Unbounded energy growth in Hamiltonian systems with a slowly varying parameter

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Abstract

We study Hamiltonian systems which depend slowly on time. We show that if the corresponding frozen system has a uniformly hyperbolic invariant set with chaotic behaviour, then the full system has orbits with unbounded energy growth (under very mild genericity assumptions). We also provide formulas for the calculation of the rate of the fastest energy growth. We apply our general theory to non-autonomous perturbations of geodesic flows and Hamiltonian systems with billiard-like and homogeneous potentials. In these examples, we show the existence of orbits with the rates of energy growth that range, depending on the type of perturbation, from linear to exponential in time. Our theory also applies to non-Hamiltonian systems with a first integral.

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1 Setting the problem

Consider a Hamiltonian system

$$H = H(p, q, \varepsilon t) \tag{1}$$

with ε small. It is natural to compare its dynamics with the frozen system

$$H = H(p, q, \nu), \tag{2}$$

where ν is now treated as a constant parameter. The Hamiltonian H is a first integral of the frozen system but not of the non-autonomous system described by (1). Let $(p(t), q(t))$ be a trajectory of (1) and $H(t) \equiv H(p(t), q(t), \varepsilon t)$. Differentiating with respect to time and using the Hamilton equations we see that the rate of energy change is small

$$\dot{H}(t) = \varepsilon \frac{\partial H}{\partial \nu}(p(t), q(t), \varepsilon t)\,.$$

Adiabatic invariants play an important role in description of dynamics for this class of systems [10]. It is also known that if the frozen system is integrable, then under certain assumptions the energy H may oscillate at a bounded distance from its initial value for a very long time.

On the other hand, in the case of chaotic dynamics in the frozen system the behaviour of the energy may be drastically different. Indeed, in the mid nineties Mather discovered that adding a time-periodic perturbation to the Hamiltonian of a uniformly hyperbolic flow creates orbits with an unbounded energy growth. Moreover, the energy on the Mather's trajectories tends to infinity linearly, i.e., it changes at a much faster rate than it could do if the unperturbed system were integrable. This result and its generalisations were studied in [4, 22, 15, 6, 7, 9], where the reader can e.g. find more detailed discussion on the history of the problem. While some papers treat the problem of estimating the energy growth in nonautonomous Hamiltonian systems mostly as a simplified model for Arnold diffusion, we think it has an independent interest and a wide range of applications (see e.g. [13, 14] where billiards with time-dependent boundaries were discussed in connection with Fermi acceleration).

In our paper we establish that the existence of orbits of unbounded and rapid energy growth is a very general phenomenon, typical for practically arbitrary slow non-autonomous perturbation of a Hamiltonian system with chaotic behaviour. The construction we employ is different from most of those used by the previous authors and is applied to a wider class of systems. Thus, we do not use variational constructions, nor KAM theory, we do not build heteroclinic chains, and we do not assume any kind of periodicity for the time-dependence. Finally, we provide formulas for calculating the energy growth rates, and provide examples for which the growth rates vary from linear to exponential one.

In short, the acceleration mechanism we discuss here is as follows. First, by saying that the frozen system has a chaotic behaviour, we mean that there exists h^* such that the frozen system has a uniformly-hyperbolic, compact, transitive, invariant set $\Lambda_{h\nu}$ in every energy level $H = h \geq h^*$ for all $\nu \geq 0$. In every given energy level, the set $\Lambda_{h\nu}$ is in the closure of a set of hyperbolic periodic orbits each of which has an orbit of a transverse heteroclinic connection to any of the others. This means that orbits of (2) may stay close to any of the periodic orbits for an arbitrary number of periods, then come close to another periodic orbit and stay there, and so on. Recall also that periodic orbits of (2) form families parametrized by the value of H and by ν. By a standard averaging procedure (see e.g. [3, 12]), one establishes that for the orbits of the original system (1) close to a periodic family of (2) there exists an adiabatic invariant – a function $J(h, \nu)$ such that $J(H(t), \varepsilon t)$ stays almost constant for a very long time $\gg \varepsilon^{-1}$. Now we take two periodic families, L_a and L_b, of the frozen system. For the orbits that stay near L_a the value of $J_a(H(t), \varepsilon t)$ will remain almost constant while $J_b(H(t), \varepsilon t)$ may grow or decrease, and for the orbits that stay near L_b we will have J_b nearly constant while J_a changes. In this paper we show that under some natural conditions one can arrange jumps between L_a and L_b in such a way that one of the functions J_a or J_b will always grow while the second one rests. Then the sum $J_a(H(t), \varepsilon t) + J_b(H(t), \varepsilon t)$ may tend to infinity. Note that both J_a and J_b are monotonically increasing functions of h, therefore the unbounded growth of $J_a(H(t), \varepsilon t) + J_b(H(t), \varepsilon t)$ implies, typically, the unbounded growth of $H(t)$.

These considerations do not depend on how the Hamiltonian depends on εt. Indeed, we treat the cases of periodic and non-periodic perturbations simultaneously, and the results hold true for periodic, quasiperiodic and other settings. We note that the invariant set $\Lambda_{h\nu}$ with the desired properties exists, provided the frozen system has a family of hyperbolic periodic orbits, each one with a transversal homoclinic trajectory.

Technically, for the most part of this paper we do not use the Hamiltonian structure of the system. Therefore, for a greater generality, instead of (1) we consider

$$\dot{x} = G(x, \varepsilon t)\,, \tag{3}$$

where $x \in \mathbb{R}^m$. The corresponding frozen system has the form

$$\dot{x} = G(x, \nu)\,. \tag{4}$$

We assume that a function $H(x, \nu)$ is an integral, i.e.

$$\frac{\partial H}{\partial x}(x, \nu) \cdot G(x, \nu) \equiv 0. \tag{5}$$

We will continue calling H the energy.

Assume that for all $\nu \geq 0$ in every energy level $H = h \geq h^*$ the frozen system (4) has a heteroclinic cycle composed of a pair of hyperbolic periodic orbits L_a and L_b, and a pair of transverse heteroclinic orbits, Γ_{ab} and Γ_{ba}, the first corresponds to a transverse intersection of $W^u(L_a)$ and $W^s(L_b)$, while the second one corresponds to a transverse intersection of $W^u(L_b)$ and $W^s(L_a)$.

We note that the set of all orbits that stay in a small neighbourhood of the heteroclinic cycle in a given energy level is a locally maximal, uniformly hyperbolic, compact, transitive, invariant set [19]. We denote this set by Λ. It is well known that a hyperbolic periodic orbit continues in a unique way as a smooth function of parameters h and ν. The same holds true for a transverse heteroclinic.

So, L_a and L_b, as well as the transverse heteroclinic orbits Γ_{ab} and Γ_{ba}, depend on h and ν in a smooth way. Let $T_c(h,\nu)$ (where $c=a$ or $c=b$) be the period of the orbit $L_c(h,\nu) : x = x_c(t;h,\nu)$. Let us consider the average of H'_ν over the periodic orbit L_c:

$$v_c(h,\nu) = \frac{1}{T_c}\int_0^{T_c} \frac{\partial H}{\partial \nu}(x,\nu)\bigg|_{x=x_c(t;h,\nu)} dt\,. \tag{6}$$

Theorem 1 *Assume that the differential equation*

$$\frac{dh}{d\nu} = \max\{v_a(h,\nu), v_b(h,\nu)\} \tag{7}$$

has a solution $h(\nu)$ such that $h(\nu) \geq h^*$ for all $\nu \geq 0$ and $h(\nu) \to +\infty$ as $\nu \to +\infty$. Then given any $h_1 \geq h_0 \equiv h(0)$ there exists $t_1 > 0$ such that for every sufficiently small ε there is a solution $x(t)$ of system (3) *such that $H(x(0),0) = h_0$ and $H(x(t),\varepsilon t) = h_1$ at a time $t \leq t_1/\varepsilon$.*

In Section 1.2 we show that in the Hamiltonian setup (i.e. in the case where system (3) is Hamiltonian) equation (7) possesses tending to infinity solutions under very mild assumptions. Thus, for the case of periodic or quasiperiodic dependence of the Hamiltonian on εt we show (Proposition 2) that the boundedness of solutions of (7) is a codimension infinity event. Simple sufficient conditions for the unbounded energy growth are given for special classes of Hamiltonian systems in Section 4.

Theorem 1 does not directly imply that system (3) has an orbit with unbounded energy. In order to prove the existence of such an orbit we need an information on the behaviour of the system near the hyperbolic set $\Lambda_{h\nu}$ at ν and h tending to infinity, i.e., for a non-compact set of values of h and ν. Therefore, certain uniformity assumptions are necessary. As they are quite technical, we postpone their precise statements till Sections 2 and 3. In short, in condition [UA1] we require that the set $\Lambda_{h\nu}$ has a cross-section and that the right-hand sides of the corresponding Poincaré map, when written in the so-called "cross-form" [21], are uniformly bounded, along with their first derivatives, for all sufficiently large h and ν and all small ε. In condition [UA2] we require a certain uniformity for the times of the first return to the cross-section and for the change in the energy between two consecutive returns. In Section 4 we check these uniformity assumptions for several classes of examples.

For a greater generality, we allow for the right-hand side of system (3) under consideration to depend explicitly on ε, i.e., the system takes the form

$$\dot{x} = G(x, \varepsilon t; \varepsilon)\,, \tag{8}$$

with G depending on ε continuously. Thus, the frozen system (4) and the integral H will also depend on ε, as well as the functions $v_{a,b}$ in (6).

Theorem 2 *Assume the uniformity assumptions* [UA1] *and* [UA2] *hold true. Consider a differential equation*

$$\frac{dh}{d\nu} = \max\{v_a(h,\nu), v_b(h,\nu)\} - \delta\beta(h,\nu) \tag{9}$$

where the smooth function β is defined by condition (46)*. Suppose there exists $\delta > 0$ such that equation* (9) *has, for all small ε, a solution $h_\delta(\nu)$ that satisfies $h_\delta(\nu) \geq h^*$ for all ν and tends to $+\infty$ as $\nu \to +\infty$. Then for all sufficiently small ε system* (8) *has an orbit $x(t)$ for which $H(x(t), \varepsilon t; \varepsilon) \to +\infty$ as $t \to= \infty$.*

Theorem 2 above is an immediate corollary of the following comparison theorem.

Theorem 3 *Assume the uniformity assumptions* [UA1] *and* [UA2] *hold true, let $\delta > 0$ and denote as h_δ a solution of differential equation* (9)*. Then for all sufficiently small ε system* (8) *has a solution $x(t)$ such that $H(x(0), 0) = h_\delta(0)$ and*

$$H(x(t), \varepsilon t) \geq h_\delta(\varepsilon t)$$

for all $t \geq 0$.

The proof of Theorem 3 is given in Sections 2 and 3. Note that the uniformity assumptions are automatically fulfilled for any compact set of h and ν, hence Theorem 1 is indeed extracted from Theorem 3 by modifying, if necessary, the equations outside a neighbourhood of the region $H(x,\nu) \in [h_0, h_1]$ and $\nu \in [0, t_1]$.

Note also that although the function β in Theorem 2 is defined in technical terms, in the examples which we consider in Section 4 this function is asymptotically (as $h \to +\infty$) of the same order as the functions $v_{a,b}$. Therefore, the contribution of the second term of equation (9) is not very important (recall that δ in (9) can be taken arbitrarily small). In other words, the energy growth rate is, essentially, given by the solution of equation (7).

1.1 Scheme of the proof

Let us now describe the scheme of the proof of Theorem 3. Consider the family $L_c : x = x_c(t; h, \nu)$ of the hyperbolic periodic orbits of the frozen system (4) (as usual, $c = a$ or $c = b$). This is a three-dimensional invariant manifold of system (4). Importantly, this manifold is normally-hyperbolic, because of the hyperbolicity of the periodic orbits which comprise it. Therefore, it persists for all small ε [8] (the set of values of (h, ν) under consideration is not compact and therefore we also need our uniformity assumptions to make such claim). Thus, system (8) has an invariant manifold $x = \tilde{x}_c(\tau; h, \nu; \varepsilon)$ such that $\tilde{x}_c = x_c$ at $\varepsilon = 0$; moreover, $\tilde{x}_c$ depend periodically on the first argument τ (with the period depending on h, ν and ε). The dynamics on this manifold is close to the dynamics of the frozen system, therefore the evolution of h and ν is slow, while the first argument τ is a fast rotating phase. Hence, in the first order with respect to ε, the evolution of h and ν on the invariant manifold is described by the system averaged with respect to the fast time

$$\dot{h} = \varepsilon v_c(h,\nu), \qquad \dot{\nu} = \varepsilon\,, \tag{10}$$

where v_c is defined by equation (6). Therefore if a trajectory stays close to L_c its energy changes following the equation

$$\frac{dh}{d\nu} = v_c(h,\nu) + \text{h.o.t.}$$

As we see, for given values of h and ν the velocity of the change of h depends on the periodic orbit L_c. We will prove that the full system has a trajectory which switches between small neighbourhoods of L_a and L_b always choosing the periodic orbit which gives larger velocity at the moment; clearly, this is the trajectory which implements the optimal strategy for the acceleration. For this trajectory the rate of energy change is described by the differential equation:

$$\frac{dh}{d\nu} = \max\{\, v_a(h,\nu), v_b(h,\nu)\,\} + \text{ h.o.t.}$$

Hence, equation (7) correctly describes the evolution of h along the trajectory of the fastest energy growth. The small δ term in (9) takes care of all higher order corrections (we can neglect this term in the framework of Theorem 1, where the time of acceleration is finite).

As we see, in order to prove Theorem 3, we just need to construct an orbit which actually jumps between L_a and L_b in the above described way. In order to do this, we code the orbits of the frozen system that stay in a small neighbourhood of the heteroclinic cycle $L_a \cup L_b \cup \Gamma_{ab} \cup \Gamma_{ba}$ by sequences of a's and b's. Given any such sequence, the corresponding orbit depends smoothly on h and ν, i.e. we have a normally-hyperbolic invariant manifold corresponding to any of these sequences. Because of the uniform normal hyperbolicity, all of these manifolds persist for all small ε (we supply a proof in Section 2; as a matter of fact, our approach is similar to that of [20]). We repeat that every sequence of a and b is a valid code, i.e. for every itinerary of the jumps between L_a and L_b system (8) has an invariant manifold, orbits on which implement this particular itinerary. In particular, it has an invariant manifold for the orbits on which the growth of the energy is estimated from below by equation (9). The rigorous construction is in Section 3.

1.2 Adiabatic invariant revisited

In this subsection we discuss the meaning of equation (7) in the Hamiltonian setup and conditions which imply that all its solutions tend to infinity. This section is of independent interest and the proofs of our main theorems do not rely on its results.

Note that Theorems 1, 2 and 3 do not assume that system (3) is Hamiltonian. However, as we will show in a moment, in the Hamiltonian case equation (7) indeed has a tending to infinity solution under almost no assumptions.

We first recall that hyperbolic periodic solutions L_c comprise, at every fixed ν, a one-parameter family parametrized by the energy h. Thus, they fill a certain two-dimensional cylinder in the phase space. As usual in the theory of slow perturbations, we may introduce "action-angle" variables on this surface. The "action" is defined by

$$J_c(h,\nu) = \oint_{L_c} p\, dq \tag{11}$$

in the case of the standard symplectic form. In a more general case, where the Hamiltonian system (2) is defined on a manifold with a symplectic form Ω, let us assume that the symplectic form is exact, i.e. $\Omega = d\vartheta$ where ϑ is an 1-form. Then the action is defined as

$$J_c(h,\nu) = \oint_{L_c} \vartheta. \tag{12}$$

It is well-known that

$$\frac{\partial J_c}{\partial h}(h,\nu) = T_c(h,\nu), \qquad \frac{\partial J_c}{\partial \nu}(h,\nu) = -\int_0^{T_c} \left.\frac{\partial H}{\partial \nu}\right|_{x=x_c(t;h,\nu)} dt. \tag{13}$$

In order to see this, note that, by definition of the action, the difference between the actions corresponding to two close closed curves $L_c(h,\nu)$ and $L_c(h+\Delta h, \nu+\Delta\nu)$ is, essentially, the area of the surface spanned by these two curves. Therefore in the case of standard symplectic form Ω we obtain

$$\begin{aligned}
\frac{\partial J_c}{\partial \nu} &= \int_0^{T_c(h,\nu)} \left(\dot q_c(t;h,\nu) \frac{\partial p_c(t;h,\nu)}{\partial \nu} - \dot p_c(t;h,\nu) \frac{\partial q_c(t;h,\nu)}{\partial \nu} \right) dt,\\
\frac{\partial J_c}{\partial h} &= \int_0^{T_c(h,\nu)} \left(\dot q_c(t;h,\nu) \frac{\partial p_c(t;h,\nu)}{\partial h} - \dot p_c(t;h,\nu) \frac{\partial q_c(t;h,\nu)}{\partial h} \right) dt.
\end{aligned}$$

In the general case we have

$$\begin{aligned}
\frac{\partial J_c}{\partial \nu} &= \int_0^{T_c} \Omega\left(\dot x_c(t;h,\nu)\,,\ \frac{\partial x_c(t;h,\nu)}{\partial \nu} \right) dt,\\
\frac{\partial J_c}{\partial h} &= \int_0^{T_c} \Omega\left(\dot x_c(t;h,\nu)\,,\ \frac{\partial x_c(t;h,\nu)}{\partial h} \right) dt.
\end{aligned}$$

Taking into account the definition of the frozen Hamiltonian vector field we see $\Omega(\dot x, \cdot) = dH(\cdot)$, which implies in the coordinates

$$\begin{aligned}
\frac{\partial J_c}{\partial \nu} &= \int_0^{T_c} \left.\frac{\partial H(x,\nu)}{\partial x}\right|_{x=x_c(t;h,\nu)} \frac{\partial x_c(t;h,\nu)}{\partial \nu} dt,\\
\frac{\partial J_c}{\partial h} &= \int_0^{T_c} \left.\frac{\partial H(x,\nu)}{\partial x}\right|_{x=x_c(t;h,\nu)} \frac{\partial x_c(t;h,\nu)}{\partial h} dt.
\end{aligned}$$

Since $H(x_c(t;h,\nu),\nu) \equiv h$ for all h and ν due to the definition, these formulas imply (13) immediately.

Let us consider the Hamiltonian system with one degree of freedom defined by the Hamiltonian function $J_c(h,\nu)$:

$$h' = -\frac{\partial J_c}{\partial \nu}(h,\nu)\,, \qquad \nu' = \frac{\partial J_c}{\partial h}(h,\nu)\,.$$

Taking into account (13), we conclude that

$$h' = \int_0^{T_c} \frac{\partial H}{\partial \nu}(x,\nu)|_{x=x_c(t;h,\nu)}dt, \qquad \nu' = T_c(h,\nu)\,, \tag{14}$$

which coincides with equation (10) up to a time change. Consequently $J_c(h,\nu)$ is an integral of (10), i.e., $J_c(h(\nu),\nu) = J(h(0),0)$ for every its solution.

This gives us a leading order model for an orbit of the full system which stays close to L_c: the action J_c is an adiabatic invariant and the energy oscillates like a trajectory of a Hamiltonian system with one degree of freedom described by the Hamilton function J_c.

Thus, when the orbit is close to the invariant manifold that corresponds to L_a, the function $J_a(H(t),\varepsilon t)$ remains almost constant for a long time, while the evolution of $J_b(H(t),\varepsilon t)$ is, in the first order, described by the equation

$$T_a \dot{J}_b = \varepsilon\{J_b, J_a\} = \varepsilon\left(\frac{\partial J_b}{\partial h}\frac{\partial J_a}{\partial \nu} - \frac{\partial J_b}{\partial \nu}\frac{\partial J_a}{\partial h}\right)\,, \tag{15}$$

where the factor T_a is due to the change of the time variable. Analogously, when the orbit is near the invariant manifold that corresponds to L_b, the function $J_b(H(t),\varepsilon t)$ remains nearly constant, while the evolution of J_a is, in the first order, given by

$$T_b \dot{J}_a = \varepsilon\{J_a, J_b\} = \varepsilon\left(\frac{\partial J_a}{\partial h}\frac{\partial J_b}{\partial \nu} - \frac{\partial J_a}{\partial \nu}\frac{\partial J_b}{\partial h}\right)\,. \tag{16}$$

As we see from (15) and (16), by virtue of the anti-symmetricity of the Poisson bracket, if $\{J_a, J_b\}$ is not identically zero one can always choose between L_a and L_b in such a way that one of the functions J_a or J_b will be increasing, while the other is constant. Thus, for an orbit of (1) that stays near the invariant manifold corresponding to L_a when $\{J_b, J_a\} > 0$ and near the invariant manifold corresponding to L_b when $\{J_b, J_a\} < 0$, we will have the "total action" $J := J_a + J_b$ steadily growing with time (in the first order of our approximations). Since $\partial J/\partial h \equiv T_a + T_b$ is always positive, the growth of J allows $h(t) \equiv H(x(t),\varepsilon t)$ to grow (we will make this statement more precise below; see Propositions 1 and 2).

It is remarkable that the above described itinerary of the switching between L_a and L_b coincides with that employed in equation (7), due to the following relation:

$$\{J_a, J_b\} = T_a T_b (v_a - v_b)\,, \tag{17}$$

which directly follows from equations (13) and (6) and implies, obviously, that the Poisson bracket changes its sign at the same time as $(v_a - v_b)$ does.

Note that v_c defined by (6) has a simple geometrical meaning: since (13) implies

$$v_c(h,\nu) = -\left.\frac{\partial J_c}{\partial \nu}(h,\nu) \middle/ \frac{\partial J_c}{\partial h}(h,\nu)\right. , \tag{18}$$

$v_c(h,\nu)$ describes the angle between the direction of the Hamiltonian vector field of J_c at a point (h,ν) and the direction of the ν-axis. Now, equation (7) can be interpreted in the following way. On the plane (h,ν) there are two Hamiltonian vector fields generated by the Hamiltonian functions J_a and J_b. A solution of (7) follows the level lines of J_a and J_b: at every point there are two level lines and the solution chooses the one which leads to larger h in the immediate future.

Because of the monotone dependence of J_c on h, such choice implies that both functions $J_c(h,\nu)$ are non-decreasing along the solutions of equation (7). Indeed, by (13) and (6), if $h(\nu)$ is a solution of (7), then

$$\frac{1}{T_c}\frac{d}{d\nu}J_c(h(\nu),\nu) = \max\{v_a, v_b\} - v_c \geq 0.$$

Now we are ready to formulate a general criterion for the unbounded growth of the solutions of (7). Note that since $\partial J_c/\partial h \neq 0$, every level line of $J_c(h,\nu)$ is a graph of a certain function h of ν. We will say that a certain level line of J_a is asymptotic to a level line of J_b if the difference in h between these two lines tends to zero as $\nu \to +\infty$.

Proposition 1 *Let the actions J_a and J_b be defined at $(h \geq h^*, \nu \geq 0)$. Assume that $\lim_{h\to+\infty} J(h,\nu) = +\infty$ uniformly for all $\nu \geq 0$. Suppose also that the actions $J_{a,b}(h,\nu)$ remain bounded from above and the periods $T_{a,b}(h,\nu) \equiv \frac{\partial J_{a,b}}{\partial h}(h,\nu)$ remain bounded away from zero on any bounded set of values of h, uniformly for all $\nu \geq 0$. Under these assumptions, if none of the level lines of J_a is asymptotic to a level line of J_b, then every solution of equation (7) that starts with a sufficiently large h_0 stays in the region $h \geq h^*$ and, if defined for all $\nu > 0$, tends to infinity as $\nu \to +\infty$.*

Proof. Let $c = a$ or $c = b$. Since $J_c(h,\nu)$ uniformly tends to infinity as $h \to +\infty$, for every finite value of J_c the corresponding level line is defined for all ν and the corresponding values of h remain uniformly bounded. Let ℓ be a level line which corresponds to the value of J_c greater than $\sup_{\nu\geq 0} J(h^*,\nu)$, so this line stays entirely above $h = h^*$ (recall that $J(h,\nu)$ is an increasing function of h for a fixed ν). Since J_c is non-decreasing along the orbits of equation (7), any solution of (7) that starts above ℓ at $\nu = 0$ remains above it for all $\nu \geq 0$, i.e. it remains above $h = h^*$. If $h(\nu)$ is such solution, then the monotonicity of $J_c(h(\nu),\nu)$ implies that there exists $\lim_{\nu\to+\infty} J_c(h(\nu),\nu)$, finite or infinite.

Now suppose that $h(\nu)$ does not tend to infinity as $\nu \to +\infty$. Then there exists at least a sequence of values of $\nu = \nu_k \to +\infty$ such that the corresponding values of $h(\nu_k)$ remain all bounded from above by the same constant. By assumption, the values of, say, $J_a(h(\nu_k), \nu_k)$ also remain uniformly bounded for all k, therefore $\bar{J}_a := \lim_{\nu \to +\infty} J_a(h(\nu), \nu)$ is finite. Moreover, the line $h = h(\nu)$ stays entirely below the level line $J_a(h, \nu) = \bar{J}_a$, hence $h(\nu)$ is uniformly bounded for all ν. Since both curves $J_a(h, \nu) = \bar{J}_a$ and $h = h(\nu)$ stay in the region of bounded h, the value of $\partial J_a/\partial h = T_a$ remains bounded away from zero between these curves. Therefore, the fact that $J_a(h(\nu), \nu) \to \bar{J}_a$ as $\nu \to +\infty$ implies that the line $h = h(\nu)$ tends to the level line $J_a(h, \nu) = \bar{J}_a$. As the same arguments are equally applied to the action J_b, we find that by assuming that $h(\nu)$ does not tend to infinity we obtain the existence of two level lines, $J_a(h, \nu) = \bar{J}_a$ and $J_b(h, \nu) = \bar{J}_b$, that are asymptotic to each other. □

In the case of periodic or quasiperiodic dependence of H on εt the periodic orbits L_c of the frozen system do not necessarily depend periodically, or quasiperiodically, on ν. However, let us assume that L_a and L_b are periodic or quasiperiodic functions of ν, or at least the corresponding actions $J_{a,b}$ and, hence, the periods $T_{a,b}$ are (this is always the case in many settings, e.g. for the classes of systems we consider in the Examples Section). Then all the uniformity assumptions of Proposition 1 are automatically fulfilled. Moreover, two level lines of (quasi)periodic functions may be asymptotic to each other only if these lines coincide (this is, of course, a very rare event). Thus, we arrive at the following result.

Proposition 2 *Let the actions $J_a(h, \nu)$ and $J_b(h, \nu)$ be defined at $(h \geq h^*, \nu \geq 0)$, and let them depend periodically or quasiperiodically on ν. Assume that $\lim_{h \to +\infty} J(h, \nu) = +\infty$ uniformly for all $\nu \geq 0$. Then, if J_a and J_b do not have a common level line, then every solution of equation (7) that starts with a sufficiently large h_0 stays in the region $h \geq h^*$ and tends to infinity as $\nu \to +\infty$.*

As we see, conditions of Theorem 1 are almost always fulfilled if the system under consideration is Hamiltonian. Thus, the phenomenon of an unbounded energy growth in slowly perturbed chaotic Hamiltonian systems has a universal nature, practically independent of a particular perturbation shape, or of the structure of the frozen system. It is caused by some basic properties of Hamiltonian dynamics, namely by the existence of adiabatic invariants for slowly perturbed one-degree-of-freedom systems and by the fact that the adiabatic invariant is the Hamiltonian of the corresponding averaged motion. In other words, this phenomenon is a direct consequence of the Hamiltonian structure of the problem.

2 Description of a horseshoe and normally-hyperbolic invariant manifolds

Consider the frozen system

$$\dot{x} = G(x, \nu; \varepsilon), \tag{19}$$

and assume that a function $H(x,\nu;\varepsilon)$ is an integral of system (19), i.e.

$$\frac{\partial H}{\partial x} \cdot G(x,\nu) \equiv 0 \tag{20}$$

(we suppress, notationally, the dependence on ε in the frozen system from now on).

Let system (19) have a pair of saddle periodic orbits $L_a : x = x_a(t;h,\nu)$ and $L_b : x = x_b(t,h,\nu)$ at all $\nu \geq \nu^*$ (for some $\nu^* < 0$) in every energy level $H = h \geq h^*$. Take a pair of small smooth cross-sections, Σ_a and Σ_b, to L_a and L_b respectively. As L_a and L_b depend smoothly on h and ν, the cross-sections $\Sigma_{a,b}$ can also be taken to depend smoothly on h and ν. Denote the Poincaré map on Σ_c near L_c as Π_{cc} ($c = a,b$); the Poincaré map is smooth and depends smoothly on h and ν.

We assume that the frozen system has, at all $\nu \geq \nu^*$ in every energy level $H = h \geq h^*$, a pair of heteroclinic orbits: $\Gamma_{ab} \subseteq W^u(L_a) \cap W^s(L_b)$ and $\Gamma_{ba} \subseteq W^u(L_b) \cap W^s(L_a)$. Let Π_{ab} and Π_{ba} be maps on Σ_a and on Σ_b defined by the orbits close to Γ_{ab} and Γ_{ba}, respectively; Π_{ab} acts from some open set in Σ_a into an open set in Σ_b, while Π_{ba} acts from an open set in Σ_b into an open set in Σ_a. There is a certain freedom in the definition of the maps Π_{ab} and Π_{ba}: each of these maps acts from a neighbourhood of one point of a heteroclinic orbit to a neighbourhood of another point of the same orbit, and different choices of the pairs of points lead to different maps. When a definite choice of the maps is made (we will do it in a moment), we find for every orbit that lies entirely in a sufficiently small neighbourhood of the heteroclinic cycle $L_a \cup L_b \cup \Gamma_{ab} \cup \Gamma_{ba}$ a uniquely defined sequence of points $M_i \in \Sigma_a \cup \Sigma_b$ such that

$$M_{i+1} = \Pi_{\xi_i \xi_{i+1}} M_i$$

where

$$\xi_i = c \ \text{ if } \ M_i \in \Sigma_c \ \ (c = a,b).$$

The sequence $\{\xi_i\}_{i=-\infty}^{i=+\infty}$ is called the code of the orbit.

The periodic orbits L_a and L_b are saddle, and the intersections of the stable and unstable manifolds of L_a and L_b that create the heteroclinic orbits are transverse, by the assumption of the theorem.

This implies (cf. [1]) that one can choose the maps Π_{ab} and Π_{ba} and define coordinates (u,w) in Σ_a and Σ_b in such a way that the following holds.

- In the given coordinates, $\Sigma_c = U_c \times W_c$ where $U_{a,b}$ and $W_{a,b}$ are certain balls in $\mathbb{R}^{m-1}$ (we assume that the dimension of the x-space equals to $2m$); so we may choose some constant R such that

$$\max\left\{ \operatorname{diam} U_a, \operatorname{diam} U_b, \operatorname{diam} W_a, \operatorname{diam} W_b \right\} \leq R. \tag{21}$$

- For each pair c and c' the Poincaré map $\Pi_{cc'}$ can be written in the "cross-form" [21]; namely, there exist smooth functions $f_{cc'}, g_{cc'} : U_c \times W_{c'} \to$

$U_{c'} \times W_c$ such that a point $M(u,w) \in \Sigma_c$ is mapped into $\bar{M}(\bar{u},\bar{w}) \in \Sigma_{c'}$ by the map $\Pi_{cc'}$ if and only if

$$\bar{u} = f_{cc'}(u,\bar{w}), \qquad w = g_{cc'}(u,\bar{w}). \tag{22}$$

- There exists $\lambda < 1$ such that

$$\left\| \frac{\partial(f_{\sigma\sigma'}, g_{\sigma\sigma'})}{\partial(u,\bar{w})} \right\| \leq \lambda < 1 \tag{23}$$

  (where we define the norm in $U \times W$ as $\max\{\|u\|, \|w\|\}$).

Inequality (23) means that the set $\Lambda_{h\nu}$ of all the orbits that lie entirely in a sufficiently small neighbourhood of the heteroclinic cycle $L_a \cup L_b \cup \Gamma_{ab} \cup \Gamma_{ba}$ in the energy level $H = h$ at the given value of ν is hyperbolic, a horseshoe. Thus, one can show that $\Lambda_{h\nu}$ is in one-to-one correspondence with the set of all sequences of a's and b's, i.e. for every sequence $\{\xi_i\}_{i=-\infty}^{i=+\infty}$ there exists one and only one orbit in $\Lambda_{h\nu}$ which has this sequence as its code. Indeed, by (22), an orbit from $\Lambda_{h\nu}$ has code $\{\xi_i\}_{i=-\infty}^{i=+\infty}$ if and only if the intersection points $M_i(u_i, w_i)$ of the orbit with the cross-section satisfy

$$u_{i+1} = f_{\xi_i\xi_{i+1}}(u_i, w_{i+1}), \qquad w_i = g_{\xi_i\xi_{i+1}}(u_i, w_{i+1}),$$

i.e. the sequence $\{\,(u_i,w_i)\,\}_{i=-\infty}^{+\infty}$ is a fixed point of the operator

$$\{\,(u_i,w_i)\,\}_{i=-\infty}^{+\infty} \mapsto \{\,(f_{\xi_{i-1}\xi_i}(u_{i-1},w_i), g_{\xi_i\xi_{i+1}}(u_i,w_{i+1})\,\}_{i=-\infty}^{+\infty}.$$

By (23), this operator is a contracting map of the space $\prod_{i=-\infty}^{+\infty} U_{\xi_i} \times W_{\xi_i}$, hence the existence and uniqueness of the orbit with the code $\{\xi_i\}_{i=-\infty}^{i=+\infty}$ follows (see e.g. [19]). Moreover, as the fixed point of a smooth contracting map depends smoothly on parameters, the orbit depends smoothly on h and ν, so the derivatives of $(u_i(h,\nu,\xi), w_i(h,\nu,\xi))$ with respect to (h,ν) are bounded uniformly for all i and ξ.

It also follows from (21),(23) that

$$\|(u_i(h,\nu,\xi^{(1)}) - u_i(h,\nu,\xi^{(2)}), w_i(h,\nu,\xi^{(1)}) - w_i(h,\nu,\xi^{(2)}))\| \leq R\lambda^{n-|i|} \tag{24}$$

for any two code sequences $\xi^{(1)} = \{\xi_i^{(1)}\}_{i=-\infty}^{+\infty}$, $\xi^{(2)} = \{\xi_i^{(2)}\}_{i=-\infty}^{+\infty}$ which coincide at $|i| \leq n$ (i.e. $\xi_i^{(1)} = \xi_i^{(2)}$ at $|i| \leq n$); the constants $R > 0$ and $\lambda \in (0,1)$ are given by (21) and (23) and are independent of $\xi^{(1,2)}$.

Let us now switch to the system with a slowly changing parameter $\nu = \varepsilon t$. This means that we augment system (19) by the equation

$$\dot{\nu} = \varepsilon, \tag{25}$$

while (19) remains unchanged. Although relation (20) still holds true, the conservation of energy no longer follows: indeed, by (19),(20),(25)

$$\frac{d}{dt} H(x(t),\nu(t)) = \varepsilon \frac{\partial H}{\partial \nu}(x(t),\nu(t)). \tag{26}$$

By continuity, for system (19),(25) the Poincaré maps $\Pi_{cc'} : \cup_{h,\nu}\Sigma_c \to \cup_{h,\nu}\Sigma_{c'}$ are still defined at small ε. Denoting $z = (h, \nu)$, for any compact set of z values we may write the maps in the following form:

$$\begin{cases} \bar{u} = f_{cc'}(u, \bar{w}, z, \varepsilon), \qquad w = g_{cc'}(u, \bar{w}, z, \varepsilon) \\ \bar{z} = z + \varepsilon\phi_{cc'}(u, \bar{w}, z, \varepsilon), \end{cases} \tag{27}$$

where f, g, ϕ are bounded along with the first derivatives and f, g satisfy (23). Clearly, any smooth transformation of the z-variables will not change the form of map (27).

As the set of values of ν and h under consideration is not compact (we are interested in the behaviour of the system for ν and h tending to infinity), we need certain uniformity assumptions. We require the following.

[UA1] *For all $h \geq h^*$ and $\nu \geq \nu^*$, one can introduce coordinates (u, w) on Σ_a and Σ_b and define $z = (\alpha(h, \varepsilon), \nu)$ with a smooth function α such that $\alpha'(h) > 0$, in such a way that for all small ε*

(i) *formula (27) holds for the Poincaré maps $\Pi_{cc'}$, and the functions f, g, ϕ along with the first derivatives are uniformly bounded and uniformly continuous with respect to ε, for all $h \geq h^*$ and $\nu \geq \nu^*$;*

(ii) *estimate (23) holds with the constant $\lambda < 1$ the same for all $h \geq h^*$, $\nu \geq \nu^*$ and all small ε;*

(iii) *the diameter of the balls U_c and W_c is uniformly bounded, i.e. (21) holds with the constant R the same for all $h \geq h^*$, $\nu \geq \nu^*$ and all small ε.*

It also makes no harm to assume that $\phi \equiv 0$ if $h = h^*$ or $\nu = \nu^*$, i.e. the region $\{h \geq h^*, \nu \geq \nu^*\}$ is invariant with respect to the Poincaré map. If this is not the case, then we can modify ϕ in a small neighbourhood of $h = h^*$ and in a small neighbourhood of $\nu = \nu^*$: as we are interested in the orbits for which $h \to +\infty$, they will never enter the region of h close to h^*; and $\nu = \varepsilon t$ is a growing function of t anyway.

Now we are ready to formulate the main technical result beneath Theorems 1-3. It has a general nature and has little to do with the Hamiltonian structure of the equations. Rather we notice that by fixing any code ξ and varying h and ν we obtain at $\varepsilon = 0$ a sequence of smooth two-dimensional surfaces, the i-th surface is the set run, as h and ν change, by the point M_i of the uniquely defined orbit with the code ξ; this sequence is invariant with respect to the corresponding Poincaré maps and is uniformly normally-hyperbolic — hence it persists at all ε sufficiently small.

Lemma 1 *Given any sequence ξ of a's and b's, there exists a uniquely defined sequence of smooth surfaces*

$$\mathcal{L}_i(\xi, \varepsilon) : (u, w) = (u_i(z, \xi, \varepsilon), w_i(z, \xi, \varepsilon)) \tag{28}$$

such that

$$\Pi_{\xi_i \xi_{i+1}} \mathcal{L}_i = \mathcal{L}_{i+1}. \tag{29}$$

The functions (u_i, w_i) *are defined for all small* ε *and all* $h \geq h^*, \nu \geq \nu^*$*, they are uniformly bounded along with their derivatives with respect to* z *and satisfy* (24).

Proof. Take a sufficiently large K and consider any sequence of surfaces of form (28) with

$$\left\| \frac{\partial(u_i, w_i)}{\partial z} \right\| \leq K \tag{30}$$

(we further suppress notationally the dependence of u_i and w_i of ξ and ε). Define functions $\eta_i(z)$ and $\bar{\eta}_i(z)$ by the relations

$$z = \eta_i + \varepsilon \phi_{\xi_i \xi_{i+1}}(u_i(\eta_i), w_{i+1}(z), \eta_i, \varepsilon), \tag{31}$$

and

$$\bar{\eta}_i = z + \varepsilon \phi_{\xi_i \xi_{i+1}}(u_i(z), w_{i+1}(\bar{\eta}_i), z, \varepsilon). \tag{32}$$

As all the derivatives of ϕ, u_i and w_{i+1} are uniformly bounded, equations (31) and (32) define the functions $\eta_i(z)$ and $\bar{\eta}_i(z)$ uniquely.

By (27), the sequence of surfaces will satisfy (29) if and only if the sequence of functions $\{u_i(z), w_i(z)\}$ is a fixed point of the operator

$$\mathcal{F} : \{u_i(z), w_i(z)\} \mapsto \{\tilde{u}_i(z), \tilde{w}_i(z)\}$$

defined by

$$\begin{aligned} \tilde{u}_{i+1}(z) &= f_{\xi_i \xi_{i+1}}(u_i(\eta_i(z)), w_{i+1}(z), \eta_i(z), \varepsilon), \\ \tilde{w}_i(z) &= g_{\xi_i \xi_{i+1}}(u_i(z), w_{i+1}(\bar{\eta}_i(z)), z, \varepsilon). \end{aligned} \tag{33}$$

Let Ψ_K be the space of sequences of functions $\psi = \{u_i(z), w_i(z)\}_{i=-\infty}^{+\infty}$ satisfying (30). Endow Ψ_K with the norm

$$\|\psi\| = \sup_{i,z} \max\{\|u_i(z)\|, \|w_i(z)\|\}. \tag{34}$$

It is easy to see that $\mathcal{F}(\Psi_K) \subset \Psi_K$ provided K is large enough, and that $\mathcal{F}$ is contracting on Ψ_K, for all small ε. Indeed, let us check this claim at $\varepsilon = 0$. In this case we have $\eta_i \equiv \bar{\eta}_i \equiv z$ (see (31),(32)). Therefore,

$$\frac{\partial \tilde{u}_{i+1}}{\partial z} = \frac{\partial f}{\partial(u,w)} \frac{\partial(u_i, w_{i+1})}{\partial z} + \frac{\partial f}{\partial z}$$

$$\frac{\partial \tilde{w}_i}{\partial z} = \frac{\partial g}{\partial(u,w)} \frac{\partial(u_i, w_{i+1})}{\partial z} + \frac{\partial g}{\partial z},$$

which gives

$$\left\| \frac{\partial(\tilde{u}_{i+1}, \tilde{w}_i)}{\partial z} \right\| \leq \lambda \left\| \frac{\partial(u_i, w_{i+1})}{\partial z} \right\| + \sup \left\| \frac{\partial(f,g)}{\partial z} \right\| \tag{35}$$

(see (23)). Thus, for any

$$K > \frac{1}{1-\lambda} \sup \left\| \frac{\partial(f,g)}{\partial z} \right\|,$$

we have $\mathcal{F}(\Psi_K) \subset \Psi_K$ indeed. To prove the contractivity of $\mathcal{F}$ at $\varepsilon = 0$ just note that it follows immediately from (33),(23) that

$$\|(\tilde{u}_{i+1}^{(1)} - \tilde{u}_{i+1}^{(2)}, \tilde{w}_i^{(1)} - \tilde{w}_i^{(2)})\| \leq \lambda \|(u_i^{(1)} - \tilde{u}_i^{(2)}, w_{i+1}^{(1)} - w_{i+1}^{(2)})\|. \tag{36}$$

At $\varepsilon \neq 0$ inequalities (35),(36) change to

$$\left\| \frac{\partial(\tilde{u}_{i+1}, \tilde{w}_i)}{\partial z} \right\| \leq (\lambda + O(\varepsilon)) \left\| \frac{\partial(u_i, w_{i+1})}{\partial z} \right\| + \sup \left\| \frac{\partial(f,g)}{\partial z} \right\| + O(\varepsilon) \tag{37}$$

and

$$\|(\tilde{u}_{i+1}^{(1)} - \tilde{u}_{i+1}^{(2)}, \tilde{w}_i^{(1)} - \tilde{w}_i^{(2)})\| \leq (\lambda + O(\varepsilon)) \; \|(u_i^{(1)} - \tilde{u}_i^{(2)}, w_{i+1}^{(1)} - w_{i+1}^{(2)})\|. \tag{38}$$

Hence, at all small ε the operator $\mathcal{F}$ remains a contracting map $\Psi_K \to \Psi_K$. Thus, it has a fixed point in the closure of Ψ_K in the norm (34). This gives us the existence of the invariant sequence of Lipshitz continuous invariant surfaces — the smoothness is standard (see e.g. Theorem 4.4 of [21]).

Finally, the estimate (24) follows immediately from (38) and (21). □

According to this lemma, for all sufficiently small ε, for every code $\xi = \{\xi_i\}_{i=-\infty}^{i=+\infty}$ system (19),(25) in the space of (x, ν) has a smooth three-dimensional invariant manifold $\mathcal{M}_\xi$ that corresponds to this code, i.e. the manifold depends continuously on ε and, at $\varepsilon = 0$, it is the union, over all $h \geq h^*, \nu \geq \nu^*$ of the orbits with the code ξ (recall that for each h, ν we have exactly one such orbit). The intersection of $\mathcal{M}_\xi$ with the cross-section $\cup_{h,\nu}(\Sigma_a \cup \Sigma_b)$ is exactly the sequence of surfaces $\{\mathcal{L}_i(\xi, \varepsilon)\}_{i=-\infty}^{+\infty}$. Thus, dynamics on $\mathcal{M}_\xi$ is described by the Poincaré map on the cross-section. The Poincaré map is obtained by plugging $u = u_i(z, \varepsilon; \xi), w = w_i(z, \varepsilon; \xi)$ into (27). Namely, z_i is the sequence of the points of intersection with the cross-section of an orbit on the invariant manifold $\mathcal{M}_\xi$ if and only if

$$z_{i+1} = z_i + \varepsilon \phi_{\xi_i \xi_{i+1}}(u_i(z_i, \varepsilon; \xi), w_{i+1}(z_{i+1}, \varepsilon; \xi), z_i, \varepsilon). \tag{39}$$

Recall that in our notations z is a vector of two components: $y := \alpha(h, \varepsilon)$ and ν. So we will write

$$\begin{aligned} y_{i+1} &= y_i + \varepsilon \theta_{\xi_i \xi_{i+1}}(u_i(z_i, \varepsilon; \xi), w_{i+1}(z_{i+1}, \varepsilon; \xi), z_i, \varepsilon), \\ \nu_{i+1} &= \nu_i + \varepsilon \tau_{\xi_i \xi_{i+1}}(u_i(z_i, \varepsilon; \xi), w_{i+1}(z_{i+1}, \varepsilon; \xi), z_i, \varepsilon), \end{aligned} \tag{40}$$

i.e., θ and τ denote the two components of the function ϕ in (39).

Note that for the codes $\xi = a^\omega$ (i.e. $\xi_i = a$ for all i) and $\xi = b^\omega$ we have $u_{i+1} \equiv u_i$ and $w_{i+1} \equiv w_i$ for all i. We denote $u_c(z, \varepsilon) := u_i(z, \varepsilon; c^\omega)$

and $w_c(z,\varepsilon) := w_i(z,\varepsilon;c^\omega)$ (where $c = a$ or b). By construction, the manifold $(u,w) = (u_c(z,0), w_c(z,0))$ is the set of the intersection points of the periodic orbit $L_c : \{x = x_c(t;h,\nu)\}$ of the frozen system with the cross-section (we have one intersection point for every value of $z = (\alpha(h,\varepsilon),\nu)$).

By (25), the function τ in (40) is just the time of one return onto the cross-section. Therefore, for the orbits on the manifold $\mathcal{M}_{c^\omega}$ we have

$$\tau_{cc}|_{\varepsilon=0} = T_c(h,\nu), \tag{41}$$

where T_c is the period of L_c. Analogously, for the function θ_{cc} in (40) we have

$$\theta_{cc}|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{y_{i+1} - y_i}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\tau \dot{y} dt,$$

where $\dot{y}$ is the time derivative of y by virtue of system (19),(25). As $y = \alpha(h)$, we find from (26),(25) that

$$\dot{y} = \varepsilon \alpha'(H) H'_\nu, \tag{42}$$

so

$$\theta_{cc}|_{\varepsilon=0} = \alpha'(h) \int_0^{T_c} \frac{\partial H}{\partial \nu}(x,\nu)\bigg|_{x=x_c(t;h,\nu)} dt \tag{43}$$

for the orbits on the manifold $\mathcal{M}_{c^\omega}$.

3 Proof of Theorem 3

Before proceeding to the proof we will formulate the second uniformity assumption. It is automatically satisfied for any compact set of values of $z = (y,\nu)$, hence for any compact set of values of h and ν.

Denote

$$\begin{aligned} \Theta_\rho(z,\varepsilon) &= \sup_{u,w,c,c',\zeta} \left\{ |\alpha'(H)H'_\nu|\, \tau_{cc'}(u,w,\zeta,\varepsilon) + \left\| \frac{\partial}{\partial(u,w)} \theta_{cc'}(u,w,\zeta,\varepsilon) \right\| \right\}, \\ \mathcal{T}_\rho(z,\varepsilon) &= \sup_{u,w,c,c',\zeta} \left\{ \tau_{cc'}(u,w,\zeta,\varepsilon) + \left\| \frac{\partial}{\partial(u,w)} \tau_{cc'}(u,w,\zeta,\varepsilon) \right\| \right\}, \end{aligned} \tag{44}$$

where ζ runs a centered at z ball of some small radius ρ – we take ρ as small as we want, but independent of ε. The supremum of $|\alpha'(H)H'_\nu|$ is taken over the piece of the orbit that starts on the cross-section at the point (u,w,ζ) and continues until the next hit with the cross-section (i.e. $\tau_{cc'}(u,w,\zeta,\varepsilon)$ gives the length of the corresponding time-interval). Thus, by (42), $\varepsilon\Theta_\rho$ estimates the maximal change in y between the two intersections with the cross-section. In particular,

$$\Theta_\rho(z,\varepsilon) \geq |\theta_{cc'}(u,w,\zeta,\varepsilon)|.$$

The function $\mathcal{T}$, obviously, estimates the return times to the cross-section.

Recall that $T_c(h,\nu)$ denotes the period of the saddle periodic orbit L_c ($c = a, b$) of the frozen system (19) for a given value of $z = (\alpha(h,\varepsilon),\nu)$. Assume that the following holds.

[UA2] (i) *There exist a constant $C > 0$ and a small $\rho > 0$ such that*

$$\max\{\Theta_\rho(\alpha(h),\nu,\varepsilon),\ \mathcal{T}_\rho(\alpha(h),\nu,\varepsilon)\} \le C\min\{T_a(h,\nu),T_b(h,\nu)\} \tag{45}$$

for all $h \ge h^*$ and $\nu \ge \nu^*$, and for all small ε.

(ii) *The functions*

$$\frac{1}{T_c(h,\nu)}\phi_{cc}(u_c(z,\varepsilon),w_c(z,\varepsilon),z,\varepsilon) \quad (c=a,b)$$

are uniformly continuous with respect to ε and $z = (y,\nu)$ for all $h = \alpha^{-1}(y) \ge h^*$ and $\nu \ge \nu^*$.

Denote as $\beta(h,\nu,\varepsilon)$ any smooth function such that

$$\beta(h,\nu,\varepsilon) \ge \frac{\Theta_\rho(\alpha(h),\nu,\varepsilon)}{\alpha'(h)\min\{T_a(h,\nu),T_b(h,\nu)\}}. \tag{46}$$

By construction, $\Theta/T_{a,b}$ estimates the velocity of the change of $y = \alpha(h)$, therefore the function β estimates the velocity of the change of h. Indeed, in the examples we consider in Section 4, it is of the same order as $v_{a,b}$.

Now we can prove Theorem 3. Take any independent of ε, arbitrarily large N and consider any code ξ such that for some i we have $\xi_i = \xi_{i+1} = \ldots = \xi_{i+N-1} = c$ (where $c = a$ or b). By (24),

$$\|(u_{i+j}(z,\varepsilon,\xi) - u_c(z,\varepsilon),\ w_{i+j}(z,\varepsilon,\xi) - w_c(z,\varepsilon,\xi))\| \le R\lambda^{\min\{j,N-1-j\}}. \tag{47}$$

Take any orbit on the invariant manifold $\mathcal{M}_\xi$ that corresponds to this code ξ, and let z_i be the sequence of intersection points of the orbit with the cross-section $\cup_{h,\nu}(\Sigma_a \cup \Sigma_b)$. It follows from (47) and from the uniform ε-closeness of the right-hand side of (39) to identity that for all sufficiently small ε

$$\begin{aligned}\|z_{i+N} - z_i - \varepsilon N\phi_{cc}(u_c(z_i,\varepsilon),w_c(z_i,\varepsilon),z_i,\varepsilon)\| \le \\ \le K(\Theta_\rho(z_i,\varepsilon) + \mathcal{T}_\rho(z_i,\varepsilon))\,(\varepsilon + (\varepsilon N)^2),\end{aligned} \tag{48}$$

where K is a constant.

By (48),(41) and the uniformity assumption,

$$\lim_{\varepsilon\to 0}(\nu_{i+N} - \nu_i)/\varepsilon = NT_c(h_i,\nu_i) + O(1), \tag{49}$$

uniformly for all z. It also follows immediately that uniformly for all z

$$\lim_{\varepsilon\to 0}\frac{y_{i+N} - y_i}{\nu_{i+N} - \nu_i} = \hat{v}_c(z_i) + O(N^{-1}\beta(h_i,\nu_i)\alpha'(h_i)), \tag{50}$$

where we denote

$$\hat{v}_c(z) = \frac{\theta_{cc}(z,c^\omega,0)}{T_c(h,\nu)} \equiv \alpha'(h)v_c(h,\nu) \tag{51}$$

(see (40),(43),(45),(46),(6)).

Thus, for every $\delta > 0$ there exists N such that for all sufficiently small ε and for every code ξ such that $[\xi_i \xi_{i+1} \dots \xi_{i+N-1}] = c^N$ the change in y after N iterations of the Poincaré map is greater than the change in the solution y of the equation

$$\frac{dy}{d\nu} = \hat{v}_c(y, \nu) - \delta\beta\alpha' \tag{52}$$

on the interval $\nu \in [\nu_i, \nu_{i+N}]$, with the initial conditions $y(\nu_i) = y_i$. Indeed, as the function $\hat{v}_c$ is uniformly bounded and uniformly continuous (according to [UA2]), we have for the solution $y(\nu)$ of (52)

$$\lim_{\varepsilon \to 0} \frac{y(\nu_{i+N}) - y(\nu_i)}{\nu_{i+N} - \nu_i} = \hat{v}_c(z_i) - \delta\beta\alpha', \tag{53}$$

uniformly for all z, so the claim follows from the comparison of (53) with (50).

It follows immediately that if N is taken sufficiently large, then for all sufficiently small ε, given any code sequence ξ built of length N blocks of equal symbols, i.e. $\xi_{jN} = \xi_{jN+1} = \ldots = \xi_{jN+N-1} = c_j$ where $\{c_j\}_{j=-\infty}^{+\infty}$ is an arbitrary sequence of a's and b's, for every orbit ℓ in the invariant manifold $\mathcal{M}_\xi$

$$y_{jN} > y(\nu_{jN}) \text{ for all } j > 0, \tag{54}$$

where $\{(y_i, \nu_i)\}_{i=-\infty}^{+\infty}$ is the sequence of the intersection points of ℓ with the cross-section, and $y(\nu)$ is the solution of the equation

$$\frac{dy}{d\nu} = \hat{v}_{c_j}(y, \nu) - \frac{1}{2}\delta\beta(h, \nu)\alpha'(h) \text{ at } \nu \in [\nu_{jN}, \nu_{(j+1)N}] \tag{55}$$

with the initial condition $y(\nu_0) = y_0$.

Let us now construct a particular code sequence ξ^* by the following rule. Fix some (y_0, ν_0) such that $y_0 > \alpha(h^*)$, $\nu_0 > \nu^*$. At $i < 0$ we put $\xi_i^* = a$. At $i \geq 0$ we put $\xi_{jN}^* = \xi_{jN+1}^* = \ldots = \xi_{jN+N-1}^* = c_j^*$, where the symbols c_j^* are defined inductively, as follows. Denote as $\xi^{*(j)}$ the code sequence such that $\xi_i^{*(j)} = \xi_i^*$ at $i < jN$ and $\xi_i^{*(j)} = a$ at $i \geq jN$. Denote as $\mathcal{M}_j$ the invariant manifold with the code $\xi^{*(j)}$. Let $\ell^{*(j)}$ be the orbit on $\mathcal{M}_j$ with the initial conditions $z_0^{*(j)} = (y_0, \nu_0)$. Let $z_i^{*(j)} = (y_i^{*(j)}, \nu_i^{*(j)})$ denote the i-th point of intersection of the orbit $\ell^{*(j)}$ with the cross-section. Define

$$c_j^* = \begin{cases} a & \text{if } \hat{v}_a(z_{jN}^{*(j)}) > \hat{v}_b(z_{jN}^{*(j)}), \\ b & \text{if } \hat{v}_a(z_{jN}^{*(j)}) \leq \hat{v}_b(z_{jN}^{*(j)}). \end{cases} \tag{56}$$

By construction, the value of c_j^* is completely determined by the segment of ξ^* with $i < jN$, so we indeed can inductively define ξ^* in this way.

Let ℓ^* be the orbit on the manifold $\mathcal{M}_{\xi^*}$ with the same initial values of (y_0, ν_0), as we have chosen for the orbits $\ell^{*(j)}$, and let z_i^* be the points of the intersection of ℓ^* with the cross-section. As the code ξ^* coincides with the code $\xi^{*(j)}$ for all $i < jN$, it follows from (24) that

$$\|(u_i(z, \xi^*) - u_i(z, \xi^{*(j)}), w_i(z, \xi^*) - w_i(z, \xi^{*(j)})\| \leq R\lambda^{jN-i}.$$

Plugging this into (39) gives, uniformly for all j,

$$z^*_{jN} - z^{*(j)}_{jN} = O(\varepsilon)$$

(we use here that $z^*_0 - z^{*(j)}_0 = 0$). Therefore, it follows from the uniform continuity of $\hat{v}(z)$ that $\hat{v}_{c^*_j}(z^{*(j)})$ is uniformly close to $\max\{\,\hat{v}_a(z^{*(j)}), \hat{v}_b(z^{*(j)})\,\}$ (see (56)). This implies (see (54),(55)) that

$$y^*_{jN} > y(\nu^*_{jN}) \text{ for all } j > 0, \tag{57}$$

where $y(\nu)$ denotes here the solution of the equation

$$\frac{dy}{d\nu} = \max\{\hat{v}_a(y,\nu), \hat{v}_b(y,\nu)\} - \delta\beta(h,\nu)\alpha'(h) \tag{58}$$

with the initial condition $y(\nu_0) = y_0$. As the change in y between N intersections with the cross-section is $O(\varepsilon N \Theta_\rho)$, i.e. it is uniformly small in comparison with $\beta\alpha' \min\{T_a, T_b\}$ (see (46)), we find from (57) that for every point on the orbit ℓ^* the value of y is larger than the value of y for the solution of (58) at the same value of ν.

Now recall that $y = \alpha(h)$ with an increasing function α. Thus, it follows from (58) that for every point on the orbit ℓ^* the value of h is larger than the value of h for the solution of

$$\frac{dh}{d\nu} = \frac{1}{\alpha'(h)} \max\{\hat{v}_a(y,\nu), \hat{v}_b(y,\nu)\} - \delta\beta(h,\nu,\varepsilon)$$

at the same value of ν, which completes the proof of Theorem 3 (see (51)).

4 Examples

4.1 Non-autonomous perturbation of a geodesic flow

We begin with the Mather problem: a geodesic flow on an m-dimensional manifold ($m \geq 2$), with the Hamiltonian H_g, subject to a non-autonomous perturbation $V(q,t)$. Here q denotes position in the configuration space, i.e. V does not depend on momenta. Assume the uniform hyperbolicity for the geodesic flow (i.e. strictly negative curvature; recall that the uniform hyperbolicity implies that periodic trajectories are dense in the phase space of the geodesic flow [2]), and assume uniform boundedness and continuity for V and its first and second derivatives.

The trajectories of the unperturbed geodesic flow are the same in every energy level, just the velocity of motion grows as the square root of the energy. Namely, the flow does not change with the following scaling of time, energy, and momenta:

$$t \to t/\sqrt{s}, \quad H \to Hs, \quad p \to p\sqrt{s}\,. \tag{59}$$

At the same time, this transformation changes the perturbation. If we let $s = \varepsilon^{-2}$, then the perturbation term $V(q,t)$ is replaced by $\varepsilon^2 V(q, \varepsilon t)$. Therefore at large energies, adding $V(q,t)$ to H_g is, effectively, a small and slow perturbation of the geodesic flow. Thus, this example belongs to the class of systems (1).

Theorem 4 *Let $L_a : q = q_a(t)$ and $L_b : q = q_b(t)$ be two periodic trajectories of the geodesic flow $H = H_g$ in the energy level $H_g = 1$. Denote*

$$\bar{V}_c(\nu) = \frac{1}{T_c}\int_0^{T_c} V(q_c(t),\nu)dt, \tag{60}$$

where T_c $(c = a,b)$ is the period of L_c. Assume

$$\liminf_{s\to+\infty} \frac{1}{s}\int_0^s \left|\frac{d}{d\nu}(\bar{V}_a(\nu) - \bar{V}_b(\nu))\right| d\nu > 0. \tag{61}$$

Then the Hamiltonian system

$$H = H_g + V(q,t) \tag{62}$$

has orbits for which H linearly grows to infinity.

Proof. After the scaling transformation (59) with $s = \varepsilon^{-2}$ the Hamiltonian function recasts as

$$\hat{H} = H_g(p,q) + \varepsilon^2 V(q,\varepsilon t)\,, \tag{63}$$

which has the form (1). We check that the assumptions of Theorem 3 are all satisfied. First we note that if ε is sufficiently small, the frozen system

$$\hat{H} = H_g + \varepsilon^2 V(q,\nu)$$

has a heteroclinic cycle close to the heteroclinic cycle of the geodesic flow in every energy level $\hat{H} = h \geq 1$. Indeed, after the scaling transformation (59) with $s = h$, the system in the energy level $\hat{H} = h$ is transformed into

$$\tilde{H} = H_g + \frac{\varepsilon^2}{h} V(q,\nu)$$

in the energy level $\tilde{H} = 1$, i.e. it is uniformly close to the geodesic flow in the level $H_g = 1$. Therefore the heteroclinic cycle of the frozen system exists for all $h \geq 1$.

It is useful to note that according to (59) the periods of the periodic orbits $L_a(h,\nu)$ and $L_b(h,\nu)$ in the heteroclinic cycle behave as

$$T_{a,b}(h,\nu) = h^{-1/2}(T^{(g)}_{a,b} + O(\varepsilon^2)), \tag{64}$$

where $T^{(g)}_{a,b}$ denotes here the ν-independent period of the corresponding orbit of the geodesic flow in the level $H_g = 1$.

In order to apply Theorem 2 we have to check that Poincaré maps near the heteroclinic cycle satisfy the uniformity assumptions [UA1] and [UA2]. Let (u_0, w_0) be the coordinates for which the Poincaré map for the geodesic flow in the level $H_g = 1$ has the form which satisfies (22) and (23). Then we define coordinates (u, w) on the cross-section in the following way:

$$(u, w) = (u_0(q, p/\sqrt{\hat{H}(p, q, \nu)}), w_0(q, p/\sqrt{\hat{H}(p, q, \nu)})) \tag{65}$$

and $z = (h, \nu)$ with $h = \hat{H}(p, q, \nu)$.

Now we need uniform estimates for the Poincaré map represented in these coordinates. Let us take any sufficiently large s and consider the part of the phase space that corresponds to

$$s \le \hat{H}(p, q, \nu) \le 2s. \tag{66}$$

The scaling (59) transforms the system to $\tilde{H} \equiv H_g + \frac{\varepsilon^2}{s} V(q, \varepsilon t/\sqrt{s})$ in the energy levels $1 \le \tilde{H} \le 2$. The scaled system is uniformly $O(\varepsilon^2/s)$-close to the geodesic flow in these energy levels. Therefore, in the coordinates

$$(u, w) = (u_0(q, p/\sqrt{s}), w_0(q, p/\sqrt{s})), \quad h = \hat{H}(p, q, \nu),\ \nu = \varepsilon t, \tag{67}$$

the following formulas hold for the Poincaré map

$$\begin{array}{rclrcl} \bar{u} & = & f_{cc'}(u, \bar{w}) + O(\varepsilon^2/h), & w & = & g_{cc'}(u, \bar{w}) + O(\varepsilon^2/h), \\ \bar{h} & = & h + O(\varepsilon^3/h^{1/2}), & \bar{\nu} & = & \nu + O(\varepsilon/h^{1/2})\,, \end{array} \tag{68}$$

uniformly for all $h \ge 1$ and ν; the functions f, g in (68) define the Poincaré map for the geodesic flow. The equations for $\bar{h}$ and $\bar{\nu}$ are obtained immediately from the fact that the time of return to the cross-section behaves as $O(h^{-1/2})$ (see e.g. (64)), while the time derivative of $\hat{H}$ along the orbit is given by $\varepsilon^3 \partial V/\partial \nu$, i.e it is uniformly $O(\varepsilon^3)$.

As $s/\hat{H}$ is uniformly bounded and separated from zero, it is easy to check that the Poincaré map written in the coordinates (u, w) defined by (65) also has the form (68). Recall that (u, w) run over balls of finite radii by construction, so the validity of the uniformity assumptions (with $\alpha(h, \varepsilon) \equiv h$) follows immediately from (68) and (64).

From the last line of (68) we see that the function β defined by (46) is uniformly $O(\varepsilon^2)$. Now, according to Theorem 3, it remains to check that for some sufficiently small $\delta > 0$ solutions of the equation

$$h'(\nu) = \max\{v_a(h, \nu), v_b(h, \nu)\} - \varepsilon^2 \delta$$

tend to infinity asymptotically linearly with time. Recall that v_c is the average change in H along the periodic solution of the frozen system (6). As the frozen system is $O(\varepsilon^2)$-close to the geodesic flow in our case, we find that

$$v_c(h, \nu) = \varepsilon^2 \frac{d}{d\nu} \bar{V}_c(\nu) + O(\varepsilon^4)$$

(see (60),(64)). For small ε the $O(\varepsilon^4)$-term is absorbed by $\varepsilon^2 \delta$, so after scaling h we are left to examine the behaviour of solutions of

$$h'(\nu) = \max\left\{ \bar{V}_a'(\nu), \bar{V}_b'(\nu) \right\} - \delta.$$

By taking integral of both parts we find

$$h(\nu)-h_0=\frac{1}{2}\left(\bar{V}_a(\nu)+\bar{V}_b(\nu)-\bar{V}_a(0)-\bar{V}_b(0)+\int_0^{\nu}\left|\frac{d}{d\nu}\left(\bar{V}_a-\bar{V}_b\right)\right|d\nu-2\delta\,\nu\right),$$

i.e. condition (61) ensures the existence of linearly tending to $+\infty$ solutions indeed. $\square$

Note that in (61) we take an integral of a non-negative function, therefore (61) is not very restrictive. For example, if $V(q,t)$ is periodic or quasi-periodic in time, condition (61) is equivalent to

$$\bar{V}_a(\nu)-\bar{V}_b(\nu)\neq \text{const.} \tag{69}$$

Thus, for a (quasi)periodic in t potential $V(q,t)$ the only case where there may be no trajectories of unbounded energy is that when the average (60) of $V(q(t),\nu)$ is the same (up to a constant) function of ν for every periodic trajectory of the geodesic flow.

Note also that we do not, in fact, need the hyperbolicity of the flow in the whole phase space. It is sufficient to have a locally-maximal, uniformly-hyperbolic, transitive, compact, invariant set Λ in the energy level $H_g=1$. Theorem 4 then holds true, provided the periodic orbits L_a and L_b belong to Λ.

4.2 Exponential energy growth

In the previous example we found trajectories with the energy growth which is asymptotically linear in time. Such estimate is essentially optimal in that case: because of the uniform boundedness of the time derivative of the perturbation $V(q,t)$ there can be no trajectories with energy growing faster than linearly.

In this section we describe a different class of perturbations for geodesic flows which have trajectories whose energy tends to infinity at a much faster rate. These are obtained by a "parametric" perturbation of the geodesic flow. Namely, consider the following Hamiltonian

$$H=\frac{1}{2}\,p\,g(q,t)\,p, \tag{70}$$

where g^{-1} is the metric tensor. We assume that the corresponding curvature remains strictly negative for all t. We also assume the uniform boundedness and continuity of g with all the derivatives up to the second order. The scaling (59) with $s=\varepsilon^{-2}$ changes the Hamiltonian to

$$\tilde{H}=\frac{1}{2}\,p\,g(q,\varepsilon t)\,p\,.$$

We see that at large energies the original system belongs to the class of Hamiltonian systems with slowly varying parameter.

By the assumed strict negativity of the curvature at every fixed t, the frozen system is hyperbolic in every energy level except for $H = 0$. Therefore, we may take a pair of saddle periodic orbits $L_a(h,\nu)$ and $L_b(h,\nu)$ and a heteroclinic cycle that contains them. As in the case of Theorem 4, the period of $L_{a,b}$ behaves as $\sim \varepsilon h^{-1/2}$, while the change of h during one period is given by

$$\Delta h \sim \varepsilon\sqrt{h}\,.$$

Indeed, by the scaling invariance of the frozen system, we find that

$$\int_0^{T_c(h,\nu)} \frac{\partial H}{\partial \nu}(p,q,\nu)\bigg|_{(p,q)=(p_c,q_c)(t;h,\nu)} dt =$$
$$= \frac{1}{2}\sqrt{h}\int_0^{T_c(1,\nu)} p_c(t;1,\nu)g'_\nu(q_c(t;1,\nu),\nu)p_c(t;1,\nu)dt,$$

where $(p,q) = (p_c,q_c)(t;h,\nu)$ is the equation of the periodic orbit $L_c(h,\nu)$.

Like we did it in the proof of Theorem 4, by using the fact that the scaling (59) can map a neighbourhood of an arbitrarily high energy level into a neighbourhood of the energy level $\hat{H} = 1$, we find that the Poincaré map has the following form (uniformly for all h and ν)

$$\begin{array}{ll} \bar{u} = f_{cc'}(u,\bar{w},\nu) + O(\varepsilon/\sqrt{h}), & w = g_{cc'}(u,\bar{w},\nu) + O(\varepsilon/\sqrt{h}), \\ \bar{h} = h + O(\varepsilon\sqrt{h}), & \bar{\nu} = \nu + O(\varepsilon/\sqrt{h}) \end{array} \tag{71}$$

where the functions f, g define the Poincaré map for the frozen geodesic flow in the energy level $H = 1$. It follows immediately from (71),(64) that the uniformity assumptions are fulfilled with $\alpha(h) = \ln h$, and $\beta(h) = O(h)$.

Thus, by Theorem 3, there exist orbits bounded from below by a solution of the equation

$$h'(\nu) = h(\nu)\left(\max\{\hat{v}_a(\nu), \hat{v}_b(\nu)\} - \delta\right) \tag{72}$$

where

$$\hat{v}_c(\nu) = \frac{1}{2T_c(1,\nu)}\int_0^{T_c(1,\nu)} p_c(t;1,\nu)g'_\nu(q_c(t;1,\nu),\nu)p_c(t;1,\nu)dt. \tag{73}$$

Let us estimate the solutions of (72). Denote $\hat{T}_c(\nu) \equiv T_c(1,\nu)$ (i.e. this is the period of the orbit L_c of the frozen system in the energy level $H = 1$). By the invariance of the frozen system

$$H = \frac{1}{2}\,p\,g(q,\nu)\,p \tag{74}$$

with respect to energy scaling, we find that

$$T_c(h,\nu) = \hat{T}_c(\nu)/\sqrt{h}.$$

Let us introduce the action variable (see Section 1.2)

$$J_c(h,\nu) = \int_0^{T_c(h,\nu)} p_c(t;h,\nu)\dot{q}_c(t;h,\nu)dt.$$

As $\dot{q} = g(q,\nu)\, p$ in system (74), this gives us the following explicit formula for the action

$$J_c(h,\nu) = \int_0^{T_c(h,\nu)} p_c(t;h,\nu)\; g(q_c(t;h,\nu),\nu)\; p_c(t;h,\nu) dt = \\ = 2hT_c(h,\nu) = 2\sqrt{h}\hat{T}_c(\nu). \tag{75}$$

Now recall that by general formula (13),

$$\frac{\partial}{\partial \nu}\hat{J}_c(1,\nu) = -\hat{v}_c(\nu)\hat{T}_c(\nu)\,. \tag{76}$$

Equations (75) with $h = 1$ and (76) imply that in the case of Hamiltonian (70) there is a closed formula which expresses $\hat{v}_c(\nu)$ in terms of $\hat{T}_c(\nu)$:

$$\hat{v}_c(\nu) = -2\frac{d}{d\nu}\ln \hat{T}_c(\nu)\,. \tag{77}$$

Plugging this in (72) we find

$$\frac{d}{d\nu}\ln h(\nu) = -\frac{d}{d\nu}(\ln \hat{T}_a(\nu) + \ln \hat{T}_b(\nu)) + \left|\frac{d}{d\nu}(\ln \hat{T}_a(\nu) - \ln \hat{T}_b(\nu))\right| - \delta,$$

which gives us

$$\ln h(\nu) - \ln h_0 = \ln \hat{T}_a(0)\hat{T}_b(0) - \ln \hat{T}_a(\nu)\hat{T}_b(\nu) + \int_0^\nu \left|\frac{d}{d\nu}\ln(\hat{T}_a(\nu)/\hat{T}_b(\nu))\right| d\nu - \delta\nu.$$

As we see, solutions of (72) tend exponentially to infinity for all sufficiently small δ, provided

$$\liminf_{s\to+\infty} \frac{1}{s}\int_0^s \left|\frac{d}{d\nu}\ln(\hat{T}_a(\nu)/\hat{T}_b(\nu))\right| d\nu > 0, \tag{78}$$

and the functions $\hat{T}_{a,b}$ are bounded away from zero and infinity uniformly for all $\nu \geq 0$

Thus, we arrive at the following result.

Theorem 5 *If the periods $\hat{T}_a(\nu)$ and $\hat{T}_b(\nu)$ of the periodic orbits $L_{a,b}$ of the frozen system in the energy level $H = 1$ are bounded away from zero and infinity for all $\nu \geq 0$, and if condition (78) is satisfied, then Hamiltonian system (70) has orbits for which H exponentially grows to infinity.*

Note that similar to the previous section, if $g(q,t)$ is periodic or quasi-periodic in time, condition (78) is equivalent to

$$\hat{T}_a(\nu)/\hat{T}_b(\nu) \neq \text{const.} \tag{79}$$

4.3 Time-dependent billiard-like potentials

Another example: Let $D \in \mathbb{R}^m$, $m \geq 2$, be a bounded region whose boundary is composed of a finite number of strictly concave (when looking from inside of D) smooth $(m-1)$-dimensional manifolds which intersect by non-zero angles. Let $V_0(q)$ $(q \in D)$ be a positive function such that

$$V_0(q) \to +\infty \quad \text{as} \quad q \to \partial D.$$

Consider the Hamiltonian system

$$H = \sum_{i=1}^{m} \frac{p_i^2}{2} + V_0(q) + V_1(q,t), \tag{80}$$

where $V_1(q,\nu)$ is uniformly bounded, along with the first and second derivatives, for all $q \in D$ and $\nu \in \mathbb{R}$. By scaling time, energy and momenta by the rule (59) with $s = \varepsilon^{-2}$, this system transforms into

$$H = \sum_{i=1}^{m} \frac{p_i^2}{2} + \varepsilon^2 V_0(q) + \varepsilon^2 V_1(q, \varepsilon t). \tag{81}$$

Thus, at large energies, it is a slow perturbation of the singular Hamiltonian

$$H = H_b = \sum_{i=1}^{m} \frac{p_i^2}{2} + \left\{ \begin{array}{ll} 0 & \text{at } \ q \in D, \\ +\infty & \text{at } \ q \notin D. \end{array} \right.$$

This defines a billiard in D: inertial motion inside D and reflection at the boundary. In order to ensure the actual closeness of (81) to a billiard with the standard reflection law ("the angle of reflection equals the angle of incidence") we need certain assumptions. Namely, let $S \subset \partial D$ be the set of "corner" points, i.e. those where ∂D is not smooth (these are the points where the smooth boundary components intersect). Assume that there exists an open neighbourhood U of $\partial D \backslash S$ such that

$$V_0(q) = W(Q(q)) \tag{82}$$

for all $q \in U$. Here $Q(q)$ is the so-called pattern function: it is at least C^2-smooth function defined for all $q \in U$, its first derivative $\partial Q / \partial q$ does not vanish in U, and the smooth boundary components of D are given by the equation $Q = 0$. The function Q thus defines the shape of the billiard region D. The function W defines the growth of the potential as the boundary is approached. We assume that $W' \neq 0$ for all small Q and that its inverse function W^{-1} satisfies

$$W^{-1}(hV) \underset{C^2}{\longrightarrow} 0 \quad \text{as} \quad h \to +\infty, \tag{83}$$

on any interval $V \geq C > 0$. Roughly speaking, by representing the potential in form (82) we achieve that its gradient ("the reaction force") is, in the limit $h = +\infty$, normal to the billiard boundary, which is an obvious necessary condition for the validity of the standard reflection law; condition (83) ensures the C^1-closeness of Poincaré map for the smooth flow (81) at large h to the Poincaré map for the billiard flow, outside the set of singular trajectories, i.e. those which hit S or which are tangent to a smooth component of the billiard boundary at some point.

As the boundary components are strictly concave, the billiard in D is dispersing. This implies [17] the hyperbolicity of the billiard flow (outside the set of singular trajectories); moreover, periodic orbits are dense in the phase space [11, 5]. We call a billiard orbit regular, if it stays bounded away from the singularities, i.e. from the set of points in the phase space which correspond to corner or to a tangency to the billiard boundary.

Theorem 6 *Let $L_a : q = q_a(t)$ and $L_b : q = q_b(t)$ be two regular periodic trajectories of the billiard in D, corresponding to kinetic energy equal to* 1*. Denote*

$$\bar{V}_c(\nu) = \frac{1}{T_c} \int_0^{T_c} V_1(q_c(t), \nu) dt, \tag{84}$$

where T_c ($c = a, b$) is the period of L_c. Assume that condition (61) (or condition (69) in the case of periodic or quasiperiodic dependence of V of t) is fulfilled. Then system (80) has orbits for which H linearly grows to infinity.

Proof. It follows from [17, 18] that for any two regular periodic orbits L_a and L_b in the strictly dispersing billiard there exists a pair of transverse heteroclinic orbits Γ_{ab} and Γ_{ba}, which are also regular. Take a sufficiently small neighbourhood of $L_a \cup L_b \cup \Gamma_{ab} \cup \Gamma_{ba}$ in the intersection of the phase space with the level $H_b = 1$. The hyperbolic set Λ of the orbits that stay in this neighbourhood consists of regular orbits only; as a whole, Λ stays bounded away from the singularity. One can take two small cross-sections, Σ_a and Σ_b, to the orbits L_a and L_b in the intersection of the phase space with the level $H_b = 1$, such that every orbit of Λ returns to $\Sigma_a \cup \Sigma_b$ at a finite time; the corresponding Poincaré maps are smooth, as the orbits of Λ undergo only regular collisions with the billiard boundary. We have the same picture in every other energy level because of the invariance of the billiard flow with respect to energy scaling.

According to [23, 16], under conditions (82),(83), at small ε a finite-time flow map of the smooth system (80) near a regular orbit of the billiard flow is close, along with the first derivatives, to the corresponding map for the billiard flow (while only an autonomous case was considered in the quoted papers, the results and proofs do not change for our case where a bounded and slow non-autonomous term $\varepsilon^2 V_1$ is added to the Hamiltonian). Therefore, for any compact interval of the energy values, the Poincaré maps defined on the cross-section $\Sigma_a \cup \Sigma_b$ by the smooth system (80) is close to the Poincaré map of the billiard flow. Now, applying the scaling transformation (59) exactly as we did it in the proof of Theorem 4, we find that the uniformity assumptions [UA1] and [UA2] are fulfilled with $\alpha(h) = h$ and the Poincaré map can be written in the form (68). The equation for $\bar{h}$ in the last line of (68) is found from the relation

$$\bar{h} = h + \varepsilon^3 \int \frac{\partial}{\partial \nu} V_1(q(t), \varepsilon t) dt \tag{85}$$

(see (81)) where the integral is taken over the corresponding orbit of the smooth system. As the orbits of the smooth system are close to the orbits of the billiard after an appropriate time-parametrization [16], the integral in (85) tends, as $\varepsilon \to 0$, to the integral over the limit billiard orbit, i.e. it remains uniformly bounded along with the first derivatives with respect to initial conditions for any compact set of energy values. Now, by using the scaling transformation (59) with $s = h$ we find that the integral in (85) behaves as $O(h^{-1/2})$, which gives the equation (68) for $\bar{h}$ indeed. After formula (68) is established for the Poincaré map, the rest of the proof follows exactly like in Theorem 4. $\square$

Note that like in Theorem 4, the conditions on the billiard domain D may be relaxed. We do not need the billiard to be dispersing; it is enough to have a locally-maximal, uniformly-hyperbolic, transitive, compact, invariant set Λ composed of regular orbits. Theorem 6 holds true, provided the regular orbits L_a and L_b belong to Λ.

4.4 Nonautonomous perturbation of a homogeneous potential

In the last example, we consider the Hamiltonian system

$$H = T(p) + V_0(q) + V_1(q,t) + V_2(q,t), \tag{86}$$

where T is a quadratic polynomial of momenta $p \in R^m$ $(m \geq 2)$, V_0 is a degree $d \geq 3$ homogeneous polynomial of the coordinates $q \in \mathbb{R}^m$, V_1 is a degree $d-1$ homogeneous polynomial of q, and V_2 is a polynomial of q of degree less than $d-1$; the coefficients of V_1 and V_2 are smooth functions of time, uniformly bounded, along with the first derivative, for all t.

By scaling time, energy, momenta and coordinates by the rule

$$t \to t/s^{1/2-1/d}, \quad H \to Hs, \quad p \to p\sqrt{s}, \quad q \to qs^{1/d} \tag{87}$$

with $s = \varepsilon^{-2d/(d-2)}$ this system transforms into

$$H = T(p) + V_0(q) + \varepsilon^{\frac{2}{d-2}} V_1(q, \varepsilon t) + O(\varepsilon^{\frac{4}{d-2}}). \tag{88}$$

It is a small and slow perturbation of the homogeneous Hamiltonian

$$H = T(p) + V_0(q). \tag{89}$$

This system is invariant with respect to the scaling (87), hence its behaviour is the same in every energy level. Assume that system (89) has a locally-maximal, uniformly-hyperbolic, compact, transitive, invariant set Λ in the energy level $H = 1$. Take any two periodic trajectories $L_a : \{q = q_a(t), p = p_a(t)\}$ and $L_b : \{q = q_b(t), p = p_b(t)\}$ from Λ. Denote

$$\bar{V}_c(\nu) = \frac{1}{T_c}\int_0^{T_c} V_1(q_c(t), \nu)dt \qquad (c = a, b). \tag{90}$$

By the scaling invariance, we obtain that a pair of saddle periodic orbits $L_{a,b}(h)$ exists in every energy level with $h > 0$; since the orbits belong to a transitive hyperbolic set, they are connected by transverse heteroclinic orbits Γ_{ab} and Γ_{ba}. As the frozen system for (88) is close to (89) (recall that $d > 2$), the former also possesses a heteroclinic cycle in every energy level. By applying scaling transformation (87) with $s = h$, one can immediately see that the change of the energy along an orbit of system (88) for one round near $L_c(h)$ is given by

$$\bar{h} - h = \varepsilon^{\frac{d}{d-2}} h^{1/2} (\int_0^{T_c} V_1(q_c(t), \nu) dt + \ldots), \tag{91}$$

while the return time to the cross-section behaves as

$$\bar{\nu} - \nu = \varepsilon h^{-\frac{d-2}{2d}} (T_c + \ldots), \tag{92}$$

where the dots stand for the terms which tend to zero as the distance to $L_c(h)$ diminishes and $\varepsilon \to 0$.

Now, like we did it in the examples above, by using the scaling transformation (87) one can check that the Poincaré map satisfies the uniformity assumptions [UA1] and [UA2] with

$$\alpha(h) = h^{\frac{1}{d}}$$

and

$$\beta = O(\varepsilon^{\frac{2}{d-2}} h^{1-\frac{1}{d}}).$$

This, along with equations (91),(92) implies that system (88) has, for any $\delta > 0$, orbits for which the energy grows not slower than the solution of the equation

$$h'(\nu) = \varepsilon^{\frac{2}{d-2}} h^{1-\frac{1}{d}} (\max\{\frac{d}{d\nu}\bar{V}_a(\nu), \frac{d}{d\nu}\bar{V}_b(\nu)\} - \delta) \tag{93}$$

(see Theorem 3). By scaling energy back in order to return to the original system (86): $h \to h\varepsilon^{2d/(d-2)}$, we rewrite this equation as

$$h'(\nu) = h^{1-\frac{1}{d}} (\max\{\frac{d}{d\nu}\bar{V}_a(\nu), \frac{d}{d\nu}\bar{V}_b(\nu)\} - \delta).$$

This is solved as

$$\begin{aligned} h(\nu)^{1/d} - h(0)^{1/d} &= \\ &= \frac{1}{2d} \left(\bar{V}_a(\nu) + \bar{V}_b(\nu) - \bar{V}_a(0) - \bar{V}_b(0) + \int_0^{\nu} \left| \frac{d}{d\nu}(\bar{V}_a - \bar{V}_b) \right| d\nu - 2\delta\, \nu \right). \end{aligned}$$

Thus, we arrive at the following result.

Theorem 7 *If* (61) *is fulfilled, then system* (86) *has orbits for which H grows to infinity as t^d.*

Like in Theorems 4 and 6 above, in the case of periodic or quasiperiodic dependence of V_1 of t, condition (61) reduces to (69).

As we see, every time we have a chaotic Hamiltonian system which is invariant with respect to a scaling of energy, its non-autonomous perturbation creates orbits of growing to infinity energy, provided very non-restrictive conditions of type (61), (69) or (79) are fulfilled. The rate of the energy growth with time depends on how the perturbation term rescales, and is determined by solving the corresponding equation (7).

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