Phase Transitions and Quantum Stabilization in Quantum Anharmonic Crystals

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Abstract

A unified theory of phase transitions and quantum effects in quantum anharmonic crystals is presented. In its framework, the relationship between these two phenomena is analyzed. The theory is based on the representation of the model Gibbs states in terms of path measures (Euclidean Gibbs measures). It covers the case of crystals without translation invariance, as well as the case of asymmetric anharmonic potentials. The results obtained are compared with those known in the literature.

Contents

1 Introduction and Setup

In recent years, there appeared a number of publications describing infuence of quantum effects on phase transitions in quantum anharmonic crystals, where the results were obtained by means of path integrals, see [3, 5, 8, 9, 45, 48, 50, 56, 63, 82]. Their common point is a statement that the phase transition (understood in one or another way) is suppressed if the model parameters obey a sufficient condition (more or less explicitely formulated). The existence of phase transitions in quantum crystals of certain types was proven earlier, see [15, 16, 23, 44, 60], also mostly by means of path integral methods. At the same time, by now only two works, [6] and [54], have apperared where both these phenomena are studied in one and the same context. In the latter paper, a more complete and extended version of the theory of interacting systems of quantum anharmonic oscillators based on path integral methods has been elaborated, see also [10, 11, 12] for more recent development, and [53] where the results of [54] were announced. The aim of the present article is to refine and extend the previous results and to develop a unified and more or less complete theory of phase transitions and quantum effects in quantum anharmonic crystals, also in the light of the results of [53, 54]. Note that, in particular, with the help of these results we prove here phase transitions in quantum crystals with asymmetric anharmonic potentials[1], what could hardly be done by other methods.

[1] This result was announced in [39].

The quantum crystal studied in this article is a system of interacting quantum anharmonic oscillators indexed by the elements of a crystal lattice $\mathbb{L}$, which for simplicity we assume to be a d-dimensional simple cubic lattice $\mathbb{Z}^d$. The quantum anharmonic oscillator is a mathematical model of a quantum particle moving in a potential field with possibly multiple minima, which has a sufficient growth at infinity and hence localizes the particle. Most of the models of interacting quantum oscillators are related with solids such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes, or quantum crystals consisting entirely of such particles. For instance, a potential field with multiple minima is seen by a helium atom located at the center of the crystal cell in bcc helium, see page 11 in [43]. The same situation exists in other quantum crystals, He, H_2 and to some extent Ne. An example of the ionic crystal with localized quantum particles moving in a double-well potential field is a KDP-type ferroelectric with hydrogen bounds, in which such particles are protons or deuterons performing one-dimensional oscillations along the bounds, see [18, 72, 78, 79]. It is believed that in such substances phase transitions are triggered by the ordering of protons. Another relevant physical object of this kind is a system of apex oxygen ions in YBaCuO-type high-temperature superconductors, see [27, 57, 74, 75]. Quantum anharmonic oscillators are also used in models describing interaction of vibrating quantum particles with a radiation (photon) field, see [32, 33, 58], or strong electron-electron correlations caused by the interaction of electrons with vibrating ions, see [25, 26], responsible for such phenomena as superconductivity, charge density waves etc. Finally, we mention systems of light atoms, like Li, doped into ionic crystals, like KCl. The quantum particles in this system are not necessarily regularly distributed. For more information on this subject, we refer to the survey [34].

To be more concrete we assume that our model describes an ionic crystal and thus adopt the ferroelectric terminology. In the corresponding physical substances, the quantum particles carry electric charge; hence, the displacement of the particle from its equilibrium point produces dipole moment. Therefore, the main contribution into the two-particle interaction is proportional to the product of the displacements of particles and is of long range. According to these arguments our model is described by the following formal Hamiltonian

$$H = -\frac{1}{2}\sum_{\ell,\ell'} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell} H_\ell. \tag{1.1}$$

Here the sums run through the lattice $\mathbb{L} = \mathbb{Z}^d$, $d \in \mathbb{N}$, the displacement, q_ℓ, of the oscillator attached to a given $\ell \in \mathbb{L}$ is a ν-dimensional vector. In general, we do not assume that the interaction intensities $J_{\ell\ell'}$ have finite range. By $(\cdot,\cdot)$ and $|\cdot|$ we denote the scalar product and norm in $\mathbb{R}^\nu$, $\mathbb{R}^d$. The one-site Hamiltonian

$$H_\ell = H_\ell^{\mathrm{har}} + V_\ell(q_\ell) \ \stackrel{\mathrm{def}}{=}\ \frac{1}{2m}|p_\ell|^2 + \frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell), \quad a > 0, \tag{1.2}$$

describes an isolated quantum anharmonic oscillator. Its part H_ℓ^{har} corresponds to a ν-dimensional harmonic oscillator of rigidity a. The mass parameter m includes Planck's constant, that is,

$$m = m_{\mathrm{ph}}/\hbar^2, \tag{1.3}$$

where $m_{\rm ph}$ is the physical mass of the particle. Therefore, the commutation relation for the components of the momentum and displacement takes the form

$$p_\ell^{(j)} q_{\ell'}^{(j')} - q_{\ell'}^{(j')} p_\ell^{(j)} = -\imath \delta_{\ell\ell'}\delta_{jj'}, \quad j, j' = 1, \ldots, \nu. \tag{1.4}$$

For a detailed discussion on how to derive a model like (1.1), (1.2) from physical models of concrete substances, we refer the reader to the survey [72].

The theory of phase transitions is one of the most important and spectacular parts of equilibrium statistical mechanics. For classical lattice models, a complete description of the equilibrium thermodynamic properties is given by constructing their Gibbs states as probability measures on appropriate configuration spaces. Usually, it is made in the Dobrushin-Lanford-Ruelle (DLR) approach which is now well-elaborated, see Georgii's monograph [30] and the references therein. In general, the quantum case does not permit such a universal description. For some systems with bounded one-site Hamiltonians, e.g., quantum spin models, the Gibbs states are defined as positive normalized functionals on algebras of quasi-local observables obeyng the condition of equilibrium between the dynamic and thermodynamic behavior of the model (KMS condition), see [19]. However, this algebraic way cannot be applied to the model (1.1), (1.2) since the construction of its dynamics in the whole crystal $\mathbb{L}$ is beyond the technical possibilities available by this time. In 1975, an approach employing path integral methods to describe thermodynamic properties of models like (1.1), (1.2) has been initiated in [1]. Its main idea was to pass from real to imaginary values of time, similarly as it was done in Euclidean quantum field theory, see [31, 67], and thereby to describe the dynamics of the model in terms of stochastic processes. Afterwards, this approach, also called Euclidean, has been developed in a number of works. Its latest and most general version is presented in [53, 54], where the reader can also find an extensive bibliography on this subject. The methods developed in these works will be extensively used in the present study.

Phase transitions are very important phenomena in the substances modeled by the Hamiltonians (1.1), (1.2). According to their commonly adopted physical interpretation, at low temperatures the oscillations of the particles become strongly correlated that produces macroscopic ordering. The mathematical theory of phase transitions in models like (1.1), (1.2) is based on quantum versions of the method of infrared estimates developed in [28]. The first publication where the infrared estimates were applied to quantum spin models seems to be [24]. After certain modifications this method, combined with path integral techniques, was applied in [15, 16, 23, 44, 60] to particular versions of our model. The main characteristic feature of these versions was a symmetry, broken by the phase transition.

In classical systems, ordering is achieved in competition with thermal fluctuations only. However, in quantum systems quantum effects play a significant disordering role, especially at low temperatures. This role was first discussed in [65]. Later on a number of publications dedicated to the study of quantum effects in such systems had appeared, see e.g., [56, 82] and the references therein. For better understanding, illuminating exactly solvable models of systems of interacting quantum anharmonic oscillators were introduced and studied, see [62, 73, 80, 81]. In these works, the quantity $m^{-1} = \hbar^2/m_{\rm ph}$ was used as a parameter describing the rate of quantum effects. Such effects became strong in the small mass limit, which was in agreement with the experimental data, e.g., on the isotopic effect in the ferroelectrics with hydrogen bounds, see [18, 79], see also [57] for the data on the isotopic effect in the YBaCuO-type high-temperature superconductors. However, in those works no other quantum effects, e.g., those connected with special properties of the anharmonic potentials, were discussed. At the same time, experimental data, see e.g., the table on page 11 in the monograph [18] or the article [76], show that high hydrostatic pressure applied to KDP-type ferroelectrics prevents them from ordering. It is believed that the pressure shortens the hydrogen bounds and thereby changes the anharmonic potential. This makes the tunneling motion of the quantum particles more intensive, which is equivalent to diminishing the particle mass. In [5, 8, 9], a theory of such quantum effects in the model (1.1), (1.2), which explains both mentioned mechanisms, was buit up. Its main conclusion is that the quantum dynamical properties, which depend on the mass m, the interaction intensities $J_{\ell\ell'}$, and the anharmonic potentials V_ℓ, can be such that the model is stable with respect to phase transitions at all temperatures.

As was mentioned above, the aim of this article is to present a unified description of phase transitions and quantum stabilization in the model (1.1), (1.2), mostly by means of methods developed in [53, 54]. We also give here complete proofs of a number of statements announced in our previous publications. The article is organized as follows. In Section 2, we briefly describe those elements of the theory developed in [53, 54] which we then apply in the subsequent sections. In Section 3, we present the theory of phase transitions in the model (1.1), (1.2). We begin by introducing three definitions of a phase transition in this model and study the relationships between them. Then we develop a version of the method of infrared estimates adapted to our model, which is more transpatrent and appropriate than the one employed in [6]. Afterwards, we obtain a sufficient conditions for the phase transitions to occur in a number of versions of the model (1.1), (1.2). This includes also the case of asymmetric anharmonic potentials V_ℓ which was never studied before. At the end of the section we make some comments on the results obtained and compare them with similar results known in the literature. Section 4 is dedicated to the study of quantum stabilization, which we understand as the suppression of phase transitions by quantum effects. Here we discuss the problem of stability of quantum crystals and the ways of its description. In particular, we introduce a parameter (quantum rigidity), responsible for the stability and prove a number of statements about its properties. Then we show that under the stability condition which we introduce here the correlations decay 'in a proper way', that means the absence of phase transitions. The relationship between the quantum stabilization and phase transitions are also analyzed. In the simplest case, where the model is translation invariant, scalar ($\nu = 1$), and with the interaction of nearest neighbor type, this relation looks as follows. The key parameter is $8dmJ\vartheta_*^2$, where d is the lattice dimension, $J > 0$ is the interaction intensity, and $\vartheta_* > 0$ is determined by the anharmonic potential V (the steeper is V the smaller is ϑ_*). Then the quantum stabilization condition (respectively, the phase transition condition) is $8dmJ\vartheta_*^2 < 1$, see (4.32), (respectively, $8dmJ\vartheta_*^2 > \phi(d)$, see (3.71) and (4.33)). Here ϕ is a function, such that $\phi(d) > 1$ and $\phi(d) \to 1$ as $d \to +\infty$. We conclude the section by commenting the results obtained therein.

2 Euclidean Gibbs States

The main element of the Euclidean approach is the description of the equilibrium thermodynamic properties of the model (1.1), (1.2) by means of Euclidean Gibbs states, which are probability measures on certain configuration spaces. In this section, we briefly describe the main elements of this approach which are then used in the subsequent parts of the article. For more details, we refer to [54].

2.1 Local Gibbs states

Let us begin by specifying the properties of the model described by the Hamiltonian (1.1). The general assumptions regarding the interaction intensities $J_{\ell\ell'}$ are

$$J_{\ell\ell'} = J_{\ell'\ell} \geq 0, \quad J_{\ell\ell} = 0, \quad \hat{J}_0 \stackrel{\mathrm{def}}{=} \sup_{\ell} \sum_{\ell'} J_{\ell\ell'} < \infty. \tag{2.1}$$

In view of the first of these properties the model is *ferroelectric*. Regarding the anharmonic potentials we assume that each $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$ is a continuous function, which obeys

$$A_V |x|^{2r} + B_V \leq V_\ell(x) \leq V(x), \tag{2.2}$$

with a continuous function V and constants $r > 1$, $A_V > 0$, $B_V \in \mathbb{R}$. In certain cases, we shall include an external field term in the form

$$V_\ell(x) = V_\ell^0(x) - (h, x), \qquad h \in \mathbb{R}^\nu, \tag{2.3}$$

where V_ℓ^0 is an appropriate function.

Definition 2.1 *The model is translation invariant if $V_\ell = V$ for all ℓ, and the interaction intensities $J_{\ell\ell'}$ are invariant under the translations of $\mathbb{L}$. The model is rotation invariant if for every orthogonal transformation $U \in O(\nu)$ and every ℓ, $V_\ell(Ux) = V_\ell(x)$. The interaction has finite range if there exists $R > 0$ such that $J_{\ell\ell'} = 0$ whenever $|\ell - \ell'| > R$.*

If $V_\ell \equiv 0$ for all ℓ, one gets a quantum harmonic crystal. It is stable if $\hat{J}_0 < a$, see Remark 2.15 below.

By Λ we denote subsets of the lattice $\mathbb{L}$; we write $\Lambda \Subset \mathbb{L}$ if Λ is non-void and finite. For such Λ, by $|\Lambda|$ we denote its cardinality. A sequence of subsets $\Lambda \Subset \mathbb{L}$ is called *cofinal* if it is ordered by inclusion and exhausts the lattice $\mathbb{L}$. If we say that something holds for all ℓ, we mean it holds for all $\ell \in \mathbb{L}$; sums like $\sum_\ell$ mean $\sum_{\ell \in \mathbb{L}}$. We also use the notations $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}$ being the set of positive integers.

Given $\Lambda \Subset \mathbb{L}$, the local Hamiltonian of the model is

$$H_\Lambda = -\frac{1}{2} \sum_{\ell, \ell' \in \Lambda} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell \in \Lambda} H_\ell, \tag{2.4}$$

which by the assumptions made above is a self-adjoint and lower bounded operator in the physical Hilbert space $L^2(\mathbb{R}^{\nu|\Lambda|})$. For every $\beta = 1/k_{\mathrm{B}}T$, T being absolute temperature, the local Gibbs state in $\Lambda \Subset \mathbb{L}$ is

$$\varrho_\Lambda(A) = \mathrm{trace}[A \exp(-\beta H_\Lambda)]/Z_\Lambda, \quad A \in \mathfrak{C}_\Lambda, \tag{2.5}$$

where

$$Z_\Lambda = \mathrm{trace}[\exp(-\beta H_\Lambda)] < \infty \tag{2.6}$$

is the partition function, and $\mathfrak{C}_\Lambda$ is the algebra of all bounded linear operators on $L^2(\mathbb{R}^{\nu|\Lambda|})$. Note that adjective *local* will always stand for a property related with a certain $\Lambda \Subset \mathbb{L}$, whereas *global* will characterize the whole infinite system.

The dynamics of the subsystem located in Λ is described by the time automorphisms

$$\mathfrak{C}_\Lambda \ni A \mapsto \mathfrak{a}_t^\Lambda(A) = \exp(\imath t H_\Lambda) A \exp(-\imath t H_\Lambda), \tag{2.7}$$

where $t \in \mathbb{R}$ is time. Given $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathfrak{C}_\Lambda$, the corresponding *Green function* is

$$G^\Lambda_{A_1,\ldots A_n}(t_1, \ldots, t_n) = \varrho_\Lambda \left[\mathfrak{a}^\Lambda_{t_1}(A_1) \cdots \mathfrak{a}^\Lambda_{t_n}(A_n)\right], \tag{2.8}$$

which is a complex valued function on $\mathbb{R}^n$. Each such a function can be looked upon, see [1, 6], as the restriction of a function $G^\Lambda_{A_1,\ldots A_n}$ analytic in the domain

$$\mathcal{D}^n_\beta = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid 0 < \Im(z_1) < \cdots < \Im(z_n) < \beta\}, \tag{2.9}$$

and continuous on its closure. The corresponding statement is known as *the multiple-time analyticity theorem*, see [1, 6], as well as [41] for a more general consideration. For every $n \in \mathbb{N}$, the subset

$$\{(z_1, \ldots, z_n) \in \mathcal{D}^n_\beta \mid \Re(z_1) = \cdots = \Re(z_n) = 0\} \tag{2.10}$$

is an inner uniqueness set for functions analytic in $\mathcal{D}^n_\beta$, see pages 101 and 352 in [66]. This means that two such functions which coincide on this set should coincide everywhere on $\mathcal{D}^n_\beta$.

For a bounded continuous function $F : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$, the corresponding multiplication operator $F \in \mathfrak{C}_\Lambda$ acts as follows

$$(F\psi)(x) = F(x)\psi(x), \qquad \psi \in L^2(\mathbb{R}^{\nu|\Lambda|}).$$

Let $\mathfrak{F}_\Lambda \subset \mathfrak{C}_\Lambda$ be the set of all such operators. One can prove (the density theorem, see [51, 52]) that the linear span of the products

$$\mathfrak{a}^\Lambda_{t_1}(F_1) \cdots \mathfrak{a}^\Lambda_{t_n}(F_n),$$

with all possible choices of $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$, and $F_1, \ldots, F_n \in \mathfrak{F}_\Lambda$, is dense in $\mathfrak{C}_\Lambda$ in the σ-weak topology in which the state (2.5) is continuous as a linear functional. Thus, the latter is determined by the set of Green functions $G^\Lambda_{F_1,\ldots F_n}$ with $n \in \mathbb{N}$ and $F_1, \ldots, F_n \in \mathfrak{F}_\Lambda$. The restriction of the Green functions $G^\Lambda_{F_1,\ldots F_n}$ to the imaginary-time sets (2.10) are called *Matsubara functions.* For

$$\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \beta, \tag{2.11}$$

they are

$$\Gamma^\Lambda_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) = G^\Lambda_{F_1,\ldots,F_n}(\imath\tau_1, \ldots, \imath\tau_n). \tag{2.12}$$

Since (2.10) is an inner uniqueness set, the collection of the Matsubara functions (2.12) with all possible choices of $n \in \mathbb{N}$ and $F_1, \ldots, F_n \in \mathfrak{F}_\Lambda$ determines the state (2.5). The extensions of the functions (2.12) to $[0, \beta]^n$ are defined as

$$\Gamma^\Lambda_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) = \Gamma^\Lambda_{F_{\sigma(1)},\ldots,F_{\sigma(n)}}(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)})$$

where σ is the permutation such that $\tau_{\sigma(1)} \leq \tau_{\sigma(2)} \leq \cdots \leq \tau_{\sigma(n)}$. One can show that for every $\theta \in [0, \beta]$,

$$\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1 + \theta, \ldots, \tau_n + \theta) = \Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n), \tag{2.13}$$

where addtion is modulo β.

2.2 Path spaces

By (2.8), the Matsubara function (2.12) can be written as

$$\begin{aligned} \Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) &= \\ &= \mathrm{trace}\left[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda} \cdot F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}\right]/Z_\Lambda, \end{aligned} \tag{2.14}$$

where $\tau_{n+1} = \beta + \tau_1$ and the arguments obey (2.11). This expression can be rewritten in an integral form

$$\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) = \int_{\Omega_\Lambda} F_1(\omega_\Lambda(\tau_1)) \cdots F_n(\omega_\Lambda(\tau_n))\nu_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2.15}$$

that is the main point of the Euclidean approach. Here ν_Λ is a probability measure on the path space Ω_Λ which we introduce now. The main single-site path space is the space of continuous periodic paths (temperature loops)

$$C_\beta = \{\phi \in C([0, \beta] \to \mathbb{R}^\nu) \mid \phi(0) = \phi(\beta)\}. \tag{2.16}$$

It is a Banach space with the usual sup-norm $\|\cdot\|_{C_\beta}$. For an appropriate $\phi \in C_\beta$, we set

$$K_\sigma(\phi) = \beta^\sigma \cdot \sup_{\tau,\tau' \in [0,\beta]\ \tau \neq \tau'} \frac{|\phi(\tau) - \phi(\tau')|}{|\tau - \tau'|^\sigma_\beta}, \quad \sigma > 0, \tag{2.17}$$

where

$$|\tau - \tau'|_\beta = \min\{|\tau - \tau'|; \beta - |\tau - \tau'|\} \tag{2.18}$$

is the periodic distance on the circle $S_\beta \sim [0, \beta]$. Then the set of Hölder-continuous periodic functions,

$$C^\sigma_\beta = \{\phi \in C_\beta \mid K_\sigma(\phi) < \infty\}, \tag{2.19}$$

can be equipped with the norm

$$\|\phi\|_{C^\sigma_\beta} = |\phi(0)| + K_\sigma(\phi), \tag{2.20}$$

which turns it into a Banach space. Along with the spaces C_β, C^σ_β, we shall use the Hilbert space $L^2_\beta = L^2(S_\beta \to \mathbb{R}^\nu, \mathrm{d}\tau)$, equipped with the inner product $(\cdot, \cdot)_{L^2_\beta}$ and norm $\|\cdot\|_{L^2_\beta}$. By $\mathcal{B}(C_\beta)$, $\mathcal{B}(L^2_\beta)$ we denote the corresponding Borel σ-algebras. In a standard way, see page 21 of [59] and the corresponding discussion in [54], it follows that

$$C_\beta \in \mathcal{B}(L^2_\beta) \quad \text{and} \quad \mathcal{B}(C_\beta) = \mathcal{B}(L^2_\beta) \cap C_\beta. \tag{2.21}$$

Given $\Lambda \subseteq \mathbb{L}$, we set

$$\begin{aligned} \Omega_\Lambda &= \{\omega_\Lambda = (\omega_\ell)_{\ell \in \Lambda} \mid \omega_\ell \in C_\beta\}, \\ \Omega = \Omega_{\mathbb{L}} &= \{\omega = (\omega_\ell)_{\ell \in \mathbb{L}} \mid \omega_\ell \in C_\beta\}. \end{aligned} \tag{2.22}$$

These path spaces are equipped with the product topology and with the Borel σ-algebras $\mathcal{B}(\Omega_\Lambda)$. Thereby, each Ω_Λ is a complete separable metric space, called *Polish space*, its elements are called *configurations in* Λ. For $\Lambda \subset \Lambda'$, the juxtaposition $\omega_{\Lambda'} = \omega_\Lambda \times \omega_{\Lambda' \setminus \Lambda}$ defines an embedding $\Omega_\Lambda \hookrightarrow \Omega_{\Lambda'}$ by identifying $\omega_\Lambda \in \Omega_\Lambda$ with $\omega_\Lambda \times 0_{\Lambda' \setminus \Lambda} \in \Omega_{\Lambda'}$. By $\mathcal{P}(\Omega_\Lambda)$, $\mathcal{P}(\Omega)$ we denote the sets of all probability measures on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$, $(\Omega, \mathcal{B}(\Omega))$ respectively.

2.3 Local Euclidean Gibbs measures

Now we construct the measure ν_Λ which appears in (2.15). A single harmonic oscillator is described by the Hamiltonian, c.f., (1.2),

$$H_\ell^{\rm har} = -\frac{1}{2m} \sum_{j=1}^{\nu} \left(\frac{\partial}{\partial x_\ell^{(j)}} \right)^2 + \frac{a}{2} |x_\ell|^2. \tag{2.23}$$

It is a self-adjoint operator in the space $L^2(\mathbb{R}^\nu)$, the properties of which are well-known. The operator semigroup $\exp(-\tau H_\ell^{\rm har})$, $\tau \in S_\beta$, defines a β-periodic Markov process, see [42]. In quantum statistical mechanics, it first appeared in R. Høegh-Krohn's paper [35]. The canonical realization of this process on $(C_\beta, \mathcal{B}(C_\beta))$ is described by the path measure which can be introduced as follows. In the space L^2_β, we define the following self-adjoint Laplace-Beltrami type operator

$$A = \left(-m \frac{{\rm d}^2}{{\rm d}\tau^2} + a \right) \otimes \mathbf{I}, \tag{2.24}$$

where $\mathbf{I}$ is the identity operator in $\mathbb{R}^\nu$. Its spectrum consists of the eigenvalues

$$\lambda_l = m (2\pi l/\beta)^2 + a, \qquad l \in \mathbb{Z}. \tag{2.25}$$

Therefore, the inverse A^{-1} is a trace-class operator on L^2_β and the Fourier transform

$$\int_{L^2_\beta} \exp\left[\imath (\psi, \phi)_{L^2_\beta} \right] \chi({\rm d}\phi) = \exp\left\{ -\frac{1}{2} (A^{-1}\psi, \psi)_{L^2_\beta} \right\} \tag{2.26}$$

defines a zero mean Gaussian measure χ on $(L^2_\beta, \mathcal{B}(L^2_\beta))$. Employing the eigenvalues (2.25) one can show that, for any $p \in \mathbb{N}$,

$$\int_{C_\beta} |\omega(\tau) - \omega(\tau')|^{2p} \chi({\rm d}\omega) \leq \frac{\Gamma(\nu/2 + p)}{\Gamma(\nu/2)} \left(\frac{2}{m} \right)^p \cdot |\tau - \tau'|^p_\beta. \tag{2.27}$$

Therefrom, by Kolmogorov's lemma (see page 43 of [68]) it follows that

$$\chi(C^\sigma_\beta) = 1, \qquad \text{for all} \ \ \sigma \in (0, 1/2). \tag{2.28}$$

Thereby, $\chi(C_\beta) = 1$; hence, with the help of (2.21) we redefine χ as a measure on $(C_\beta, \mathcal{B}(C_\beta))$, possessing the property (2.28). We shall call it *Høegh-Krohn's measure*. An account of the properties of χ can be found in [6]. Here we present the following two of them. The first property is obtained directly from Fernique's theorem (see Theorem 1.3.24 in [22]).

Proposition 2.2 (Fernique) *For every $\sigma \in (0, 1/2)$, there exists $\lambda_\sigma > 0$, which can be estimated explicitely, such that*

$$\int_{L^2_\beta} \exp\left(\lambda_\sigma \|\phi\|^2_{C^\sigma_\beta}\right) \chi(\mathrm{d}\phi) < \infty. \tag{2.29}$$

The second property follows from the estimate (2.27) by the Garsia-Rodemich-Rumsey lemma, see [29]. For fixed $\sigma \in (0, 1/2)$, we set

$$\Xi_\vartheta(\omega) = \sup_{\tau,\tau':\ 0<|\tau-\tau'|_\beta<\vartheta} \left\{ \frac{|\omega(\tau) - \omega(\tau')|^2}{|\tau - \tau'|^{2\sigma}_\beta} \right\}, \qquad \vartheta \in (0, \beta/2), \quad \omega \in C^\sigma_\beta. \tag{2.30}$$

One can show that, for each σ and ϑ, it can be extended to a measurable map $\Xi_\vartheta : C_\beta \to [0, +\infty]$.

Proposition 2.3 (Garsia-Rodemich-Rumsey estimate) *Given $\sigma \in (0, 1/2)$, let $p \in \mathbb{N}$ be such that $(p-1)/2p > \sigma$. Then*

$$\int_{C_\beta} \Xi^p_\vartheta(\omega)\chi(\mathrm{d}\omega) \leq D(\sigma, p, \nu) m^{-p} \vartheta^{p(1-2\sigma)}, \tag{2.31}$$

where m is the mass (1.3) and

$$D(\sigma, p, \nu) = \frac{2^{3(2p+1)}(1 + 1/\sigma p)^{2p}}{(p - 1 - 2\sigma p)(p - 2\sigma p)} \cdot \frac{2^p \Gamma(\nu/2 + 1)}{\Gamma(\nu/2)}. \tag{2.32}$$

The Høegh-Krohn measure is the local Euclidean Gibbs measure for a single harmonic oscillator. The measure $\nu_\Lambda \in \mathcal{P}(\Omega_\Lambda)$, which is the Euclidean Gibbs measure corresponding to the system of interacting anharmonic oscillators located in $\Lambda \Subset \mathbb{L}$, is defined by means of the Feynman-Kac formula as a Gibbs modification

$$\nu_\Lambda(\mathrm{d}\omega_\Lambda) = \exp\left[-I_\Lambda(\omega_\Lambda)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda)/N_\Lambda \tag{2.33}$$

of the 'free measure'

$$\chi_\Lambda(\mathrm{d}\omega_\Lambda) = \prod_{\ell\in\Lambda} \chi(\mathrm{d}\omega_\ell). \tag{2.34}$$

Here

$$I_\Lambda(\omega_\Lambda) = -\frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda} \int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau \tag{2.35}$$

is *the energy functional* which describes the interaction of the paths ω_ℓ, $\ell \in \Lambda$. The normalizing factor

$$N_\Lambda = \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda) \tag{2.36}$$

is the relative partition function, whereas the Feynman-Kac representation of the partition function (2.6) is

$$Z_\Lambda = N_\Lambda Z^{\mathrm{har}}_\Lambda, \tag{2.37}$$

where

$$\begin{aligned} Z_\Lambda^{\rm har} &\stackrel{\rm def}{=} \ {\rm trace}\exp\left[-\beta\sum_{\ell\in\Lambda} H_\ell^{\rm har}\right] \\ &= \left\{\frac{\exp\left[-(\beta/2)\sqrt{a/m}\right]}{1-\exp\left(-\beta\sqrt{a/m}\right)}\right\}^{\nu|\Lambda|}. \end{aligned}$$

Now let us summarize the connections between the description of the subsystem located in $\Lambda \Subset \mathbb{L}$ in terms of the states (2.5) and of the Euclidean Gibbs measures (2.33). By the density theorem, the state ϱ_Λ is fully determined by the Green functions (2.8) corresponding to all choices of $n \in \mathbb{N}$ and $F_1, \dots, F_n \in \mathfrak{F}_\Lambda$. Then the multiple-time analyticity theorem leads us from the Green functions to the Matsubara functions (2.12), which then are represented as integrals over path spaces with respect to the local Euclidean Gibbs measures, see (2.15). On the other hand, these integrals taken for all possible choices of bounded continuous functions $F_1, \dots, F_n$ fully determine the measure ν_Λ. Thereby, we have a one-to-one correspondence between the local Gibbs states (2.5) and the states on the algebras of bounded continuous functions determined by the local Euclidean Gibbs measures (2.33). Our next aim is to extend this approach to the global states. To this end we make more precise the definition of the path spaces in infinite Λ, e.g., in $\Lambda = \mathbb{L}$.

2.4 Tempered configurations

To describe the global thermodynamic properties we need the conditional distributions $\pi_\Lambda(\mathrm{d}\omega|\xi)$, $\Lambda \Subset \mathbb{L}$. For models with infinite-range interactions, the construction of such distributions is a nontrivial problem, which can be solved by imposing a priori restrictions on the configurations defining the corresponding conditions. In this and in the subsequent subsections, we present the construction of such distributions performed [54].

The distributions $\pi_\Lambda(\mathrm{d}\omega|\xi)$ are defined by means of the energy functionals $I_\Lambda(\omega|\xi)$ describing the interaction of the configuration ω with the configuration ξ, fixed outside of Λ. Given $\Lambda \Subset \mathbb{L}$, such a functional is

$$I_\Lambda(\omega|\xi) = I_\Lambda(\omega_\Lambda) - \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta}, \quad \omega \in \Omega, \tag{2.38}$$

where I_Λ is given by (2.35). Recall that $\omega = \omega_\Lambda \times \omega_{\Lambda^c}$; hence,

$$I_\Lambda(\omega|\xi) = I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|0_\Lambda \times \xi_{\Lambda^c}). \tag{2.39}$$

The second term in (2.38) makes sense for all $\xi \in \Omega$ only if the interaction has finite range, see Definition 2.1. Otherwise, one has to impose appropriate restrictions on the configurations ξ, such that, for all ℓ and $\omega \in \Omega$,

$$\sum_{\ell'} J_{\ell\ell'} \cdot |(\omega_\ell, \xi_{\ell'})_{L^2_\beta}| < \infty. \tag{2.40}$$

These restrictions are formulated by means of special mappings (weights), which define the scale of growth of $\{\|\xi_\ell\|_{L^2_\beta}\}_{\ell\in\mathbb{L}}$. Their choice depends on the asymptotic properties of $J_{\ell\ell'}$, $|\ell-\ell'|\to+\infty$, see (2.1). If for a certain $\alpha>0$,

$$\sup_{\ell}\sum_{\ell'} J_{\ell\ell'}\exp(\alpha|\ell-\ell'|)<\infty, \tag{2.41}$$

then the weights $\{w_\alpha(\ell,\ell')\}_{\alpha\in\mathcal{I}}$ are chosen as

$$w_\alpha(\ell,\ell')=\exp(-\alpha|\ell-\ell'|), \qquad \mathcal{I}=(0,\overline{\alpha}), \tag{2.42}$$

where $\overline{\alpha}$ is the supremum of $\alpha>0$, for which (2.41) holds. If the latter condition does not hold for any $\alpha>0$, we assume that

$$\sup_{\ell}\sum_{\ell'} J_{\ell\ell'}\cdot(1+|\ell-\ell'|)^{\alpha d}, \tag{2.43}$$

for a certain $\alpha>1$. Then we set $\overline{\alpha}$ to be the supremum of $\alpha>1$ obeying (2.43) and

$$w_\alpha(\ell,\ell')=(1+\varepsilon|\ell-\ell'|)^{-\alpha d}, \tag{2.44}$$

where $\varepsilon>0$ is a technical parameter. In the sequel, we restrict ourselves to these two kinds of $J_{\ell\ell'}$. For more details on this item, we refer the reader to [54].

Given $\alpha\in\mathcal{I}$ and $\omega\in\Omega$, we set

$$\|\omega\|_\alpha=\left[\sum_{\ell}\|\omega_\ell\|^2_{L^2_\beta}w_\alpha(0,\ell)\right]^{1/2}, \tag{2.45}$$

and

$$\Omega_\alpha=\{\omega\in\Omega \mid \|\omega\|_\alpha<\infty\}. \tag{2.46}$$

Thereby, we endow Ω_α with the metric

$$\rho_\alpha(\omega,\omega')=\|\omega-\omega'\|_\alpha+\sum_{\ell}2^{-|\ell|}\frac{\|\omega_\ell-\omega'_\ell\|_{C_\beta}}{1+\|\omega_\ell-\omega'_\ell\|_{C_\beta}}, \tag{2.47}$$

which turns it into a Polish space. The set of tempered configurations is defined to be

$$\Omega^{\rm t}=\bigcap_{\alpha\in\mathcal{I}}\Omega_\alpha. \tag{2.48}$$

We endow it with the projective limit topology, which turns it into a Polish space as well. For every $\alpha\in\mathcal{I}$, the embeddings $\Omega^{\rm t}\hookrightarrow\Omega_\alpha\hookrightarrow\Omega$ are continuous; hence, $\Omega_\alpha,\Omega^{\rm t}\in\mathcal{B}(\Omega)$ and the Borel σ-algebras $\mathcal{B}(\Omega_\alpha)$, $\mathcal{B}(\Omega^{\rm t})$ coincide with the ones induced on them by $\mathcal{B}(\Omega)$.

2.5 Local Gibbs specification

Let us turn to the functional (2.38). By standard methods, one proves that, for every $\alpha\in\mathcal{I}$, the map $\Omega_\alpha\times\Omega_\alpha\mapsto I_\Lambda(\omega|\xi)$ is continuous. Furthermore, for any ball $B_\alpha(R)=\{\omega\in\Omega_\alpha \mid \rho_\alpha(0,\omega)<R\}$, $R>0$, one has

$$\inf_{\omega\in\Omega,\ \xi\in B_\alpha(R)} I_\Lambda(\omega|\xi)>-\infty, \qquad \sup_{\omega,\xi\in B_\alpha(R)}|I_\Lambda(\omega|\xi)|<+\infty.$$

Therefore, for $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, the conditional relative partition function

$$N_\Lambda(\xi) = \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda) \tag{2.49}$$

is continuous in ξ. Furthermore, for any $R > 0$ and $\alpha \in \mathcal{I}$,

$$\inf_{\xi \in B_\alpha(R)} N_\Lambda(\xi) > 0.$$

For such ξ and Λ, and for $B \in \mathcal{B}(\Omega)$, we set

$$\pi_\Lambda(B|\xi) = \frac{1}{N_\Lambda(\xi)} \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \mathbb{I}_B(\omega_\Lambda \times \xi_{\Lambda^c}) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2.50}$$

where $\mathbb{I}_B$ stands for the indicator of B. We also set

$$\pi_\Lambda(\cdot|\xi) \equiv 0, \quad \text{for} \ \ \xi \in \Omega \setminus \Omega^{\mathrm{t}}. \tag{2.51}$$

From these definitions one readily derives a consistency property

$$\int_\Omega \pi_\Lambda(B|\omega)\pi_{\Lambda'}(\mathrm{d}\omega|\xi) = \pi_{\Lambda'}(B|\xi), \quad \Lambda \subset \Lambda', \tag{2.52}$$

which holds for all $B \in \mathcal{B}(\Omega)$ and $\xi \in \Omega$.

The local Gibbs specification is the family $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$. Each π_Λ is a measure kernel, which means that, for a fixed $\xi \in \Omega$, $\pi(\cdot|\xi)$ is a measure on $(\Omega, \mathcal{B}(\Omega))$, which is a probability measure whenever $\xi \in \Omega^{\mathrm{t}}$. For any $B \in \mathcal{B}(\Omega)$, $\pi_\Lambda(B|\cdot)$ is $\mathcal{B}(\Omega)$-measurable.

By $C_{\mathrm{b}}(\Omega_\alpha)$ (respectively, $C_{\mathrm{b}}(\Omega^{\mathrm{t}})$) we denote the Banach spaces of all bounded continuous functions $f : \Omega_\alpha \to \mathbb{R}$ (respectively, $f : \Omega^{\mathrm{t}} \to \mathbb{R}$) equipped with the supremum norm. For every $\alpha \in \mathcal{I}$, one has a natural embedding $C_{\mathrm{b}}(\Omega_\alpha) \hookrightarrow C_{\mathrm{b}}(\Omega^{\mathrm{t}})$. Given $\alpha \in \mathcal{I}$, by $\mathcal{W}_\alpha$ we denote the usual weak topology on the set of all probability measures $\mathcal{P}(\Omega_\alpha)$ defined by means of $C_{\mathrm{b}}(\Omega_\alpha)$. By $\mathcal{W}^{\mathrm{t}}$ we denote the weak topology on $\mathcal{P}(\Omega^{\mathrm{t}})$. With these topologies the sets $\mathcal{P}(\Omega_\alpha)$ and $\mathcal{P}(\Omega^{\mathrm{t}})$ become Polish spaces (Theorem 6.5, page 46 of [59]).

By standard methods one proves the following, see Lemma 2.10 in [54],

Proposition 2.4 (Feller Property) *For every $\alpha \in \mathcal{I}$, $\Lambda \Subset \mathbb{L}$, and any $f \in C_{\mathrm{b}}(\Omega_\alpha)$, the function*

$$\begin{aligned} \Omega_\alpha \ni \xi \mapsto & \ \pi_\Lambda(f|\xi) \\ \stackrel{\mathrm{def}}{=} & \ \frac{1}{N_\Lambda(\xi)} \int_{\Omega_\Lambda} f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{2.53}$$

belongs to $C_{\mathrm{b}}(\Omega_\alpha)$. The linear operator $f \mapsto \pi_\Lambda(f|\cdot)$ is a contraction on $C_{\mathrm{b}}(\Omega_\alpha)$.

Note that by (2.50), for $\xi \in \Omega^{\mathrm{t}}$, $\alpha \in \mathcal{I}$, and $f \in C_{\mathrm{b}}(\Omega_\alpha)$,

$$\pi_\Lambda(f|\xi) = \int_\Omega f(\omega)\pi_\Lambda(\mathrm{d}\omega|\xi). \tag{2.54}$$

Recall that the particular cases of our model were specified by Definition 2.1. For $B \in \mathcal{B}(\Omega)$ and $U \in O(\nu)$, we set

$$U\omega = (U\omega_\ell)_{\ell \in \mathbb{L}} \qquad UB = \{U\omega \mid \omega \in B\}.$$

Furthermore, for a given ℓ_0, we set

$$t_{\ell_0}(\omega) = (\omega_{\ell-\ell_0})_{\ell\in\mathbb{L}}, \qquad t_{\ell_0}(B) = \{t_{\ell_0}(\omega) \mid \omega \in B\}.$$

Then if the model possesses the corresponding symmetry, one has

$$\pi_\Lambda(UB|U\xi) = \pi_\Lambda(B|\xi), \qquad \pi_{\Lambda+\ell}(t_\ell(B)|t_\ell(\xi)) = \pi_\Lambda(B|\xi), \tag{2.55}$$

which ought to hold for all U, ℓ, B, and ξ.

2.6 Tempered Euclidean Gibbs measures

Definition 2.5 *A measure $\mu \in \mathcal{P}(\Omega)$ is called a tempered Euclidean Gibbs measure if it satisfies the Dobrushin-Lanford-Ruelle (equilibrium) equation*

$$\int_\Omega \pi_\Lambda(B|\omega)\mu(\mathrm{d}\omega) = \mu(B), \quad \text{for all} \quad \Lambda \Subset \mathbb{L} \quad \text{and} \quad B \in \mathcal{B}(\Omega). \tag{2.56}$$

By $\mathcal{G}^{\mathrm{t}}$ we denote the set of all tempered Euclidean Gibbs measures of our model existing at a given β. The elements of $\mathcal{G}^{\mathrm{t}}$ are supported by Ω^{t}. Indeed, by (2.50) and (2.51) $\pi_\Lambda(\Omega \setminus \Omega^{\mathrm{t}}|\xi) = 0$ for every $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega$. Then by (2.56),

$$\mu(\Omega \setminus \Omega^{\mathrm{t}}) = 0. \tag{2.57}$$

Furthermore,

$$\mu\left(\left\{\omega \in \Omega^{\mathrm{t}} \mid \forall \ell \in \mathbb{L}: \ \omega_\ell \in C_\beta^\sigma\right\}\right) = 1, \tag{2.58}$$

which follows from (2.28), (2.29). If the model is translation and/or rotation invariant, then, for every $U \in O(\nu)$ and $\ell \in \mathbb{L}$, the corresponding transformations preserve $\mathcal{G}^{\mathrm{t}}$. That is, for any $\mu \in \mathcal{G}^{\mathrm{t}}$,

$$\Theta_U(\mu) \stackrel{\mathrm{def}}{=} \mu \circ U^{-1} \in \mathcal{G}^{\mathrm{t}}, \qquad \theta_\ell(\mu) \stackrel{\mathrm{def}}{=} \mu \circ t_\ell^{-1} \in \mathcal{G}^{\mathrm{t}}. \tag{2.59}$$

In particular, if $\mathcal{G}^{\mathrm{t}}$ is a singleton, its unique element should be invariant in the same sense as the model. From Proposition 2.4 one readily gets the following important fact.

Proposition 2.6 *For each $\alpha \in \mathcal{I}$, every $\mathcal{W}_\alpha$-accumulation point $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ of the family $\{\pi_\Lambda(\cdot|\xi) \mid \Lambda \Subset \mathbb{L}, \ \xi \in \Omega^{\mathrm{t}}\}$ is an element of $\mathcal{G}^{\mathrm{t}}$.*

Now let us pay some attention to the case where the model (1.1), (1.2) is translation invariant. Recall that the lattice $\mathbb{L} = \mathbb{Z}^d$ is considered as an additive group. For $\ell_0 \in \mathbb{L}$, $\Lambda \Subset \mathbb{L}$, and $\omega \in \Omega$, we set

$$\Lambda + \ell_0 = \{\ell + \ell_0 \mid \ell \in \Lambda\}; \quad t_{\ell_0}(\omega) = (\xi_\ell^{\ell_0})_{\ell\in\mathbb{L}}, \quad \xi_\ell^{\ell_0} = \omega_{\ell-\ell_0}. \tag{2.60}$$

Furthermore, for $B \in \mathcal{B}(\Omega)$, we set

$$t_\ell(B) = \{t_\ell(\omega) \mid \omega \in B\}. \tag{2.61}$$

Clearly, $t_\ell(B) \in \mathcal{B}(\Omega)$ and $t_\ell(\Omega^{\mathrm{t}}) = \Omega^{\mathrm{t}}$ for all ℓ.

Definition 2.7 *A probability measure $\mu \in \mathcal{P}(\Omega)$ is said to be translation invariant if for every ℓ and $B \in \mathcal{B}(\Omega)$, one has $\mu(t_\ell(B)) = \mu(B)$.*

As was mentioned above, the Gibbs specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ of the translation invariant model is translation invariant, that is, it has the property (2.55).

Remark 2.8 *The translation invariance of the Gibbs specification does not mean that each probability kernel π_Λ as a measure is translation invariant. Moreover, it does not mean that all the Euclidean Gibbs measures defined by this specification are translation invariant. One can only claim that if the set $\mathcal{G}^{\rm t}$ consists of one element only, this element ought to translation invariant.*

Set

$$\mathcal{B}^{\rm inv} = \{B \in \mathcal{B}(\Omega) \mid \forall \ell : \quad t_\ell(B) = B\}, \tag{2.62}$$

which is the set of all translation invariant events. By construction, $\Omega^{\rm t} \in \mathcal{B}^{\rm inv}$. We say that $\mu \in \mathcal{P}(\Omega)$ is trivial on $\mathcal{B}^{\rm inv}$ if for every $B \in \mathcal{B}^{\rm inv}$, one has $\mu(B) = 0$ or $\mu(B) = 1$. By $\mathcal{P}^{\rm inv}(\Omega)$ we denote the set of translation invariant probability measures on $(\Omega, \mathcal{B})$.

Definition 2.9 *A probability measure $\mu \in \mathcal{P}^{\rm inv}(\Omega)$ is said to be ergodic (with respect to the group $\mathbb{L}$) if it is trivial on $\mathcal{B}^{\rm inv}(\Omega)$.*

Ergodic measures are characterized by a mixing property, which we formulate here according to [69], see Theorem III.1.8 on page 244. For $L \in \mathbb{N}$, we set

$$\Lambda_L = (-L, L]^d \cap \mathbb{Z}^d, \tag{2.63}$$

which is called *a box*. For a measure μ and an appropriate function f, we write

$$\langle f \rangle_\mu = \int f {\rm d}\mu \tag{2.64}$$

Proposition 2.10 (Von Neumann Ergodic Theorem) *Given $\mu \in \mathcal{P}^{\rm inv}(\Omega)$, the following statements are equivalent:*

(i) μ *is ergodic;*

(ii) *for all $f, g \in L^2(\Omega, \mu)$,*

$$\lim_{L \to +\infty} \frac{1}{|\Lambda_L|} \left\{ \sum_{\ell \in \Lambda_L} \left(\int_\Omega f(\omega) g(t_\ell(\omega)) \mu({\rm d}\omega) - \langle f \rangle_\mu \cdot \langle g \rangle_\mu \right) \right\} = 0. \tag{2.65}$$

Proposition 2.11 *If the model is translation invariant and $\mathcal{G}^{\rm t}$ is a singleton, its unique element is ergodic.*

Now we give a number of statements describing the properties of $\mathcal{G}^{\rm t}$. More details can be found in [54].

Proposition 2.12 *For every $\beta > 0$, the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\rm t}$ is non-void, convex, and $\mathcal{W}^{\rm t}$- compact.*

Recall that the Hölder norm $\| \cdot \|_{C^\sigma_\beta}$ was defined by (2.20).

Proposition 2.13 *For every $\sigma \in (0, 1/2)$ and $\varkappa > 0$, there exists a positive constant C such that, for any ℓ and for all $\mu \in \mathcal{G}^{\rm t}$,*

$$\int_\Omega \exp\left(\lambda_\sigma \|\omega_\ell\|^2_{C^\sigma_\beta} + \varkappa \|\omega_\ell\|^2_{L^2_\beta} \right) \mu({\rm d}\omega) \leq C, \tag{2.66}$$

where λ_σ is the same as in (2.29).

In view of (2.66), the one-site projections of each $\mu \in \mathcal{G}^{\rm t}$ are sub-Gaussian. The constant C does not depend on ℓ and is the same for all $\mu \in \mathcal{G}^{\rm t}$, though it may depend on σ and $\varkappa$. The estimate (2.66) plays a crucial role in the theory of the set $\mathcal{G}^{\rm t}$.

According to [30] certain Gibbs states correspond to the thermodynamic phases of the underlying physical system. Thus, in our context multiple phases exist only if $\mathcal{G}^{\rm t}$ has more than one element for appropriate values of β and the model parameters. On the other hand, a priori one cannot exclude that this set always has multiple elements, which would make it useless for describing phase transitions. The next statement which we present here[2] clarifies the situation. Let us decompose

$$V_\ell = V_{1,\ell} + V_{2,\ell}, \tag{2.67}$$

where $V_{1,\ell} \in C^2(\mathbb{R}^\nu)$ is such that

$$-a \leq b \stackrel{\rm def}{=} \inf_\ell \inf_{x,y\in\mathbb{R}^\nu,\ y\neq 0} \left(V''_{1,\ell}(x)y, y\right)/|y|^2 < \infty. \tag{2.68}$$

As for the second term, we set

$$0 \leq \delta \stackrel{\rm def}{=} \sup_\ell \left\{ \sup_{x\in\mathbb{R}^\nu} V_{2,\ell}(x) - \inf_{x\in\mathbb{R}^\nu} V_{2,\ell}(x) \right\} \leq \infty. \tag{2.69}$$

Its role is to produce multiple minima of the potential energy responsible for eventual phase transitions. Clearly, the decomposition (2.67) is not unique; its optimal realizations for certain types of V_ℓ are discussed in section 6 of [13]. Recall that the interaction parameter $\hat{J}_0$ was defined in (2.1).

Proposition 2.14 *The set $\mathcal{G}^{\rm t}$ is a singleton if*

$$e^{\beta\delta} < (a+b)/\hat{J}_0. \tag{2.70}$$

Remark 2.15 *The latter condition surely holds at all β if*

$$\delta = 0 \quad \text{and} \quad \hat{J}_0 < a + b. \tag{2.71}$$

If the oscillators are harmonic, $\delta = b = 0$, which yields the stability condition

$$\hat{J}_0 < a. \tag{2.72}$$

The condition (2.70) does not contain the particle mass m; hence, the property stated holds also in the quasi-classical limit[3] $m \to +\infty$.

By the end of this subsection we consider the scalar case $\nu = 1$. Let us introduce the following order on $\mathcal{G}^{\rm t}$. As the components of the configurations $\omega \in \Omega$ are continuous functions $\omega_\ell : S_\beta \to \mathbb{R}^\nu$, one can set $\omega \leq \tilde{\omega}$ if $\omega_\ell(\tau) \leq \tilde{\omega}_\ell(\tau)$ for all ℓ and τ. Thereby,

$$K_+(\Omega^{\rm t}) \stackrel{\rm def}{=} \{f \in C_{\rm b}(\Omega^{\rm t}) \mid f(\omega) \leq f(\tilde{\omega}), \quad \text{if} \ \ \omega \leq \tilde{\omega}\}, \tag{2.73}$$

which is a cone of bounded continuous functions.

[2] C.f., Theorem 3.4 in [54], Theorem 2.1 in [14], and Theorem 4.1 in [13].
[3] More details on this limit can be found in [5].

Proposition 2.16 *If for given $\mu, \tilde{\mu} \in \mathcal{G}^{\rm t}$, one has*

$$\langle f \rangle_\mu = \langle f \rangle_{\tilde{\mu}}, \qquad \text{for all} \quad f \in K_+(\Omega^{\rm t}), \tag{2.74}$$

then $\mu = \tilde{\mu}$.

This fact allows for introducing the FKG-order.

Definition 2.17 *For $\mu, \tilde{\mu} \in \mathcal{G}^{\rm t}$, we say that $\mu \leq \tilde{\mu}$, if*

$$\langle f \rangle_\mu \leq \langle f \rangle_{\tilde{\mu}}, \qquad \text{for all} \quad f \in K_+(\Omega^{\rm t}). \tag{2.75}$$

Proposition 2.18 *The set $\mathcal{G}^{\rm t}$ possesses maximal μ_+ and minimal μ_- elements in the sense of Definition 2.17. These elements are extreme; they also are translation invariant if the model is translation invariant. If $V_\ell(-x) = V_\ell(x)$ for all ℓ, then $\mu_+(B) = \mu_-(-B)$ for all $B \in \mathcal{B}(\Omega)$.*

The proof of this statement follows from the fact that, for $f \in K_+(\Omega^{\rm t})$ and any $\Lambda \Subset \mathbb{L}$,

$$\langle f \rangle_{\pi_\Lambda(\cdot|\xi)} \leq \langle f \rangle_{\pi_\Lambda(\cdot|\xi')}, \qquad \text{whenever} \quad \xi \leq \xi', \tag{2.76}$$

which one obtaines by the FKG inequality, see [54]. By means of this inequality, one also proves the following

Proposition 2.19 *The family $\{\pi_\Lambda(\cdot|0)\}_{\Lambda \Subset \mathbb{L}}$ has only one $\mathcal{W}^{\rm t}$-accumulation point, μ_0, which is an element of $\mathcal{G}^{\rm t}$.*

2.7 Periodic Euclidean Gibbs measures

If the model is translation invariant, there should exist $\phi : \mathbb{N}_0^d \to \mathbb{R}^+$ such that

$$J_{\ell\ell'} = \phi(|\ell_1 - \ell'_1|, \ldots, |\ell_d - \ell'_d|). \tag{2.77}$$

For the box (2.63), we set

$$J^\Lambda_{\ell\ell'} \stackrel{\rm def}{=} \phi(|\ell_1 - \ell'_1|_L, \ldots, |\ell_d - \ell'_d|_L), \tag{2.78}$$

where

$$|\ell_j - \ell'_j|_L \stackrel{\rm def}{=} \min\{|\ell_j - \ell'_j| \ ; \ L - |\ell_j - \ell'_j|\}, \quad j = 1, \ldots, d. \tag{2.79}$$

For $\ell, \ell' \in \Lambda$, we introduce the periodic distance

$$|\ell - \ell'|_\Lambda = \sqrt{|\ell_1 - \ell'_1|^2_L + \cdots + |\ell_d - \ell'_d|^2_L}. \tag{2.80}$$

With this distance the box Λ turns into a torus, which one can obtained by imposing periodic conditions on its boundaries. Now we set, c.f., (2.35),

$$I^{\rm per}_\Lambda(\omega_\Lambda) = -\frac{1}{2}\sum_{\ell,\ell' \in \Lambda} J^\Lambda_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell \in \Lambda}\int_0^\beta V_\ell(\omega_\ell(\tau)){\rm d}\tau, \tag{2.81}$$

and thereby, c.f., (2.33),

$$\begin{aligned} \nu^{\rm per}_\Lambda({\rm d}\omega_\Lambda) &= \exp\left[-I^{\rm per}_\Lambda(\omega_\Lambda)\right]\chi_\Lambda({\rm d}\omega_\Lambda)/N^{\rm per}_\Lambda, \\ N^{\rm per}_\Lambda &= \int_{\Omega_\Lambda}\exp\left[-I^{\rm per}_\Lambda(\omega_\Lambda)\right]\chi_\Lambda({\rm d}\omega_\Lambda). \end{aligned} \tag{2.82}$$

By means of (2.78) we introduce the periodic Hamiltonian

$$H_\Lambda^{\rm per} = H_\Lambda = -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}^\Lambda \cdot (q_\ell, q_{\ell'}) + \sum_{\ell\in\Lambda} H_\ell, \tag{2.83}$$

and the corresponding periodic local Gibbs state

$$\varrho_\Lambda^{\rm per}(A) = {\rm trace}[A\exp(-\beta H_\Lambda^{\rm per})]/{\rm trace}[\exp(-\beta H_\Lambda^{\rm per})], \quad A \in \mathfrak{C}_\Lambda. \tag{2.84}$$

The relationship between the measure $\nu_\Lambda^{\rm per}$ and this state is the same as in the case of ν_Λ and ϱ_Λ.

Set, c.f., (2.50),

$$\pi_\Lambda^{\rm per}(B) = \frac{1}{N_\Lambda^{\rm per}}\int_{\Omega_\Lambda} \exp\left[-I_\Lambda^{\rm per}(\omega_\Lambda)\right] \mathbb{I}_B(\omega_\Lambda \times 0_{\Lambda^c})\chi_\Lambda({\rm d}x_\Lambda), \tag{2.85}$$

which is a probability measure on $\Omega^{\rm t}$. Then

$$\pi_\Lambda^{\rm per}({\rm d}(\omega_\Lambda \times \omega_{\Lambda^c})) = \nu_\Lambda^{\rm per}({\rm d}\omega_\Lambda)\prod_{\ell'\in\Lambda^c} \delta_{0_{\ell'}}({\rm d}x_{\ell'}), \tag{2.86}$$

where $0_{\ell'}$ is the zero element of the Banach space C_β. Note that the projection of $\pi_\Lambda^{\rm per}$ onto Ω_Λ is $\nu_\Lambda^{\rm per}$.

Let $\mathcal{L}_{\rm box}$ be the sequence of all boxes (2.63). Arguments similar to those used in the proof of Lemma 4.4 in [54] yield the following

Lemma 2.20 *For every $\alpha \in \mathcal{I}$ and $\sigma \in (0, 1/2)$, there exists a constant $C > 0$ such that, for all boxes Λ,*

$$\int_{\Omega^{\rm t}} \left(\sum_\ell \|\omega_\ell\|^2_{C^\sigma_\beta} w_\alpha(0,\ell)\right)^2 \pi_\Lambda^{\rm per}({\rm d}\omega) \leq C. \tag{2.87}$$

Thereby, the family $\{\pi_\Lambda^{\rm per}\}_{\Lambda\in\mathcal{L}_{\rm box}}$ is $\mathcal{W}^{\rm t}$-relatively compact.

Let $\mathcal{M}$ be the family of $\mathcal{W}^{\rm t}$-accumulation points of $\{\pi_\Lambda^{\rm per}\}_{\Lambda\in\mathcal{L}_{\rm box}}$.

Proposition 2.21 *It follows that $\mathcal{M} \subset \mathcal{G}^{\rm t}$. The elements of $\mathcal{M}$, called periodic Euclidean Gibbs measures, are translation invariant.*

The proof of this statement is similar to the proof of Proposition 2.6. It can be done by demonstrating that each $\mu \in \mathcal{M}$ solves the DLR equation (2.56). To this end, for chosen $\Lambda \Subset \mathbb{L}$, one picks the box Δ containing this Λ, and shows that

$$\int_\Omega \pi_\Lambda(\cdot|\xi)\pi_\Delta^{\rm per}({\rm d}\xi) \Rightarrow \mu(\cdot), \qquad \text{if} \ \ \pi_\Delta^{\rm per} \Rightarrow \mu \ \ \text{in} \ \ \mathcal{W}^{\rm t}.$$

Here both convergence are taken along a subsequence of $\mathcal{L}_{\rm box}$.

2.8 The pressure

In the translation invariant case, one can introduce a thermodynamic function, which contains important information about the thermodynamic properties of the model. This is the pressure, which in our case up to a factor coincides with the free energy density. As our special attention will be given to the dependence of the pressure on the external field h, c.f. (2.3), we indicate this dependence explicitely. For $\Lambda \Subset \mathbb{L}$, we set, see (2.49),

$$p_\Lambda(h, \xi) = \frac{1}{|\Lambda|} \log N_\Lambda(h, \xi), \quad \xi \in \Omega^{\mathrm{t}}. \tag{2.88}$$

To simplify notations we write $p_\Lambda(h) = p_\Lambda(h, 0)$. Thereby, for $\mu \in \mathcal{G}^{\mathrm{t}}$, we set

$$p^\mu_\Lambda(h) = \int_\Omega p_\Lambda(h, \xi)\mu(\mathrm{d}\xi). \tag{2.89}$$

Furthermore, we set

$$p^{\mathrm{per}}_\Lambda(h) = \frac{1}{|\Lambda|} \log N^{\mathrm{per}}_\Lambda(h). \tag{2.90}$$

If, for a cofinal sequence $\mathcal{L}$, the limit

$$p^\mu(h) \ \stackrel{\mathrm{def}}{=}\ \lim_{\mathcal{L}} p^\mu_\Lambda(h), \tag{2.91}$$

exists, we call it pressure in the state μ. We shall also consider

$$p(h) \ \stackrel{\mathrm{def}}{=}\ \lim_{\mathcal{L}} p_\Lambda(h), \qquad p^{\mathrm{per}}(h) \ \stackrel{\mathrm{def}}{=}\ \lim_{\mathcal{L}_{\mathrm{box}}} p^{\mathrm{per}}_\Lambda(h). \tag{2.92}$$

Given $l = (l_1, \ldots l_d)$, $l' = (l'_1, \ldots l'_d) \in \mathbb{L} = \mathbb{Z}^d$, such that $l_j < l'_j$ for all $j = 1, \ldots, d$, we set

$$\Gamma = \{\ell \in \mathbb{L} \mid l_j \leq \ell_j \leq l'_j, \ \ \text{for all } j = 1, \ldots, d\}. \tag{2.93}$$

For this parallelepiped, let $\mathfrak{G}(\Gamma)$ be the family of all pair-wise disjoint translates of Γ which cover $\mathbb{L}$. Then for $\Lambda \Subset \mathbb{L}$, we let $N_-(\Lambda|\Gamma)$ (respectively, $N_+(\Lambda|\Gamma)$) be the number of the elements of $\mathfrak{G}(\Gamma)$ which are contained in Λ (respectively, which have non-void intersections with Λ).

Definition 2.22 *A cofinal sequence $\mathcal{L}$ is a van Hove sequence if for every Γ,*

$$(a) \quad \lim_{\mathcal{L}} N_-(\Lambda|\Gamma) = +\infty; \qquad (b) \quad \lim_{\mathcal{L}} \left(N_-(\Lambda|\Gamma)/N_+(\Lambda|\Gamma)\right) = 1. \tag{2.94}$$

One observes that $\mathcal{L}_{\mathrm{box}}$ is a van Hove sequence. It is known, see Theorem 3.10 in [54], that

Proposition 2.23 *For every $h \in \mathbb{R}$ and any van Hove sequence $\mathcal{L}$, it follows that the limits (2.91) and (2.92) exist, do not depend on the particular choice of $\mathcal{L}$, and are equal, that is $p(h) = p^{\mathrm{per}}(h) = p^\mu(h)$ for each $\mu \in \mathcal{G}^{\mathrm{t}}$.*

Let the model be rotation invariant, see Definition 2.1. Then the pressure depends on the norm of the vector $h \in \mathbb{R}^\nu$. Therefore, without loss of generality one can choose the external field to be $(h, 0, \ldots, 0)$, $h \in \mathbb{R}$. For the measure (2.33), by $\nu^{(0)}_\Lambda$ we denote its version with $h = 0$. Then

$$N_\Lambda(h) = N_\Lambda(0) \int_{\Omega_\Lambda} \exp\left(h \sum_{\ell \in \Lambda} \int_0^\beta \omega^{(1)}_\ell(\tau)\mathrm{d}\tau \right) \nu^{(0)}_\Lambda(\mathrm{d}\omega_\Lambda). \tag{2.95}$$

The same representation can also be written for $N_\Lambda^{\rm per}(h)$. One can show that the pressures $p_\Lambda(h)$ and $p_\Lambda^{\rm per}(h)$, as functions of h, are analytic in a subset of $\mathbb{C}$, which contains $\mathbb{R}$. Thus, one can compute the derivatives and obtain

$$\frac{\partial}{\partial h} p_\Lambda(h) = \beta M_\Lambda(h), \qquad \frac{\partial}{\partial h} p_\Lambda^{\rm per}(h) = \beta M_\Lambda^{\rm per}(h), \tag{2.96}$$

where

$$M_\Lambda(h) \stackrel{\rm def}{=} \frac{1}{|\Lambda|} \sum_{\ell\in\Lambda} \varrho_\Lambda[q_\ell^{(1)}], \quad M_\Lambda^{\rm per}(h) \stackrel{\rm def}{=} \varrho_\Lambda^{\rm per}[q_\ell^{(1)}] \tag{2.97}$$

are local *polarizations*, corresponding to the zero and periodic boundary conditions respectively. Furthermore,

$$\begin{aligned} &\frac{\partial^2}{\partial h^2} p_\Lambda(h) \qquad\qquad (2.98)\\ &= \frac{1}{2|\Lambda|}\int_{\Omega_\Lambda}\int_{\Omega_\Lambda}\left[\sum_{\ell\in\Lambda}\int_0^\beta \left(\omega_\ell^{(1)}(\tau) - \tilde{\omega}_\ell^{(1)}(\tau)\right){\rm d}\tau\right]^2 \nu_\Lambda({\rm d}\omega_\Lambda)\nu_\Lambda({\rm d}\tilde{\omega}_\Lambda) \geq 0. \end{aligned}$$

The same can be said about the second derivative of $p_\Lambda^{\rm per}(h)$. Therefore, both $p_\Lambda(h)$ and $p_\Lambda^{\rm per}(h)$ are convex functions. For the reader convenience, we present here the corresponding properties of convex functions following [69], pages 34 - 37.

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, by $\varphi'_\pm(t)$ we denote its one-side derivatives at a given $t \in \mathbb{R}$. By *at most countable set* we mean the set which is void, finite, or countable.

Proposition 2.24 *For a convex function $\varphi : \mathbb{R} \to \mathbb{R}$, it follows that:*

(a) *the derivatives $\varphi'_\pm(t)$ exist for every $t \in \mathbb{R}$;*
the set $\{t \in \mathbb{R} \mid \varphi'_+(t) \neq \varphi'_-(t)\}$ is at most countable;

(b) *for every $t \in \mathbb{R}$ and $\theta > 0$,*

$$\varphi'_-(t) \leq \varphi'_+(t) \leq \varphi'_-(t+\theta) \leq \varphi'_+(t+\theta); \tag{2.99}$$

(c) *the point-wise limit φ of a sequence of convex functions $\{\varphi_n\}_{n\in\mathbb{N}}$ is a convex function; if φ and all φ_n's are differentiable at a given t, $\varphi'_n(t) \to \varphi'(t)$ as $n \to +\infty$.*

Proposition 2.25 *The pressure $p(h)$, see Proposition 2.23, is a convex function of $h \in \mathbb{R}$. Therefore, the set*

$$\mathcal{R} \stackrel{\rm def}{=} \{h \in \mathbb{R} \mid p'_-(h) < p'_+(h)\} \tag{2.100}$$

is at most countable. For any $h \in \mathcal{R}^c$ and any van Hove sequence $\mathcal{L}$, it follows that

$$\lim_{\mathcal{L}} M_\Lambda(h) = \lim_{\mathcal{L}_{\rm box}} M_\Lambda^{\rm per}(h) = \beta^{-1}p'(h) \stackrel{\rm def}{=} M(h). \tag{2.101}$$

By this statement, for any $h \in \mathcal{R}^c$, the limiting periodic state is unique. In the scalar case, one can tell more on this item. The following result is a consequence of Propositions 2.23 and 2.18.

Proposition 2.26 *If $\nu = 1$ and $p(h)$ is differentiable at a given $h \in \mathbb{R}$, then $\mathcal{G}^{\rm t}$ is a singleton at this h.*

Returning to the general case $\nu \in \mathbb{N}$ we note that by Proposition 2.25 the global polarization $M(h)$ is a nondecreasing function of $h \in \mathcal{R}^c$; it is continuous on each open connected component of $\mathcal{R}^c$. That is, $M(h)$ is continuous on the intervals $(a_-, a_+) \subset \mathcal{R}^c$, where $a_\pm$ are two consecutive elements of $\mathcal{R}$. At each such $a_\pm$, the global magnetization is discontinuous. One observes however that the set $\mathcal{R}^c$ may have empty interior; hence, $M(h)$ may be nowhere continuous.

In the sequel, to study phase transitions in the model with the anharmonic potentials V of general type, we use the regularity of the temperature loops and Proposition 2.3. Let the model be just translation invariant. i.e., the anharmonic potential has the form (2.3), where V^0 is independent of ℓ. Let us consider the following measure on C_β:

$$\begin{aligned} \lambda(\mathrm{d}\omega) &= \frac{1}{N_\beta} \exp\left(- \int_0^\beta V^0(\omega(\tau))\mathrm{d}\tau \right) \chi(\mathrm{d}\omega), \qquad (2.102) \\ N_\beta &= \int_{C_\beta} \exp\left(- \int_0^\beta V^0(\omega(\tau))\mathrm{d}\tau \right) \chi(\mathrm{d}\omega), \end{aligned}$$

where χ is Høegh-Krohn's measure. For a box Λ, we introduce the following functions on Ω_Λ

$$\begin{aligned} Y_\Lambda(\omega_\Lambda) &= \frac{1}{2} \sum_{\ell,\ell' \in \Lambda} J^\Lambda_{\ell\ell'} \sum_{j=1}^{\nu} \int_0^\beta \omega_\ell^{(j)}(\tau)\omega_{\ell'}^{(j)}(\tau)\mathrm{d}\tau, \qquad (2.103) \\ X_\Lambda^{(j)}(\omega_\Lambda) &= \sum_{\ell \in \Lambda} \int_0^\beta \omega_\ell^{(j)}(\tau)\mathrm{d}\tau, \qquad j = 1, \dots, \nu. \end{aligned}$$

Then from (2.90) one gets

$$\begin{aligned} p_\Lambda^{\rm per}(h) &= \log N_\beta \\ &+ \frac{1}{|\Lambda|} \log \left\{ \int_{\Omega_\Lambda} \exp\left[Y_\Lambda(\omega_\Lambda) + \sum_{j=1}^{\nu} h^{(j)} X_\Lambda^{(j)}(\omega_\Lambda) \right] \prod_{\ell \in \Lambda} \lambda(\mathrm{d}\omega_\ell) \right\} \qquad (2.104) \end{aligned}$$

As the measure (2.102) is a perturbation of the Høegh-Krohn measure, we can study the regularity of the associated stochastic process by means of Proposition 2.3. Fix some $p \in \mathbb{N} \setminus \{1\}$ and $\sigma \in (0, 1/2 - 1/2p)$. Thereby, for $\vartheta \in (0, \beta)$, one obtains

$$\int_{C_\beta} \Xi_\vartheta^p(\omega)\lambda(\mathrm{d}\omega) \leq e^{-\beta B_V} \cdot \langle \Xi_\vartheta^p \rangle_\chi / N_\beta,$$

B_V being as in (2.2). By Proposition 2.3 this yields

$$\langle \Xi_\vartheta^p \rangle_\lambda \leq D_V(\sigma, \nu, p) m^{-p} \vartheta^{p(1-2\sigma)}, \qquad (2.105)$$

where, see (2.32),

$$D_V(\sigma, \nu, p) \stackrel{\rm def}{=} \frac{2^{3(2p+1)}(1 + 1/\sigma p)^{2p}}{(p - 1 - 2p\sigma)(p - 2p\sigma)} \cdot \frac{2^p \exp\left(-\beta B_V\right) \Gamma(\nu/2 + p)}{N_\beta \Gamma(\nu/2)}.$$

For $c > 0$ and $n \in \mathbb{N}$, $n \geq 2$, we set

$$C^{\pm}(n;c) = \{\omega \in C_\beta \mid \pm\omega^{(j)}(k\beta/n) \geq c, \ j = 1, \ldots, \nu; \ k = 0, 1, \ldots n\}. \quad (2.106)$$

For every $n \in \mathbb{N}$, $j_1, \ldots, j_n \in \{1, \ldots, \nu\}$, and $\tau_1, \ldots, \tau_n \in [0, \beta]$, the joint distribution of $\omega^{(j_1)}(\tau_1), \ldots, \omega^{(j_n)}(\tau_n)$ induced by Høegh-Krohn's measure χ is Gaussian. Therefore, $\chi(C^{\pm}(n;c)) > 0$. Clearly, the same property has the measure (2.102). Thus, we have

$$\Sigma(n;c) \stackrel{\rm def}{=} \min\left\{\lambda\left(C^{+}(n;c)\right); \lambda\left(C^{-}(n;c)\right)\right\} > 0. \quad (2.107)$$

For $\varepsilon \in (0, c)$, we set

$$\begin{array}{rcl} A(c;\varepsilon) & = & \{\omega \in C_\beta \mid \Xi_{\beta/n}(\omega) \leq (c-\varepsilon)^2(\beta/n)^{-2\sigma}\}, \\ B^{\pm}(\varepsilon, c) & = & A(c;\varepsilon)\bigcap C^{\pm}(n;c). \end{array} \quad (2.108)$$

Then for any $\tau \in [0, \beta]$, one finds $k \in \mathbb{N}$ such that $|\tau - k\beta/n| \leq \beta/n$, and hence, for any $j = 1, \ldots, \nu$,

$$|\omega^{(j)}(\tau) - \omega^{(j)}(k\beta/n)| \leq \left[\Xi_{\beta/n}(\omega)\right]^{1/2}(\beta/n)^{\sigma},$$

which yields $\pm\omega^{(j)}(\tau) \geq \varepsilon$ if $\omega \in B^{\pm}(\varepsilon, c)$. Let us estimate $\lambda[B^{\pm}(\varepsilon, c)]$. By (2.105) and Chebyshev's inequality, one gets

$$\begin{array}{rcl} \lambda\left(C_\beta \setminus A(c;\varepsilon)\right) & \leq & \dfrac{\beta^{2\sigma p}}{n^{2\sigma p}(c-\varepsilon)^{2p}}\langle \Xi^{p}_{\beta/n}\rangle_\lambda \\ & \leq & \dfrac{\beta^p D_V(\sigma, \nu, p)}{[mn(c-\varepsilon)^2]^p}. \end{array}$$

Thereby,

$$\begin{array}{rcl} \lambda\left[B^{\pm}(\varepsilon, c)\right] & = & \lambda\left[C^{\pm}(n;c) \setminus (C_\beta \setminus A(c;\varepsilon))\right] \\ & \geq & \Sigma(n;c) - \lambda\left(C_\beta \setminus A(c;\varepsilon)\right) \\ & \geq & \Sigma(n;c) - \dfrac{\beta^p D_V(\sigma, \nu, p)}{[mn(c-\varepsilon)^2]^p} \\ & \stackrel{\rm def}{=} & \gamma(m), \end{array} \quad (2.109)$$

which is positive, see (2.107), for all

$$m \geq m_* \stackrel{\rm def}{=} \frac{\beta}{n(c-\varepsilon)^2} \cdot \left(\frac{D_V(\sigma, \nu, p)}{\Sigma(n;c)}\right)^{1/p}. \quad (2.110)$$

This result will be used for estimating the integrals in (2.104).

3 Phase Transitions

There exist several approaches to describe phase transitions. Their common point is that the macroscopic equilibrium properties of a statistical mechanical model can be different at the same values of the model parameters. That is, one speaks about the possibility for the multiple states to exist rather than the transition (as a process) between these states or between their uniqueness and multiplicity.

3.1 Phase transitions and order parameters

We begin by introducing the main notion of this section.

Definition 3.1 *The model described by the Hamiltonians (1.1), (1.2) has a phase transition if $|\mathcal{G}^{\rm t}| > 1$ at certain values of β and the model parameters.*

Note that here we demand the existence of multiple *tempered* Euclidean Gibbs measures. For models with finite range interactions, there may exist Euclidean Gibbs measures, which are not tempered. Such measures should not be taken into account. Another observation is that in Definition 3.1 we do not assume any symmetry of the model, the translation invariance including. If the model is rotation invariant (symmetric for $\nu = 1$, see Definition 2.1), the unique element of $\mathcal{G}^{\rm t}$ should have the same symmetry. If $|\mathcal{G}^{\rm t}| > 1$, the symmetry can be 'distributed' among the elements of $\mathcal{G}^{\rm t}$. In this case, the phase transition is connected with *a symmetry breaking*. In the sequel, we consider mostly phase transitions of this type. However, in subsection 3.5 we study the case where the anharmonic potentials V_ℓ have no symmetry and hence there is no symmetry breaking connected with the phase transition.

If the model is translation invariant, the multiplicity of its Euclidean Gibbs states is equivalent to the existence of non-ergodic elements of $\mathcal{G}^{\rm t}$, see Corollary 2.11. Thus, to prove that the model has a phase transition it is enough to show that there exists an element of $\mathcal{G}^{\rm t}$, which fails to obey (2.65). In the case where the model is not translation invariant, we employ a comparison method, based on correlation inequalities. Its main idea is that the model has a phase transition if the translation invariant model with which we compare it has a phase transition.

Let us consider first the translation and rotation invariant case. Given ℓ and $j = 1, \dots, \nu$, we set

$$D^{\Lambda}_{\ell\ell'} = \beta \int_0^\beta \left\langle (\omega_\ell(\tau), \omega_{\ell'}(\tau')) \right\rangle_{\nu_\Lambda^{\rm per}} {\rm d}\tau'. \tag{3.1}$$

The right-hand side in (3.1) does not depend on τ due to the property (2.13). To introduce the Fourier transformation in the box Λ we employ the conjugate set Λ_* (Brillouin zone), consisting of the vectors $p = (p_1, \dots, p_d)$, such that

$$p_j = -\pi + \frac{\pi}{L} s_j, \ \ s_j = 1, \dots, 2L, \ \ j = 1, \dots, d. \tag{3.2}$$

Then the Fourier transformation is

$$\begin{aligned} \omega^{(j)}_\ell(\tau) &= \frac{1}{|\Lambda|^{1/2}} \sum_{p \in \Lambda_*} \hat{\omega}^{(j)}_p(\tau) e^{\imath(p,\ell)}, \\ \hat{\omega}^{(j)}_p(\tau) &= \frac{1}{|\Lambda|^{1/2}} \sum_{\ell \in \Lambda} \omega^{(j)}_\ell(\tau) e^{-\imath(p,\ell)}. \end{aligned} \tag{3.3}$$

In order that $\omega^{(j)}_\ell(\tau)$ be real, the Fourier coefficients should satisfy

$$\overline{\hat{\omega}^{(j)}_p(\tau)} = \hat{\omega}^{(j)}_{-p}(\tau).$$

By the rotation invariance of the state $\langle\cdot\rangle_{\nu_\Lambda^{\rm per}}$, as well as by its invariance with respect to the translations of the torus Λ, it follows that

$$\langle \hat{\omega}_p^{(j)}(\tau)\hat{\omega}_{p'}^{(j')}(\tau')\rangle_{\nu_\Lambda^{\rm per}} = \delta_{jj'}\delta(p+p')\sum_{\ell'\in\Lambda}\langle \omega_\ell^{(j)}(\tau)\omega_{\ell'}^{(j)}(\tau')\rangle_{\nu_\Lambda^{\rm per}} e^{\imath(p,\ell'-\ell)}. \quad (3.4)$$

Thus, we set

$$\begin{aligned}\widehat{D}_p^\Lambda &= \sum_{\ell'\in\Lambda} D_{\ell\ell'}^\Lambda e^{\imath(p,\ell'-\ell)}, \\ D_{\ell\ell'}^\Lambda &= \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*}\widehat{D}_p^\Lambda e^{\imath(p,\ell-\ell')}.\end{aligned} \quad (3.5)$$

One observes that $\widehat{D}_p^\Lambda$ can be extended to all $p\in(-\pi,\pi]^d$. Furthermore,

$$\widehat{D}_p^\Lambda = \widehat{D}_{-p}^\Lambda = \sum_{\ell'\in\Lambda} D_{\ell\ell'}^\Lambda \cos(p,\ell'-\ell), \quad (3.6)$$

and

$$D_{\ell\ell'}^\Lambda = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*}\widehat{D}_p^\Lambda e^{\imath(p,\ell-\ell')} = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*}\widehat{D}_p^\Lambda \cos(p,\ell-\ell'). \quad (3.7)$$

For $u_\Lambda = (u_\ell)_{\ell\in\Lambda}$, $u_\ell\in\mathbb{R}$,

$$\begin{aligned}\left(u_\Lambda, D^\Lambda u_\Lambda\right)_{l^2(\Lambda)} &\stackrel{\rm def}{=} \sum_{\ell,\ell'\in\Lambda} D_{\ell\ell'}^\Lambda u_\ell u_{\ell'} \\ &= \sum_{j=1}^{\nu}\left\langle \left[\sum_{\ell\in\Lambda} u_\ell\int_0^\beta \omega_\ell^{(j)}(\tau){\rm d}\tau\right]^2\right\rangle_{\nu_\Lambda^{\rm per}} \geq 0.\end{aligned} \quad (3.8)$$

Thereby, the operator $D^\Lambda : l^2(\Lambda)\to l^2(\Lambda)$ is strictly positive; hence, all its eigenvalues $\widehat{D}_p^\Lambda$ are also strictly positive.

Suppose now that we are given a continuous function $\widehat{B}:(-\pi,\pi]^d\to(0,+\infty]$ with the following properties:

$$\begin{aligned}&\text{(i)} \qquad \int_{(-\pi,\pi]^d}\widehat{B}(p){\rm d}p < \infty, \\ &\text{(ii)} \qquad \widehat{D}_p^\Lambda \leq \widehat{B}(p), \quad \text{for all } p\in\Lambda_*\setminus\{0\},\end{aligned} \quad (3.9)$$

holding for all boxes Λ. Then we set

$$B_{\ell\ell'} = \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d}\widehat{B}(p)\cos(p,\ell-\ell'){\rm d}p, \quad \ell,\ell'\in\mathbb{L}, \quad (3.10)$$

and

$$B_{\ell\ell'}^\Lambda = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*\setminus\{0\}}\widehat{B}(p)\cos(p,\ell-\ell'), \quad \ell,\ell'\in\Lambda. \quad (3.11)$$

We also set $B_{\ell\ell'}^\Lambda = 0$ if either of ℓ,ℓ' belongs to Λ^c.

Proposition 3.2 *For every ℓ,ℓ', it follows that $B_{\ell\ell'}^{\Lambda_L}\to B_{\ell\ell'}$ as $L\to+\infty$.*

Proof: By (3.9), $\widehat{B}(p)\cos(p, \ell - \ell')$ is an absolutely integrable function in the sense of improper Riemann integral. The right-hand side of (3.11) is its integral sum; thereby, the convergence stated is obtained in a standard way. □

From claim (i) of (3.9) by the Riemann-Lebesgue lemma, see page 116 in [55], one obtains

$$\lim_{|\ell-\ell'|\to+\infty} B_{\ell\ell'} = 0. \tag{3.12}$$

Lemma 3.3 *For every box Λ and any $\ell, \ell' \in \Lambda$, it follows that*

$$D^{\Lambda}_{\ell\ell'} \geq \left(D^{\Lambda}_{\ell\ell} - B^{\Lambda}_{\ell\ell}\right) + B^{\Lambda}_{\ell\ell'}. \tag{3.13}$$

Proof: By (3.7), (3.11), and claim (ii) of (3.9), one has

$$\begin{aligned} D^{\Lambda}_{\ell\ell} - D^{\Lambda}_{\ell\ell'} &= \frac{2}{|\Lambda|}\sum_{p\in\Lambda_*\setminus\{0\}} \widehat{D}^{\Lambda}_p \sin^2(p, \ell-\ell') \\ &\leq \frac{2}{|\Lambda|}\sum_{p\in\Lambda_*\setminus\{0\}} \widehat{B}(p) \sin^2(p, \ell-\ell') \\ &= B^{\Lambda}_{\ell\ell} - B^{\Lambda}_{\ell\ell'}, \end{aligned}$$

which yields (3.13). □

For $\mu \in \mathcal{G}^{\mathrm{t}}$, we set, c.f., (3.1),

$$D^{\mu}_{\ell\ell'} = \beta\int_0^\beta \langle(\omega_\ell(\tau), \omega_{\ell'}(\tau'))\rangle_\mu \mathrm{d}\tau'. \tag{3.14}$$

Corollary 3.4 *For every periodic $\mu \in \mathcal{G}^{\mathrm{t}}$, it follows that*

$$D^{\mu}_{\ell\ell'} \geq (D^{\mu}_{\ell\ell} - B_{\ell\ell}) + B_{\ell\ell'}, \tag{3.15}$$

holding for any ℓ, ℓ'.

Proof: For periodic $\mu \in \mathcal{G}^{\mathrm{t}}$, one finds the sequence $\{L_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$, such that $\pi^{\mathrm{per}}_{\Lambda_{L_n}} \Rightarrow \mu$ as $n \to +\infty$, see Proposition 2.21. This fact alone does not yet mean that $D^{\Lambda_{L_n}}_{\ell\ell'} \to D^{\mu}_{\ell\ell'}$, what we would like to get. To prove the latter convergence one employs Lemma 2.20 and proceeds as in the proof of claim (b) of Lemma 5.2 in [54]. Then (3.15) follows from (3.13) and Proposition 3.2. □

One observes that the first summand in (3.15) is independent of ℓ, whereas the second one may be neither positive nor summable. Suppose now that there exists a positive ϑ such that, for any box Λ,

$$D^{\Lambda}_{\ell\ell} \geq \vartheta. \tag{3.16}$$

Then, in view of (3.12), the phase transition occurs if

$$\vartheta > B_{\ell\ell}. \tag{3.17}$$

For certain versions of our model, we find the function $\widehat{B}$ obeying the conditions (3.9) and the bound (3.17). Note that under (3.16) and (3.17) by (3.15) it follows that

$$\lim_{L\to+\infty}\frac{1}{|\Lambda_L|}\sum_{\ell'\in\Lambda_L} D^{\mu}_{\ell\ell'} = \lim_{L\to+\infty}\frac{1}{|\Lambda_L|^2}\sum_{\ell,\ell'\in\Lambda_L} D^{\mu}_{\ell\ell'} > 0. \tag{3.18}$$

Let us consider now another possibilities to define phase transitions in translation invariant versions of our model. For a box Λ, see (2.63), we introduce

$$\begin{aligned} P_\Lambda &= \frac{1}{(\beta|\Lambda|)^2}\sum_{\ell,\ell'\in\Lambda} D^\Lambda_{\ell\ell'} \\ &= \int_{\Omega_\Lambda}\left|\frac{1}{\beta|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta \omega_\ell(\tau)\mathrm{d}\tau\right|^2 \nu^{\mathrm{per}}_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{3.19}$$

and set

$$P \stackrel{\mathrm{def}}{=} \limsup_{L\to+\infty} P_{\Lambda_L}. \tag{3.20}$$

Definition 3.5 *The above P is called the order parameter. If $P > 0$ for given values of β and the model parameters, then there exists a long range order.*

By standard arguments one proves the following

Proposition 3.6 *If (3.16) and (3.17) hold, then $P > 0$.*

The appearance of the long range order, which in a more 'physical' context is identified with a phase transition, does not imply the phase transition in the sense of Definition 3.1. At the same time, Definition 3.1 describes models without translation invariance. On the other hand, Definition 3.5 is based upon the local states only and hence can be formulated without employing $\mathcal{G}^{\mathrm{t}}$. Yet another 'physical' approach to phase transitions in translation invariant models like (1.1), (1.2) is based on the properties of the pressure $p(h)$, which by Proposition 2.23 exists and is the same in every state. It does not employ the set $\mathcal{G}^{\mathrm{t}}$ and is based on the continuity of the global polarization (2.101), that is, on the differentiability of $p(h)$.

Definition 3.7 (Landau Classification) *The model has a first order phase transition if $p'(h)$ is discontinuous at a certain h_*. The model has a second order phase transition if there exists $h_* \in \mathbb{R}^\nu$ such that $p'(h)$ is continuous but $p''(h)$ is discontinuous at $h = h_*$.*

Remark 3.8 *Like in Definition 3.1, here we do not assume any symmetry of the model (except for the translation invariance). As $p(h)$ is convex, $p'(h)$ is increasing; hence, $p''(h) \geq 0$. The discontinuity of the latter mentioned in Definition 3.7 includes the case $p''(h_*) = +\infty$, where the polarization $M(h)$ at $h = h_*$ grows infinitely fast, but still is continuous.*

The relationship between the first order phase transition and the long range order is established with the help of the following result, the proof of which can be done by a slight modification of the arguments used in [24], see Theorem 1.1 and its corollaries. Let $\{\mu_n\}_{N\in\mathbb{N}}$ (respectively, $\{M_n\}_{n\in\mathbb{N}}$) be a sequence of probability measures on $\mathbb{R}$ (respectively, positive real numbers, such that $\lim M_n = +\infty$). We also suppose that, for any $y \in \mathbb{R}$,

$$f(y) = \lim_{n\to+\infty}\frac{1}{M_n}\log\int e^{yu}\mu_n(\mathrm{d}u) \tag{3.21}$$

exists and is finite. As the function f is convex, it has one-sided derivatives $f'_\pm(0)$, see Proposition 2.24.

Proposition 3.9 (Griffiths) *Let the sequence of measures $\{\mu_n\}_{N\in\mathbb{N}}$ be as above. If $f'_+(0) = f'_-(0) = \phi$ (i.e., f is differentiable at $y = 0$), then*

$$\lim_{n\to+\infty} \int g(u/M_n)\mu_n(\mathrm{d}u) = g(\phi), \tag{3.22}$$

for any continuous $g : \mathbb{R} \to \mathbb{R}$, such that $|g(u)| \leq \lambda e^{\varkappa|u|}$ with certain $\lambda, \varkappa > 0$. Furthermore, for each such a function g,

$$\limsup_{n\to+\infty} \int g(u/M_n)\mu_n(\mathrm{d}u) \leq \max_{z\in[f'_-(0), f'_+(0)]} g(z). \tag{3.23}$$

In particular, if $f'_-(0) = -f'_+(0)$, then for any $k \in \mathbb{N}$,

$$f'_+(0) \geq \limsup_{n\to+\infty} \left(\int (u/M_n)^{2k} \mu_n(\mathrm{d}u) \right)^{1/2k}. \tag{3.24}$$

Write, c.f., (2.95),

$$N^{\mathrm{per}}_\Lambda(h) = N^{\mathrm{per}}_\Lambda(0) \int_{\Omega_\Lambda} \exp\left(h \sum_{\ell\in\Lambda} \int_0^\beta \omega^{(1)}_\ell(\tau)\mathrm{d}\tau \right) \nu^{0,\mathrm{per}}_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.25}$$

where $\nu^{0,\mathrm{per}}_\Lambda$ is the local periodic Euclidean Gibbs measure with $h = 0$. Now let $\{L_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ be the sequence such that the sequences of local measures $\{\nu^{0,\mathrm{per}}_{\Lambda_{L_n}}\}$ and $\{\nu^{\mathrm{per}}_{\Lambda_{L_n}}\}$ converge to the corresponding periodic Euclidean Gibbs measures μ^0 and μ respectively. Set

$$\mathcal{X}_n = \left\{ \omega_{\Lambda_{L_n}} \in \Omega_{\Lambda_{L_n}} \;\middle|\; \exists u \in \mathbb{R} : \sum_{\ell\in\Lambda_{L_n}} \int_0^\beta \omega^{(1)}_\ell(\tau)\mathrm{d}\tau = u \right\}. \tag{3.26}$$

Clearly, each such $\mathcal{X}_n$ is measurable and isomorphic to $\mathbb{R}$. Let μ_n, $n \in \mathbb{N}$, be the projection of $\{\nu^{0,\mathrm{per}}_{\Lambda_{L_n}}\}$ onto this $\mathcal{X}_n$. Then

$$p(h) = p(0) + f(h), \tag{3.27}$$

where f is given by (3.21) with such μ_n and $M_n = |\Lambda_{L_n}| = (2L_n)^d$. Thereby, we apply (3.24) with $k = 2$ and obtain

$$p'_+(0) \geq \beta \limsup_{n\to+\infty} \sqrt{P_{\Lambda_{L_n}}}.$$

Thus, in the case where the model is just rotation and translation invariant, the existence of the long range order implies the first order phase transition.

Consider now the second order phase transitions in the rotation invariant case. For $\alpha \in [0, 1]$, we set, c.f., (3.19),

$$P^{(\alpha)}_\Lambda = \frac{\beta^{-2}}{|\Lambda|^{1+\alpha}} \int_{\Omega_\Lambda} \left| \sum_{\ell\in\Lambda} \int_0^\beta \omega_\ell(\tau)\mathrm{d}\tau \right|^2 \nu^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda), \tag{3.28}$$

where Λ is a box. Then $P^{(1)}_\Lambda = P_\Lambda$ and, as we just have shown, the existence of a positive limit (3.20) yields a first order phase transition.

Proposition 3.10 *If there exists $\alpha \in (0,1)$, such that for a sequence $\{L_n\}$, there exists a finite limit*

$$\lim_{n\to+\infty} P^{(\alpha)}_{\Lambda_{L_n}} \stackrel{\mathrm{def}}{=} P^{(\alpha)} > 0. \tag{3.29}$$

Then the model has at $h = 0$ a second order phase transition.

Proof: We observe that

$$P^{(\alpha)}_{\Lambda} = \nu p''_{\Lambda}(0)/\beta^2 |\Lambda|^{\alpha}.$$

Then there exists $c > 0$, such that

$$p''_{\Lambda_{L_n}}(0) \geq c|\Lambda_{L_n}|^{\alpha}, \qquad \text{for all} \quad n \in \mathbb{N}.$$

As each p''_{Λ} is continuous, one finds the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_n \downarrow 0$ and

$$p''_{\Lambda_{L_n}}(h) \geq \frac{1}{2}c|\Lambda_{L_n}|^{\alpha}, \qquad \text{for all} \quad h \in [0, \delta_n] \quad \text{and} \quad n \in \mathbb{N}. \tag{3.30}$$

If $p''(0)$ were finite, see Remark 3.8, one would get

$$p''(0) = \lim_{n\to+\infty} \left[p'_{\Lambda_{L_n}}(\delta_n) - p'_{\Lambda_{L_n}}(0) \right] / \delta_n,$$

which contradicts (3.30). $\square$

Proposition 3.10 remains true if one replaces in (3.28) the periodic local measure $\nu^{\mathrm{per}}_{\Lambda}$ by the one corresponding to the zero boundary condition, i.e., by ν_{Λ}. Then the limit in (3.29) can be taken along any van Hove sequence $\mathcal{L}$. We remind that Proposition 3.10 describes the rotation invariant case. The existence of a positive $P^{(\alpha)}$ with $\alpha > 0$ may be interpreted as follows. According to the central limit theorem for independent identically distributed random variables, for our model with $J_{\ell\ell'} = 0$ and $V_\ell = V$, the only possibility to have a finite positive limit in (3.29) is to set $\alpha = 0$. If $P^{(0)} < \infty$ for nonzero interaction, one can say that the dependence between the temperature loops is weak; it holds for small $\hat{J}_0$. Of course, in this case $P^{(\alpha)} = 0$ for any $\alpha > 0$. If $P^{(\alpha)}$ gets positive for a certain $\alpha \in (0,1)$, one says that a strong dependence between the loops appears. In this case, the central limit theorem holds with an abnormal normalization. However, this dependence is not so strong to make p' discontinuous, which occurs for $\alpha = 1$, where a new law of large numbers comes to power. In statistical physics, the point at which $P^{(\alpha)} > 0$ for $\alpha \in (0,1)$ is called *a critical point.* The quantity $P^{(0)}$ is called *susceptibility,* it gets discontinuous at the critical point. Its singularity at this point is connected with the value of α for which $P^{(\alpha)} > 0$. The above analysis opens the possibility to extend the notion of the critical point to the models which are not translation invariant.

Definition 3.11 *The rotation invariant model has a critical point if there exist a van Hove sequence $\mathcal{L}$ and $\alpha \in (0,1)$ such that*

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|^{1+\alpha}} \int_{\Omega_\Lambda} \left| \sum_{\ell\in\Lambda} \int_0^{\beta} \omega_\ell(\tau)\mathrm{d}\tau \right|^2 \nu_\Lambda(\mathrm{d}\omega_\Lambda) > 0 \tag{3.31}$$

at certain values of the model parameters, including h and β.

Note that by Proposition 3.10, it follow that in the translation invariant case the notions of the critical point and of the second order phase transition coincide.

3.2 Infrared bound

Here, for the translation and rotation invariant version of our model, we find the function $\widehat{B}$ obeying (3.9).

For a box Λ, let E be the set of all unordered pairs $\langle \ell, \ell' \rangle$, $\ell, \ell' \in \Lambda$, such that $|\ell - \ell'|_\Lambda = 1$, see (2.80). Suppose also that the interaction intensities (2.78) are such that $J^\Lambda_{\ell\ell'} = J > 0$ if and only if $\langle \ell, \ell' \rangle \in E$ and hence the measure (2.82) can be written

$$\nu^{\mathrm{per}}_\Lambda(\mathrm{d}\omega_\Lambda) = \frac{1}{Y_\Lambda(0)} \exp\left(-\frac{J}{2} \sum_{\langle \ell, \ell' \rangle \in E} \|\omega_\ell - \omega_{\ell'}\|^2_{L^2_\beta} \right) \sigma_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.32}$$

where

$$\begin{aligned} &\sigma_\Lambda(\mathrm{d}\omega_\Lambda) \\ &\quad = \exp\left(Jd \sum_{\ell \in \Lambda} \|\omega_\ell\|^2_{L^2_\beta} - \sum_{\ell \in \Lambda} \int_0^\beta V(\omega_\ell(\tau))\mathrm{d}\tau \right) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{3.33}$$

and

$$Y_\Lambda(0) = \int_{\Omega_\Lambda} \exp\left(-\frac{J}{2} \sum_{\langle \ell, \ell' \rangle \in E} \|\omega_\ell - \omega_{\ell'}\|^2_{L^2_\beta} \right) \sigma_\Lambda(\mathrm{d}\omega_\Lambda). \tag{3.34}$$

With every edge $\langle \ell, \ell' \rangle \in E$ we associate $b_{\ell\ell'} \in L^2_\beta$ and consider

$$Y_\Lambda(b) = \int_{\Omega_\Lambda} \exp\left(-\frac{J}{2} \sum_{\langle \ell, \ell' \rangle \in E} \|\omega_\ell - \omega_{\ell'} - b_{\ell\ell'}\|^2_{L^2_\beta} \right) \sigma_\Lambda(\mathrm{d}\omega_\Lambda). \tag{3.35}$$

By standard arguments, see [46] and the references therein, one proves the following

Lemma 3.12 (Gaussian Domination) *For every $b = (b_{\ell\ell'})_{\langle \ell, \ell' \rangle \in E}$, $b_{\ell\ell'} \in L^2_\beta$, it follows that*

$$Y_\Lambda(b) \le Y_\Lambda(0). \tag{3.36}$$

Let $\mathcal{X}_E$ be the real Hilbert space

$$\mathcal{X}_E = \{ b = (b_{\ell\ell'})_{\langle \ell, \ell' \rangle \in E} \mid b_{\ell\ell'} \in L^2_\beta \}, \tag{3.37}$$

with scalar product

$$(b, c)_{\mathcal{X}_E} = \sum_{\langle \ell, \ell' \rangle \in E} (b_{\ell\ell'}, c_{\ell\ell'})_{L^2_\beta}. \tag{3.38}$$

To simplify notations we write $e = \langle \ell, \ell' \rangle$. A bounded linear operator $Q : \mathcal{X}_E \to \mathcal{X}_E$ may be defined by means of its kernel $Q^{jj'}_{ee'}(\tau, \tau')$, $j, j' = 1, \ldots, \nu$, $e, e' \in E$, and $\tau, \tau' \in [0, \beta]$. That is

$$(Qb)^{(j)}_e(\tau) = \sum_{j'=1}^d \sum_{e' \in E} \int_0^\beta Q^{jj'}_{ee'}(\tau, \tau') b^{(j')}_{e'}(\tau')\mathrm{d}\tau'. \tag{3.39}$$

Let us study the operator with the following kernel

$$Q^{jj'}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(\tau,\tau') = \left\langle \left[\omega^{(j)}_{\ell_1}(\tau) - \omega^{(j)}_{\ell_1'}(\tau)\right]\cdot\left[\omega^{(j')}_{\ell_2}(\tau') - \omega^{(j')}_{\ell_2'}(\tau')\right] \right\rangle_{\nu_\Lambda^{\rm per}}, \quad (3.40)$$

where the expectation is taken with respect to the measure (3.32). This operator in positive. Indeed,

$$(b, Qb)_{\mathcal{X}_E} = \left\langle \left[\sum_{\langle \ell,\ell'\rangle\in E} (\omega_\ell - \omega_{\ell'}, b_{\ell\ell'})_{L^2_\beta} \right]^2 \right\rangle_{\nu_\Lambda^{\rm per}} \geq 0.$$

The kernel (3.40) can be expressed in terms of the Matsubara functions; thus, as a function of τ, τ', it has the property (2.13). We employ the latter by introducing yet another Fourier transformation. Set

$$\mathcal{K} = \{k = (2\pi/\beta)\kappa \mid \kappa \in \mathbb{Z}\}, \quad (3.41)$$

$$e_k(\tau) = \begin{cases} \beta^{-1/2}\cos k\tau, & \text{if } k > 0; \\ -\beta^{-1/2}\sin k\tau, & \text{if } k < 0; \\ \sqrt{2/\beta}, & \text{if } k = 0. \end{cases} \quad (3.42)$$

The transformation we need is

$$\hat{\omega}^{(j)}_\ell(k) = \int_0^\beta \omega^{(j)}_\ell(\tau) e_k(\tau)\mathrm{d}\tau, \quad (3.43)$$

$$\omega^{(j)}_\ell(\tau) = \sum_{k\in\mathcal{K}} \hat{\omega}^{(j)}_\ell(k) e_k(\tau). \quad (3.44)$$

Then the property (2.13) yields, c.f., (3.4)

$$\langle \hat{\omega}^{(j)}_\ell(k)\hat{\omega}^{(j')}_{\ell'}(k')\rangle_{\nu_\Lambda^{\rm per}} = 0 \quad \text{if} \quad k \neq k', \quad \text{or} \quad j \neq j'.$$

Taking this into account we employ in (3.40) the transformation (3.43) and obtain

$$Q^{jj'}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(\tau,\tau') = \delta_{jj'} \sum_{k\in\mathcal{K}} \widehat{Q}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(k) e_k(\tau) e_k(\tau'), \quad (3.45)$$

with

$$\widehat{Q}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(k) = \left\langle \left[\hat{\omega}^{(j)}_{\ell_1}(k) - \hat{\omega}^{(j)}_{\ell_1'}(k)\right]\cdot\left[\hat{\omega}^{(j)}_{\ell_2}(k) - \hat{\omega}^{(j)}_{\ell_2'}(k)\right] \right\rangle_{\nu_\Lambda^{\rm per}}. \quad (3.46)$$

In view of the periodic conditions imposed on the boundaries of the box Λ the latter kernel, as well as the one given by (3.40), are invariant with respect to the translations of the corresponding torus. This allows us to 'diagonalize' the kernel (3.46) by means of a spatial Fourier transformation (3.2), (3.3). Then the spacial periodicity of the state $\langle\cdot\rangle_{\nu_\Lambda^{\rm per}}$ yields

$$\langle \hat{\omega}^{(j)}(p,k)\hat{\omega}^{(j)}(p',k)\rangle_{\nu_\Lambda^{\rm per}} = 0 \quad \text{if} \quad p + p' \neq 0. \quad (3.47)$$

Taking this into account we obtain

$$\begin{aligned}\widehat{Q}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(k) &= \sum_{p\in\Lambda_*}\langle\hat{\omega}^{(j)}(p,k)\hat{\omega}^{(j)}(-p,k)\rangle_{\nu^{\rm per}_{\beta,\Lambda}} \\ &\times \left(e^{\imath(p,\ell_1)} - e^{\imath(p,\ell_1')}\right)/|\Lambda|^{1/2} \\ &\times \left(e^{-\imath(p,\ell_2)} - e^{\imath(-p,\ell_2')}\right)/|\Lambda|^{1/2}.\end{aligned} \tag{3.48}$$

Since the summand corresponding to $p = 0$ equals zero, the sum can be restricted to $\Lambda_* \setminus \{0\}$. This representation however cannot serve as a spectral decomposition similar to (3.45) because the eigenfunctions here are not normalized. Indeed,

$$\sum_{\langle \ell,\ell'\rangle\in E}\left(e^{\imath(p,\ell)} - e^{\imath(p,\ell')}\right)/|\Lambda|^{1/2} \times \left(e^{-\imath(p,\ell)} - e^{-\imath(p,\ell')}\right)/|\Lambda|^{1/2} = 2\mathcal{E}(p)$$

where

$$\mathcal{E}(p) \stackrel{\rm def}{=} \sum_{j=1}^{d}[1-\cos p_j]. \tag{3.49}$$

Then we set

$$\sigma_{\ell\ell'}(p) = \left(e^{\imath(p,\ell)} - e^{\imath(p,\ell')}\right)/\sqrt{2|\Lambda|\mathcal{E}(p)}, \quad p\in\Lambda_*\setminus\{0\}, \tag{3.50}$$

and

$$\widehat{Q}(p,k) = 2\mathcal{E}(p)\langle\hat{\omega}^{(j)}(p,k)\hat{\omega}^{(j)}(-p,k)\rangle_{\nu^{\rm per}_{\Lambda}}, \quad p\in\Lambda_*\setminus\{0\}. \tag{3.51}$$

Thereby,

$$\begin{aligned}&Q_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(\tau,\tau') = \\ &= \sum_{p\in\Lambda_*\setminus\{0\}}\sum_{k\in\mathcal{K}}\widehat{Q}(p,k)\sigma_{\ell_1\ell_1'}(p)\sigma_{\ell_2\ell_2'}(-p)e_k(\tau)e_k(\tau'),\end{aligned} \tag{3.52}$$

which is the spectral decomposition of the operator (3.39). Now we show that the eigenvalues (3.51) have a specific upper bound[4].

Lemma 3.13 *For every $p \in \Lambda_* \setminus \{0\}$ and $k \in \mathcal{K}$, the eigenvalues (3.51) obey the estimate*

$$\widehat{Q}(p,k) \leq 1/J, \tag{3.53}$$

where J is the same as in (3.32). From this estimate one gets

$$\langle\hat{\omega}^{(j)}(p,k)\hat{\omega}^{(j)}(-p,k)\rangle_{\nu^{\rm per}_{\Lambda}} \leq \frac{1}{2J\mathcal{E}(p)}, \quad p\in\Lambda_*\setminus\{0\}. \tag{3.54}$$

Proof: The estimate in question will be obtained from the Gaussian domination (3.36). For $t \in \mathbb{R}$ and a given $b \in \mathcal{X}_E$, we consider the function $\phi(t) = Y_\Lambda(tb)$. By Lemma 3.12, $\phi''(0) \leq 0$. Computing the derivative from (3.35) we get

$$\phi''(0) = J(b,Qb)_{\mathcal{X}_E} - \|b\|^2_{\mathcal{X}_E},$$

[4] Their natural lower bound is zero as the operator (3.39) is positive

where the operator Q is defined by its kernel (3.40). Then the estimate (3.53) is immediate. $\square$

By (3.3), (3.45), and (3.51), we readily obtain

$$\langle (\hat{\omega}_p(\tau), \hat{\omega}_{-p}(\tau')) \rangle_{\nu_\Lambda^{\rm per}} = \frac{\nu}{2\beta \mathcal{E}(p)} \sum_{k \in \mathcal{K}} \widehat{Q}(p,k) \cos[k(\tau - \tau')], \quad p \neq 0,$$

which yields, see (3.5) and (3.53),

$$\widehat{D}_p^\Lambda = \frac{\beta\nu}{2\mathcal{E}(p)} \widehat{Q}(p,0) \leq \frac{\beta\nu}{2J\mathcal{E}(p)}, \quad p \neq 0. \tag{3.55}$$

Comparing this estimate with (3.9) we have the following

Corollary 3.14 *If the model is translation and rotation invariant with the nearest neighbor interaction, then the infrared estimate (3.9) holds with*

$$\widehat{B}(p) = \frac{\beta\nu}{2J\mathcal{E}(p)}, \quad p \in (-\pi, \pi]^d \setminus \{0\}, \qquad \widehat{B}(0) = +\infty. \tag{3.56}$$

3.3 Phase transition in the translation and rotation invariant model

In this subsection, we consider the model described by Corollary 3.14. First we obtain the lower bounds for

$$\langle (\omega_\ell(\tau), \omega_\ell(\tau)) \rangle_{\nu_\Lambda^{\rm per}},$$

from which we then obtain the bounds (3.16). In the case where the anharmonic potential has the form

$$V(u) = -b|u|^2 + b_2 |u|^4, \quad b > a/2, \quad b_2 > 0, \tag{3.57}$$

a being the same as in (1.1), the bound (3.16) can be found explicitly. We begin by considering this special case.

Lemma 3.15 *Let V be as in (3.57). Then, for every $\Lambda \Subset \mathbb{L}$,*

$$\langle (\omega_\ell(\tau), \omega_\ell(\tau)) \rangle_{\nu_\Lambda^{\rm per}} \geq \frac{(2b-a)\nu}{4b_2(\nu+2)} \stackrel{\rm def}{=} \vartheta_*. \tag{3.58}$$

Proof: Let A be a self-adjoint operator, such that the expressions below make sense. Then

$$\begin{aligned} &\varrho_\Lambda^{\rm per}\left([A, [H_\Lambda^{\rm per}, A]]\right) \qquad (3.59)\\ &= \varrho_{\beta,\Lambda}^{\rm per}\left(AH_\Lambda^{\rm per}A + AH_\Lambda^{\rm per}A - AAH_\Lambda^{\rm per} - H_\Lambda^{\rm per}AA\right) \\ &= \frac{1}{Z_{\beta,\Lambda}^{\rm per}} \sum_{s,s' \in \mathbb{N}} |A_{ss'}|^2 \left(E_{s'}^{\rm per} - E_s^{\rm per}\right) \left\{\exp\left[-\beta E_s^{\rm per}\right] - \exp\left[-\beta E_{s'}^{\rm per}\right]\right\} \\ &\geq 0. \end{aligned}$$

Here $E_s^{\rm per}$, $s \in \mathbb{N}$ are the eigenvalues of the periodic Hamiltonian (2.83), $A_{ss'}$ are the corresponding matrix elements of A, and $\varrho_\Lambda^{\rm per}$ is the periodic local Gibbs state (2.84). By the Euclidean representation,

$$\langle (\omega_\ell(\tau), \omega_\ell(\tau)) \rangle_{\nu_\Lambda^{\rm per}} = \sum_{j=1}^{\nu} \left\langle \left(\omega_\ell^{(j)}(0)\right)^2 \right\rangle_{\nu_{\beta,\Lambda}^{\rm per}} = \sum_{j=1}^{\nu} \varrho_\Lambda^{\rm per}\left[\left(q_\ell^{(j)}\right)^2\right].$$

Then we take in (3.59) $A = p_\ell^{(j)}$, $j = 1, \ldots, \nu$, make use of the commutation relation (1.4), take into account the rotation invariance, and arrive at

$$\begin{aligned} \varrho_\Lambda^{\mathrm{per}}\left([A, [H_\Lambda^{\mathrm{per}}, A]]\right) &= \varrho_{\beta,\Lambda}^{\mathrm{per}}\left(-2b + a + 2b_2|q_\ell|^2 + 4b_2(q_\ell^{(j)})^2\right) \\ &= -2b + a + 4b_2(\nu+2)\left\langle \left[\omega_\ell^{(j)}(0)\right]^2 \right\rangle_{\nu_{\beta,\Lambda}^{\mathrm{per}}} \\ &\geq 0, \end{aligned} \tag{3.60}$$

which yields (3.58). □

Now we consider the case where V is more general as to compare with (3.57).

Lemma 3.16 *Let the model be translation and rotation invariant, with nearest neighbor interaction. Then, for every $\theta > 0$, there exist positive m_* and J_*, which may depend on β, θ, and on the potential V, such that, for $m > m_*$ and $J > J_*$,*

$$\langle (\omega_\ell(\tau), \omega_\ell(\tau)) \rangle_{\nu_\Lambda^{\mathrm{per}}} \geq \theta. \tag{3.61}$$

Proof: Let us rewrite (2.104)

$$\begin{aligned} p_\Lambda^{\mathrm{per}}(J) &= \log N_\beta \\ &+ \frac{1}{|\Lambda|}\log\left\{\int_{\Omega_\Lambda} \exp\left[Y_\Lambda(\omega_\Lambda)\right] \prod_{\ell\in\Lambda} \lambda(\mathrm{d}\omega_\ell)\right\}, \end{aligned} \tag{3.62}$$

where we indicate the dependence of the pressure on the interaction intensity and have set $h = 0$ since the potential V should be rotation invariant. Clearly, $p_\Lambda^{\mathrm{per}}(J)$ is convex; its derivative can be computed from (3.62). Then we get

$$\begin{aligned} \frac{J}{|\Lambda|}\sum_{\langle \ell,\ell'\rangle\in E} \left\langle (\omega_\ell, \omega_{\ell'})_{L^2_\beta}\right\rangle_{\nu_\Lambda^{\mathrm{per}}} &= J\frac{\partial}{\partial J} p_\Lambda^{\mathrm{per}}(J) \\ &\geq p_\Lambda^{\mathrm{per}}(J) - p_\Lambda^{\mathrm{per}}(0) \\ &= \frac{1}{|\Lambda|}\log\left\{\int_{\Omega_\Lambda} \exp\left[Y_\Lambda(\omega_\Lambda)\right] \prod_{\ell\in\Lambda} \lambda(\mathrm{d}\omega_\ell)\right\}, \end{aligned} \tag{3.63}$$

where E is the same as in (3.37). By the translation invariance and (2.13), one gets

$$\begin{aligned} \left\langle (\omega_\ell, \omega_{\ell'})_{L^2_\beta}\right\rangle_{\nu_\Lambda^{\mathrm{per}}} &\leq \left(\left\langle (\omega_\ell, \omega_\ell)_{L^2_\beta}\right\rangle_{\nu_\Lambda^{\mathrm{per}}} + \left\langle (\omega_{\ell'}, \omega_{\ell'})_{L^2_\beta}\right\rangle_{\nu_\Lambda^{\mathrm{per}}}\right)/2 \\ &= \left\langle (\omega_\ell, \omega_\ell)_{L^2_\beta}\right\rangle_{\nu_\Lambda^{\mathrm{per}}} = \beta\left\langle (\omega_\ell(\tau), \omega_\ell(\tau))\right\rangle_{\nu_\Lambda^{\mathrm{per}}}. \end{aligned}$$

Then we choose ε, c, and n as in (2.109), apply this estimate in (3.63), and obtain

$$\begin{aligned} \beta J d\left\langle (\omega_\ell(\tau), \omega_\ell(\tau))\right\rangle_{\nu_\Lambda^{\mathrm{per}}} &\geq \frac{1}{|\Lambda|}\log\left\{\int_{[B^+(\varepsilon;c)]^{\nu|\Lambda|}} \exp\left[Y_\Lambda(\omega_\Lambda)\right] \prod_{\ell\in\Lambda} \lambda(\mathrm{d}\omega_\ell)\right\} \\ &\geq \beta J \nu d \varepsilon^2 + \nu \log\gamma(m). \end{aligned} \tag{3.64}$$

For $m > m_*$ given by (2.110), $\gamma(m) > 0$ and the latter estimate makes sense. Given $\theta > 0$, one picks $\varepsilon > \sqrt{\theta/\nu}$ and then finds J_* such that the right-hand side of the latter estimate equals θ for $J = J_*$. □

To convert (3.58) and (3.61) into the bound (3.16) we need the function $f : [0, +\infty) \to [0, 1)$ defined implicitly by

$$f(u \tanh u) = u^{-1} \tanh u, \quad \text{for} \quad u > 0; \quad \text{and} \quad f(0) = 1. \tag{3.65}$$

It is differentiable, convex, monotone decreasing on $(0, +\infty)$, such that $tf(t) \to 1$. For $t \geq 6$, $f(t) \approx 1/t$ to five-place accuracy, see Theorem A.2 in [24]. By direct calculation,

$$\frac{f'(u\tau)}{f(u\tau)} = -\frac{1}{u\tau} \cdot \frac{\tau - u(1-\tau^2)}{\tau + u(1-\tau^2)}, \qquad \tau = \tanh u. \tag{3.66}$$

Proposition 3.17 *For every fixed $\alpha > 0$, the function*

$$\phi(t) = t\alpha f(t/\alpha), \quad t > 0 \tag{3.67}$$

is differentiable and monotone increasing to α^2 as $t \to +\infty$.

Proof: By (3.66),

$$\phi'(t) = \frac{2\alpha\tau(1-\tau^2)}{\tau + u(1-\tau^2)} > 0, \qquad u\tau = u \tanh u = t/\alpha.$$

The limit α^2 is obtained from the corresponding asymptotic property of f. $\square$

Next, we need the following fact, known as Inequality of Bruch and Falk, see Theorem IV.7.5 on page 392 of [69] or Theorem 3.1 in [24].

Proposition 3.18 *Let A be as in (3.59). Let also*

$$b(A) = \beta^{-1} \int_0^\beta \varrho_\Lambda^{\rm per} \left\{ A \exp[-\tau H_\Lambda^{\rm per}] A \exp[\tau H_\Lambda^{\rm per}] \right\} {\rm d}\tau,$$
$$g(A) = \varrho_\Lambda^{\rm per}\left(A^2\right); \qquad c(A) = \varrho_\Lambda^{\rm per}\left\{ [A, [\beta H_\Lambda^{\rm per}, A]] \right\},$$

Then

$$b(A) \geq g(A) f\left(\frac{c(A)}{4g(A)}\right), \tag{3.68}$$

where f is the same as in (3.65).

Set

$$\mathcal{J}(d) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{{\rm d}p}{\mathcal{E}(p)}, \tag{3.69}$$

where $\mathcal{E}(p)$ is given by (3.49). The exact value of $\mathcal{J}(3)$ can be expressed in terms of complete elliptic integrals, see [83] and also [37] for more recent developments. For our aims, it is enough to have the following property, see Theorem 5.1 in [23].

Proposition 3.19 *For $d \geq 4$, one has*

$$\frac{1}{d - 1/2} < \mathcal{J}(d) < \frac{1}{d - \alpha(d)} < \frac{1}{d-1}, \tag{3.70}$$

where $\alpha(d) \to 1/2$ as $d \to +\infty$.

Recall that m is the reduced particle mass (1.3).

Theorem 3.20 *Let $d \geq 3$, the interaction be of nearest neighbor type, and the anharmonic potential be of the form (3.57), which defines the parameter ϑ_*. Let also the following condition be satisfied*

$$8m\vartheta_*^2 J > \mathcal{J}(d). \tag{3.71}$$

Then for every $\beta > \beta_*$, where the latter is the unique solution of the equation

$$2\beta J\vartheta_* f(\beta/4m\vartheta_*) = \mathcal{J}(d), \tag{3.72}$$

the model has a phase transition in the sense of Definition 3.1.

Proof: One observes that

$$[q_\ell^{(j)}, [H_\Lambda^{\rm per}, q_\ell^{(j)}]] = 1/m, \quad \ell \in \Lambda. \tag{3.73}$$

Then we take in (3.68) $A = q_\ell^{(j)}$ and obtain

$$b(A) \geq \left\langle \left(\omega_\ell^{(j)}(0)\right)^2 \right\rangle_{\nu_\Lambda^{\rm per}} f\left(\frac{\beta}{4m\left\langle \left(\omega_\ell^{(j)}(0)\right)^2 \right\rangle_{\nu_\Lambda^{\rm per}}}\right).$$

By Proposition 3.17, $\vartheta f(\beta/4m\vartheta)$ is an increasing function of ϑ. Thus, by (3.58) and (3.1),

$$D_{\ell\ell}^\Lambda \geq \beta^2 \nu\vartheta_* f(\beta/4m\vartheta_*), \tag{3.74}$$

which yields the bound (3.16). Thereby, the condition (i) in (3.17) takes the form

$$\vartheta_* f\left(\beta/4m\vartheta_*\right) > \mathcal{J}(d)/2\beta J. \tag{3.75}$$

By Proposition 3.17, the function

$$\phi(\beta) = 2\beta J\vartheta_* f(\beta/4m\vartheta_*)$$

is monotone increasing and hits the level $\mathcal{J}(d)$ at certain β_*. For $\beta > \beta_*$, the estimate (3.75) holds, which yields $|\mathcal{G}_\beta^{\rm t}| > 1$. $\square$

One observes that $f(\beta/4m\vartheta_*) \to 1$ as $m \to +\infty$. In this limit, the condition (3.71) turns into the corresponding condition for a classical model of ϕ^4 anharmonic oscillators, Now let us turn to the general case.

Theorem 3.21 *Let $d \geq 3$, the interaction be of nearest neighbor type, and the anharmonic potential be rotation invariant. Then, for every $\beta > 0$, there exist m_* and $J_* > 0$, which may depend on β and on the anharmonic potential, such that $|\mathcal{G}^{\rm t}| > 1$ for $m > m_*$ and $J > J_*$.*

Proof: Given positive β and θ, the estimate (3.61) for big enough m and J. Then one applies Proposition 3.18, which yields that the condition (i) in (3.17) is satisfied if

$$\theta f(\beta/4m\theta) > \mathcal{J}(d)/2\beta J.$$

Then one sets m_* to be as in (2.110) and J_* to be the smallest value of J for which both (3.61) and the latter inequality hold. $\square$

3.4 Phase transition in the symmetric scalar models

In the case $\nu = 1$, we can extend the above results to the models without translation invariance and with much more general $J_{\ell\ell'}$ and V_ℓ. However, certain assumptions beyond (2.1) and (2.2) should be made. Suppose also that the interaction between the nearest neighbors is uniformly nonzero, i.e.,

$$\inf_{|\ell-\ell'|=1} J_{\ell\ell'} \stackrel{\mathrm{def}}{=} J > 0. \tag{3.76}$$

Next we suppose that all V_ℓ's are even continuous functions and the upper bound in (2.79) can be chosen to obey the following conditions:

(a) for every ℓ,

$$V(u_\ell) - V_\ell(u_\ell) \leq V(\tilde{u}_\ell) - V_\ell(\tilde{u}_\ell), \quad \text{whenever } \ u_\ell^2 \leq \tilde{u}_\ell^2; \tag{3.77}$$

(b) the function V has the form

$$V(u_\ell) = \sum_{s=1}^{r} b^{(s)} u_\ell^{2s}; \quad 2b^{(1)} < -a; \quad b^{(s)} \geq 0, \ s \geq 2, \tag{3.78}$$

where a is as in (1.1) and $r \geq 2$ is either positive integer or infinite;

(c) if $r = +\infty$, the series

$$\varPhi(\vartheta) = \sum_{s=2}^{+\infty} \frac{(2s)!}{2^{s-1}(s-1)!} b^{(s)} \vartheta^{s-1}, \tag{3.79}$$

converges at some $\vartheta > 0$.

Since $2b^{(1)} + a < 0$, the equation

$$a + 2b^{(1)} + \varPhi(\vartheta) = 0, \tag{3.80}$$

has a unique solution $\vartheta_* > 0$. By the above assumptions, all V_ℓ are 'uniformly double-welled'. If $V_\ell(u_\ell) = v_\ell(u_\ell^2)$ and v_ℓ are differentiable, the condition (3.77) can be formulated as an upper bound for v'_ℓ. Note that the pressure as a unified characteristics of all Euclidean Gibbs states makes senses for translation invariant models only. Thus, the notions mentioned in Definition 3.7 are not applicable to the versions of the model which do not possess this property.

The main result of this subsection is contained in the following statement.

Theorem 3.22 *Let the model be as just described. Let also the condition (3.71) with ϑ_* defined by the equation (3.77) and J defined by (3.76) be satisfied. Then for every $\beta > \beta_*$, where β_* is defined by the equation (3.71), the model has a phase transition in the sense of Definition 3.1. If the model is translation invariant, the long range order and the first order phase transition take place at such β.*

Proof: The proof is made by comparing the model under consideration with a reference model, which is the scalar model with the nearest neighbor interaction of intensity (3.76) and with the anharmonic potential (3.78). Thanks to the condition (3.77), the reference model is more stable; hence, the phase transition in this model implies the same for the model considered. The comparison is conducted by means of the correlation inequalities.

The reference model is translation invariant and hence can be defined by its local periodic Hamiltonians

$$H_\Lambda^{\rm low} = \sum_{\ell\in\Lambda}\left[H_\ell^{\rm har} + V(q_\ell)\right] - J\sum_{\langle \ell,\ell'\rangle\in E} q_\ell q_{\ell'}, \tag{3.81}$$

where for a box Λ, E is the same as in (3.32); $H_\ell^{\rm har}$ is as in (1.1). For this model, we have the infrared estimate (3.55) with $\nu = 1$. Let us obtain the lower bound, see (3.58). To this end we use the inequalities (3.59), (3.60) and obtain

$$\begin{aligned} 0 \quad &\leq \quad a + 2b^{(1)} + \sum_{s=2}^{r} 2s(2s-1)b^{(s)}\left\langle\, [\omega_\ell(0)]^{2(s-1)}\,\right\rangle_{\nu_\Lambda^{\rm low}} && (3.82)\\ &\leq \quad a + 2b^{(1)} + \sum_{s=2}^{r} 2s(2s-1)\frac{(2s-2)!}{2^{s-1}(s-1)!}\cdot b^{(s)}\left[\left\langle\, (\omega_\ell(0))^2\,\right\rangle_{\nu_\Lambda^{\rm low}}\right]^{s-1}. \end{aligned}$$

Here $\nu_\Lambda^{\rm low}$ is the periodic Gibbs measure for the model (3.81). To get the second line we used the Gaussian upper bound inequality, see page 1031 in [54] and page 1372 in [6], which is possible since all $b^{(s)}$, $s \geq 2$ are nonnegative. The solution of the latter inequality is

$$\left\langle\, (\omega_\ell(0))^2\,\right\rangle_{\nu_\Lambda^{\rm low}} \geq \vartheta_*. \tag{3.83}$$

Then the proof of the phase transitions in the model (3.81) goes along the line of arguments used in proving Theorem 3.20. Thus, for $\beta > \beta_*$, $\langle\omega_\ell(0)\rangle_{\mu_+^{\rm low}} > 0$, where $\mu_+^{\rm low}$ is the corresponding maximal Euclidean Gibbs measure, see Proposition 2.18. But,

$$\langle\omega_\ell(0)\rangle_{\mu_+} > \langle\omega_\ell(0)\rangle_{\mu_+^{\rm low}}, \tag{3.84}$$

see Lemma 7.7 in [54]. At the same time $\langle\omega_\ell(0)\rangle_\mu = 0$ for any periodic $\mu \in \mathcal{G}^{\rm t}$, which yields the result to be proven. □

3.5 Phase transition in the scalar model with asymmetric potential

The phase transitions proven so far have a common feature – the spontaneous symmetry breaking. This means that the symmetry, e.g., rotation invariance, possessed by the model and hence by the unique element of $\mathcal{G}^{\rm t}$ is no longer possessed by the multiple Gibbs measures appearing as its result. In this subsection, we show that the translation invariant scalar version o the model (1.1), (1.2) has a phase transition without symmetry breaking. However, we restrict ourselves to the case of first order phase transitions, see Definition 3.7. The reason for this can be explained as follows. The fact that $D^\mu_{\ell\ell'}$ does not decay to zero as $|\ell-\ell'| \to +\infty$, see (3.18), implies that μ is non-ergodic only if μ is symmetric. Otherwise, to show that μ is non-ergodic one should prove that the difference $D^\mu_{\ell\ell'} - \langle f_\ell\rangle_\mu \cdot \langle f_{\ell'}\rangle_\mu$ does not decay to zero, which cannot be done by means of our methods based on the infrared estimate.

In what follows, we consider the translation invariant scalar version of the model (1.1), (1.2) with the nearest neighbor interaction. The only condition imposed on the anharmonic potential is (2.2). Obviously, we have to include the external field, that is the anharmonic potential is now $V(u) - hu$. Since we are not going to impose any conditions on the odd part of V, we cannot apply the GKS inequalities, see [6, 54], the comparison methods are based on, see (3.84). In view of this fact we suppose that the interaction is of nearest neighbor type. Thus, for a box Λ, the periodic local Hamiltonian of the model has the form (3.81).

In accordance with Definition 3.7, our goal is to show that the model parameters (except for h) and the inverse temperature β can be chosen in such a way that the set $\mathcal{R}$, defined by (2.100), is non-void. The main idea on how to do this can be explained as follows. First we find a condition, independent of h, under which $D^{\mu}_{\ell\ell'}$ does not decay to zero for a certain periodic μ. Next we prove the following

Lemma 3.23 *There exist $h_{\pm}$, $h_- < h_+$, which may depend on the model parameters and β, such that the magnetization (2.101) has the property:*

$$M(h) < 0, \quad \text{for} \quad h \in \mathcal{R}^c \cap (-\infty, h_-); \qquad M(h) > 0, \quad \text{for} \quad h \in \mathcal{R}^c \cap (h_+ + \infty).$$

Thereby, if $\mathcal{R}$ were void, one would find $h_* \in (h_-, h_+)$ such that $M(h_*) = 0$. At such h_*, the aforementioned property of D^{μ} would yield the non-ergodicity of μ and hence the first order phase transition, see Theorem 3.22.

In view of Corollary 3.4, $D^{\mu}_{\ell\ell'}$ does not decay to zero if (3.16) holds with big enough ϑ. By Proposition 3.18, the lower bound (3.16) can be obtained from the estimate (3.61). The only problem with the latter estimate is that it holds for $h = 0$.

Lemma 3.24 *For every $\beta > 0$ and θ, there exist positive m_* and J_*, which may depend on $\beta > 0$ and θ but are independent of h, such that, for any box Λ and any $h \in \mathbb{R}$,*

$$\left\langle [\omega_\ell(0)]^2 \right\rangle_{\nu_\Lambda^{\rm per}} \geq \theta, \qquad \text{if} \quad J > J_* \quad \text{and} \quad m > m_*. \tag{3.85}$$

Proof: For $h \in \mathbb{R}$, we set

$$\begin{aligned} \lambda^h(\mathrm{d}\omega) &= \frac{1}{N_\beta^h} \exp\left(h \int_0^\beta \omega(\tau)\mathrm{d}\tau \right) \lambda(\mathrm{d}\omega), \\ N_\beta^h &= \int_{C_\beta} \exp\left(h \int_0^\beta \omega(\tau)\mathrm{d}\tau \right) \lambda(\mathrm{d}\omega), \end{aligned} \tag{3.86}$$

where λ is as in (2.102). Then for $\pm h > 0$, we get the estimate (3.64) in the following form

$$\beta J d \left\langle [\omega_\ell(0)]^2 \right\rangle_{\nu_\Lambda^{\rm per}} \geq \beta J d \varepsilon^2 + \log \lambda^h \left[B^{\pm}(\varepsilon, c) \right], \tag{3.87}$$

where $B^{\pm}(\varepsilon, c)$ is as in (2.108), (2.109). Let us show now that, for $\pm h \geq 0$,

$$\lambda^h \left[B^{\pm}(\varepsilon, c) \right] \geq \lambda \left[B^{\pm}(\varepsilon, c) \right]. \tag{3.88}$$

For $h \geq 0$, let $I(\omega)$ be the indicator function of the set $C^+_\beta(n; c)$, see (2.106). For $\delta > 0$ and $t \in \mathbb{R}$, we set

$$\iota_\delta(t) = \begin{cases} 0 & t \leq c, \\ (t - c)/\delta & t \in (c, c + \delta], \\ 1 & c \geq c + \delta. \end{cases}$$

Thereby,

$$I_\delta(\omega) \stackrel{\rm def}{=} \prod_{k=0}^{n} \iota_\delta\left[\omega(k\beta/n)\right].$$

By Lebesgue's dominated convergence theorem

$$\begin{aligned} N_\beta^h \lambda^h \left[C_\beta^+(n;c)\right] &= \int_{C_\beta} I(\omega) \exp\left(h\int_0^\beta \omega(\tau){\rm d}\tau\right)\lambda({\rm d}\omega) \qquad (3.89)\\ &= \lim_{\delta\downarrow 0}\int_{C_\beta} I_\delta(\omega)\exp\left(h\int_0^\beta \omega(\tau){\rm d}\tau\right)\lambda({\rm d}\omega). \end{aligned}$$

As the function I_δ is continuous and increasing, by the FKG inequality, see Theorem 6.1 in [6], it follows that

$$\int_{C_\beta} I_\delta(x)\exp\left(h\int_0^\beta \omega(\tau){\rm d}\tau\right)\lambda({\rm d}\omega) \geq N_\beta^h\int_{C_\beta} I_\delta(\omega)\lambda({\rm d}\omega).$$

Passing here to the limit we obtain from (3.89)

$$\lambda^h\left[C_\beta^+(n;c)\right] \geq \lambda\left[C_\beta^+(n;c)\right],$$

which obviously yields (3.88). For $h \leq 0$, one just changes the signs of h and ω. Thereby, we can rewrite (3.87) as follows, c.f., (3.64),

$$\left\langle\, [\omega_\ell(0)]^2\,\right\rangle_{\nu_\Lambda^{\rm per}} \geq \varepsilon^2 + [\log\gamma(m)]/\beta J d.$$

Then one applies the arguments from the very end of the proof of Lemma 3.16. □

Proof of Lemma 3.23: Suppose that $h > 0$. Then restricting the integration in (2.104) to $[B^+(\varepsilon,c)]^\Lambda$, we get

$$\begin{aligned} p_\Lambda^{\rm per}(h) &\geq h\beta\varepsilon + \log N_\beta + \frac{1}{2}\beta\varepsilon^2\sum_{\ell'\in\Lambda} J_{\ell\ell'}^\Lambda + \log\lambda[B^+(\varepsilon,c)] \qquad (3.90)\\ &\geq h\beta\varepsilon + \log N_\beta + \log\gamma(m). \end{aligned}$$

As the right-hand side of the latter estimate is independent of Λ, it can be extended to the limiting pressure $p(h)$. For any positive $h \in \mathcal{R}^c$, by the convexity of $p(h)$ one has

$$\begin{aligned} M(h) &\geq [p(h) - p(0)]/\beta h\\ &\geq \varepsilon + \frac{1}{\beta h}\left\{-p(0) + \log N_\beta + \log\gamma(m)\right\}. \end{aligned}$$

Picking big enough h we get the positivity stated. The negativity can be proven in the same way. □

Now we are at a position to prove the main statement of this subsection.

Theorem 3.25 *Let the model be scalar, translation invariant, and with the nearest-neighbor interaction. Let also $d \geq 3$. Then for every β, there exist $m_* > 0$ and $J_* > 0$ such that, for all $m > m_*$ and $J > J_*$, there exists $h_* \in \mathbb{R}$, possibly dependent on m, β, and J, such that $p'(h)$ gets discontinuous at h_*, i.e., the model has a first order phase transition.*

Proof: Let m_* be as in (2.110) and J_*, θ be as in Lemma 3.24. Fix any $\beta > 0$ and $m > m_*$. Then, for $J > J_*$, the estimate (3.85) holds, which yields the validity of (3.74) for all boxes Λ with such β, m, and $\nu = 1$. Thereby, we increase J, if necessary, up to the value at which (3.75) holds. Afterwards, all the parameters, except for h, are set fixed. In this case, there exists a periodic state $\mu \in \mathcal{G}^{\rm t}$ such that the first summand in (3.15) is positive; hence, $D^{\mu}_{\ell\ell'}$ does not decay to zero as $|\ell - \ell'| \to +\infty$, see (3.12) and (3.15). If $p(h)$ is everywhere differentiable, i.e., if $\mathcal{R} = \emptyset$, then by Lemma 3.23 there exists h_* such that $M(h_*) = 0$; hence, the state μ with such h_* is non-ergodic, which yields $|\mathcal{G}^{\rm t}| > 1$ and hence a first order phase transition. Otherwise, $\mathcal{R} \neq \emptyset$. $\square$

3.6 Comments

- *Subsection 3.1:* According to Definition 3.1, the phase transition corresponds to the existence of multiple equilibrium phases at the same values of the model parameters and temperature. This is a standard definition for theories, which employ Gibbs states, see [30]. In the translation invariant case, a way of proving phase transitions can be to show the existence of non-ergodic elements of $\mathcal{G}^{\rm t}$. For classical lattice systems, it was realized in [28] by means of infrared estimates. More or less at the same time, an alternative rigorous theory of phase transitions in classical lattice spin models based on contour estimates has been proposed. This is the Pirogov-Sinai theory elaborated in [61], see also [70]. Later on, this theory was essentially extended and generalized into an abstract sophisticated method, applicable also to classical (but not quantum) models with unbounded spins, see [84] and the references therein.

  For quantum lattice models, the theory of phase transitions has essential peculiarities, which distinguish it from the corresponding theory of classical systems. Most of the results in this domain were obtained by means of quantum versions of the method of infrared estimates. The first publication in which such estimates were applied to quantum spin models seems to be the article [24]. After certain modifications this method was applied to a number of models with unbounded Hamiltonians [5, 15, 16, 23, 44, 60]. In our approach, the quantum crystal is described as a system of 'classical' infinite dimensional spins. This allows for applying here the original version of the method of infrared estimates elaborated in [28] adapted to the infinite dimensional case, which has been realized in the present work. Among others, the adaptation consists in employing such tools as the Garsia-Rodemich-Rumsey lemma, see [29]. This our approach is more effective and transparent than the one used in [5, 15, 16, 44]. It also allows for comparing the conditions (3.16), (3.17) with the stability conditions obtained in the next section.

  In the physical literature, there exist definitions of phase transitions alternative to Definition 3.1, based directly on the thermodynamic properties of the system. These are the definition employing the differentiability of the pressure (Definition 3.7, which is applicable to translation invariant models only), and the definition based on the long range order. The relationship between the latter two notions is established by means of the Griffiths theorem, Proposition 3.9, the proof of which can be found in [24]. For translation invariant models with bounded interaction, non-differentiability of the pressure corresponds to the non-uniqueness of the Gibbs states, see [36, 69]. We failed to prove this for our model.

In the language of limit theorems of probability theory, the appearance of the long range order corresponds to the fact that a new law of large numbers comes to power, see Theorem 3.9 and the discussion preceding Definition 3.11. The critical point of the model corresponds to the case where the law of large numbers still holds in its original form (in the translation invariant case this means absence of the first order phase transitions), but the central limit theorem holds true with an abnormal normalization. For a hierarchical version of the model (1.1), (1.2), the critical point was described in [4]. Algebras of abnormal fluctuation operators were studied in [20]. In application to quantum crystals, such operators were discussed in [80, 81], where the reader can find a more detailed discussion of this subject as well as the corresponding bibliography.

- *Subsection 3.2:* As was mentioned above, the method of infrared estimates was originated in [28]. The version employed here is close to the one presented in [46]. We note that, in accordance with the conditions (3.9),(3.16), and (3.17), the infrared bound was obtained for the Duhamel function, see (3.55), rather than for

$$\sum_{\ell' \in \Lambda} \langle (\omega_\ell(\tau), \omega_{\ell'}(\tau)) \rangle_{\nu_\Lambda^{\rm per}} \cdot \cos(p, \ell - \ell'),$$

which was used in [6, 15, 16, 44].

- *Subsection 3.3:* The lower bound (3.58) was obtained in the spirit of [23, 60]. The estimate stated in Lemma 3.16 is completely new; the key element of its proving is the estimate (2.105), obtained by means of Proposition 2.3. The sufficient condition for the phase transition obtained in Theorem 3.20 is also new. Its significant feature is the appearance of a universal parameter responsible for the phase transition, which includes the particle mass m, the anharmonicity parameter ϑ_*, and the interaction strength J. This is the parameter on the left-hand side of (3.71). The same very parameter will describe the stability of the model studied in the next section. Theorem 3.21 is also new.

- *Subsection 3.4:* Here we mostly repeat the corresponding results of [54], announced in [53].

- *Subsection 3.5:* The main characteristic feature of the scalar model studied in [5, 15, 16, 23, 44, 60], as well the the one described by Theorem 3.22, was the Z_2-symmetry broken by the phase transition. This symmetry allowed for obtaining estimates like (3.83), crucial for the method. However, in classical models, for proving phase transitions by means of the infrared estimates, symmetry was not especially important, see Theorem 3.5 in [28] and the discussion preceding this theorem. There might be two explanations of such a discrepancy: (a) the symmetry was the key element but only of the methods employed therein, and, like in the classical case, its lack does not imply the lack of phase transitions; (b) the symmetry is crucial in view of e.g. quantum effects, which stabilize the system, see the next section. So far, there has been no possibility to check which of these explanations is true. Theorem 3.25 solves this dilemma in favor of explanation (a). Its main element is again an estimate, obtained by means of the Garsia-Rodemich-Rumsey lemma. The corresponding result was announced in [39].

4 Quantum Stabilization

In physical substances containing light quantum particles moving in multi-welled potential fields phase transitions are experimentally suppressed by application of strong hydrostatic pressure, which makes the wells closer to each other and increases the tunneling of the particles. The same effect is achieved by replacing the particles with the ones having smaller mass. The aim of this section is to obtain a description of such effects in the framework of the theory developed here and to compare it with the theory of phase transitions presented in the previous section.

4.1 The stability of quantum crystals

Let us look at the scalar harmonic version of the model (1.1) – a quantum harmonic crystal. For this model, the one-particle Hamiltonian includes first two terms of (1.2) only. Its spectrum consists of the eigenvalues $E_n^{\rm har} = (n + 1/2)\sqrt{a/m}$, $n \in \mathbb{N}_0$. The parameter $a > 0$ is the oscillator rigidity. For reasons, which become clear in a while, we consider the following gap parameter

$$\varDelta^{\rm har} = \min_{n\in\mathbb{N}} (E_n^{\rm har} - E_{n-1}^{\rm har}). \tag{4.1}$$

Then

$$\varDelta^{\rm har} = \sqrt{a/m}; \qquad a = m\varDelta_{\rm har}^2. \tag{4.2}$$

The set of tempered Euclidean Gibbs measures of the harmonic crystal can be constructed similarly as it was done in section 2, but with one exception. Such measures exist only under the stability condition (2.71), which might now be rewritten

$$\hat{J}_0 < m\varDelta_{\rm har}^2. \tag{4.3}$$

In this case, $\mathcal{G}^{\rm t}$ is a singleton at all β, that readily follows from Theorem 2.14. As the right-hand side of (4.3) is independent of m, this stability condition is applicable also to the classical harmonic crystal which is obtained in the classical limit $m \to +\infty$, see [6]. According to (2.2) the anharmonic potentials V_ℓ have a super-quadratic growth due to which the tempered Euclidean Gibbs measures of anharmonic crystals exist for all $\hat{J}_0$. In this case, the instability of the crystal is connected with phase transitions. A sufficient condition for some of the models described in the previous section to have a phase transition may be derived from the equation (3.75). It is

$$2\beta J\vartheta_* f(\beta/4m\vartheta_*) > \mathcal{J}(d), \tag{4.4}$$

which in the classical limit $m \to +\infty$ takes the form

$$2\beta J\vartheta_* > \mathcal{J}(d).$$

The latter condition can be satisfied by picking big enough β. Therefore, the classical anharmonic crystals always have phase transitions – no matter how small is the interaction intensity. For finite m, the left-hand side of (4.4) is bounded by $8m\vartheta_*^2 J$, and the bound is achieved in the limit $\beta \to +\infty$. If for given values of the interaction parameter J, the mass m, and the parameter ϑ_* which characterizes the anharmonic potential, this bound does not exceed $\mathcal{J}(d)$, the condition (4.4) will never be satisfied. Although this condition is only sufficient, one might expect that the phase transition can be eliminated at all β if the compound parameter $8m\vartheta_*^2 J$ is small enough. Such an effect, if really exists, could be called *quantum stabilization* since it is principally impossible in the classical analog of the model.

4.2 Quantum rigidity

In the harmonic case, big values of the rigidity a ensure the stability. In this subsection, we introduce and stugy *quantum rigidity*, which plays a similar role in the anharmonic case

Above the sufficient condition (4.4) for a phase transition to occur was obtained for a simplified version of the model (1.1), (1.2) – nearest neighbor interactions, polynomial anharmonic potentials of special kind (3.78), ect. Then the results were extended to more general models via correlation inequalities. Likewise here, we start with a simple scalar version of the one-particle Hamiltonian (1.1), which we take in the form

$$H_m = \frac{1}{2m}p^2 + \frac{a}{2}q^2 + V(q), \tag{4.5}$$

where the anharmonic potential is, c.f., (3.78),

$$V(q) = b^{(1)}q^2 + b^{(2)}q^4 + \cdots + b^{(r)}q^{2r}, \qquad b^{(r)} > 0, \quad r \in \mathbb{N} \setminus \{1\}. \tag{4.6}$$

The subscript m in (4.5) indicates the dependence of the Hamiltonian on the mass. Recall that H_m acts in the physical Hilbert space $L^2(\mathbb{R})$. Its relevant properties are summarized in the following

Proposition 4.1 *The Hamiltonian H_m is essentially self-adjoint on the set $C_0^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support. The spectrum of H_m has the following properties: (a) it consists of eigenvalues E_n, $n \in \mathbb{N}_0$ only; (b) to each E_n there corresponds exactly one eigenfunction $\psi_n \in L^2(\mathbb{R})$; (c) there exists $\gamma > 1$ such that*

$$n^{-\gamma}E_n \to +\infty, \qquad \text{as} \quad n \to +\infty. \tag{4.7}$$

Proof: The essential self-adjointness of H_m follows from the Sears theorem, see Theorem 1.1, page 50 of [17] or Theorem X.29 of [64]. The spectral properties follow from Theorem 3.1, page 57 (claim (a)) and Proposition 3.3, page 65 (claim (b)), both taken from the book [17]. To prove claim (c) we employ a classical formula, see equation (7.7.4), page 151 of the book [77], which in our context reads

$$\frac{2}{\pi}\sqrt{2m}\int_0^{u_n} \sqrt{E_n - V(u)}\,\mathrm{d}u = n + \frac{1}{2} + O\left(\frac{1}{n}\right), \tag{4.8}$$

where n, and hence E_n, are big enough so that the equation

$$V(u) = E_n \tag{4.9}$$

have the unique positive solution u_n. Then

$$u_n^{r+1} \int_0^1 \sqrt{\phi_n(t) - t^{2r}}\, \mathrm{d}t = \frac{\pi}{2\sqrt{2mb^{(r)}}} \left(n + \frac{1}{2}\right) + O\left(\frac{1}{n}\right), \tag{4.10}$$

where

$$\phi_n(t) = \frac{E_n}{b^{(r)} u_n^{2r}} - \frac{u_n^{2-2r}}{b^{(r)}}(b^{(1)} + a/2)t^2 - \cdots - \frac{u_n^{-2}}{b^{(r)}} b^{(r-1)} t^{2(r-1)}.$$

Note that $\phi_n(1) = 1$ for all n, which follows from (4.9). Thus,

$$\frac{E_n}{b^{(r)} u_n^{2r}} \to 1, \qquad \text{as} \ \ n \to +\infty. \tag{4.11}$$

Thereby, we have

$$\begin{aligned} c_n \ &\stackrel{\text{def}}{=} \ \int_0^1 \sqrt{\phi_n(t) - t^{2r}}\, \mathrm{d}t \to \int_0^1 \sqrt{1 - t^{2r}}\, \mathrm{d}t \\ &= \ \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2r}\right)}{2r\Gamma\left(\frac{3}{2} + \frac{1}{2r}\right)}. \end{aligned} \tag{4.12}$$

Then combining (4.12) with (4.9) and (4.11) we get

$$E_n = \left[\frac{b^{(r)}}{(2m)^r}\right]^{1/(r+1)} \cdot \left[\frac{\pi r \Gamma\left(\frac{3}{2} + \frac{1}{2r}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2r}\right)} \cdot \left(n + \frac{1}{2}\right)\right]^{\frac{2r}{r+1}} + o\,(1)\,, \tag{4.13}$$

which readily yields (4.7) with any $\gamma \in (1, 2r/(r+1))$. $\square$

Thus, in view of the property (4.13) we introduce the gap parameter

$$\varDelta_m = \min_{n \in \mathbb{N}} (E_n - E_{n-1}), \tag{4.14}$$

and thereby, c.f., (4.2),

$$\mathcal{R}_m = m \varDelta_m^2, \tag{4.15}$$

which can be called *quantum rigidity* of the oscillator. One might expect that the stability condition for quantum anharmonic crystals, at least for their scalar versions with the anharmonic potentials independent of ℓ, is similar to (4.3). That is, it has the form

$$\hat{J}_0 < \mathcal{R}_m. \tag{4.16}$$

4.3 Properties of quantum rigidity

Below $f \sim g$ means that $\lim(f/g) = 1$.

Theorem 4.2 *For every $r \in \mathbb{N}$, the gap parameter $\varDelta_m$, and hence the quantum rigidity $\mathcal{R}_m$ corresponding to the Hamiltonian (4.5), (4.6), are continuous functions of m. Furthermore,*

$$\varDelta_m \sim \varDelta_0 m^{-r/(r+1)}, \qquad \mathcal{R}_m \sim \varDelta_0^2 m^{-(r-1)/(r+1)}, \qquad m \to 0, \tag{4.17}$$

with a certain $\varDelta_0 > 0$.

Proof: Given $\alpha > 0$, let $U_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the following unitary operator

$$(U_\alpha \psi)(x) = \sqrt{\alpha}\psi(\alpha x). \tag{4.18}$$

Then by (1.4)

$$U_\alpha^{-1} p U_\alpha = \alpha p, \qquad U_\alpha^{-1} q U_\alpha = \alpha^{-1} q.$$

Fix any $m_0 > 0$ and set $\rho = (m/m_0)^{1/(r+1)}$, $\alpha = \rho^{1/2}$. Then

$$\widetilde{H}_m \stackrel{\text{def}}{=} U_\alpha^{-1} H_m U_\alpha = \rho^{-r} T(\rho), \tag{4.19}$$

where

$$\begin{aligned} T(\rho) &= H_{m_0} + Q(\rho) && (4.20)\\ &= \frac{1}{2m_0}p^2 + \rho^{r-1}(b^{(1)} + a/2)q^2 + \rho^{r-2}b^{(2)}q^4 + \cdots + b^{(r)}q^{2r}, \\ Q(\rho) &= (\rho - 1)\Big[p_{r-1}(\rho)(b^{(1)} + a/2)q^2 && (4.21)\\ &+ p_{r-2}(\rho)b^{(2)}q^4 + \cdots + p_{r-s}(\rho)b^{(s)}q^{2s} + \cdots + b^{(r-1)}q^{2(r-1)}\Big], \end{aligned}$$

and

$$p_k(\rho) = 1 + \rho + \rho^2 + \cdots + \rho^{k-1}. \tag{4.22}$$

As the operators H_m, $\widetilde{H}_m$, are unitary equivalent, their gap parameters (4.14) coincide. The operators $\widetilde{H}_m$ and $T(\rho)$, $\rho > 0$ possess the properties established by Proposition 4.1. In particular, they have the property (4.7) with one and the same γ. Therefore, there exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for $|\rho - 1| < \varepsilon$, the gap parameters (4.14) for $\widetilde{H}_m$ and $T(\rho)$ are defined by the first k eigenvalues of these operators. As an essentially self-adjoint operator, $T(\rho)$ possesses a unique self-adjoint extension $\hat{T}(\rho)$, the eigenvalues of which coincide with those of $T(\rho)$. Furthermore, for complex ρ, $\hat{T}(\rho)$ is a closed operator, its domain $Dom[\hat{T}(\rho)]$ does not depend on ρ. For every $\psi \in Dom[\hat{T}(\rho)]$, the map $\mathbb{C} \ni \zeta \mapsto \hat{T}(\zeta)\psi \in L^2(\mathbb{R})$ is holomorphic. Therefore, $\{\hat{T}(\rho) \mid |\rho - 1| < \varepsilon\}$ is a self-adjoint holomorphic family. Hence, the eigenvalues $\Theta_n(\rho)$, $n \in \mathbb{N}_0$ of $\hat{T}(\rho)$ are continuous functions of $\rho \in (1 - \varepsilon, 1 + \varepsilon)$, see Chapter VII, §3 in the book [40]. At $\rho = 1$ they coincide with those of $\hat{H}_{m_0}$. Since we have given $k \in \mathbb{N}$ such that, for all $\rho \in (1 - \varepsilon, 1 + \varepsilon)$,

$$\min_{n\in\mathbb{N}} [\Theta_n(\rho) - \Theta_{n-1}(\rho)] = \min_{n\in\{1,2,\ldots,k\}} [\Theta_n(\rho) - \Theta_{n-1}(\rho)],$$

the function

$$\widetilde{\Delta}(\rho) \stackrel{\text{def}}{=} \min_{n\in\mathbb{N}} \rho^{-r}[\Theta_n(\rho) - \Theta_{n-1}(\rho)] \tag{4.23}$$

is continuous. But by (4.19)

$$\Delta_m = \widetilde{\Delta}\left((m/m_0)^{1/(r+1)}\right), \tag{4.24}$$

which proves the continuity stated since $m_0 > 0$ has been chosen arbitrarily.

To prove the second part of the theorem we rewrite (4.20) as follows

$$T(\rho) = H_{m_0}^{(0)} + R(\rho), \tag{4.25}$$

where

$$H^{(0)}_{m_0} = \frac{1}{2m_0}p^2 + b^{(r)}q^{2r},$$

and

$$R(\rho) = \rho\left(\rho^{r-2}(b^{(1)} + a/2)q^2 + \rho^{r-3}b^{(2)}q^4 + \cdots + b^{(r-1)}q^{2(r-1)}\right).$$

Repeating the above perturbation arguments one concludes that the self-adjoint family $\{\hat{T}(\rho) \mid |\rho| < \varepsilon\}$ is holomorphic at zero; hence, the gap parameter of (4.25) tends, as $\rho \to 0$, to that of $H^{(0)}_{m_0}$, i.e., to $\varDelta_0$. Thereby, the asymptotics (4.17) for $\varDelta_m$ follows from (4.19) and the unitary equivalence of H_m and $\widetilde{H}_m$. $\square$

Our second result in this domain is the quasi-classical analysis of the parameters (4.14), (4.15). Here we shall suppose that the anharmonic potential V has the form (4.6) with $b^{(s)} \geq 0$ for all $s = 2, \ldots, r-1$, c.f., (3.78). We remind that in this case the parameter $\vartheta_* > 0$ is the unique solution of the equation (3.79).

Theorem 4.3 *Let V be as in (3.78). Then the gap parameter $\varDelta_m$ and the quantum rigidity $\mathcal{R}_m$ of the Hamiltonian (4.5) with such V obey the estimates*

$$\varDelta_m \leq \frac{1}{2m\vartheta_*}, \qquad \mathcal{R}_m \leq \frac{1}{4m\vartheta_*^2}. \tag{4.26}$$

Proof: Let ϱ_m be the local Gibbs state (2.5) corresponding to the Hamiltonian (4.5). Then by means of the inequality (3.59) and the Gaussian upper bound we get, see (3.82),

$$a + 2b^{(1)} + \varPhi\left(\varrho_m(q^2)\right) \geq 0,$$

by which

$$\varrho_m(q^2) \geq \vartheta_*. \tag{4.27}$$

Let ψ_n, $n \in \mathbb{N}_0$ be the eigenfunctions of the Hamiltonian H_m corresponding to the eigenvalues E_n. By Proposition 4.1, to each E_n there corresponds exactly one ψ_n. Set

$$Q_{nn'} = (\psi_n, q\psi_{n'})_{L^2(\mathbb{R})}, \quad n, n' \in \mathbb{N}_0.$$

Obviously, $Q_{nn} = 0$ for any $n \in \mathbb{N}_0$. Consider

$$\Gamma(\tau, \tau') = \varrho_m\left[q\exp\left(-(\tau' - \tau)H_m\right)q\exp\left(-(\tau - \tau')H_m\right)\right], \quad \tau, \tau' \in [0, \beta],$$

which is the Matsubara function corresponding to the state ϱ_m and the operators $F_1 = F_2 = q$. Set

$$\hat{u}(k) = \int_0^\beta \Gamma(0, \tau)\cos k\tau \mathrm{d}\tau, \qquad k \in \mathcal{K} = \{(2\pi/\beta)\kappa \mid \kappa \in \mathbb{Z}\}. \tag{4.28}$$

Then

$$\begin{aligned} \hat{u}(k) &= \frac{1}{Z_m}\sum_{n,n'=0}^{+\infty}|Q_{nn'}|^2\frac{E_n - E_{n'}}{k^2 + (E_n - E_{n'})^2} \\ &\times \left\{\exp(-\beta E_{n'}) - \exp(-\beta E_n)\right\}, \end{aligned} \tag{4.29}$$

where $Z_m = \mathrm{trace}\exp(-\beta H_m)$. The term $(E_n - E_{n'})^2$ in the denominator can be estimated by means of (4.14), which yields

$$\begin{aligned}
\hat{u}(k) &\leq \frac{1}{k^2+\Delta_m^2}\cdot\frac{1}{Z_m}\sum_{n,n'=0}^{+\infty}\left|Q_{nn'}\right|^2(E_n - E_{n'}) \qquad (4.30)\\
&\times \{\exp(-\beta E_n) - \exp(-\beta E_{n'})\}\\
&\leq \frac{1}{k^2+\Delta_m^2}\cdot\varrho_m([q,[H_m,q]])\\
&= \frac{1}{m(k^2+\Delta_m^2)}.
\end{aligned}$$

By this estimate we get

$$\begin{aligned}
\varrho_m(q^2) &= \Gamma(0,0) = \frac{1}{\beta}\sum_{k\in\mathcal{K}} u(k) \qquad (4.31)\\
&\leq \frac{1}{\beta}\sum_{k\in\mathcal{K}}\frac{1}{m(k^2+\Delta_m^2)} = \frac{1}{2m\Delta_m}\coth(\beta\Delta_m/2)\,.
\end{aligned}$$

Combining the latter estimate with (4.27) we arrive at

$$\Delta_m \tanh(\beta\Delta_m/2) < 1/(2m\vartheta_*),$$

which yields (4.26) in the limit $\beta \to +\infty$. $\square$

Now let us analyze the quantum stability condition (4.16) in the light of the latter results. The first conclusion is that, unlike to the case of harmonic oscillators, this condition can be satisfied for all $\hat{J}_0$ by letting the mass be small enough. For the nearest-neighbor interaction, one has $\hat{J}_0 = 2dJ$; hence, if (4.16) holds, then

$$8dm\vartheta_*^2 J < 1. \qquad (4.32)$$

This can be compared with the estimate

$$8dm\vartheta_*^2 J > d\mathcal{J}(d), \qquad (4.33)$$

guaranteeing a phase transition, which one derives from (4.4).

For finite d, $d\mathcal{J}(d) > 1$, see Proposition 3.19; hence, there is a gap between the latter estimate and (4.32), which however diminishes as $d \to +\infty$ since

$$\lim_{d\to+\infty} d\mathcal{J}(d) = 1.$$

In the remaining part of this section, we show that for the quantum crystals, both scalar and vector, a stability condition like (4.16) yields a sufficient decay of the pair correlation function. In the scalar case, this decay guaranties the uniqueness of tempered Euclidean Gibbs measures. However, in the vector case it yields a weaker result – suppression of the long range order and of the phase transitions of any order in the sense of Definition 3.7. The discrepancy arises from the fact that the uniqueness criteria based on the FKG inequalities are applicable to scalar models only.

4.4 Decay of correlations in the scalar case

In this subsection, we consider the model (1.1), (1.2) which is (a) translation invariant; (b) scalar; (c) the anharmonic potential is $V(q) = v(q^2)$ with v being convex on $\mathbb{R}_+$.

Let Λ be the box (2.63) and Λ_* be its conjugate (3.2). For this Λ, let

$$K^{\Lambda}_{\ell\ell'}(\tau,\tau') \stackrel{\rm def}{=} \left\langle \omega_\ell(\tau)\omega_{\ell'}(\tau')\right\rangle_{\nu^{\rm per}_\Lambda} \tag{4.34}$$

be the periodic correlation function. Recall that the periodic interaction potential $J^{\Lambda}_{\ell\ell'}$ was defined by (2.78). For the one-particle Hamiltonian (1.2), let $\hat{u}(k)$ be as in (4.28).

Theorem 4.4 *Let the model be as just describes. If*

$$\hat{u}(0)\hat{J}_0 < 1, \tag{4.35}$$

then

$$K^{\Lambda}_{\ell\ell'}(\tau,\tau') \leq \frac{1}{\beta|\Lambda|}\sum_{p\in\Lambda_*}\sum_{k\in\mathcal{K}} \frac{\exp\left[\imath(p,\ell-\ell') + \imath k(\tau-\tau')\right]}{[\hat{u}(k)]^{-1} - \hat{J}^{\Lambda}_0 + \Upsilon^{\Lambda}(p)}, \tag{4.36}$$

where

$$\hat{J}^{\Lambda}_0 = \sum_{\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}, \qquad \Upsilon^{\Lambda}(p) = \hat{J}^{\Lambda}_0 - \sum_{\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}\exp[\imath(p,\ell-\ell')]. \tag{4.37}$$

Proof: Along with the periodic local Gibbs measure (2.82) we introduce

$$\begin{aligned}&\nu^{\rm per}_\Lambda({\rm d}\omega_\Lambda|t) \\ &\quad = \frac{1}{N^{\rm per}_\Lambda(t)}\exp\left\{\frac{t}{2}\sum_{\ell,\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} - \int_0^\beta \sum_{\ell\in\Lambda} V(\omega_\ell(\tau)){\rm d}\tau\right\}\chi_\Lambda({\rm d}\omega_\Lambda),\end{aligned} \tag{4.38}$$

where $t\in[0,1]$ and $N^{\rm per}_\Lambda(t)$ is the corresponding normalization factor. Thereby, we set

$$X_{\ell\ell'}(\tau,\tau'|t) = \langle\omega_\ell(\tau)\omega_{\ell'}(\tau')\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)}, \quad \ell,\ell'\in\Lambda. \tag{4.39}$$

By direct calculation

$$\begin{aligned}&\frac{\partial}{\partial t}X_{\ell\ell'}(\tau,\tau'|t) \\ &\quad = \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda} J^{\Lambda}_{\ell_1\ell_2}\int_0^\beta R_{\ell\ell'\ell_1\ell_2}(\tau,\tau',\tau'',\tau''|t){\rm d}\tau'' \\ &\quad + \sum_{\ell_1,\ell_2\in\Lambda} J^{\Lambda}_{\ell_1\ell_2}\int_0^\beta X_{\ell\ell_1}(\tau,\tau''|t)X_{\ell_2\ell'}(\tau'',\tau'|t){\rm d}\tau'',\end{aligned} \tag{4.40}$$

where

$$\begin{aligned} R_{\ell_1\ell_2\ell_3\ell_4}(\tau_1,\tau_2,\tau_3,\tau_4|t) &= \langle\omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\omega_{\ell_3}(\tau_3)\omega_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)} \\ &- \langle\omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)}\cdot\langle\omega_{\ell_3}(\tau_3)\omega_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)} \\ &- \langle\omega_{\ell_1}(\tau_1)\omega_{\ell_3}(\tau_3)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)}\cdot\langle\omega_{\ell_2}(\tau_2)\omega_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)} \\ &- \langle\omega_{\ell_1}(\tau_1)\omega_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)}\cdot\langle\omega_{\ell_2}(\tau_2)\omega_{\ell_3}(\tau_3)\rangle_{\nu^{\rm per}_\Lambda(\cdot|t)}.\end{aligned}$$

By the Lebowitz inequality, see [6], we have

$$R_{\ell_1\ell_2\ell_3\ell_4}(\tau_1,\tau_2,\tau_3,\tau_4|t) \leq 0, \tag{4.41}$$

holding for all values of the arguments. Let us consider (4.40) as an integro-differential equation subject to the initial condition

$$X_{\ell\ell'}(\tau,\tau'|0) = \delta_{\ell\ell'}\Gamma(\tau,\tau') = (\delta_{\ell\ell'}/\beta)\sum_{k\in\mathcal{K}}\hat{u}(k)\cos k(\tau-\tau'). \tag{4.42}$$

Besides, we also have

$$X_{\ell\ell'}(\tau,\tau'|1) = K^{\Lambda}_{\ell\ell'}(\tau,\tau'|p). \tag{4.43}$$

Along with the Cauchy problem (4.40), (4.42) let us consider the following equation

$$\frac{\partial}{\partial t}Y_{\ell\ell'}(\tau,\tau'|t) = \sum_{\ell_1,\ell_2\in\Lambda}\left[J^{\Lambda}_{\ell_1\ell_2} + \frac{\varepsilon}{|\Lambda|}\right]\int_0^{\beta} Y_{\ell\ell_1}(\tau,\tau''|t)Y_{\ell_2\ell'}(\tau'',\tau'|t)\mathrm{d}\tau'', \tag{4.44}$$

where $\varepsilon > 0$ is a parameter, subject to the initial condition

$$\begin{aligned} Y_{\ell\ell'}(\tau,\tau'|0) &= X_{\ell\ell'}(\tau,\tau'|0) \\ &= (\delta_{\ell\ell'}/\beta)\sum_{k\in\mathcal{K}}\hat{u}(k)\cos k(\tau-\tau'). \end{aligned} \tag{4.45}$$

Let us show that under the condition (4.35) there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0,\varepsilon_0)$, the problem (4.44), (4.45), $t\in[0,1]$, has the unique solution

$$Y_{\ell\ell'}(\tau,\tau'|t) = \frac{1}{\beta|\Lambda|}\sum_{p\in\Lambda_*}\sum_{k\in\mathcal{K}}\frac{\exp\left[\imath(p,\ell-\ell')+\imath k(\tau-\tau')\right]}{[\hat{u}(k)]^{-1} - t[\hat{J}^{\Lambda}_0 + \varepsilon\delta_{p,0}] + t\,\Upsilon^{\Lambda}(p)}, \tag{4.46}$$

where $\hat{J}_0$, $\Upsilon^{\Lambda}(p)$ are the same as in (4.37) and $\delta_{p,0}$ is the Kronecker symbol with respect to each of the components of p. By means of the Fourier transformation

$$\begin{aligned} Y_{\ell\ell'}(\tau,\tau'|t) &= \frac{1}{\beta|\Lambda|}\sum_{p\in\Lambda_*}\sum_{k\in\mathcal{K}}\widehat{Y}(p,k|t)\exp\left[\imath(p,\ell-\ell')+\imath k(\tau-\tau')\right], \\ \widehat{Y}(p,k|t) &= \sum_{\ell'\in\Lambda}\int_0^{\beta}Y_{\ell\ell'}(\tau,\tau'|t)\exp\left[-\imath(p,\ell-\ell')-\imath k(\tau-\tau')\right]\mathrm{d}\tau', \end{aligned} \tag{4.47}$$

we bring (4.44), (4.45) into the following form

$$\frac{\partial}{\partial t}\widehat{Y}(p,k|t) = \left[\hat{J}^{\Lambda}(p) + \varepsilon\delta_{p,0}\right]\cdot\left[\widehat{Y}(p,k|t)\right]^2, \quad \widehat{Y}(p,k|0) = \hat{u}(k), \tag{4.48}$$

where, see (4.37),

$$\hat{J}^{\Lambda}(p) = \sum_{\ell'\in\Lambda}J^{\Lambda}_{\ell\ell'}\exp\left[\imath(p,\ell-\ell')\right] = \hat{J}^{\Lambda}_0 - \Upsilon^{\Lambda}(p). \tag{4.49}$$

Clearly, $\hat{J}^{\Lambda}_0 \leq \hat{J}_0$, $|\hat{J}^{\Lambda}(p)| \leq \hat{J}^{\Lambda}_0$, and $\hat{u}(k) \leq \hat{u}(0)$. Then in view of (4.35), one finds $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, the following holds

$$\left[\hat{J}^{\Lambda}(p) + \varepsilon\delta_{p,0}\right]\hat{u}(k) < 1,$$

for all $p \in \Lambda_*$ and $k \in \mathcal{K}$. Thus, the problem (4.48) can be solved explicitly, which via the transformation (4.47) yields (4.46).

Given $\theta \in (0,1)$, we set

$$Y^{(\theta)}_{\ell\ell'}(\tau,\tau'|t) = Y_{\ell\ell'}(\tau,\tau'|t+\theta), \quad t \in [0, 1-\theta]. \tag{4.50}$$

Obviously, the latter function obeys the equation (4.44) on $t \in [0, 1-\theta]$ with the initial condition

$$Y^{(\theta)}_{\ell\ell'}(\tau,\tau'|0) = Y_{\ell\ell'}(\tau,\tau'|\theta) > Y_{\ell\ell'}(\tau,\tau'|0) = X_{\ell\ell'}(\tau,\tau'|0). \tag{4.51}$$

The latter inequality is due to the positivity of both sides of (4.44). Therefore,

$$Y^{(\theta)}_{\ell\ell'}(\tau,\tau'|t) > 0, \tag{4.52}$$

for all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0,\beta]$, and $t \in [0, 1-\theta]$.

Let us show now that under the condition (4.35), for all $\theta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_0)$,

$$X_{\ell\ell'}(\tau,\tau'|t) < Y^{(\theta)}_{\ell\ell'}(\tau,\tau'|t), \tag{4.53}$$

also for all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0,\beta]$, and $t \in [0, 1-\theta]$. To this end we introduce

$$Z^{\pm}_{\ell\ell'}(\tau,\tau'|t) \ \stackrel{\mathrm{def}}{=}\ Y^{(\theta)}_{\ell\ell'}(\tau,\tau'|t) \pm X_{\ell\ell'}(\tau,\tau'|t), \quad t \in [0, 1-\theta]. \tag{4.54}$$

Then one has from (4.40), (4.44)

$$\begin{aligned}
&\frac{\partial}{\partial t} Z^{-}_{\ell\ell'}(\tau,\tau'|t) \qquad\qquad (4.55)\\
&\quad = \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda} J^{\Lambda}_{\ell_1\ell_2}\int_0^\beta \left\{ Z^{+}_{\ell\ell_1}(\tau,\tau''|t) Z^{-}_{\ell'\ell_2}(\tau',\tau''|t)\right.\\
&\quad \left. + Z^{-}_{\ell\ell_1}(\tau,\tau''|t) Z^{+}_{\ell'\ell_2}(\tau',\tau''|t)\right\} \mathrm{d}\tau''\\
&\quad + \frac{\varepsilon}{|\Lambda|}\sum_{\ell_1,\ell_2\in\Lambda}\int_0^\beta Y^{(\theta)}_{\ell\ell_1}(\tau,\tau''|t) Y^{(\theta)}_{\ell'\ell_2}(\tau',\tau''|t)\mathrm{d}\tau'' - S_{\ell\ell'}(\tau,\tau'|t),
\end{aligned}$$

where $S_{\ell\ell'}(\tau,\tau'|t)$ stands for the first term on the right-hand side of (4.40). By (4.54) and (4.51)

$$Z^{-}_{\ell\ell'}(\tau,\tau'|0) = Y_{\ell\ell'}(\tau,\tau'|\theta) - X_{\ell\ell'}(\tau,\tau'|0) > 0, \tag{4.56}$$

which holds for all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0,\beta]$. For every $\ell, \ell' \in \Lambda$, both $Y_{\ell\ell'}(\tau,\tau'|t)$, $X_{\ell\ell'}(\tau,\tau'|t)$ and, hence, $Z^{\pm}_{\ell\ell'}(\tau,\tau'|t)$ are continuous functions of their arguments. Set

$$\zeta(t) = \inf\left\{ Z^{-}_{\ell\ell'}(\tau,\tau'|t) \mid \ell,\ell' \in \Lambda, \quad \tau,\tau' \in [0,\beta]\right\}. \tag{4.57}$$

By (4.56), it follows that $\zeta(0) > 0$. Suppose now that $\zeta(t_0) = 0$ at some $t_0 \in [0, 1-\theta]$ and $\zeta(t) > 0$ for all $t \in [0, t_0)$. Then by the continuity of $Z^{-}_{\ell\ell'}$, there exist $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0,\beta]$ such that

$$Z^{-}_{\ell\ell'}(\tau,\tau'|t_0) = 0 \qquad \text{and} \qquad Z^{-}_{\ell\ell'}(\tau,\tau'|t) > 0 \qquad \text{for all } t < t_0.$$

For these $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0,\beta]$, the derivative $(\partial/\partial t) Z^{-}_{\ell\ell'}(\tau,\tau'|t)$ at $t = t_0$ is positive since on the right-hand side of (4.55) the third term is positive and the remaining terms are non-negative. But a differentiable function, which is positive at $t \in [0, t_0)$ and zero at $t = t_0$, cannot increase at $t = t_0$. Thus, $\zeta(t) > 0$ for all $t \in [0, 1-\theta]$, which yields (4.53). By the latter estimate, we have

$$\begin{aligned} X_{\ell\ell'}(\tau,\tau'|1-\theta) &< Y_{\ell\ell'}(\tau,\tau'|1) \\ &= \frac{1}{\beta|\Lambda|}\sum_{p\in\Lambda_*}\sum_{k\in\mathcal{K}}\frac{\exp\left[\imath(p,\ell-\ell')+\imath k(\tau-\tau')\right]}{[\hat{u}(k)]^{-1}-t[\hat{J}_0^{\Lambda}+\varepsilon\delta_{p,0}]+t\,\Upsilon^{\Lambda}(p)}. \end{aligned}$$

All the function above depend on θ and ε continuously. Hence, passing here to the limit $\theta = \varepsilon \downarrow 0$ and taking into account (4.43) we obtain (4.36). $\square$

By means of Proposition 2.21, the result just proven can be extended to all periodic elements of $\mathcal{G}^{\mathrm{t}}$. For $\mu \in \mathcal{G}^{\mathrm{t}}$, we set

$$K^{\mu}_{\ell\ell'}(\tau,\tau') = \left\langle \omega_\ell(\tau)\omega_{\ell'}(\tau')\right\rangle_\mu. \tag{4.58}$$

Theorem 4.5 *Let the stability condition (4.16) be satisfied. Then for every periodic $\mu \in \mathcal{G}^{\mathrm{t}}$, the correlation function (4.58) has the bound*

$$\begin{aligned} K^{\mu}_{\ell\ell'}(\tau,\tau') &\leq Y_{\ell\ell'}(\tau,\tau') \qquad (4.59)\\ &\stackrel{\mathrm{def}}{=} \frac{1}{\beta(2\pi)^d}\sum_{k\in\mathcal{K}}\int_{(-\pi,\pi]^d}\frac{\exp\left[\imath(p,\ell-\ell')+\imath k(\tau-\tau')\right]}{[\hat{u}(k)]^{-1}-\hat{J}_0+\Upsilon(p)}\mathrm{d}p, \end{aligned}$$

where

$$\Upsilon(p) = \hat{J}_0 - \sum_{\ell'} J_{\ell\ell'}\exp[\imath(p,\ell-\ell')], \quad p \in (-\pi,\pi]^d. \tag{4.60}$$

The same bound has also the correlation function $K^{\mu_0}_{\ell\ell'}(\tau,\tau')$, where $\mu_0 \in \mathcal{G}^{\mathrm{t}}$ is the same as in Proposition 2.19.

Remark 4.6 *By (4.30), $[\hat{u}(k)]^{-1} \geq m(\varDelta_m^2 + k^2)$. The upper bound in (4.59) with $[\hat{u}(k)]^{-1}]$ replaced by $m([\varDelta^{\mathrm{har}}]^2 + k^2)$ turns into the infinite volume correlation function for the quantum harmonic crystal discussed at the beginning of subsection 4.1. Thus, under the condition (4.35) the decay of the correlation functions in the periodic states is not less than it is in the stable quantum harmonic crystal. As we shall see in the next subsection, such a decay stabilizes also anharmonic ones.*

For $\Upsilon(p) \sim \Upsilon_0|p|^2$, $\Upsilon_0 > 0$, as $p \to 0$, the asymptotics of the bound in (4.59) as $\sqrt{|\ell-\ell'|^2 + |\tau-\tau'|^2} \to +\infty$ will be the same as for the $d+1$-dimensional free field, which is well known, see claim (c) of Proposition 7.2.1, page 162 of [31]. Thus, we have the following

Proposition 4.7 *If the function (4.60) is such that $\Upsilon(p) \sim \Upsilon_0|p|^2$, $\Upsilon_0 > 0$, as $p \to 0$, the upper bound in (4.59) has an exponential spacial decay.*

4.5 Decay of correlations in the vector case

In the vector case, the eigenvalues of the Hamiltonian (4.5) are no longer simple; hence, the parameter (4.14) definitely equals zero. Therefore, one has to pick another parameter, which can describe the quantum rigidity in this case. If the model is rotation invariant, its dimensionality ν is just a parameter. Thus, one can compare the stability of such a model with the stability of the model with $\nu = 1$. This approach was developed in [49], see also [6, 38]. Here we present the most general result in this domain, which is then used to study the quantum stabilization in the vector case.

We begin by introducing the corresponding class of functions. A function $f : \mathbb{R} \to \mathbb{R}$ is called polynomially bounded if $f(x)/(1+|x|^k)$ is bounded for some $k \in \mathbb{N}$. Let $\mathcal{F}$ be the set of continuous polynomially bounded $f : \mathbb{R} \to \mathbb{R}$ which are either odd and increasing or even and positive.

Proposition 4.8 *Suppose that the model is rotation invariant and for all $\ell \in \Lambda$, $\Lambda \Subset \mathbb{L}$, $V_\ell(x) = v_\ell(|x|^2)$ with v_ℓ being convex on $\mathbb{R}_+$. Then for any $\tau_1, \ldots, \tau_n \in [0, \beta]$, $\ell_1, \ldots, \ell_n \in \Lambda$, $j = 1, \ldots, \nu$, $f_1, \ldots f_n \in \mathcal{F}$,*

$$\langle f_1(\omega_{\ell_1}^{(j)}(\tau_1)) \cdots f_n(\omega_{\ell_n}^{(j)}(\tau_n))\rangle_{\nu_\Lambda} \leq \langle f_1(\omega_{\ell_1}(\tau_1)) \cdots f_n(\omega_{\ell_n}(\tau_n))\rangle_{\tilde{\nu}_\Lambda}, \qquad (4.61)$$

where $\tilde{\nu}_\Lambda$ is the Euclidean Gibbs measure (2.33) of the scalar model with the same $J_{\ell\ell'}$ as the model considered and with the anharmonic potentials $V_\ell(q) = v_\ell(q^2)$.

By this statement one immediately gets the following fact.

Theorem 4.9 *Let the model be translation invariant and such as in Proposition 4.8. Let also $\varDelta_m$ be the gap parameter (4.14) of the scalar model with the same interaction intensities $J_{\ell\ell'}$ and with the anharmonic potentials $V(q) = v(q^2)$. Then if the stability condition (4.16) is satisfied, the longitudinal correlation function*

$$K^\mu_{\ell\ell'}(\tau, \tau') = \langle \omega_{\ell'}^{(j)}(\tau)\omega_\ell^{(j)}(\tau')\rangle_\mu, \qquad j = 1, 2, \ldots, \nu, \qquad (4.62)$$

corresponding to any of the periodic states $\mu \in \mathcal{G}^{\mathrm{t}}$, as well as to any of the accumulation points of the family $\{\pi_\Lambda(\cdot|0)\}_{\Lambda \Subset \mathbb{L}}$, obeys the estimate (4.59) in which $\hat{u}(k)$ is calculated according to (4.29) for the one-dimensional anharmonic oscillator of mass m and the anharmonic potential $v(q^2)$.

4.6 Suppression of phase transitions

From the 'physical' point of view, the decay of correlations (4.59) already corresponds to the lack of any phase transition. However, in the mathematical theory, one should show this as a mathematical fact basing on the definition of a phase transition. The most general one is Definition 3.1 according to which the suppression of phase transitions corresponds to the uniqueness of tempered Euclidean Gibbs states. Properties like the differentiability of the pressure, c.f., Definition 3.7, or the lack of the order parameter, see Definition 3.5, may also indicate the suppression of phase transitions, but in a weaker sense. The aim of this section is to demonstrate that the decay of correlations caused by the quantum stabilization yields the two-times differentiability of the pressure, which in the scalar case yields the uniqueness. This result is then extended to the models which are not necessarily translation invariant.

In the scalar case, the most general result is the following statement, see Theorem 3.13 in [54].

Theorem 4.10 *Let the anharmonic potentials V_ℓ be even and such that there exists a convex function $v : \mathbb{R}_+ \to \mathbb{R}$, such that, for any V_ℓ,*

$$V_\ell(x_\ell) - v(x_\ell^2) \leq V_\ell(\tilde{x}_\ell) - v(\tilde{x}_\ell^2) \quad \text{whenever} \quad x_\ell^2 < \tilde{x}_\ell^2. \qquad (4.63)$$

For such v, let $\varDelta_m$ be the gap parameter of the one-particle Hamiltonian (1.1) with the anharmonic potential $v(q^2)$. Then the set of tempered Euclidean Gibbs measures of this model is a singleton if the stability condition (4.16) involving $\varDelta_m$ and the interaction parameter $\hat{J}_0$ of this model is satisfied.

The proof of this theorem is conducted by comparing the model with the translation invariant reference model with the anharmonicity potential $V(q) = v(q^2)$. By Proposition 2.18, for the model considered and the reference model, there exist maximal elements, μ_+ and μ_+^{ref}, respectively. By means of the symmetry $V_\ell(q) = V_\ell(-q)$ and the FKG inequality, one proves that, for both models, the uniqueness occurs if

$$\langle \omega_\ell(0) \rangle_{\mu_+^{\mathrm{ref}}} = 0, \qquad \langle \omega_\ell(0) \rangle_{\mu_+} = 0, \qquad \text{for all } \ \ell. \tag{4.64}$$

By the GKS inequalities, the condition (4.63) implies

$$0 \leq \langle \omega_\ell(0) \rangle_{\mu_+} \leq \langle \omega_\ell(0) \rangle_{\mu_+^{\mathrm{ref}}}, \tag{4.65}$$

which means that the reference model is less stable with respect to the phase transitions than the initial model. The reference model is translation invariant. By means of a technique employing this fact, one proves that the decay of correlations in the reference model which occurs under the stability condition (4.16) yields, see Theorem 4.4,

$$\langle \omega_\ell(0) \rangle_{\mu_+^{\mathrm{ref}}} = 0,$$

and therefrom (4.64) by (4.65). The details can be found in [54].

As was mentioned above, in the vector case we did not manage to prove that the decay of correlations implies the uniqueness. The main reason for this is that the proof of Theorem 4.10 was based on the FKG inequality, which can be proven for scalar models only. In the vector case, we get a weaker result, by which the decay of correlations yields the normality of thermal fluctuations. To this end we introduce *the fluctuation operators*

$$Q_\Lambda^{(j)} = \frac{1}{\sqrt{|\Lambda|}} \sum_{\ell \in \Lambda} q_\ell^{(j)}, \qquad \Lambda \Subset \mathbb{L}, \quad j = 1, \ldots, \nu. \tag{4.66}$$

Such operators correspond to *normal fluctuations*.

Definition 4.11 *The fluctuations of the displacements of oscillators are called normal if the Matsubara functions (2.12) for the operators $F_1 = Q^{(j_1)}, \ldots, F_n = Q^{(j_n)}$, remain bounded as $\Lambda \nearrow \mathbb{L}$.*

If Λ is a box, the parameter (3.28) can be written

$$P_\Lambda^{(\alpha)} = \frac{1}{\beta^2 |\Lambda|^\alpha} \sum_{j=1}^{\nu} \int_0^\beta \int_0^\beta \Gamma^{\beta,\Lambda}_{Q_\Lambda^{(j)}, Q_\Lambda^{(j)}}(\tau, \tau') \mathrm{d}\tau \mathrm{d}\tau'. \tag{4.67}$$

Thus, if the fluctuations are normal, phase transitions of the second order (and all the more of the first order) do not occur.

Like in the proof of Theorem 4.9, the model is compared with the scalar ferromagnetic model with the same mass and the anharmonic potential $v(q^2)$. Then the gap parameter $\varDelta_m$ is the one calculated for the latter model.

Theorem 4.12 *Let the model be the same as in Theorem 4.9 and let the stability condition involving the interaction parameter $\hat{J}_0$ of the model and the gap parameter Δ_m corresponding to its scalar analog be satisfied. Then the fluctuations of the displacements of the oscillators remain normal at all temperatures.*

4.7 Comments

- *Subsection 4.1:* In an ionic crystal, the ions usually form massive complexes the dynamics of which determine the physical properties of the crystal, including its instability with respect to structural phase transitions, see [21]. Such massive complexes can be considered as classical particles; hence, the phase transitions are described in the framework of classical statistical mechanics. At the same time, in a number of ionic crystals containing localized light ions certain aspects of the phase transitions are apparently unusual from the point of view of classical physics. Their presence can only be explained in a quantum-mechanical context, which points out on the essential role of the light ions. This influence of the quantum effects on the phase transition was detected experimentally already in the early 1970's. Here we mention the data presented in [18, 76] on the KDP-type ferroelectrics and in [57] on the YBaCuO-type superconductors. These data were then used for justifying the corresponding theoretical models and tools of their study. On a theoretical level, the influence of quantum effects on the structural phase transitions in ionic crystals was first discussed in the paper [65], where the particle mass was chosen as the only parameter responsible for these effects. The conclusion, obtained there was that the long range order, see Definition 3.5, gets impossible at all temperatures if the mass is sufficiently small. Later on, a number of rigorous studies of quantum effects inspired by this result as well as by the corresponding experimental data have appeared, see [56, 82] and the references therein. Like in [65], in these works the reduced mass (1.3) was the only parameter responsible for the effects. The result obtained was that the long range order is suppressed at all temperatures in the light mass limit $m \to 0$. Based on the study of the quantum crystals performed in [2, 3, 5, 7, 9], a mechanism of quantum effects leading to the stabilization against phase transitions was proposed, see [8].

- *Subsection 4.2:* According to [8] the key parameter responsible for the quantum stabilization is $\mathcal{R}_m = m\Delta_m^2$, see (4.15). In the harmonic case, $m\Delta_m^2$ is merely the oscillator rigidity and the stability of the crystal corresponds to large values of this quantity. That is why the parameter $m\Delta_m^2$ was called quantum rigidity and the effect was called quantum stabilization. If the tunneling between the wells gets more intensive (closer minima), or if the mass diminishes, $m\Delta_m^2$ gets bigger and the particle 'forgets' about the details of the potential energy in the vicinity of the origin (including instability) and oscillates as if its equilibrium at zero is stable, like in the harmonic case.

- *Subsection 4.3:* Theorems 4.2 and 4.3 are new. Preliminary results of this kind were obtained in [3, 50].

- *Subsection 4.4:* Theorems 4.4, 4.5, 4.7 were proven in [45].

- *Subsection 4.5:* Various scalar domination estimates were obtained in [47, 48, 49].
- *Subsection 4.6:* Theorem 4.10 was proven in [54]. The proof of Theorem 4.12 was done in [49]. The suppression of abnormal fluctuations in the hierarchical version of the model (1.1), (1.2) was proven in [2].

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