t)$. There is an ambiguity concerning the coordinates of the vertices, but this does not influence the constructions below.

The above graph can be embedded into $\mathbb{R}^d$, if one identifies $\mathbb{Z}^d \ni m \sim p(m) := \sum_{j=1}^{d} m_j\, l_j h_j \in \mathbb{R}^d$, $(m, k) \sim \big[p(m), p(m) + l_k h_k\big]$, but this will not be used.

The quantum state space of the system is $\mathcal{H} := \bigoplus_{m \in \mathbb{Z}^d} \bigoplus_{j \in \{1,\dots,d\}} \mathcal{H}_{m,j}$ where $\mathcal{H}_{m,j} = L^2([0, l_j])$, and the elements of $\mathcal{H}$ will be denoted by $f = (f_{m,j})$, $f_{m,j} \in \mathcal{H}_{m,j}$, $m \in \mathbb{Z}^d$, $j = 1, \dots, d$, or $f = (f_\epsilon)$, $f_\epsilon \in \mathcal{H}_\epsilon$, $\epsilon \in \mathbb{Z}^d \times \{1, \dots, d\}$. In what follows, we denote by $P_\epsilon = P_{m,j}$ the orthogonal projection from $\mathcal{H}$ to $\mathcal{H}_\epsilon = \mathcal{H}_{m,j}$, $\epsilon = (m, j)$. We say that a function $f = (f_{m,j})$ is *concentrated on an edge* (m, j) if $P_{m,j} f = f$, i.e. if all components of f but $f_{m,j}$ vanish.

Let us describe the Schrödinger operator acting in $\mathcal{H}$. Fix real-valued potentials $U_j \in L^2([0, l_j])$, $j = 1, \dots, d$, and real constants $\alpha(m)$, $m \in \mathbb{Z}^d$. Set $A := \operatorname{diag}\big(\alpha(m)\big)$; this is a self-adjoint operator in $l^2(\mathbb{Z}^d)$. Denote by H_A the operator acting as

$$(f_{m,j}) \mapsto \Big((-\frac{d^2}{dt^2} + U_j) f_{m,j} \Big) \tag{1a}$$

on functions $(f_{m,j}) \in \bigoplus\limits_{m,j} H^2([0,l_j])$ satisfying the following boundary conditions:

$$f_{m,j}(0) = f_{m-h_k,k}(l_k) =: f(m), \quad j,k = 1,\ldots,d \tag{1b}$$

(which means the continuity at all vertices) and

$$f'(m) = \alpha(m) f(m), \quad m \in \mathbb{Z}^d, \tag{1c}$$

where

$$f'(m) := \sum_{j=1}^{d} f'_{m,j}(0) - \sum_{j=1}^{d} f'_{m-h_j,j}(l_j). \tag{2}$$

The constants $\alpha(m)$ are usually referred to as *Kirchhoff coupling constants*. The boundary conditions corresponding to zero Kirchhoff coupling constants are usually called the Kirchhoff boundary conditions. Non-zero Kirchhoff coupling constants are usually interpreted as measuring the impurities at the vertices (zero coupling constants correspond to the *ideal coupling*). Later we will assume that $\alpha(m)$ are independent identically distributed random variables, but here we treat first the deterministic case. For convenience, for $\alpha \in \mathbb{R}$ we denote by H_α the above operator H_A with the diagonal A, $A = \alpha\,\mathrm{id}$.

Our aim now is to provide a reduction of the spectral problem for H_A to a family of discrete spectral problems. We will do this using the machinery of self-adjoint extensions; a self-contained presentation of this technique in the abstract setting can be found e.g. in the recent preprint [6].

Denote by S the operator acting as (1a) on the functions f satisfying only the boundary conditions (1b). On the domain of S, one can define linear maps

$$f \mapsto \Gamma f := \big(f(m)\big)_{m\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d), \quad f \mapsto \Gamma' f := \big(f'(m)\big)_{m\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$$

where f' is defined in (2). By the Sobolev embedding theorems, the maps Γ, Γ' are well-defined, and the map $(\Gamma,\Gamma') : \operatorname{dom} S \to l^2(\mathbb{Z}^d) \times l^2(\mathbb{Z}^d)$ is surjective. Moreover, by a simple computation, for any f, g in $\operatorname{dom} S$, one has

$$\langle f, Sg\rangle - \langle Sf, g\rangle = \langle \Gamma f, \Gamma' g\rangle - \langle \Gamma' f, \Gamma g\rangle$$

(see e.g. proposition 1 in [29]). In the abstract language, $(\mathbb{Z}^d, \Gamma, \Gamma')$ form a *boundary triple* for S. This permits to write a useful formula for the resolvent of H_A, which will play a crucial role below.

First, denote by H^0 the restriction of S to $\ker\Gamma$. Clearly, H^0 acts as (1a) on functions $(f_{m,j})$ with $f_{m,j} \in H^2([0,l_j])$ satisfying the Dirichlet boundary conditions, $f_{m,j}(0) = f_{m,j}(l_j) = 0$ for all m, j, and the spectrum of H^0 is just the union of the Dirichlet spectra of the operators $-\dfrac{d^2}{dt^2} + U_j$ on the segments $[0, l_j]$.

Denote by φ_j and ϑ_j the solutions to $-y'' + U_j y = Ey$ satisfying $\varphi(0;E) = \vartheta'(0;E) = 0$ and $\varphi'(0;E) = \vartheta(0;E) = 1$. For short, we denote $\phi_j(t;E) := \varphi_j(l_j;E)\vartheta_j(t;E) - \vartheta_j(l_j;E)\varphi_j(t;E)$. Clearly, ϕ_j is the solution to the above differential equation satisfying $\phi_j(l_j;E) = 0$ and $-\phi_j'(l_j;E) = 1$.

For E outside $\operatorname{spec} H^0$, consider the operator $\gamma(E) : l^2(\mathbb{Z}^d) \to \mathcal{H}$ defined as follows: for $\xi \in l^2(\mathbb{Z}^d)$, $\gamma(E)\xi$ is the unique solution to $(S-E)f = 0$ with $\Gamma f = \xi$. For each E, $\gamma(E)$ is a linear topological isomorphism between $l^2(\mathbb{Z}^d)$ and $\ker(S-E)$. Clearly, in terms of the functions $\phi_j, \varphi_j, \vartheta_j$ introduced above, one has

$$\big(\gamma(E)\xi\big)_{m,j}(t) = \frac{1}{\varphi_j(l_j;E)}\Big(\xi(m+h_j)\varphi_j(t;E) + \xi(m)\phi_j(t;E)\Big). \tag{3}$$

Furthermore, for $E \notin \sigma(H^0)$, define the operator $M(E) : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ by $M(E) := \Gamma'\gamma(E)$. In our case,

$$M(E)\xi(m) = \sum_{j=1}^{d} \frac{1}{\varphi_j(l_j;E)}\Big(\xi(m-h_j) + \xi(m+h_j)\Big) - \Big(\sum_{j=1}^{d} \frac{\vartheta_j(l_j;E) + \varphi_j'(l_j;E)}{\varphi_j(l_j;E)}\Big)\xi(m).$$

We denote for clarity

$$a(E) := \sum_{j=1}^{d} \frac{\eta_j(E)}{\varphi_j(l_j;E)}, \quad b_j(E) := \frac{1}{\varphi_j(l_j;E)}, \quad \eta_j(E) := \vartheta_j(l_j;E) + \varphi_j'(l_j;E).$$

then

$$M(E)\xi(m) = \sum_{j=1}^{d} b_j(E)\big(\xi(m-h_j) + \xi(m+h_j)\big) - a(E)\xi(m). \tag{4}$$

The maps γ and M satisfy a number of important properties. In particular, γ and M depend analytically on their argument (outside $\operatorname{spec} H^0$), and for any admissible real E one has

$$\frac{dM(E)}{dE} = \gamma^*(E)\gamma(E), \tag{5}$$

and for any non-real E there is $c_E > 0$ such that

$$\frac{\Im M(E)}{\Im E} \geq c_E. \tag{6}$$

The resolvents of H^0 and H_A are related by the Krein resolvent formula,

$$(H_A - E)^{-1} = (H^0 - E)^{-1} - \gamma(E)\big(M(E) - A\big)^{-1}\gamma^*(\bar{E}), \qquad E \notin \operatorname{spec} H^0 \cup \operatorname{spec} H_A. \tag{7}$$

Moreover, the set $\operatorname{spec} H_A \setminus \operatorname{spec} H^0$ coincides with $\{E \notin \operatorname{spec} H^0 : 0 \in \operatorname{spec}\big(M(E) - A\big)\}$, and the same correspondence holds for the eigenvalues with $\gamma(E)$ being an isomorphism of the corresponding eigensubspaces.

We note that for special quantum graphs one can perform the complete reduction of the spectral problem to the spectral problem for the discrete Laplacian on the underlying combinatorial graph [6, 7, 30]. In general, the spectrum is rather complicated and depends on various geometric and arithmetic parameters, see e.g. [10].

Eq. (7) shows that $(H_A - E)^{-1}$ is an integral operator whose kernel (the Green function) G_A has the following form:

$$
\begin{aligned}
G_A\big((m,j,t),(m',j',t')\big) &= \delta_{mm'}\delta_{jj'}G_j(t,t';E) \\
&\quad - \frac{1}{\varphi_j(t;E)\varphi_{j'}(t';E)}\Big[\big(M(E)-A\big)^{-1}(m,m')\phi_j(t;E)\phi_{j'}(t';E) \\
&\quad + \big(M(E)-A\big)^{-1}(m+h_j,m')\varphi_j(t;E)\phi_{j'}(t';E) \\
&\quad + \big(M(E)-A\big)^{-1}(m,m'+h_{j'})\phi_j(t;E)\varphi_{j'}(t';E) \\
&\quad + \big(M(E)-A\big)^{-1}(m+h_j,m'+h_{j'})\varphi_j(t;E)\varphi_{j'}(t';E)\Big],
\end{aligned}
\tag{8}
$$

where G_j is the Green function for $-d^2/dx^2 + U_j$ on $L^2([0,l_j])$ with the Dirichlet boundary conditions, i.e.

$$
G_j(t,t';E) = \begin{cases} \dfrac{\varphi_j(t;E)\phi(t';E)}{W_j(E)}, & t<t', \\[2mm] \dfrac{\varphi_j(t';E)\phi(t;E)}{W_j(E)}, & t>t', \end{cases} \qquad W_j(E) := \varphi_j(t;E)\phi_j'(t;E) - \varphi_j'(t;E)\phi_j(t;E).
\tag{9}
$$

1.2. **Random Hamiltonians.** On $(\Omega,\mathbb{P})$ a probability space, let $(\alpha_\omega(m))_{m\in\mathbb{Z}^d}$ be a family of independent identically distributed (i.i.d.) random variables whose common distribution has a bounded density ρ with support $[\alpha_-,\alpha_+]$.

By a random Hamiltonian acting on the quantum graph, we mean the family of operators given by Eqs. (1) corresponding to the parameterizing operator $A_\omega := \{\lambda\alpha_\omega(m)\}$ of Kirchhoff coupling constants at the vertices, where $\alpha_\omega(m)$ are described above. This family of Hamiltonians will be denoted by $H_{\lambda,\omega}$ or $H_{A,\omega}$.

For the moment we can set without loss of generality $\lambda = 1$ and denote the Hamiltonians simply by H_ω.

The shifts τ_m, defined by $(\tau_m\omega)_{m'} = \omega_{m+m'}$, $m,m'\in\mathbb{Z}^d$, act as a measure preserving ergodic family on Ω. For any τ_m, there exists a unitary map U_m on $\mathcal{H}$, $(U_m f)_{m',j'} = f_{m+m',j'}$, $m,m'\in\mathbb{Z}^d$, $j'\in\{1,\dots,d\}$, with $H_{\tau_m\omega} = U_m^* H_\omega U_m$, which implies the following standard result from the theory of random operators, the existence of an almost sure spectrum and of almost sure spectral components (see e.g. [31]), i.e. the existence of closed subsets $\Sigma_\bullet \subset \mathbb{R}$ and a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that $\operatorname{spec}_\bullet H_\omega = \Sigma_\bullet$, $\bullet \in \{\mathrm{pp},\mathrm{ac},\mathrm{sc}\}$, for any $\omega\in\Omega'$. Let $\Sigma = \Sigma_{\mathrm{pp}}\cup\Sigma_{\mathrm{ac}}\cup\Sigma_{\mathrm{sc}}$ be the almost sure spectrum of H_ω.

By Eq. (7) and the discussion thereafter, for any $E\notin \operatorname{spec} H^0$ one has the equivalence $E \in \operatorname{spec} H_\omega$ if and only if $0\in\operatorname{spec}\big(M(E)-A_\omega\big)$. At the same time, $M(E)-A_\omega$ is a usual metrically transitive operator in $l^2(\mathbb{Z}^d)$ and hence possesses an almost sure spectrum $\Sigma_M(E)$ which satisfies (see [31])

$$
\begin{aligned}
\Sigma_M(E) &= \operatorname{spec} M(E) - [\alpha_-,\alpha_+] \\
&= \left[-2\sum_{j=1}^d \big|b_j(E)\big| - a(E) - \alpha_+,\, 2\sum_{j=1}^d \big|b_j(E)\big| - a(E) - \alpha_-\right].
\end{aligned}
\tag{10}
$$

Hence, the characteristic equation for $E \notin \operatorname{spec} H^0$ to be in the almost sure spectrum of H_ω reads

$$\Big(2\sum_{j=1}^{d}\big|b_j(E)\big| - a(E) - \alpha_-\Big)\cdot\Big(2\sum_{j=1}^{d}\big|b_j(E)\big| + a(E) + \alpha_+\Big) \geq 0. \tag{11}$$

So, the spectrum of H_ω outside the Dirichlet eigenvalues is a union of bands.

Let us turn to the dependence of $H_{\lambda,\omega}$ on λ. The characteristic equation (11) for the spectrum becomes

$$\Big(2\sum_{j=1}^{d}\big|b_j(E)\big| - a(E) - \lambda\alpha_-\Big)\cdot\Big(2\sum_{j=1}^{d}\big|b_j(E)\big| + a(E) + \lambda\alpha_+\Big) \geq 0. \tag{12}$$

Let us describe the behavior of the almost sure spectrum as $\lambda \to +\infty$. Recall the well-known asymptotics [28]:

$$\begin{aligned} \eta_j(E) &\sim 2\cosh l_j\sqrt{-E}, \quad \varphi_j(l_j,E) \sim \frac{\sinh l_j\sqrt{-E}}{\sqrt{-E}}, \quad E\to-\infty,\\ \eta_j(E) &\sim 2\cos l_j\sqrt{E}, \quad \varphi_j(l_j,E) \sim \frac{\sin l_j\sqrt{E}}{\sqrt{E}}, \quad E\to+\infty. \end{aligned} \tag{13}$$

In particular, $b_j(E) = O(e^{-\alpha\sqrt{-E}})$, $\alpha > 0$, and $a(E) \sim 2d\sqrt{-E}$ for $E \to -\infty$.

If $\alpha_- < 0 < \alpha_+$, then condition (12) can be satisfied for any E if λ is chosen sufficiently large, i.e. the spectrum tends to cover the whole real axis. The edges of the spectrum are situated in the domains where the expressions $2\sum_{j=1}^{d}\big|b_j(E)\big| \pm a(E)$ are of order λ; so, these edges lie in $O(\lambda^{-1})$-neighborhoods of the Dirichlet eigenvalues and close to $-\infty$.

If $0 \in [\alpha_-,\alpha_+]$, then (12) will be satisfied for any λ if

$$\Big(2\sum_{j=1}^{d}\big|b_j(E)\big| - a(E)\Big)\cdot\Big(2\sum_{j=1}^{d}\big|b_j(E)\big| + a(E)\Big) \geq 0,$$

i.e. the spectrum contains a part which does not depend on λ; actually, this part is nothing but the spectrum of the Hamiltonian H_0 corresponding to the zero coupling constants at all vertices i.e. $\alpha_\omega(m) =$, $\forall m$.

If α_- and α_+ are both positive or both negative, for (12) to be satisfied, the expressions $2\sum_{j=1}^{d}\big|b_j(E)\big| \pm a(E)$ must be of the same order as λ, i.e. must be large. Therefore, for $\lambda \to +\infty$ the condition (11) can be satisfied only in the following cases:

- $\varphi(l_j,E) \sim \lambda^{-1}$ for some j,
- $\alpha_+ < 0$ and $\sqrt{-E} \sim \lambda$.

In other words, for $\lambda \to +\infty$ the spectrum on the positive half-line concentrates in $O(\lambda^{-1})$ neighborhoods of the Dirichlet eigenvalues. For $\alpha_+ < 0$ there is a band going to infinity on the negative half-line.

Finally, if $\alpha_- < 0$ then, there is some spectrum on the negative half-axis at the energies of order $\sqrt{-E} \sim -\lambda$.

2. Localization conditions for quantum graphs

In this section we again set $\lambda = 1$ and study the operator H_ω. The following spectral characteristics of H_ω will be of crucial importance for us.

Let $f, g \in \mathcal{H}$. Let $\mu^{f,g}$ denote the spectral measure for H_A associated with H_A and $|\mu^{f,g}|$ denote its absolute value. For any measurable set F and two edges (m, j), (m', j') we set

$$\mu^{(m,j),(m',j')}(F) := \sup_{\substack{f = P_{m,j} f, \\ g = P_{m',j'} g \\ \|f\| = \|g\| = 1}} |\mu^{f,g}|(F)$$

and call $\mu^{(m,j),(m',j')}$ the *upper spectral measure* associated with the edges (m, j) and (m', j') and H_A. For the random Hamiltonian H_ω, the corresponding quantities get an additional subindex ω. Recall that for μ a complex valued regular Borel measure and F a Borel set, one defines

$$|\mu|(F) = \sup_{f \in \mathcal{C}_0(\mathbb{R}),\ |f|_\infty \le 1} \left| \int_F f(E) d\mu(E) \right|. \tag{14}$$

We provide localization criteria for H_A in terms of the upper spectral measures; this extends to the quantum graph case the localization criteria known for discrete Hamiltonians, cf. theorem IV.4 and corollary IV.5 in [27].

Theorem 1. *Let $F \subset \mathbb{R}$. Assume that, for any (m, j), one has*

$$\sum_{m' \in \mathbb{Z}^d} \sum_{j'=1}^{d} \mu^{(m,j),(m',j')}(F) < \infty, \tag{15}$$

then H_A has only pure point spectrum in F.

Proof of theorem 1. We use the following result from [4] (theorem on p. 642):

Proposition 2. *Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and F_r be a family of orthogonal projections such that $\operatorname{s-lim}_{r \to +\infty} F_r = 1$. Suppose that there exists a family $\{S_n\}$ of linear operators, such that each S_n is bounded, defined everywhere, and commutes with H, and the strong limit $S := \operatorname{s-}\lim_{n \to \infty} S_n$ exists and $\overline{\operatorname{ran} S} = \mathcal{H}$. Assume additionally that $F_r S_n$ is compact for any r and n. Then, the invariant subspace $\mathcal{H}_{pp}$ of H corresponding to the pure point spectrum admits the following description:*

$$\mathcal{H}_{pp} = \big\{ f \in \mathcal{H} : \lim_{r \to \infty} \sup_{t \in \mathbb{R}} (1 - F_r) e^{itH} f = 0 \big\}.$$

and the the technical result

Proposition 3. *Let Λ be a subset of $\mathbb{Z}^d$. Denote by P_Λ the orthogonal projection from $\mathcal{H}$ to the span of the functions $(f_{m,j})$ with $f_{m,j} = 0$ for $m \notin \Lambda$. For any finite Λ and any $E \notin \operatorname{spec} H_A$, the operator $T := P_\Lambda (H_A - E)^{-1}$ is Hilbert-Schmidt, hence compact.*

that we prove in the appendix A.

Denote by P_F denote the spectral projection onto F corresponding to H_A. It is sufficient to show that $P_F f$ belongs to the invariant space of H_A associated with the point spectrum for any $f \in \mathcal{H}$. Clearly, it suffices to consider only functions f concentrated on a single edge.

Let us use proposition 2. Take $S = S_n = (H_A - i)^{-1}$. As F_r we take the orthogonal projections from $\mathcal{H}$ to the functions $(f_{m,j})$ with $f_{m,j} = 0$ for $|m| > r$. Clearly, S is bounded, commutes with H_A, $\operatorname{ran} S = \operatorname{dom} H_A$ is dense in $\mathcal{H}$, $F_r S$ is compact for any r due to proposition 3, and F_r strongly converge to the identity operator. Hence, the assumptions of proposition 2 are satisfied.

Take any f with $f = P_{m,j} f$. Clearly, in our setting,

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|(1-F_r)e^{-itH_A}P_F f)\|^2 &= \sup_{t\in\mathbb{R}} \sum_{|m'|>r}\sum_{j'=1}^{d} \left\|\left(e^{-itH_A}P_F f\right)_{m',j'}\right\|^2\\ &= \sup_{t\in\mathbb{R}} \sum_{|m'|>r}\sum_{j'=1}^{d} \langle e^{-itH_A}P_F f, P_{m',j'}e^{-itH_A}P_F f\rangle\\ &\le \sum_{|m'|>r}\sum_{j'=1}^{d} \sup_{t,s\in\mathbb{R}} \left|\left\langle e^{-isH_A}P_F f, P_{m',j'}e^{-itH_A}P_F f\right\rangle\right|.\end{aligned}$$

Due to the definition of the absolute value of a measure one has

$$\sup_{s\in\mathbb{R}} \left|\left\langle e^{-isH_A}P_F f, P_{m',j'}e^{-itH_A}P_F f\right\rangle\right| \le \left|\mu^{f, P_{m',j'}e^{-itH_A}P_F f}\right|(F).$$

Using the definition of $\mu^{(m,j),(m',j')}(F)$ one obtains

$$\sup_{t\in\mathbb{R}} \left|\mu^{f, P_{m',j'}e^{-itH_A}P_F f}\right|(F) \le \mu^{(m,j),(m',j')}(F)\|P_F f\|\,\|f\| \le \mu^{(m,j),(m',j')}(F)\|f\|^2.$$

Finally, we obtain

$$\sup_{t\in\mathbb{R}} \left\|(1-F_r)e^{-itH_A}P_F f)\right\|^2 \le \|f\|^2 \sum_{|m'|>r}\sum_{j'=1}^{d} \mu^{(m,j),(m',j')}(F),$$

and by (15), $\lim\limits_{r\to+\infty} \sup_{t\in\mathbb{R}} \left\|(1-F_r)e^{-itH_A}P_F f\right\|^2 = 0$. ☐

Theorem 1 admits a direct application to the random Hamiltonians H_ω.

Corollary 4. *Let $F \subset \mathbb{R}$. Assume that, for any edge (m,j), one has*

$$\mathbb{E}\Big(\sum_{m'\in\mathbb{Z}^d}\sum_{j'=1}^{d} \mu_\omega^{(m,j),(m',j')}(F)\Big) < \infty, \tag{16}$$

then H_ω has only pure point spectrum in F almost surely.

Proof. Eq. (16) says, in particular, that for any (m,j) there exists $\Omega_{m,j} \subset \Omega$ with $\mathbb{P}(\Omega_{m,j}) = 1$ such that, for $\omega \in \Omega_{m,j}$, $\sum\limits_{m'\in\mathbb{Z}^d}\sum\limits_{j'=1}^{d} \mu_\omega^{(m,j),(m',j')}(F) < \infty$.

Denote $\Omega' := \bigcap\limits_{m,j} \Omega_{m,j}$; as the set of all (m,j) is countable, $\mathbb{P}(\Omega') = 1$. Clearly, $\sum\limits_{m'\in\mathbb{Z}^d}\sum\limits_{j'=1}^{d} \mu_\omega^{(m,j),(m',j')}(F) < \infty$ for all (m,j) and all $\omega \in \Omega'$, and the spectrum of H_ω in F is pure point for any $\omega \in \Omega'$ by theorem 1. ☐

In the next result, we show that assumption (16) is a consequence of a finite volume criteria *à la* [2] on the discrete Hamiltonians defined in section 1. The finite volume criteria is expressed in terms of finite volume approximations of our operators that we define first.

Let Λ be a subset of $\mathbb{Z}^d$. Denote by H_A^Λ the operator acting by the same rule (1a) on functions f satisfying the boundary conditions $f'(m) = \alpha(m)f(m)$ for $m \in \Lambda$ and the Dirichlet boundary conditions $f(m) = 0$ for $m \notin \Lambda$. In other words, the functions from the domain H_A^Λ satisfy the same boundary conditions as for H_A at the vertices lying in Λ and those as for H^0 at the vertices outside Λ. One can relate the operators of H_A^Λ and H^0 by a formula similar to (7) using e.g. the construction of [32].

Namely, consider $l^2(\Lambda)$ as a subset of $l^2(\mathbb{Z}^d)$ and denote by Π_Λ the orthogonal projection from $l^2(\mathbb{Z}^d)$ to $l^2(\Lambda)$. Denote also $M_\Lambda(E) := P_\Lambda M(E)\Pi_\Lambda$, $A_\Lambda := \Pi_\Lambda A \Pi_\Lambda$; these two operators are to be considered as acting in $l^2(\Lambda)$, and $\gamma_\Lambda(E) = \gamma(E)\Pi_\Lambda$, then, for $E \notin \operatorname{spec} H^0 \cup \operatorname{spec} H_A^\Lambda$, the following resolvent formula holds:

$$(H_A^\Lambda - E)^{-1} = (H^0 - E)^{-1} - \gamma_\Lambda(E)\big(M_\Lambda(E) - A_\Lambda\big)^{-1}\gamma_\Lambda^*(\bar{E}). \tag{17}$$

As previously, for any $E \notin \operatorname{spec} H^0$ one has $\ker(H_A^\Lambda - E) = \gamma_\Lambda(E) \ker\big(M_\Lambda(E) - A_\Lambda\big)$.

In the appendix A, we prove the following auxiliary result

Proposition 5. *Denote $\Lambda_N := \{m \in \mathbb{Z}^d : \max_j |m_j| \le N\}$, $N \in \mathbb{N}$, then the operators $H_A^{\Lambda_N}$ converge to H_A in the strong resolvent sense as $N \to \infty$.*

that will be used in the proof of our localization criterion.

Proposition 6. *Let $F \subset \mathbb{R}$ be a segment containing no Dirichlet eigenvalues. Assume that there exists $A, a > 0$ and $s \in (0,1)$ such that*

$$\mathbb{E}\left|\left(M_\Lambda(E) - A_{\Lambda,\omega}\right)^{-1}(m, m')\right|^s \le Ae^{-a|m-m'|} \tag{18}$$

for all finite $\Lambda \subset \mathbb{Z}^d$ and all $E \in F$. Then, there exist $B, c > 0$ such that for any two edges (m, j) and (m', j') one has

$$\mathbb{E}\big(\mu_\omega^{(m,j),(m',j')}(F)\big) \le Be^{-c|m-m'|}. \tag{19}$$

Remark 7. By theorem 1, the result of proposition 6 clearly implies that, under the assumptions of proposition 6, the spectrum is almost surely pure point in F. By the results of [2], in particular, theorem 4.1 therein, the assumption of proposition 6 also implies that, for $E \in F$, the spectrum of $M(E) - A_\omega$ is localized in an open interval containing 0. Hence, using the remark following Krein's resolvent formula, equation (7), for E in the spectrum of H_ω and not an eigenvalue of H_0 (i.e. not a Dirichlet eigenvalue), 0 is an eigenvalue for $M(E) - A_\omega$. It is associated to an eigenfunction, say ξ, that is exponentially localized in $\mathbb{Z}^d$. The corresponding eigenfunction for H_ω at energy E, say, φ is then given by $\varphi = \gamma(E)\xi$. By (3), φ is also exponentially localized in the sense that there exists $C > 0$ such that

$$\sup_{1 \le j \le d} \|\varphi\|_{\mathcal{H}_{m,j}} \le Ce^{-|m|/C}.$$

Moreover, as in the appendix A of [2], by (14), proposition 6 implies dynamical localization bounds for the operator H_A in the following sense

$$\mathbb{E}\left(\sup_{\substack{f=P_{m,j}f,\\ g=P_{m',j'}g\\ \|f\|=\|g\|=1}} |\langle f, e^{itH_\omega}\mathbf{1}_F(H_\omega)g\rangle|\right) \le Ce^{-|m-m'|/C}. \tag{20}$$

Proof of proposition 6. In view of proposition 5, $H^{\Lambda_n}_{A,\omega}$ converges to $H_{A,\omega}$ in the strong resolvent sense for a suitable choice of finite $\Lambda_n \subset \mathbb{Z}^d$ and any ω. This implies the weak convergence $\mu^{f,g}_{\Lambda,\omega} \to \mu^{f,g}_\omega$ for any f, g, ω. Consequently, by the Fatou lemma, for any F one has $\mathbb{E}\big(\mu^{(m,j),(m',j')}_\omega(F)\big) \le \liminf \mathbb{E}\big(\mu^{(m,j),(m',j')}_{\Lambda,\omega}(F)\big)$. In other words, it is sufficient to show the existence of positive B and c such that for any (m,j) and (m',j') the estimate $\mathbb{E}\big(\mu^{(m,j),(m',j')}_{\Lambda,\omega}(F)\big) \le Be^{-c|m-m'|}$ holds for sufficiently large Λ. In proving this estimate, we follow essentially the steps of [2, theorem A.1] or [1, lemma 3.1].

Pick two edges (m,j) and (m',j') and consider $\Lambda \subset \mathbb{Z}^d$ containing m and m' and all vetrices n with $|n-m'| \le 2$.

Denote $\hat{A}_\omega := A_\omega + (\hat{v} - \alpha(m'))\Pi_{m'}$, where $\Pi_{m'}$ is the projection onto $\delta_{m'}$ and $\hat{v}$ is distributed identically to $\alpha(m')$, and consider the modified Hamiltonian $H_{\hat{A},\omega}$. Note that under our assumptions $\mathbb{E}\big(|\hat{v}|^\delta\big) < \infty$ for any $\delta > 0$. For almost every $\hat{v}$, if 0 is an eigenvalue of $M_\Lambda(E) - A_{\Lambda,\omega}$, then $M_\Lambda(E) - \hat{A}_{\Lambda,\omega}$ is invertible. Consider also the operators $\tilde{A}_\omega := A_\omega + (\tilde{v} - \alpha(m'+h_{j'}))\Pi_{m'+h_{j'}}$ with $\tilde{v}$ distributed identically to $\alpha(m'+h_{j'})$, to which the previous observations apply as well.

We note that the spectrum of $H^\Lambda_{A,\omega}$ outside the Dirichlet eigenvalues is discrete. Almost surely, each eigenvalue of $M_\Lambda(E) - A_\Lambda$ is simple. One has

$$\mu^{f,g}_{\Lambda,\omega}(F) = \sum_{E_k \in \operatorname{spec} H^\Lambda_\omega \cap F} \frac{\langle f, \gamma_\Lambda(E_k)\xi_k\rangle\,\langle \gamma_\Lambda(E_k)\xi_k, g\rangle}{\|\gamma_\Lambda(E_k)\xi_k\|^2}, \tag{21}$$

where E_k and ξ_k satisfy $(M(E_k) - A_{\Lambda,\omega})\xi_k = 0$, $\xi_k \ne 0$.

Let $E \notin \operatorname{spec} H^0$. In the space $\mathcal{H}_{m',j'} = L^2[0, l_{j'}]$ consider the subspace $L(E)$ spanned by the linearly independent functions $\varphi_{j'}(E) := \varphi_{j'}(\cdot, E)$ and $\phi_{j'}(E) := \phi_{j'}(\cdot, E)$. Denote by $P(E)$ the orthogonal projection from $\mathcal{H}_{m',j'}$ to $L(E)$. Any function $h \in L(E)$ can be uniquely represented in the form $h = \hat{h} + \tilde{h}$ with $\hat{h}, \tilde{h} \in L(E)$, $\hat{h} \perp \varphi_{j'}(E)$, $\tilde{h} \perp \phi_{j'}(E)$. Denote the corresponding projections $L(E) \ni h \mapsto \hat{h} \in L(E)$ and $L(E) \ni h \mapsto \tilde{h} \in L(E)$ by $\hat{P}(E)$ and $\tilde{P}(E)$, respectively. In view of the analytic dependence of $\varphi_{j'}(E)$ and $\phi_{j'}(E)$, the norms of the operators $\hat{P}(E)P(E)$ and $\tilde{P}(E)P(E)$ are uniformly bounded,

$$\|\hat{P}(E)P(E)\| + \|\tilde{P}(E)P(E)\| \le p, \quad p > 0, \quad E \in F. \tag{22}$$

From now on we assume that $f = P_{m,j}f$ and $g = P_{m',j'}g$. Having in mind the explicit expression for $\gamma(E)$ (see (3)), we compute

$$[\gamma^*_\Lambda(E)g](m) = \sum_{s=1}^d \frac{1}{\overline{\varphi_s(l_s;E)}}\big(\langle \varphi_s(E), g_{m-h_s,s}\rangle + \langle \phi_j(E), g_{m,s}\rangle\big), \quad m \in \Lambda, \tag{23}$$

and one concludes that, for any $E \in F$, one has $\gamma^*(E)g = \gamma^*(E)\hat{P}(E)P(E)g + \gamma^*(E)\tilde{P}(E)P(E)g$, which permits us to rewrite (21) in the form

$$\mu^{f,g}_{\Lambda,\omega}(F) = \sum_{E_k \in \operatorname{spec} H^\Lambda_\omega \cap F} \frac{\langle f, \gamma_\Lambda(E_k)\xi_k\rangle \, \langle \gamma_\Lambda(E_k)\xi_k, \hat{P}(E_k)P(E_k)g\rangle}{\|\gamma_\Lambda(E_k)\xi_k\|^2} + \sum_{E_k \in \operatorname{spec} H^\Lambda_\omega \cap F} \frac{\langle f, \gamma_\Lambda(E_k)\xi_k\rangle \, \langle \gamma_\Lambda(E_k)\xi_k, \tilde{P}(E_k)P(E_k)g\rangle}{\|\gamma_\Lambda(E_k)\xi_k\|^2}. \tag{24}$$

Denote

$$\hat\varphi_E := \frac{\big(M_\Lambda(E) - A_{\Lambda,\omega}\big)^{-1}\delta_{m'}}{\langle \delta_{m'}, \big(M_\Lambda(E) - A_{\Lambda,\omega}\big)^{-1}\delta_{m'}\rangle} = \frac{\big(M_\Lambda(E) - \hat{A}_{\Lambda,\omega}\big)^{-1}\delta_{m'}}{\langle \delta_{m'}, \big(M_\Lambda(E) - \hat{A}_{\Lambda,\omega}\big)^{-1}\delta_{m'}\rangle}.$$

Assume that ξ is an eigenvector of $M_\Lambda(E) - A_{\Lambda,\omega}$ corresponding to the eigenvalue 0. Then $0 = (M_\Lambda(E) - A_{\Lambda,\omega})\xi = (M_\Lambda(E) - \hat{A}_{\Lambda,\omega})\xi + \big(\hat{v} - \alpha(m')\big)\Pi_{m'}\xi$. Almost surely the matrix $M_\Lambda(E) - \hat{A}_{\Lambda,\omega}$ is invertible and $\Pi_{m'}\xi \neq 0$ (otherwise ξ would be an eigenvector of $M_\Lambda(E) - \hat{A}_{\Lambda,\omega}$). Hence, $\xi = \big(\alpha(m') - \hat{v}\big)\langle \delta_{m'}, \xi\rangle (M_\Lambda(E) - \hat{A}_{\Lambda,\omega})^{-1}\delta_{m'}$. This means, that $\xi = C\hat\varphi_E$ with a suitable constant C.

By a direct calculation, $\big(M_\Lambda(E) - A_{\Lambda,\omega}\big)\hat\varphi_E = (\alpha(m') - \hat{v} - \hat\Gamma(E))\delta_{m'}$, where

$$\hat\Gamma(E) = -\frac{1}{\langle \delta_{m'}, \big(M_\Lambda(E) - \hat{A}_{\Lambda,\omega}\big)^{-1}\delta_{m'}\rangle}.$$

Hence, the spectrum of $H^{\Lambda_n}_{A,\omega}$ in F is determined by the condition $\alpha(m') - \tilde{v} = \hat\Gamma(E)$, and $\hat\varphi_E$ are the corresponding (non-normalized) eigenfunctions. Clearly, one has always

$$\langle \delta_{m'}, \hat\varphi_E\rangle = 1. \tag{25}$$

Using these observations one can write almost surely

$$\frac{\langle f, \gamma_\Lambda(E_k)\xi_k\rangle \, \langle \gamma_\Lambda(E_k)\xi_k, g\rangle}{\|\gamma_\Lambda(E_k)\xi_k\|^2} = \frac{\langle f, \gamma_\Lambda(E_k)\hat\varphi_{E_k}\rangle \langle \gamma_\Lambda(E_k)\hat\varphi_{E_k}, g\rangle}{\|\gamma_\Lambda(E_k)\hat\varphi_{E_k}\|^2}. \tag{26}$$

Exactly in the same way one shows that the spectrum can be determined from the condition $\alpha(m' + h_{j'}) - \tilde{v} = \tilde\Gamma(E)$ with

$$\tilde\Gamma(E) = -\frac{1}{\langle \delta_{m'+h_{j'}}, \big(M_\Lambda(E) - \tilde{A}_{\Lambda,\omega}\big)^{-1}\delta_{m'+h_{j'}}\rangle}$$

and that

$$\frac{\langle f, \gamma_\Lambda(E_k)\xi_k\rangle \, \langle \gamma_\Lambda(E_k)\xi_k, g\rangle}{\|\gamma_\Lambda(E_k)\xi_k\|^2} = \frac{\langle f, \gamma_\Lambda(E_k)\tilde\varphi_{E_k}\rangle \langle \gamma_\Lambda(E_k)\tilde\varphi_{E_k}, g\rangle}{\|\gamma_\Lambda(E_k)\tilde\varphi_{E_k}\|^2}, \tag{27}$$

where

$$\tilde\varphi_E := \frac{\big(M_\Lambda(E) - A_{\Lambda,\omega}\big)^{-1}\delta_{m'+h_{j'}}}{\langle \delta_{m'+h_{j'}}, \big(M_\Lambda(E) - A_{\Lambda,\omega}\big)^{-1}\delta_{m'+h_{j'}}\rangle} = \frac{\big(M_\Lambda(E) - \tilde{A}_{\Lambda,\omega}\big)^{-1}\delta_{m'+h_{j'}}}{\langle \delta_{m'}, \big(M_\Lambda(E) - \tilde{A}_{\Lambda,\omega}\big)^{-1}\delta_{m'+h_{j'}}\rangle}.$$

and obviously

$$\langle \delta_{m'+h_{j'}}, \tilde\varphi_E\rangle = 1. \tag{28}$$

Combining the represention (21) and (24) for the spectral measures with the identitites (26) and (27) one obtain the following:

$$
\begin{aligned}
\mu^{f,g}_{\Lambda,\omega}(dE) &= \frac{\langle f, \gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E, g\rangle}{\|\gamma_\Lambda(E)\hat\varphi_E\|^2}\cdot\Big(\sum_k\delta(E-E_k)\Big)dE\\
&= \frac{\langle f, \gamma_\Lambda(E)\tilde\varphi_E\rangle\langle\gamma_\Lambda(E)\tilde\varphi_E, g\rangle}{\|\gamma_\Lambda(E)\tilde\varphi_E\|^2}\cdot\Big(\sum_k\delta(E-E_k)\Big)dE\\
&= \frac{\langle f, \gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E, \hat P(E)P(E)g\rangle}{\|\gamma_\Lambda(E)\hat\varphi_E\|^2}\cdot\Big(\sum_k\delta(E-E_k)\Big)dE\\
&\quad+\frac{\langle f, \gamma_\Lambda(E)\tilde\varphi_E\rangle\langle\gamma_\Lambda(E)\tilde\varphi_E, \tilde P(E)P(E)g\rangle}{\|\gamma_\Lambda(E)\tilde\varphi_E\|^2}\cdot\Big(\sum_k\delta(E-E_k)\Big)dE.
\end{aligned}
\tag{29}
$$

Now, note that

$$\sum\delta(E-E_k) = -\delta(\alpha(m')-\hat v-\hat\Gamma(E))\hat\Gamma'(E) = -\delta(\alpha(m'+h_{j'})-\tilde v-\tilde\Gamma(E))\tilde\Gamma'(E)$$

and that, using (4) and (5), one obtains,

$$\hat\Gamma'(E) = -\hat\Gamma^2(E)\langle\delta_{m'}, (M_\Lambda(E)-\hat A_{\Lambda,\omega})^{-1}M'_\Lambda(E)(M_\Lambda(E)-\hat A_{\Lambda,\omega})^{-1}\delta_{m'}\rangle = -\|\gamma_\Lambda(E)\hat\varphi_E\|^2.$$

and

$$\tilde\Gamma'(E) = -\tilde\Gamma^2(E)\langle\delta_{m'+h_{j'}}, (M_\Lambda(E)-\tilde A_{\Lambda,\omega})^{-1}M'_\Lambda(E)(M_\Lambda(E)-\tilde A_{\Lambda,\omega})^{-1}\delta_{m'+h_{j'}}\rangle = -\|\gamma_\Lambda(E)\tilde\varphi_E\|^2.$$

This allows one to rewrite (29) as

$$
\begin{aligned}
\mu^{f,g}_{\Lambda,\omega}(dE) &= \delta(\alpha(m')-\hat v-\hat\Gamma(E))\langle f,\gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E, g\rangle dE\\
&= \delta(\alpha(m'+h_{j'})-\tilde v-\tilde\Gamma(E))\langle f,\gamma_\Lambda(E)\tilde\varphi_E\rangle\langle\gamma_\Lambda(E)\tilde\varphi_E, g\rangle dE\\
&= \delta(\alpha(m')-\hat v-\hat\Gamma(E))\langle f,\gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E, \hat P(E)P(E)g\rangle dE\\
&\quad+\delta(\alpha(m'+h_{j'})-\tilde v-\tilde\Gamma(E))\langle f,\gamma_\Lambda(E)\tilde\varphi_E\rangle\langle\gamma_\Lambda(E)\tilde\varphi_E, \tilde P(E)P(E)g\rangle dE.
\end{aligned}
\tag{30}
$$

According to the general properties of spectral measures, one always has $\mu^{f,g}_{\Lambda,\omega}(dE) = \Psi^{f,g}(E)\mu^{f,f}_{\Lambda,\omega}(dE)$, where Ψ is a measurable function satisfying

$$\int_{\mathbb{R}}|\Psi^{f,g}(E)|^2\mu^{f,f}_{\Lambda,\omega}(dE)\le\|g\|^2\|f\|^2.$$

In our case, the first two equalities in (30) imply

$$\int_{\mathbb{R}}\big|\langle\gamma_\Lambda(E)\hat\varphi_E, h\rangle\big|^2\delta\big(\alpha(m')-\hat v-\hat\Gamma(E)\big)dE\le\|h\|^2 \tag{31}$$

and

$$\int_{\mathbb{R}}\big|\langle\gamma_\Lambda(E)\tilde\varphi_E, h\rangle\big|^2\delta(\alpha(m'+h_{j'})-\tilde v-\tilde\Gamma(E))dE\le\|h\|^2 \tag{32}$$

for any h.

Now we use the third respresentation in (30) for the spectral measure to estimate the upper spectral measure for the edges (m,j) and (m',j'). Clearly,

$$|\mu^{f,g}_{\Lambda,\omega}|(F) = \int_F \big|\langle f, \gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E, \hat P(E)P(E)g\rangle\big|\delta(\alpha(m') - \hat v - \hat\Gamma(E))dE$$
$$+ \int_F \big|\langle f, \gamma_\Lambda(E)\tilde\varphi_E\rangle\langle\gamma_\Lambda(E)\tilde\varphi_E, \tilde P(E)P(E)g\rangle\big|\delta(\alpha(m'+h_{j'}) - \tilde v - \tilde\Gamma(E))dE.$$

The construction of the operators $\hat P(E)P(E)$ and $\hat P(E)P(E)$ implies that $\Pi_{m'}\gamma^*(E)_\Lambda\hat P(E)P(E) = \gamma^*(E)\hat P(E)P(E)$ and $\Pi_{m'+h_{j'}}\gamma^*_\Lambda(E)\tilde P(E)P(E) = \gamma^*(E)\tilde P(E)P(E)$. Together with the normalization conditions (25) and (28), for any g, this implies

$$\begin{aligned}\Big|\Big\langle\gamma_\Lambda(E)\hat\varphi_E, \hat P(E)P(E)g\Big\rangle\Big| &= \big\|\gamma^*_\Lambda(E)\hat P(E)P(E)g\big\|,\\ \Big|\Big\langle\gamma_\Lambda(E)\tilde\varphi_E, \tilde P(E)P(E)g\Big\rangle\Big| &= \big\|\gamma^*_\Lambda(E)\tilde P(E)P(E)g\big\|.\end{aligned} \tag{33}$$

Now, we estimate

$$\begin{aligned}&\mathbb{E}\left(\sup_{\|f\|=\|g\|=1}|\mu^{f,g}_{\Lambda,\omega}|(F)\right)\\ &\le \mathbb{E}\left(\sup_{\|f\|=\|g\|=1}\int_F\big|\langle f,\gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E,\hat P(E)P(E)g\rangle\big|\delta(\alpha(m')-\hat v-\hat\Gamma(E))dE\right)\\ &+\mathbb{E}\left(\sup_{\|f\|=\|g\|=1}\int_F\big|\langle f,\gamma_\Lambda(E)\tilde\varphi_E\rangle\langle\gamma_\Lambda(E)\tilde\varphi_E,\tilde P(E)P(E)g\rangle\big|\delta(\alpha(m'+h_{j'})-\tilde v-\tilde\Gamma(E))dE\right).\end{aligned} \tag{34}$$

Using (22) and (33), one gets

$$\begin{aligned}&\mathbb{E}\left(\sup_{\|f\|=\|g\|=1}\int_F\big|\langle f,\gamma_\Lambda(E)\hat\varphi_E\rangle\langle\gamma_\Lambda(E)\hat\varphi_E,\hat P(E)P(E)g\rangle\big|\delta(\alpha(m')-\hat v-\hat\Gamma(E))dE\right)\\ &\le p\,G\,\mathbb{E}\left(\sup_{\|f\|=1}\int_F|\langle f,\gamma_\Lambda(E)\hat\varphi_E\rangle|\,\delta(\alpha(m')-\hat v-\hat\Gamma(E))dE\right),\end{aligned} \tag{35}$$

where $G := \sup_{E\in F}\|\gamma^*_\Lambda(E)\| < \infty$. Using the Hölder inequality and (31), one obtains

$$\begin{aligned}&\mathbb{E}\left(\sup_{\|f\|=1}\int_F|\langle f,\gamma_\Lambda(E)\hat\varphi_E\rangle|\,\delta\big(\alpha(m')-\hat v-\hat\Gamma(E)\big)dE\right)\\ &\le\left[\mathbb{E}\left(|\alpha(m')-\hat v|^\alpha\sup_{\|f\|=1}\int_F\left|\langle f,\gamma_\Lambda(E)(M_\Lambda(E)-\hat A_\Lambda)^{-1}\delta_{m'}\rangle\right|^\alpha\delta\left(\alpha(m')-\hat v-\hat\Gamma(E)\right)dE\right)\right]^{1/(2-\alpha)}\end{aligned}$$

for any $\alpha\in(0,1)$. Using again the Hölder inequality we get

$$\begin{aligned}&\mathbb{E}\left(|\alpha(m')-\hat v|^\alpha\sup_{\|f\|=1}\int_F\left|\langle f,\gamma_\Lambda(E)(M_\Lambda(E)-\hat A_\Lambda)^{-1}\delta_{m'}\rangle\right|^\alpha\delta\left(\alpha(m')-\hat v-\hat\Gamma(E)\right)dE\right)\\ &\le 2\mathbb{E}(|\hat v|^\alpha)^{\alpha/\delta}\left[\mathbb{E}\left(\sup_{\|f\|=1}\int_F\left|\langle f,\gamma_\Lambda(E)(M_\Lambda(E)-\hat A_\Lambda)^{-1}\delta_{m'}\rangle\right|^s\delta\left(\alpha(m')-\hat v-\hat\Gamma(E)\right)dE\right)\right]^{\alpha/s}\end{aligned}$$

with $\alpha/s + \alpha/\delta = 1$. Using (18), we estimate,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{\|f\|=1}\int_F \left|\langle f, \gamma_\Lambda(E)(M_\Lambda(E)-\hat{A}_\Lambda)^{-1}\delta_{m'}\rangle\right|^s \delta\left(\alpha(m') - \hat{v} - \hat{\Gamma}(E)\right) dE\right)\\
&\le \int_F \mathbb{E}\left(\sup_{\|f\|=1}\left|\langle f, \gamma_\Lambda(E)(M_\Lambda(E)-\hat{A}_\Lambda)^{-1}\delta_{m'}\rangle\right|^s\right)\rho\left(\hat{v}+\hat{\Gamma}(E)\right) dE\\
&\le R|F|\sup_{E\in F}\mathbb{E}\left(\left\|\left(\gamma_\Lambda(E)(M_\Lambda(E)-\hat{A}_\Lambda)^{-1}\delta_{m'}\right)_{m,j}\right\|^s\right)\\
&\le R|F|C\sup_{E\in F}\mathbb{E}\left(\left|(M_\Lambda(E)-\hat{A}_\Lambda)^{-1}(m,m')\right|^s\right)\\
&\qquad\qquad + R|F|C\sup_{E\in F}\mathbb{E}\left(\left|(M_\Lambda(E)-\hat{A}_\Lambda)^{-1}(m+h_j,m')\right|^s\right)\\
&\le R|F|C\left(Ae^{-a|m-m'|} + Ae^{-a|m+h_j-m'|}\right) \le AR|F|C(1+e^a)e^{-a|m-m'|},
\end{aligned}
$$

where $R = \sup\rho$ and

$$
C = \max\left(\sup_{E\in F}\left\|\frac{\varphi_j(\cdot,E)}{\varphi_j(l_j,E)}\right\|^s, \sup_{E\in F}\left\|\frac{\phi_j(\cdot,E)}{\varphi_j(l_j,E)}\right\|^s\right).
$$

Finally, as follows from (35), one has

$$
\mathbb{E}\left(\sup_{\|f\|=\|g\|=1}\int_F \left|\langle f,\gamma_\Lambda(E)\hat{\varphi}_E\rangle\langle\gamma_\Lambda(E)\hat{\varphi}_E, \hat{P}(E)P(E)g\rangle\right|\delta(\alpha(m')-\hat{v}-\hat{\Gamma}(E))dE\right) \le \hat{B}e^{-\hat{c}|m-m'|} \tag{36}
$$

with some $\hat{B}, \hat{c} > 0$.

One can estimate the second term on the right-hand side of (34) in exactly the same way. Using (22) and (33) and the inequality (32), after similar steps, one gets

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{\|f\|=1}\left(\int_F \left|\langle f, \gamma_\Lambda(E)(M_\Lambda(E)-\tilde{A}_\Lambda)^{-1}\delta_{m'+h_{j'}}\rangle\right|^s \delta\left(\alpha(m'+h_{j'}) - \tilde{v} - \tilde{\Gamma}(E)\right) dE\right)\right)\\
&\le \int_F \mathbb{E}\left(\sup_{\|f\|=1}\left|\langle f, \gamma_\Lambda(E)(M_\Lambda(E)-\tilde{A}_\Lambda)^{-1}\delta_{m'+h_{j'}}\rangle\right|^s\right)\rho\left(\tilde{v}+\tilde{\Gamma}(E)\right) dE\\
&\le R|F|\sup_{E\in F}\mathbb{E}\left(\left\|\left(\gamma_\Lambda(E)(M_\Lambda(E)-\tilde{A}_\Lambda)^{-1}\delta_{m'+h_{j'}}\right)_{m,j}\right\|^s\right)\\
&\le R|F|C\sup_{E\in F}\mathbb{E}\left(\left|(M_\Lambda(E)-\tilde{A}_\Lambda)^{-1}(m,m'+h_{j'})\right|^s\right)\\
&\qquad + R|F|C\sup_{E\in F}\mathbb{E}\left(\left|(M_\Lambda(E)-\tilde{A}_\Lambda)^{-1}(m+h_j,m'+h_{j'})\right|^s\right)\\
&\le R|F|C\left(Ae^{-a|m-m'-h_{j'}|} + Ae^{-a|m+h_j-m'-h_{j'}|}\right)\\
&\le AR|F|C(e^a+e^{2a})e^{-a|m-m'|},
\end{aligned}
$$

which gives, for some positive constants $\tilde{B}$ and $\tilde{c}$,

(37)
$$\mathbb{E}\left(\sup_{\|f\|=\|g\|=1}\int_F \Big|\langle f,\gamma_\Lambda(E)\tilde{\varphi}_E\rangle\langle\gamma_\Lambda(E)\tilde{\varphi}_E,\tilde{P}(E)P(E)g\rangle\Big|\,\delta(\alpha(m'+h_{j'})-\tilde{v}-\tilde{\Gamma}(E))dE\right)\le \tilde{B}e^{-\tilde{c}|m-m'|}.$$

Substituting (36) and (37) into (34) we obtain the requested inequality (19). □

3. Finite volume criteria

We now will show how the results of [2] apply in our case.

We need some constants characterizing the distribution of the coupling constants. Let $s\in(0,1)$. Define

$$C_s=\sup_{A\in\mathcal{M}_{2\times 2}(\mathbb{C})}\int\int\rho(du)\rho(dv)\Big|\Big[\big(A-\operatorname{diag}(u,v)\big)^{-1}\Big]_{jk}\Big|^s.$$

In [2], it is shown that C_s is finite. It is also shown that for any $s\in(0,1/4)$, if, for $a,b,c\in\mathbb{C}$, we define $f(V):=(V-a)^{-1}$, $g(V):=(V-b)(V-c)^{-1}$, then

$$D_s=\sup_{a,b,c}\frac{\mathbb{E}\big(|f(V)|^s|g(V)|^s\big)}{\mathbb{E}\big(|f(V)|^s\big)\mathbb{E}\big(|g(V)|^s\big)}<+\infty.$$

We set $\tilde{C}_s:=C_sD_s^2$.

In the standard basis of $l^2(\mathbb{Z}^d)$ the operator $M(E)+a(E)$ is given by the matrix $\big(\tau_{m,m'}(E)\big)_{m,m'\in\mathbb{Z}^d}$ with

$$\tau_{m,m'}(E)=\begin{cases}0 & m=m',\\ b_j(E), & m=m'\pm h_j,\\ 0, & |m-m'|>1.\end{cases}\tag{38}$$

Let $s\in(0,1/4)$. For any $\Lambda\subset\mathbb{Z}^d$ denote

$$T^s_{m,\partial\Lambda}(E):=\sum_{n\in W}|\tau_{m,n}(E)|^s,\quad m\in\mathbb{Z}^d,\quad W=\begin{cases}\mathbb{Z}^d\setminus\Lambda, & m\in\Lambda,\\ \Lambda, & m\notin\Lambda.\end{cases}$$

Furthermore, set

$$\Theta^s_\Lambda(E):=\sum_{m\in\Lambda}T^s_{m,\partial\Lambda}(E),$$

and

$$\begin{aligned}k_\Lambda(m,n;E):=&\,|\tau_{m,n}(E)|^sI_1(m,n)+T^s_{m,\partial\Lambda}(E)T^s_{n,\partial\Lambda}(E)\frac{\tilde{C}_s}{\lambda^s}I_2(m,n)\\&+T^s_{m,\partial\Lambda}(E)T^s_{n,\partial\Lambda}(E)\Big(\frac{\tilde{C}_s}{\lambda^s}\Big)^2\Theta^s_\Lambda(E)I_3(u,v),\end{aligned}$$

where

$$I_1(m,n)=\begin{cases}1 & m\in\Lambda,n\notin\Lambda,\\ 0, & \text{otherwise},\end{cases},\ I_2(m,n)=\begin{cases}1 & m\in\Lambda,\\ 0, & \text{otherwise},\end{cases},\ I_3(m,n)=\begin{cases}1 & m\in\Lambda,n\in\Lambda,\\ 0, & \text{otherwise}.\end{cases}$$

Theorem 3.2 in [2] and the remark thereafter read in our case as follows.

Proposition 8. *Take any interval $X \subset \mathbb{R}$ free of Dirichlet eigenvalues. Assume that there exist $\beta \in (0,1)$ and $s \in (0,1/4)$ such that for all $E \in X$ there exists a finite $\Lambda \subset \mathbb{Z}^d$ with $0 \in \Lambda$ obeying*

$$\sup_{W \subset \Lambda} \sum_{(m,n) \in \Lambda \times (\mathbb{Z}^d \setminus \Lambda)} \mathbb{E}\Big(\big| \big(M_W(E) - \lambda A_{W,\omega}\big)^{-1}(0,m) \big|^s \Big) k_\Lambda(m,n;E) \le \beta. \tag{39}$$

Then, there exist $B, c > 0$ such that, for any finite $\Theta \subset \mathbb{Z}^d$, any $m \in \Theta$, and any $E \in X$, one has

$$\sum_{m' \in \Theta} \mathbb{E}\Big(\big| \big(M_\Theta(E) - \lambda A_{\Theta,\omega}\big)^{-1}(m,m') \big|^s \Big) e^{c|m-m'|} \le B.$$

We note that the possibility to choose the constant B independent of E follows from Eq. (3.20) in [2].

It is also important to emphasize that in the sum (39) the coefficients $k_\Lambda(m,n;E)$ are non-zero only if simultaneously $\operatorname{dist}(n,\Lambda) = 1$ and $\operatorname{dist}(m, \mathbb{Z}^d \setminus \Lambda) = 1$.

For convenience, we formulate proposition 8 for the special case $\Lambda = \{0\}$, which will be used below.

Proposition 9. *Take any $X \subset \mathbb{R}$ free of Dirichlet eigenvalues. Assume that there exists $\beta \in (0,1)$ and $s \in (0,1/4)$ such that for all $E \in X$ one has*

$$c(E)\Big(1 + c(E)\,\frac{\tilde{C}_s}{\lambda^s}\Big) \int_{\alpha_-}^{\alpha_+} \frac{1}{\big|a(E) + \lambda V\big|^s}\,\rho(dV) < \beta, \quad c(E) := 2\sum_{j=1}^{d} \big|b_j(E)\big|^s. \tag{40}$$

Then there exist $B, c > 0$ such that for any finite $\Lambda \subset \mathbb{Z}^d$, for any $m, m' \in \Lambda$, and any $E \in X$ there holds

$$\mathbb{E}\left(\Big| \big(M_\Lambda(E) - \lambda A_{\Lambda,\omega}\big)^{-1}(m,m') \Big|^s \right) \le B e^{-c|m-m'|}.$$

The condition $s \in (0,1/4)$ is needed for the so-called decoupling property to hold (see [1]). Actually a revision of the proofs in [2] shows that the decoupling property is not necessary in our case as the operators $M(E)$ do not depend on the random variables, and one can obtain some finite volume criteria with any power $s \in (0,1)$.

The following theorem summarizes all the above localization conditions for quantum graphs.

Theorem 10. *Let $X \subset \mathbb{R}$ be free of the Dirichlet eigenvalues and have a finite Lebesgue measure. Assume that the assumptions of proposition 8 are satisfied, then $H_{\lambda,\omega}$ has only pure point spectrum in X.*

Proof. By proposition 8, there exist $B, c > 0$ such that for all finite $\Lambda \subset \mathbb{Z}^d$ and all $E \in X$ one has $\mathbb{E}\Big| \big(M_\Lambda(E) - \lambda A_{\Lambda,\omega}\big)^{-1}(m,m') \Big|^s \le B e^{-c|m-m'|}$. Then, by proposition 6, one has $\mathbb{E}\Big(\mu^{(m,j),(m',j')}(X)\Big) \le B e^{-c|m-m'|}$, $B, c > 0$. Hence, for any (m,j) the following bound holds

$$\mathbb{E}\Big(\sum_{m' \in \mathbb{Z}^d} \sum_{j'=1}^{d} \mu^{(m,j),(m',j')}(X) \Big) \le B d \sum_{m' \in \mathbb{Z}^d} e^{-c|m'|} < \infty,$$

and the spectrum of $H_{\lambda,\omega}$ in X is pure point by corollary 4. $\square$

4. Strong disorder localization

Here we are going to exhibit assumptions ensuring that one obtains dense pure point spectrum in some regions for sufficiently large constant λ. To garantee the presence of the dense pure point spectrum, it is necessary to show the overlapping of the spectrum of $H_{\lambda,\omega}$ with the region where the assumptions of proposition 8 are fulfilled.

Proposition 11. *For any $E_0 \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that the spectrum of $H_{\lambda,\omega}$ lying in $(-\infty, E_0)$ but outside the ε-neighborhoods of the Dirichlet eigenvalues is pure point for all $\lambda > \lambda_0$.*

Proof. We use the single point criterium, proposition 9. Denote by X the half-axis $(-\infty, E_0)$ without he ε-neighborhoods of the Dirichlet eigenvalues. Due to the asymptotics (13), one can estimate, for some $\delta > 0$, $|\varphi_j(l_j; E)| \geq \delta > 0$ uniformly for $E \in X$. Hence for $E \in X$ one has $|b_j(E)| \leq B$, $|c(E)| \leq B$ for some $B > 0$, and, moreover, due to (13), $b_j(E) = O(e^{-\alpha\sqrt{-E}})$, $c(E) = O(e^{-s\alpha\sqrt{-E}})$ for some $\alpha > 0$ as $E \to -\infty$.

Pick $s \in (0, 1/4)$. As the density ρ is bounded, say, $\rho \leq R$, one has

$$\int_{\alpha_-}^{\alpha_+} \bigl|\, a(E) + \lambda V \bigr|^{-s} \rho(dV) \leq R \int_{\alpha_-}^{\alpha_+} \bigl|\, a(E) + \lambda V \bigr|^{-s} dV \leq \frac{R}{\lambda} \int_{\alpha_-/\lambda - a(E)}^{\alpha_+/\lambda - a(E)} \bigl|V\bigr|^{-s} dV$$
$$\leq \frac{2R}{\lambda s} \left|\frac{\alpha_+ - \alpha_-}{\lambda}\right|^{1-s} \leq \frac{C}{\lambda^s}.$$

Therefore,

$$c(E)\Bigl(1 + c(E)\,\frac{\tilde{C}_s}{\lambda^s}\Bigr) \int \frac{1}{\bigl|\, a(E) + \lambda V \bigr|^{s}}\, \rho(dV) \leq \tilde{C}(E)(\lambda^{-s} + \lambda^{-2s}) \tag{41}$$

where $\tilde{C}(E)$ is bounded in X. Hence, the left-hand side of (41) tends to 0 uniformly in X as λ becomes large. The spectrum if $H_{\lambda,\omega}$ in any compact subset of X is then pure point by theorem 1. □

Proposition 11 does not guarantee that there is some spectrum in the set considered. To show the presence of a dense point spectrum we use the estimates of subsection 1.2 to obtain

Theorem 12. *Assume that $\alpha_- < 0$. Then, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that the spectrum of $H_{\lambda,\omega}$ in $(-\infty, \inf \operatorname{spec} H_0 - \varepsilon)$ is dense pure point for $\lambda > \lambda_0$.*

Theorem 13. *Let $0 \in [\alpha_-, \alpha_+]$. Then, for any $E_0 > \inf \operatorname{spec} H^0$ and any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that the spectrum of $H_{\lambda,\omega}$ lying in $(-\infty, E_0)$ but outside the ε-neighborhoods of the Dirichlet eigenvalues is dense pure point for all $\lambda > \lambda_0$.*

Both theorems 12 and 13 are direct consequences of proposition 11. The discussion of subsection 1.2 shows that the intersection of the spectrum of $H_{\lambda,\omega}$ with the sets considered is non-empty for large λ.

In theorems 12 and 13, we only stated the localized spectrum. Clearly by virtue of remark 7, we get also exponential decay of the eigenfunctions and dynamical localization.

We were not able to study the effect of the strong disorder in neighborhoods on the Dirichlet eigenvalues. The reason is that in these neighborhoods the expression $a(E)$ in the single point criterion, proposition 9, becomes unbounded, and hence even large λ gives no possibility to control the value of the integral in (40). Moreover, if both the constants α_- and α_+ are positive, then, by discussion of subsection 1.2, the whole spectrum is concentrated in these neighborhoods, so the above theorems do not provide any localilization result in this case. In the next section we will be able to fill this gap at least partially and to prove localization near the spectral edges independently of their location.

5. Localization at band edges

Here we are going to show the presence of the dense pure point spectrum at the edges of the spectrum of H_ω.

The starting point will the following simple observation.

Proposition 14. *Let $E_0 \in \operatorname{spec} H_\omega \setminus H^0$. If for some $\varepsilon > 0$ one has $(E_0-\varepsilon, E_0) \notin \operatorname{spec} H_\omega$ or $(E_0, E_0+\varepsilon) \notin \operatorname{spec} H_\omega$, then either $\inf\operatorname{spec}\big(M(E_0)-A_\omega\big) = 0$ or $\sup\operatorname{spec}\big(M(E_0)-A_\omega\big)=0$. In other words, if $E_0 \notin \operatorname{spec} H^0$ is at the border of the spectrum of H_ω, then 0 is a border of the spectrum of $M(E_0)-A_\omega$.*

Proof. As (10) shows, the spectrum of $M(E)-A_\omega$ is a segment $[m_-(E), m_+(E)]$ whose ends $m_-(E) := \inf \Sigma_M(E)$ and $m_+(E) := \sup\Sigma_M(E)$ depend continuously on E. As $E_0 \in \operatorname{spec} H_\omega$, one has necessarily $0 \in \Sigma_M(E)$, i.e. $m_-(E_0)m_+(E_0) \le 0$. If one had $m_-(E_0)m_+(E_0) < 0$, i.e. $m_-(E_0) < 0$ and $m_-(E_0) > 0$, then the inequality $m_-(E)m_+(E) < 0$ would hold also for $E \in (E_0-\varepsilon, E_0+\varepsilon)$ with some $\varepsilon > 0$. But this would mean that $(E_0-\varepsilon, E_0+\varepsilon) \subset \operatorname{spec} H_\omega$, which contradicts the assumptions. Therefore, the only possibility is $m_-(E_0)\cdot m_+(E_0) = 0$. □

Theorem 15. *Let $E_0 \notin \operatorname{spec} H^0$ be at the border of the spectrum of H_ω. Then the spectrum of H_ω in some neighborhood of E_0 is pure point almost surely.*

In the present case, remark 7 gives also exponential decay of the eigenfunctions and dynamical localization.

Proof. Proposition 14 shows that 0 is an edge of the spectrum of $M(E_0)-A_\omega$. To be definite, we consider only the case $\inf\operatorname{spec}\big(M(E_0)-A_\omega\big) = 0$; the other case can be studied in the same way. Note that due to the variational principle one has $M_W(E_0) - A_{W,\omega} \ge 0$ for any $W \subset \mathbb{Z}^d$.

Let us do first some preparations. For any $W \subset \mathbb{Z}^d$ and $\varepsilon > 0$ consider the following subset of Ω:

$$\Omega(\varepsilon, W) := \big\{\omega \in \Omega : \inf\operatorname{spec}\big(M_W(E_0) - A_{W,\omega}\big) \le \varepsilon\big\}.$$

Clearly, by the variational principle one has $\Omega(\varepsilon, W) \subset \Omega(\varepsilon, W')$ if $W \subset W'$.

Let $\mathcal{N}(\lambda)$ by the integrated density of states corresponding to $M(E_0)-A_\omega$. Denote $\Lambda_N := \{m \in \mathbb{Z}^d : \max_j |m_j| \le N\}$, $N \in \mathbb{N}$. It is known [23] that with some $C > 0$ one has

$$\mathbb{P}\big(\Omega(\varepsilon, \Lambda_N)\big) \le CN^d\mathcal{N}(\varepsilon) \quad \text{for any } N \ge 1.$$

At the same time, one has the Lifshitz asymptotics for $\mathcal{N}(\varepsilon)$, i.e. there exists $\varepsilon_0 > 0$ and $\eta > 0$ such that $\mathcal{N}(\varepsilon) \le e^{-\varepsilon^{-\eta}}$, $\varepsilon \in (0, \varepsilon_0)$. Indeed, by (38), the Fourier symbol of $M(E_0)$ is of the form

$\sum_{j=1}^{d} b_j \cos\theta_j - a$ with $b_j \neq 0$, hence, one can apply to $M(E_0)+A_\omega$ the techniques of [21] to obtain that $\log|\log \mathcal{N}(\varepsilon)| = -\frac{d}{2}\log\varepsilon(1+o(1))$ when $\varepsilon \to 0+$.

For any finite $W \subset \mathbb{Z}^d$ and $\varepsilon > 0$ denote

$$\tilde\Omega(\varepsilon, W) := \big\{\omega \in \Omega : \inf \operatorname{spec}\big(M_W(E) - A_{W,\omega}\big) \leq \varepsilon \text{ for some } E,\, |E-E_0| < \varepsilon\big\}.$$

Note that the condition $\inf \operatorname{spec}\big(M_W(E) - A_{W,\omega}\big) \leq \varepsilon$ is equivalent to the existence of a non-zero $\xi_E \in l^2(W)$ with

$$\big\langle \xi_E, \big(M_W(E) - A_{W,\omega}\big)\xi_E\big\rangle \leq \varepsilon \cdot \|\xi_E\|^2. \tag{42}$$

Representing $M(E) = M(E_0) + (E-E_0)B(E)$, where $\|B(E)\| \leq D$ for some $D > 0$ in a neighborhood of E_0, one immediately sees that (42) implies $\big\langle \xi_E, \big(M_W(E_0) - A_{W,\omega}\big)\xi_E\big\rangle \leq (D+1)\varepsilon\|\xi_E\|^2$, which means $\inf \operatorname{spec}\big(M_W(E_0) - A_{W,\omega}\big) \leq (D+1)\varepsilon$. This shows the inclusion $\tilde\Omega(\varepsilon, W) \subset \Omega((D+1)\varepsilon, W)$.

With the above preparations we just need to repeat the basic steps from [22, Section 2]. It is sufficient to show that there exists a neighborhood X of E_0 where the assumptions of proposition 8 are satisfied for $\Lambda = \Lambda_N$ with a suitable N.

Let us fix some $s \in (0, 1/4)$. Consider any $W \subset \Lambda_N$. As shown above, one has $\tilde\Omega(\varepsilon, W) \subset \Omega((D+1)\varepsilon, W) \subset \Omega((D+1)\varepsilon, \Lambda_N)$. Subsequently, for $\varepsilon \in (0, \varepsilon')$ with some $\varepsilon' > 0$, one has

$$\mathbb{P}\big(\tilde\Omega(\varepsilon, W)\big) \leq \mathbb{P}\big(\Omega((D+1)\varepsilon, \Lambda_N)\big) \leq CN^d e^{-\varepsilon^{-\eta}}, \quad \eta > 0. \tag{43}$$

For $\omega \notin \tilde\Omega(\varepsilon, W)$ one can use the Combes-Thomas estimates, see e.g. [22, lemma 6.1], which gives that for some $C', r > 0$ one has

$$\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m, m')\big| \leq C' e^{-r|m-m'|}. \tag{44}$$

Eq. (6.1) in [22] shows that the constants C' and r can be chosen independent of W as in our case $\inf \operatorname{spec}\big(M_W(E) - A_{W,\omega}\big) > \varepsilon$.

Take any $s' \in (s, 1)$, then for any E with $|E - E_0| < \varepsilon$ one has also an a priori estimate

$$\mathbb{E}\Big(\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m, m')\big|^{s'}\Big) \leq C_{s'}, \tag{45}$$

see [2, lemma 2.1].

Now we have

$$\begin{aligned}\mathbb{E}\Big(\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m, m')\big|^{s}\Big) &= \mathbb{E}\Big(\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m, m')\big|^{s}\mathbf{1}_{\omega\in\tilde\Omega(\varepsilon,W)}\Big)\\ &\quad + \mathbb{E}\Big(\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m, m')\big|^{s}\mathbf{1}_{\omega\notin\tilde\Omega(\varepsilon,W)}\Big).\end{aligned}$$

Using (44) we obtain easily

$$\mathbb{E}\Big(\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m, m')\big|^{s}\mathbf{1}_{\omega\notin\tilde\Omega(\varepsilon,W)}\Big) \leq Be^{-b|m-m'|}, \quad B, b > 0.$$

Using the Hölder inequality, (43) and (45), for some $C' > 0$ and $\gamma > 0$, one has

$$\mathbb{E}\Big(\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m,m')\big|^s \mathbf{1}_{\omega\notin\tilde\Omega(\varepsilon,W)}\Big) \le \Big(\mathbb{E}\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m,m')\big|^s\Big)^{s/s'} \mathbb{P}(\tilde\Omega(\varepsilon,W))^{(s'-s)/s}$$
$$\le C' N^d e^{-\varepsilon^{-\gamma}}.$$

Finally,

$$\mathbb{E}\big|\big(M_W(E) - A_{W,\omega}\big)^{-1}(m,m')\big|^s \le Be^{-b|m-m'|} + C'N^d e^{-\varepsilon^{-\gamma}}. \tag{46}$$

Now let us estimate the sum (39). We emphasize again that the coefficients $k_{\Lambda_N}(m,n;E)$ in this sum are non-zero only if simultaneously $\operatorname{dist}(n,\Lambda_N) = 1$ and $\operatorname{dist}(m,\mathbb{Z}^d\setminus\Lambda_N) = 1$. Moreover, the non-zero terms are uniformly bounded in a neighborhood of E_0, $k_{\Lambda_N}(m,n;E) \le K$, $K > 0$. Therefore, using (46),

$$\sum_{\substack{m\in\Lambda_N\\ n\in\mathbb{Z}^d\setminus\Lambda_N}} \mathbb{E}\Big(\big|\big(M_W(E) - \lambda A_{W,\omega}\big)^{-1}(0,m)\big|^s\Big) k_\Lambda(m,n;E) \le K \sum_{\substack{m\in\Lambda_N\\ \operatorname{dist}(m,\mathbb{Z}^d\setminus\Lambda_N)=1\\ n\notin\Lambda_N\\ \operatorname{dist}(n,\Lambda_N)=1}} \Big(Be^{-b|m|} + C'N^d e^{-\varepsilon^{-\gamma}}\Big)$$
$$\le K' N^{2d}\Big(Be^{-bN} + C'N^d e^{-\varepsilon^{-\gamma}}\Big).$$

Now choosing, for example, $N \sim \varepsilon^{-1}$ one can make the sum as small as needed for sufficiently small ε. The spectrum of H_ω near E_0 is then pure point by theorem 1. $\square$

Appendix A. Proofs of propositions 5 and 3

In this subsection, we prove some auxiliary results on the finite volume approximation for H_A defined in section 2.

Proof of proposition 5. To prove the convergence, we will use the following variant of theorem VIII.1.5 from [18]: *Let T_n, T be self-adjoint operators. Assume that there exists a domain D of essential self-adjointness (or a core) for T such that every function f from D belongs to $\operatorname{dom} T_m$ for m sufficiently large and $T_n f \to Tf$ for any such f. Assume that, for at least one non-real z, the sequence $\|(T_n - z)^{-1}\|$ is bounded, then T_n converges to T in the strong resolvent sense.*

In our case, take as D the set of the functions $f \in \operatorname{dom} H_A$ having a compact support. Clearly, any such f lies in $\operatorname{dom} H_A^{\Lambda_n}$ for n sufficiently large, and $H_\alpha^{\Lambda_n} f$ just coincides with $H_A f$ for such n. Let us show that D is a domain of essential self-adjointness for H_A.

Choose functions $u_j \in C^\infty[0,l_j]$ such that u_j is 1 in a neighborhood of 0 and is 0 in a neighborhood of l_j. Take an arbitrary $f \in \operatorname{dom} H_A$. For $M \in \mathbb{N}$ denote $f^M := (f^M_{m,j})$ with

$$f^M_{m,j}(t) := \begin{cases} f_{m,j}(t), & m \in \Lambda_{M-1},\\ u_j(t) f_{m,j}(t), & (m,j) = m \to m',\ m \in \Lambda_M\setminus\Lambda_{M-1}, m' \notin \Lambda_M,\\ u_j(l_j - t) f_{m,j}(t), & (m,j) = m \to m',\ m' \in \Lambda_M\setminus\Lambda_{M-1}, m \notin \Lambda_M,\\ 0, & \text{otherwise.}\end{cases}$$

Clear, $f^M \in D$ and $f^M \underset{M\to+\infty}{\to} f$. Note that $H_A(f^M - f) = (F^M_{m,j})$ with

$$\begin{aligned} F^M_{m,j} &= u_j'' f_{m,j} + 2u_j' f_{m,j}' + \big(u_j - 1\big)\big(-f_{m,j}'' + U_j f_{m,j}\big),\\ &\qquad (m,j) = m \to m', \quad m \in \Lambda_M \setminus \Lambda_{M-1}, m' \notin \Lambda_M,\\ F^M_{m,j} &= u_j''(l_j - \cdot) f_{m,j} - 2u_j'(l_j - \cdot) f_{m,j}' + \big(u_j(l_j - \cdot) - 1\big)\big(-f_{m,j}'' + U_j f_{m,j}\big),\\ &\qquad (m,j) = m \to m', \quad m' \in \Lambda_M \setminus \Lambda_{M-1}, m \notin \Lambda_M, \end{aligned}$$

and all other components $F^M_{m,j}$ equal to 0. As $(f_{m,j}) \in \mathcal{H}$, $(f'_{m,j}) \in \mathcal{H}$ and $(-f''_{m,j} + U_j f_{m,j}) \in \mathcal{H}$, one has $H_A(f^M - f) \underset{M\to\infty}{\to} 0$. This shows that H_A is essentially self-adjoint on D.

To conclude the proof of the strong resolvent convergence it remains to show that that the norms $\|(H^{\Lambda_n}_\alpha - E)^{-1}\|$ are uniformly bounded for at least one non-real E. Take an arbitrary E with $\Im E \neq 0$. By (6) one has $\Im M(E)/\Im E \geq c$ for some $c > 0$. In particular, for any n one has

$$\Big|\Big\langle \big(\Pi_{\Lambda_n}(M(E) - A)\Pi_{\Lambda_n}\big)\Pi_{\Lambda_n}\xi, \Pi_{\Lambda_n}\xi\Big\rangle\Big| = \Big|\big\langle (M_{\Lambda_n}(E) - A_{\Lambda_n})\Pi_{\Lambda_n}\xi, \Pi_{\Lambda_n}\xi\big\rangle\Big| \geq c\|\Pi_{\Lambda_n}\xi\|^2,$$

which means that $M_{\Lambda_n}(E) - A_{\Lambda_n}$ has a bounded inverse, and that $\big\|\big(M_{\Lambda_n}(E) - A_{\Lambda_n}\big)^{-1}\big\| \leq c^{-1}$. Now it follows from (17) that the norms $\|(H^{\Lambda_n}_\alpha - E)^{-1}\|$ are uniformly bounded for any non-real E. □

Proof of proposition 3. We note first that T is an integral operator whose integral kernel is

$$T\big((m,j,t),(m',j',t')\big) = \begin{cases} G_A\big((m,j,t),(m',j',t')\big), & m \in \Lambda,\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$\|T\|^2_{HS} = \sum_{m\in\Lambda}\sum_{m'\in\mathbb{Z}^d}\sum_{j,j'=1}^{d}\int_0^{l_j}\int_0^{l_{j'}} \big|G_A\big((m,j,t),(m',j',t')\big)\big|^2\,dt'\,dt.$$

Using the explicit form (8) for G_A and the bounds $\|\varphi_j\|, \|\phi_j\| \leq C$ (with C independent of j) we obtain

$$\|T\|^2_{HS} \leq 2\Big(\sum_{j=1}^{d}\|G_j\|^2_{HS}\Big)|\Lambda| + C'\sum_{m\in\tilde\Lambda}\sum_{m'\in\mathbb{Z}^d}\big|(M(E) - A)^{-1}(m,m')\big|^2 \tag{47}$$

with some $C' > 0$, where $\tilde\Lambda := \{m \in \mathbb{Z}^d : \inf_{m'\in\Lambda}|m - m'|^2 \leq 2\}$. Clearly, due to (9) the Hilbert-Schmidt norms of G_j are finite. Furthermore, as $(M(E) - A)^{-1}$ is bounded for non-real E, one has

$$\sum_{m'\in\mathbb{Z}^d}\big|(M(E) - A)^{-1}(m,m')\big|^2 < \infty$$

for any $m \in \mathbb{Z}^d$ by the Riesz theorem. Hence, due to the finiteness of Λ (and of $\tilde\Lambda$), the sum on the right-hand side of (47) is finite. □

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L.A.G.A., CNRS UMR 7539, Institut Galilée, Université Paris Nord, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France & Institut Universitaire de France

E-mail address: `klopp@math.univ-paris13.fr`

L.A.G.A., CNRS UMR 7539, Institut Galilée, Université Paris Nord, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France & Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany

E-mail address: `const@math.hu-berlin.de`