STABILITY OF SPECTRAL TYPES FOR JACOBI MATRICES UNDER DECAYING RANDOM PERTURBATIONS

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Abstract. We study stability of spectral types for semi-infinite self-adjoint tridiagonal matrices under random decaying perturbations. We show that absolutely continuous spectrum associated with bounded eigenfunctions is stable under Hilbert-Schmidt random perturbations. We also obtain some results for singular spectral types.

1. Introduction

In this paper we study semi-infinite Jacobi matrices of the form

$$J\left(\{a(n)\}_{n=1}^{\infty},\{b(n)\}_{n=1}^{\infty}\right)=\begin{pmatrix} b(1) & a(1) & 0 & 0 & \dots \\ a(1) & b(2) & a(2) & 0 & \dots \\ 0 & a(2) & b(3) & a(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

with

$$b(n)\in\mathbb{R},\ a(n)>0,$$

as operators on $\ell^2(\mathbb{Z}_+=\{1,2,\dots\})$. We shall assume throughout that $J\left(\{a(n)\},\{b(n)\}\right)$ is self-adjoint. For this to be true, $\Sigma_{n=1}^{\infty}a(n)^{-1}=\infty$ suffices [1]. In fact, we need a somewhat stronger restriction on the growth of $\{a(n)\}$ (see (1.7) below).

Such operators are a natural generalization of discrete Schrödinger operators on the half line. In particular, the discrete Laplacian on $\ell^2(\mathbb{Z}_+)$ can be described with the help of the constant sequences $\mathbf{1}\equiv\{a^\circ(n)\}$, $\mathbf{0}\equiv\{b^\circ(n)\}$, where $a^\circ(n)=1$ and $b^\circ(n)=0$ for all $n\in\mathbb{Z}_+$, so that

$$\Delta=J\left(\mathbf{1},\mathbf{0}\right).$$

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From the fact that the vector

$$\delta_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

is a cyclic vector for $J(\{a(n)\}, \{b(n)\})$, it follows ([20]) that there exists a measure μ, which coincides with the spectral measure of the vector δ_1, so that $J(\{a(n)\}, \{b(n)\})$ is unitarily equivalent to the operator of multiplication by the parameter on $L^2(\mathbb{R}, d\mu)$. μ decomposes as

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}},$$

where μ_{ac} is the part of μ that is absolutely continuous with respect to the Lebesgue measure, μ_{sc} is a continuous measure that is singular with respect to the Lebesgue measure, and μ_{pp} is a pure point measure.

We want to investigate the stability of certain continuity properties of μ under a decaying random perturbation of $J(\{a(n)\}, \{b(n)\})$. The first part of the paper deals with the stability of the essential support of the absolutely continuous spectrum. In the second part, we restrict the discussion to the case $\{a(n)\} = \mathbf{1}$ (the discrete Schrödinger case) and deal with the more delicate singular spectral types. In both cases, a principal tool in the analysis is the connection between properties of the spectral measure and the behavior at infinity of solutions of the difference equation

$$a(n)\varphi(n+1) + a(n-1)\varphi(n-1) + b(n)\varphi(n) = E\varphi(n) \tag{1.2}$$

for fixed $E \in \mathbb{R}$ and $n \geq 1$ (we set $a(0) = 1$). Such a difference equation can be regarded as an initial value problem, which makes it natural to introduce the *single-step transfer matrices*:

$$S^E(n) = \begin{pmatrix} \frac{E-b(n)}{a(n)} & -\frac{a(n-1)}{a(n)} \\ 1 & 0 \end{pmatrix}, \quad n \geq 1, \tag{1.3}$$

that satisfy

$$\begin{pmatrix} \varphi(n+1) \\ \varphi(n) \end{pmatrix} = S^E(n) \begin{pmatrix} \varphi(n) \\ \varphi(n-1) \end{pmatrix}$$

for any $\{\varphi(n)\}_{n=0}^{\infty}$ that solves (1.2). Thus, if we denote

$$\vec{\varphi}(n) = \begin{pmatrix} \varphi(n+1) \\ \varphi(n) \end{pmatrix}$$

and $T^E(n) \equiv S^E(n) \cdot \ldots \cdot S^E(1)$, then

$$\vec{\varphi}(n) = T^E(n)\vec{\varphi}(0). \tag{1.4}$$

The *essential support* of an absolutely continuous measure ν on $\mathbb{R}$ is the equivalence class $\Sigma_{\rm ac}(\nu)$ of sets $A \subseteq \mathbb{R}$ such that ν is supported on A and that the restriction of Lebesgue measure to A is absolutely continuous w.r.t. ν. We shall use $\Sigma_{\rm ac}(\{a(n)\},\{b(n)\})$ to denote the essential support of $\mu_{\rm ac}$ and refer to it as the essential support of the absolutely continuous spectrum of $J(\{a(n)\},\{b(n)\})$.

Over the past decade, there has been a significant amount of work done (see, e.g., [2, 3, 4, 5, 11, 13, 14, 22]), in the area of one-dimensional Schrödinger operators, towards determining conditions on a perturbing potential $\{\tilde{b}(n)\}$ ensuring that

$$\Sigma_{\rm ac}(\mathbf{1},\{b(n)\}) = \Sigma_{\rm ac}(\mathbf{1},\{b(n)+\tilde{b}(n)\}). \tag{1.5}$$

That such an equality exists for any $\{\tilde{b}(n)\} \in \ell^1$ is a well known result from scattering theory [21, Chapter XI.3]. For general $\{b(n)\}$, this is the best there is at present, in terms of sheer ℓ^p properties of the perturbation. For $\{b(n)\} = \mathbf{0}$, however, it has been proven by Deift-Killip [5] that (1.5) holds for $\{\tilde{b}(n)\}$ merely in ℓ^2. This result has been later extended by Killip [11] to include any periodic $\{b(n)\}$. For arbitrary background potentials $\{b(n)\}$, it has been conjectured by Kiselev-Last-Simon [16] that an ℓ^2 perturbation does not change the essential support of the absolutely continuous spectrum. For a perturbation of the off-diagonal entries as well as the diagonal entries, Killip-Simon [12] have shown that if $\{\tilde{a}(n)\},\{\tilde{b}(n)\} \in \ell^2$, then

$$\Sigma_{\rm ac}(\mathbf{1},\mathbf{0}) = \Sigma_{\rm ac}(\mathbf{1}+\{\tilde{a}(n)\},\{\tilde{b}(n)\}). \tag{1.6}$$

Our first result deals with the preservation of $\Sigma_{\rm ac}(\{a(n)\},\{b(n)\})$ for general $\{b(n)\}$ and $\{a(n)\}$ obeying

$$\limsup_{L\to\infty} \frac{1}{L}\sum_{n=1}^{L} a(n)^{-1} > 0 \tag{1.7}$$

under a *random* decaying perturbation of both the diagonal and off-diagonal entries. For a measurable set $B \subseteq \mathbb{R}$, $\Sigma_{\rm ac} \cap B$ denotes the equivalence class of sets $A \cap B$ such that $A \in \Sigma_{\rm ac}$.

Theorem 1.1. *Let $J(\{a(n)\},\{b(n)\})$ be a Jacobi matrix such that $\{a(n)\}$ obeys (1.7), and let $\tilde{a}_\omega(n) : \Omega \to \mathbb{R}$ and $\tilde{b}_\omega(n) : \Omega \to \mathbb{R}$ $(n \geq 1)$ be two sequences of independent random variables with zero mean, defined over a probability space $(\Omega, \mathcal{F}, P)$. Assume that there exists a $\delta > 0$, for which*

$$\delta^{-1} > \frac{a(n)}{a(n)+\tilde{a}_\omega(n)} > \delta \tag{1.8}$$

for every n and $\omega \in \Omega$. Let $J_0 = J(\{a(n)\}, \{b(n)\})$ and

$$J_\omega = J(\{a(n) + \tilde{a}_\omega(n)\}, \{b(n) + \tilde{b}_\omega(n)\}).$$

Then, for a.e. ω,

$$\Sigma_{\mathrm{ac}}(J_0) \cap \Gamma = \Sigma_{\mathrm{ac}}(J_\omega) \cap \Gamma, \tag{1.9}$$

where Γ is the set of all $E \in \mathbb{R}$ for which

$$\sum_{n=1}^{\infty} \left(\left\langle \tilde{a}_\omega(n)^4 \right\rangle^{1/2} + \left\langle \tilde{b}_\omega(n)^2 \right\rangle \right) \left((a(n)+1) t^E(n) \right)^4 < \infty, \tag{1.10}$$

where we denote $\langle f_\omega \rangle \equiv \int_\Omega f_\omega dP(\omega)$ for any measurable function f_ω of ω and $t^E(n) \equiv \| T^E(n) \|$ is the norm of the n'th transfer matrix corresponding to J_0.

We note that Kaluzhny-Last [10] recently studied Jacobi matrices of the form $J(\{a(n)+\tilde{a}_\omega(n)\}, \{b(n)+\tilde{b}_\omega(n)\})$, where $\{a(n)\}-\mathbf{1}$ and $\{b(n)\}$ are decaying sequences of bounded variation and $\{\tilde{a}_\omega(n)\}$, $\{\tilde{b}_\omega(n)\}$ are as in Theorem 1.1 and obey

$$\sum_{n=1}^{\infty} \left(\left\langle \tilde{a}_\omega(n)^2 \right\rangle + \left\langle \tilde{b}_\omega(n)^2 \right\rangle \right) < \infty.$$

They show that, with probability one, such operators have purely absolutely continuous spectrum on $(-2, 2)$ and moreover, this purity of the absolutely continuous spectrum is stable under changing any finite number of entries in the Jacobi matrices. Since the unperturbed $J(\{a(n)\}, \{b(n)\})$ is known (see, e.g., [24]) in this case to have purely absolutely continuous spectrum on $(-2, 2)$ with $\{t^E(n)\}_{n=1}^\infty$ being a bounded sequence for every $E \in (-2, 2)$, we see that a part of their result, namely, the fact that $\Sigma_{\mathrm{ac}}(J_0) = \Sigma_{\mathrm{ac}}(J_\omega)$, can be recovered as a special case of Theorem 1.1.

To further elucidate Theorem 1.1, consider the case $a(n) = 1$, $\tilde{a}_\omega(n) = 0$. The condition defining Γ translates into an ℓ^2 type condition on the perturbation when one studies energies for which the transfer matrices are bounded: For a given background potential $\{b(n)\}$, denote

$$\Gamma_0 \equiv \Gamma_0(\{b(n)\}) = \{E \in \mathbb{R} \mid t^E(n) \text{ is bounded}\}.$$

Then it follows from the theory of subordinacy [8] (also see [24]) that there exists a set $A \in \Sigma_{\mathrm{ac}}(\mathbf{1}, \{b(n)\})$ for which $\Gamma_0 \subseteq A$. From Theorem 1.1, it follows that

Corollary 1.2. *Assume that*

$$\sum_{n=1}^{\infty} \left\langle \tilde{b}_\omega(n)^2 \right\rangle < \infty.$$

Then, for a.e. ω,

$$\Sigma_{\mathrm{ac}}(\mathbf{1},\{b(n)\})\cap\Gamma_0=\Sigma_{\mathrm{ac}}(\mathbf{1},\{b(n)+\tilde{b}_\omega(n)\})\cap\Gamma_0.$$

Corollary 1.2 constitutes some progress towards a random version of the above mentioned conjecture of Kiselev-Last-Simon [16]. Whether actually $\Gamma_0(\{b(n)\})\in\Sigma_{\mathrm{ac}}(\{b(n)\})$ for any $\{b(n)\}$ is a long standing open problem. For some related work, see Maslov-Molchanov-Gordon [19].

The question of stability of *singular* spectral types has received much less attention than the one concerning Σ_{ac}. One of the reasons for this is the fact that singular spectral types are not stable even under rank one perturbations (see [7]). One may, however, bypass this problem by using an idea of Del-Rio-Simon-Stolz [6] to consider the union of spectral supports over the different boundary conditions. This provides a unified approach for the different spectral types, in that spectral stability is obtained for any compactly supported perturbation (see [6]). Kiselev-Last-Simon [16] have modified and extended this approach, via the theory of subordinacy, to deal with the classification of spectral types according to the singularity/continuity of the spectral measure w.r.t. α-dimensional Hausdorff measures. In our definitions, we follow their general methodology.

While it is possible, using the methods developed below, to deal with the general Jacobi case, we restrict the discussion to the case of diagonal perturbations of discrete Schrödinger operators. We take this approach in order to avoid technical difficulties which may obscure the main argument. Thus, for fixed $E\in\mathbb{R}$, we shall be looking at properties of solutions of the equations

$$\varphi(n+1)+\varphi(n-1)+b(n)\varphi(n)=E\varphi(n) \tag{1.11}$$

for $n\geq 2$,

$$\varphi(2)+(b(1)-\tan(\theta))\varphi(1)=E\varphi(1) \tag{1.12}$$

for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Such sequences are obviously eigenvectors (not necessarily in ℓ^2) of the infinite matrix

$$H_\theta=\begin{pmatrix} b(1)-\tan(\theta) & 1 & 0 & 0 & \dots \\ 1 & b(2) & 1 & 0 & \dots \\ 0 & 1 & b(3) & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.13}$$

We denote by $\varphi^E_{1,\theta}(n)$ the solution to (1.11), (1.12), normalized by

$$\varphi^E_{1,\theta}(1)=\cos(\theta). \tag{1.14}$$

We also include the case $\theta = -\pi/2$, for which (1.12) and (1.14) are replaced by $\varphi^E_{1,-\pi/2}(1) = 0$, $\varphi^E_{1,-\pi/2}(2) = 1$. We shall use the notation

$$\varphi^E_{2,\theta} \equiv \varphi^E_{1,\theta-\pi/2}. \tag{1.15}$$

Remark. One may define $\varphi^E_{1,\theta}$ by referring only to (1.11) (for $n \geq 1$) and using $\varphi^E_{1,\theta}(0) = -\sin(\theta)$, $\varphi^E_{1,\theta}(1) = \cos(\theta)$. This way $\varphi^E_{1,\theta}$ is more naturally defined on $[-\frac{\pi}{2}, \frac{\pi}{2})$, without anything special for $\theta = -\pi/2$.

A basic object in the theory of subordinacy is the L'th norm (for $L > 0$) of a function $f : \mathbb{Z}_+ \to \mathbb{C}$,

$$\| f \|_L^2 \equiv \sum_{n=1}^{\lfloor L \rfloor} |f(n)|^2 + (L - \lfloor L \rfloor)|f(\lfloor L \rfloor + 1)|^2, \tag{1.16}$$

where $\lfloor \cdot \rfloor$ denotes integer part. For a given $E \in \mathbb{R}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, $\varphi^E_{1,\theta}$ is called *subordinate* if

$$\lim_{L\to\infty} \frac{\| \varphi^E_{1,\theta} \|_L}{\| \varphi^E_{2,\theta} \|_L} = 0. \tag{1.17}$$

It is clear that a subordinate solution does not necessarily exist for every E, but whenever it does, it is unique. We denote the θ for which $\varphi^E_{1,\theta}$ is subordinate, if it exists, by $\theta(E)$. One may decompose $\mathbb{R}$ into three disjoint sets:

$$\Sigma_{\mathrm{pp}} \equiv \{E \in \mathbb{R} \mid \theta(E) \text{ exists and } \varphi^E_{1,\theta(E)} \in \ell^2\}$$
$$\Sigma_{\mathrm{sc}} \equiv \{E \in \mathbb{R} \mid \theta(E) \text{ exists and } \varphi^E_{1,\theta(E)} \notin \ell^2\}$$
$$\mathbb{R} \setminus (\Sigma_{\mathrm{pp}} \cup \Sigma_{\mathrm{sc}})$$

What makes the discussion of stability of singular spectral types interesting is the fact (see, e.g., [16]) that these three sets have the following spectral interpretation:

- $\Sigma_{\mathrm{pp}} = \cup_\theta \tilde{\sigma}_{\mathrm{pp}}(H_\theta)$, where $\tilde{\sigma}_{\mathrm{pp}}(H_\theta)$ is the set of eigenvalues of H_θ.
- For any θ, $\mu_{\theta,\mathrm{sc}}(\cdot) = \mu_\theta(\Sigma_{\mathrm{sc}} \cap \cdot)$ and any other set A with this property equals Σ_{sc} up to a set of Lebesgue measure zero.
- $\Sigma_{\mathrm{ac}} \ni \mathbb{R} \setminus (\Sigma_{\mathrm{pp}} \cup \Sigma_{\mathrm{sc}})$.

The above sets are clearly independent of θ and stable under compactly supported perturbations.

The Jitomirskaya-Last extension of subordinacy theory [9] makes it possible to investigate the stability of Hausdorff-dimensional properties of the spectral measure. It follows from their analysis that for any $\alpha \in (0,1]$, there exist sets $\Sigma_{\alpha\mathrm{c}} \subseteq \mathbb{R}$ and $\Sigma_{\alpha\mathrm{s}} \subseteq \mathbb{R}$ such that for any θ,

$$\mu_{\theta,\alpha\mathrm{c}} = \mu_\theta(\Sigma_{\alpha\mathrm{c}} \cap \cdot), \quad \mu_{\theta,\alpha\mathrm{s}} = \mu_\theta(\Sigma_{\alpha\mathrm{s}} \cap \cdot) \tag{1.18}$$

where $\mu_{\theta,\alpha c}$ is the part of μ_θ that is continuous with respect to the α-dimensional Hausdorff measure, and $\mu_{\theta,\alpha s}$ is the part which is singular with respect to it. (For the study of decompositions of a measure w.r.t. dimensional Hausdorff measures and for the significance of this analysis to quantum mechanics, see, for example, [17] and references therein.) For any $\alpha \in (0,1]$, $\Sigma_{\alpha s} \subseteq \Sigma_{\rm sc} \cup \Sigma_{\rm pp}$, and for any $E \in \Sigma_{\rm sc}$, whether $E \in \Sigma_{\alpha s}$ or not, depends on the decay of the subordinate solution at infinity:

$$E \in \Sigma_{\alpha s}$$

if and only if

$$\liminf_{L\to\infty} \frac{\| \varphi^E_{1,\theta(E)} \|_L}{\| \varphi^E_{2,\theta(E)} \|_L^{\tilde\beta(\alpha)}} = 0 \tag{1.19}$$

where $\tilde\beta(\alpha) = \frac{\alpha}{2-\alpha}$ (see [9]).

The discussion above motivates the following definition of [16]: Let $E \in \Sigma_{\rm sc}$. Define

$$\beta(E) = \liminf_{L\to\infty} \frac{\ln \| \varphi^E_{1,\theta(E)} \|_L}{\ln \| \varphi^E_{2,\theta(E)} \|_L}. \tag{1.20}$$

For any E with $\beta(E) > 0$, we also define

$$\eta(E) = \frac{1-\beta(E)}{\beta(E)}. \tag{1.21}$$

Again, it is clear that the sets $\Sigma_{\alpha s}$ and $\Sigma_{\alpha c}$ and the parameter $\beta(E)$ (where it is defined) are stable under compactly supported perturbations. To obtain more, one needs a regularity condition on the energy: Following Kiselev-Last-Simon [16], we shall call an energy E *regular* if for some θ and all $\varepsilon > 0$, we have

$$\| \varphi^E_{1,\theta} \|_L < C_\varepsilon L^{\frac{1}{2}+\varepsilon}.$$

Since almost every energy is regular both with respect to each μ_θ (see [1]) and (by spectral averaging—see Theorem 1.8 in [23]) with respect to Lebesgue measure, the demand that energies be regular is not a severe restriction.

Let

$$\Lambda_0 = \{E \in \Sigma_{\rm sc} \mid E \text{ is regular and } \beta(E) > 0\}. \tag{1.22}$$

For deterministic perturbations, Kiselev-Last-Simon [16] have shown that, for any $0 < \alpha < 1$,

$$\tilde\Lambda \cap \Sigma_{\alpha c}(\mathbf{1}, \{b(n)\}) \subseteq \Sigma_{\alpha c}\left(\mathbf{1}, \{b(n) + \tilde b(n)\}\right),$$

$$\tilde\Lambda \cap \Sigma_{\alpha s}(\mathbf{1}, \{b(n)\}) \subseteq \Sigma_{\alpha s}\left(\mathbf{1}, \{b(n) + \tilde b(n)\}\right),$$

where $\{b(n)\}$ is any background potential, and

$$\tilde{\Lambda} = \{E \in \Lambda_0 \mid |\tilde{b}(n)| < Cn^{-\gamma} \text{ for some } \gamma > \eta(E) + 1\}. \tag{1.23}$$

For random potentials we show

Theorem 1.3. *Let $\{\tilde{b}_\omega(n)\}$ be a sequence of independent real-valued random variables with zero mean on Ω. For any $E \in \Lambda_0$, $\tilde{\eta} > 0$ and $n \geq 1$, let*

$$r_{\tilde{\eta}}^E(n) \equiv |\varphi_{1,\theta(E)}^E(n)|^4 n^{2\tilde{\eta}} + |\varphi_{2,\theta(E)}^E(n)|^4, \tag{1.24}$$

and let

$$\Lambda = \left\{ E \in \Lambda_0 \;\middle|\; \sum_{n=1}^{\infty} \left(r_{\tilde{\eta}}^E(n) \left\langle \tilde{b}_\omega(n)^2 \right\rangle \right) < \infty \text{ for some } \tilde{\eta} > \eta(E) \right\}. \tag{1.25}$$

Then, for any $0 < \alpha < 1$ and any fixed measure ν on $\mathbb{R}$, for a.e. ω,

$$\Lambda \cap \Sigma_{\alpha c}(\mathbf{1}, \{b(n)\}) \subseteq \Sigma_{\alpha c}(\mathbf{1}, \{b(n) + \tilde{b}_\omega(n)\}),$$
$$\Lambda \cap \Sigma_{\alpha s}(\mathbf{1}, \{b(n)\}) \subseteq \Sigma_{\alpha s}(\mathbf{1}, \{b(n) + \tilde{b}_\omega(n)\}), \tag{1.26}$$

where the inclusion is up to a set of ν-measure zero.

Theorems 1.1 and 1.3 have the common feature of the appearance of the 4th power of the norms of the transfer matrices (in 1.3, see the definition of $r_{\tilde{\eta}}^E(n)$). The reason for this is that our basic tool is a random variation of parameters, where the perturbing potential is coupled to the square of the transfer matrices and thus, when estimating the variance of the perturbation, the 4th power enters the picture. For examples where the pointwise behavior of the solutions to (1.2) is known, this does not constitute a problem. One such example is the class of bounded sparse potentials studied by Zlatoš [25]. For this class, one has stability of $\Sigma_{\alpha \mathrm{c}}$ and $\Sigma_{\alpha \mathrm{s}}$ under random perturbations decaying like $n^{-\gamma}$ for $\gamma > \eta(E) + \frac{1}{2}$ (compare with $\gamma > \eta(E) + 1$ in (1.23)).

In light of these remarks, the general question of the pointwise behavior of the solutions of (1.2) is one that arises naturally in connection with the results presented here. The more famous question of whether or not there is almost-everywhere boundedness of solutions with respect to the absolutely continuous part of the spectral measure is only one facet of this general problem.

The rest of this paper is organized as follows. Section 2 covers some preliminaries—especially a useful characterization of Σ_{ac} due to Last-Simon [18] and a variation on a classic theorem concerning the almost everywhere convergence of random series with convergent variances. In Section 3 we introduce the main idea behind our analysis. We formulate and prove two different (but similar) lemmas which are central to the proofs of our two main theorems. These theorems are proved in Section 4. Section 5 has the worked out application of Theorem 1.3 to the above mentioned sparse potentials of Zlatoš [25].

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2. Preliminaries

As explained in the introduction, we want to exploit the connection between spectral properties of the operator $J(\{a(n)\},\{b(n)\})$ and the asymptotic properties of the solutions to the corresponding difference equation. That is, we want to compare the asymptotic properties of the solutions to the difference equation corresponding to the basic operator, with those of the solutions to the equation corresponding to the perturbed one. In the singular continuous case we will 'equate' the behavior at infinity of the perturbed and unperturbed solutions (in a sense to be precisely defined in Section 4). For the absolutely continuous case, however, we need a little less. We rely on the following characterization of $\Sigma_{\rm ac}$ due to Last-Simon [18]:

Proposition 2.1 (Last-Simon [18])**.** *Let $J(\{a(n)\},\{b(n)\})$ be a self-adjoint Jacobi matrix such that $\{a(n)\}$ satisfies* (1.7)*, and let $T^E(n)$ be the corresponding transfer matrices defined by* (1.3)-(1.4)*. Let $\{N_j\}_{j=1}^\infty$ be a sequence for which*

$$\lim_{j\to\infty}\frac{1}{N_j}\sum_{n=1}^{N_j}\frac{1}{a(n)}>0$$

and let $\Sigma_{\rm ac}\equiv\Sigma_{\rm ac}(\{a(n)\},\{b(n)\})$. Then

$$\left\{E\in\mathbb{R}\;\middle|\;\liminf_{j\to\infty}\frac{1}{N_j}\sum_{n=1}^{N_j}\parallel T^E(n)\parallel^2<\infty\right\}\in\Sigma_{\rm ac}.$$

Remark. This is actually a slight generalization of Theorem 1.1 of [18] to the general Jacobi case. Its proof is essentially the same as their proof.

The following are variants of a martingale inequality and convergence theorem which play a crucial role in the proofs of Lemmas 3.1 and 3.2.

Lemma 2.2. *Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\{x_\omega(n)\}$ be a sequence of independent random variables such that*

$$\int_\Omega x_\omega(n)dP(\omega)\equiv\langle x_\omega(n)\rangle=0$$

for all n. Let

$$z_\omega(n) = x_\omega(n) f_n(x_\omega(n+1), x_\omega(n+2), \dots)$$

where the f_n are real-valued, measurable functions on $\mathbb{R}^\infty$.

Then, for any $N_1 < N_2$ and $r \geq 0$,

$$P\left(\left\{\omega \;\middle|\; \max_{N_1 \leq n \leq N_2} |z_\omega(n) + \ldots + z_\omega(N_2)| > r\right\}\right) \leq \frac{\sum_{j=N_1}^{N_2} \langle (z_\omega(j))^2 \rangle}{r^2}. \tag{2.1}$$

Proof. Obviously, we may assume that $\left\langle (z_\omega(n))^2 \right\rangle < \infty$ for all n, since otherwise there is nothing to prove. Denote

$$Y_\omega(n) = \sum_{j=N_1}^{n-1} z_\omega(j), \qquad Q_\omega(n) = \sum_{j=n}^{N_2} z_\omega(j),$$

and let

$$A_j = \{\omega \in \Omega \mid |Q_\omega(j)| > r;\ |Q_\omega(j+1)|, \ldots, |Q_\omega(N_2)| \leq r\}.$$

Then, if $i < j$,

$$\langle z_\omega(i) Q_\omega(j) \chi_j \rangle = \langle x_\omega(i) \rangle \langle f_i (x_\omega(i+1), \ldots) Q_\omega(j) \chi_j \rangle = 0$$

where

$$\chi_j = \chi_{A_j} = \text{the characteristic function of } A_j,$$

and thus,

$$\langle \chi_j Y_\omega(j) Q_\omega(j) \rangle = 0$$

so that

$$\left\langle \chi_j Q_\omega(j)^2 \right\rangle \leq \left\langle \chi_j \left(Y_\omega(j) + Q_\omega(j)\right)^2 \right\rangle.$$

Therefore

$$r^2 \langle \chi_j \rangle \leq \left\langle \chi_j Q_\omega(j)^2 \right\rangle \leq \left\langle \chi_j \left(Y_\omega(j) + Q_\omega(j)\right)^2 \right\rangle$$

and

$$\begin{aligned}
r^2 \sum_{j=N_1}^{N_2} \langle \chi_j \rangle &\leq \sum_{j=N_1}^{N_2} \left\langle \chi_j Q_\omega(j)^2 \right\rangle \leq \sum_{j=N_1}^{N_2} \left\langle \chi_j \left(Y_\omega(j) + Q_\omega(j)\right)^2 \right\rangle \\
&= \sum_{j=N_1}^{N_2} \left\langle \chi_j \left(\sum_{l=N_1}^{N_2} z_\omega(l) \right)^2 \right\rangle \leq \left\langle \left(\sum_{j=N_1}^{N_2} z_\omega(j) \right)^2 \right\rangle \\
&= \left\langle \sum_{j=N_1}^{N_2} z_\omega(j)^2 \right\rangle
\end{aligned}$$

where in the last equality we use

$$\langle z_\omega(i) z_\omega(j) \rangle = 0 \quad \text{for } i \neq j.$$

This ends the proof. $\square$

Theorem 2.3. *Using the notation of Lemma 2.2, assume that*

$$\sum_{n=1}^{\infty} \left\langle z_\omega(n)^2 \right\rangle < \infty.$$

Then

$$\sum_{n=1}^{\infty} z_\omega(n)$$

converges almost surely. Furthermore, for any n*,*

$$\left\langle \left(\sum_{j=n}^{\infty} z_\omega(j) \right)^2 \right\rangle \leq \sum_{j=1}^{\infty} \left\langle z_\omega(j)^2 \right\rangle < \infty. \tag{2.2}$$

Proof. For any $\varepsilon > 0$ and for any $N_1 < N_2$, by (2.1),

$$P\left(\left\{ \omega \;\middle|\; \max_{N_1 \leq n \leq N_2} |z_\omega(n) + \ldots + z_\omega(N_2)| > \varepsilon \right\} \right) \leq \frac{\sum_{j=N_1}^{N_2} \langle z_\omega(j)^2 \rangle}{\varepsilon^2}.$$

Thus, the event

$$\left\{ \omega \;\middle|\; \exists N_1, N_2, \text{ arbitrarily large, } \max_{N_1 \leq n \leq N_2} |z_\omega(n) + \ldots + z_\omega(N_2)| > \varepsilon \right\}$$

has probability zero. So we get that, with probability one, for any $\varepsilon > 0$, there exists N_ε so that for any $N_1 > N_2 > N_\varepsilon$,

$$\left| \sum_{j=N_1}^{N_2} z_\omega(j) \right| < \varepsilon,$$

or, in other words, $\sum_{j=1}^{\infty} z_\omega(j)$ converges with probability one. (2.2) now follows from Fatou's lemma. $\square$

3. A Central Lemma

The idea at the basis of our analysis is that of variation of parameters. We want to obtain a 'linear' relationship between the generalized eigenfunctions of the original problem and those of the perturbed problem. Thus, for fixed $E \in \mathbb{R}$, let $T_0^E(n)$ and $S_0^E(n)$ denote the n-steps and one-step transfer matrices respectively and let $T_\omega^E(n)$ and $S_\omega^E(n)$ denote the same objects for the perturbed problem (depending on the random parameter ω). Define $\mathcal{D}_\omega^E(n)$ through the equation:

$$T_\omega^E(n) = T_0^E(n) \mathcal{D}_\omega^E(n). \tag{3.1}$$

Then

$$\begin{aligned}\mathcal{D}^E_\omega(n-1) &= T^E_0(n-1)^{-1}T^E_\omega(n-1)\\ &= T^E_0(n-1)^{-1}T^E_\omega(n-1)T^E_\omega(n)^{-1}T^E_0(n)T^E_0(n)^{-1}T^E_\omega(n)\\ &= T^E_0(n-1)^{-1}T^E_\omega(n-1)T^E_\omega(n)^{-1}T^E_0(n)\mathcal{D}^E_\omega(n)\\ &= T^E_0(n)^{-1}S^E_0(n)S^E_\omega(n)^{-1}T^E_\omega(n)T^E_\omega(n)^{-1}T^E_0(n)\mathcal{D}^E_\omega(n)\\ &= T^E_0(n)^{-1}S^E_0(n)S^E_\omega(n)^{-1}T^E_0(n)\mathcal{D}^E_\omega(n)\\ &= (I+\mathcal{U}^E_\omega(n))\mathcal{D}^E_\omega(n),\end{aligned} \tag{3.2}$$

where

$$\mathcal{U}^E_\omega(n) = T^E_0(n)^{-1}(S^E_0(n)S^E_\omega(n)^{-1}-I)T^E_0(n). \tag{3.3}$$

Almost sure convergence of $\mathcal{D}^E_\omega(n)$ to I would insure that the asymptotic properties of $T^E_\omega(n)$ would resemble those of $T^E_0(n)$. This would suffice in the absolutely continuous case. In the singular continuous case we would like to control the convergence rate of each of the column vectors of $\mathcal{D}^E_\omega(n)$ separately. The following lemma is actually a random version of a well known result on the control of the amplitudes (see for instance [16] and problem XI.97 in [21]). It is central to everything that follows.

Lemma 3.1. *Let* $\left\{\mathcal{U}(n) = \begin{pmatrix} u_{11}(n) & u_{12}(n)\\ u_{21}(n) & u_{22}(n)\end{pmatrix}\right\}_{n=1}^{\infty}$ *be a sequence of matrices in* $M_2(\mathbb{R})$*, and let* $\{\tilde{b}_\omega(n)\}$ *be a sequence of independent random variables with zero mean. Suppose that*

$$\sum_{n=1}^{\infty}\left(\left\langle \tilde{b}_\omega(n)^2\right\rangle (u_{11}(n)^2+u_{12}(n)^2+u_{22}(n)^2+u_{21}(n)^2 f_+(n)^2)\right) < \infty \tag{3.4}$$

for some monotonically increasing sequence - $f_+(n)>0$*. Then,* P*-almost surely,*

$$\mathbf{d}_\omega(n-1)-\mathbf{d}_\omega(n) = \tilde{b}_\omega(n)\mathcal{U}(n)\mathbf{d}_\omega(n) \tag{3.5}$$

has solutions - $\mathbf{d}^+_\omega(n)\equiv\begin{pmatrix} d^+_{1,\omega}(n)\\ d^+_{2,\omega}(n)\end{pmatrix}$ *and* $\mathbf{d}^-_\omega(n)\equiv\begin{pmatrix} d^-_{1,\omega}(n)\\ d^-_{2,\omega}(n)\end{pmatrix}$*, that satisfy*

$$\lim_{n\to\infty} d^+_{1,\omega}(n) = 0 \tag{3.6}$$

$$\lim_{n\to\infty} d^+_{2,\omega}(n) = 1 \tag{3.7}$$

$$\lim_{n\to\infty} d^-_{1,\omega}(n) = 1 \tag{3.8}$$

$$\lim_{n\to\infty} d_{2,\omega}^{-}(n) f_{+}(n) \quad = \quad 0 \tag{3.9}$$

Remark. Note that, for $f_{+}(n) \equiv 1$, (3.6)-(3.9) mean that the matrix equation

$$\mathcal{D}_\omega(n-1) = \left(I + \tilde{b}_\omega(n)\mathcal{U}(n) \right) \mathcal{D}_\omega(n)$$

has a solution $\mathcal{D}_\omega(n)$ such that $\lim_{n\to\infty} \mathcal{D}_\omega(n) = I$.

Proof. We start by constructing $\mathbf{d}^{+}$. Let

$$\mathbf{d}_\omega^{+,0}(n) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.10}$$

and denote

$$\tilde{\mathcal{U}}_\omega(n) = \tilde{b}_\omega(n)\mathcal{U}(n). \tag{3.11}$$

Then Theorem 2.3 says that

$$\mathbf{d}_\omega^{+,1}(n) = \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{+,0}(j) \tag{3.12}$$

is defined P-a.s. for any n and that $\langle \| \mathbf{d}_\omega^{+,1}(n) \|^2 \rangle$ is bounded in n. Note also, that $\mathbf{d}_\omega^{+,1}(n)$ is a measurable function of

$$\{\tilde{b}_\omega(n+1), \tilde{b}_\omega(n+2), ...\}.$$

Now, for $k \geq 1$, assume that $\mathbf{d}_\omega^{+,k}(n)$ is defined P-a.s. as a measurable function of

$$\{\tilde{b}_\omega(n+1), \tilde{b}_\omega(n+2), ...\}$$

and that $\left\langle \| \mathbf{d}_\omega^{+,k}(n) \|^2 \right\rangle$ is bounded in n. Then by Theorem 2.3, it is possible to define

$$\mathbf{d}_\omega^{+,(k+1)}(n) = \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{+,k}(j) \tag{3.13}$$

P-a.s. and this definition satisfies all of the properties listed above. Thus, by induction, we construct $\mathbf{d}_\omega^{+,k}(n)$ for every $k \in \mathbb{N}$. Now,

$$\begin{aligned}
&\left\langle \| \mathbf{d}_\omega^{+,k}(n) \|^2 \right\rangle \\
&= \left\langle \| \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)(j)\mathbf{d}_\omega^{+,(k-1)}(j) \|^2 \right\rangle \\
&= \left\langle \left(\sum_{j=n+1}^{\infty} \tilde{b}_\omega(j) \left(u_{11}(j) d_{1,\omega}^{+,(k-1)}(j) + u_{12}(j) d_{2,\omega}^{+,(k-1)}(j) \right) \right)^2 \right\rangle \\
&\quad + \left\langle \left(\sum_{j=n+1}^{\infty} \tilde{b}_\omega(j) \left(u_{21}(j) d_{1,\omega}^{+,(k-1)}(j) + u_{22}(j) d_{2,\omega}^{+,(k-1)}(j) \right) \right)^2 \right\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=n+1}^{\infty} \left\langle \tilde{b}_\omega(j)^2 \right\rangle \left\langle \left(u_{11}(j) d_{1,\omega}^{+,(k-1)}(j) + u_{12}(j) d_{2,\omega}^{+,(k-1)}(j) \right)^2 \right\rangle \\
&\quad + \sum_{j=n+1}^{\infty} \left\langle \tilde{b}_\omega(j)^2 \right\rangle \left\langle \left(u_{21}(j) d_{1,\omega}^{+,(k-1)}(j) + u_{22}(j) d_{2,\omega}^{+,(k-1)}(j) \right)^2 \right\rangle \\
&= \sum_{j=n+1}^{\infty} \left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,(k-1)}(j) \right\|^2 \right\rangle,
\end{aligned}$$

where, for the inequality, we used Fatou's lemma, the independence of the $\tilde{b}_\omega(j)$ and the fact that $\mathbf{d}_\omega^{+,(k-1)}(j)$ is a function of $\{\tilde{b}_\omega(j+1), \tilde{b}_\omega(j+2)\ldots\}$ only. Now, there exists a universal constant C, such that for any 2×2 matrix $\mathcal{A}$,

$$\| \mathcal{A} \|^2 \leq C \| \mathcal{A} \|_{\mathrm{HS}}^2,$$

where $\| \cdot \|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. Therefore, using independence again,

$$\begin{aligned}
&\left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,(k-1)}(j) \right\|^2 \right\rangle \leq \left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \right\|^2 \left\| \mathbf{d}_\omega^{+,(k-1)}(j) \right\|^2 \right\rangle \\
&\leq C \left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \right\|_{\mathrm{HS}}^2 \left\| \mathbf{d}_\omega^{+,(k-1)}(j) \right\|^2 \right\rangle \\
&= C \left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \right\|_{\mathrm{HS}}^2 \right\rangle \left\langle \left\| \mathbf{d}_\omega^{+,(k-1)}(j) \right\|^2 \right\rangle.
\end{aligned}$$

Thus,

$$\left\langle \left\| \mathbf{d}_\omega^{+,k}(n) \right\|^2 \right\rangle \leq \left(\sup_{j>n} \left\langle \left\| \mathbf{d}_\omega^{+,(k-1)}(j) \right\|^2 \right\rangle \right) \cdot C \sum_{j=n+1}^{\infty} \left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \right\|_{\mathrm{HS}}^2 \right\rangle,$$

and therefore, for any $N \in \mathbb{N}$,

$$\begin{aligned}
\sup_{n \geq N} \left\langle \left\| \mathbf{d}_\omega^{+,k}(n) \right\|^2 \right\rangle &\leq \left(\sup_{n \geq N} \left\langle \left\| \mathbf{d}_\omega^{+,(k-1)}(n) \right\|^2 \right\rangle \right) \\
&\quad \times C \sum_{j=N+1}^{\infty} \left(\left\langle \left\| \tilde{\mathcal{U}}_\omega(j) \right\|_{\mathrm{HS}}^2 \right\rangle \right)
\end{aligned}$$

with C independent of k and N. Thus, it follows that

$$\left\langle \left\| \mathbf{d}_\omega^{+,k}(N) \right\|^2 \right\rangle \leq \left(C \sum_{j=N+1}^{\infty} \left\langle (\tilde{b}_\omega(j))^2 \right\rangle \| \mathcal{U}(j) \|_{\mathrm{HS}}^2 \right)^k.$$

From (3.4), it thus follows that there exists some $N \in \mathbb{N}$, which we denote by $N_{1/4}$, so that for any $n \geq N_{1/4}$

$$\left\langle \left\| \mathbf{d}_\omega^{+,k}(n) \right\|^2 \right\rangle \leq \left(\frac{1}{4} \right)^k . \tag{3.14}$$

Let

$$\Omega_{k,n} = \left\{ \omega \,\middle|\, \left\| \mathbf{d}_\omega^{+,k}(n) \right\| \geq \left(\frac{1}{4} \right)^{\frac{k}{4}} \right\}.$$

Then, by Chebyshev's inequality, for $n \geq N_{1/4}$

$$P(\Omega_{k,n}) \leq \left(\frac{1}{2} \right)^k .$$

Thus, by the Borel Cantelli lemma, for n large enough, there exists a set $\Omega^0 \subseteq \Omega$ of full P measure such that for any $\omega \in \Omega^0$

$$\mathbf{d}_\omega^{+}(n) = \sum_{k=0}^{\infty} \mathbf{d}_\omega^{+,k}(n) \tag{3.15}$$

is defined.

Suppose for a while that we could show

$$\sum_{k=0}^{\infty} \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,k}(j) = \sum_{j=n+1}^{\infty} \sum_{k=0}^{\infty} \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,k}(j) \tag{3.16}$$

P-a.s. and for large enough n, in the sense that both sides converge and are equal. Then we would have

$$\begin{aligned} \mathbf{d}_\omega^{+}(n) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \sum_{k=1}^{\infty} \mathbf{d}_\omega^{+,k}(n) = \sum_{k=0}^{\infty} \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,k}(j) \\ &= \sum_{j=n+1}^{\infty} \sum_{k=0}^{\infty} \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,k}(j) = \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+}(j), \end{aligned}$$

which implies

$$\mathbf{d}_\omega^{+}(n-1) - \mathbf{d}_\omega^{+}(n) = \tilde{\mathcal{U}}_\omega(n) \mathbf{d}_\omega^{+}(n),$$

which is (3.5). Furthermore, (3.6) and (3.7) would be obvious from the convergence.

Therefore, we need to prove (3.16). We know that for $n \geq N_{\frac{1}{4}}$

$$\sum_{k=0}^{\infty} \mathbf{d}_\omega^{+,k}(n) = \sum_{k=0}^{\infty} \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j) \mathbf{d}_\omega^{+,k}(j)$$

converges P-a.s. which is precisely the convergence of the LHS. It is also obvious that $\mathbf{d}_\omega^+(n)$ is a measurable function of $\{\tilde{b}_\omega(n+1), \tilde{b}_\omega(n+2)...\}$, so if we show uniform boundedness of $\langle \| \mathbf{d}_\omega^+(n) \|^2 \rangle$, we will have the convergence of the RHS$= \sum_{j=n+1}^\infty \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^+(j)$, by Theorem 2.3. But this is true, since

$$\begin{aligned}
\left\langle \left\|\mathbf{d}_\omega^+(n)\right\|^2 \right\rangle &= \left\langle \lim_{N\to\infty} \left\| \left(\sum_{k=0}^N \mathbf{d}_\omega^{+,k}(n) \right) \right\|^2 \right\rangle \\
&\le \liminf_{N\to\infty} \left\langle \left\| \left(\sum_{k=0}^N \mathbf{d}_\omega^{+,k}(n) \right) \right\|^2 \right\rangle \\
&\le \liminf_{N\to\infty} \left\langle \left(\sum_{k=0}^N \left\|\mathbf{d}_\omega^{+,k}(n)\right\| \right)^2 \right\rangle \\
&= \liminf_{N\to\infty} \Big\langle \left(\left\|\mathbf{d}_\omega^{+,0}(n)\right\| + \ldots + \left\|\mathbf{d}_\omega^{+,N}(n)\right\| \right) \\
&\quad \times \left(\left\|\mathbf{d}_\omega^{+,0}(n)\right\| + \ldots + \left\|\mathbf{d}_\omega^{+,N}(n)\right\| \right) \Big\rangle \\
&= \liminf_{N\to\infty} \Big(\left\langle \left\|\mathbf{d}_\omega^{+,0}(n)\right\|^2 \right\rangle + 2\left\langle \left\|\mathbf{d}_\omega^{+,1}(n)\right\| \left\|\mathbf{d}_\omega^{+,0}(n)\right\| \right\rangle \\
&\quad + \left\langle \left\|\mathbf{d}_\omega^{+,1}(n)\right\|^2 \right\rangle + \ldots \Big)
\end{aligned}$$

where, in the fourth and fifth lines, each factor of the form $\|\mathbf{d}^{+,k_0}\|$ in one set of summands, is coupled to factors of the form $\|\mathbf{d}^{+,k}\|$ for $k \le k_0$ in the other set of summands. Using this way of writing the product, the Cauchy-Schwarz inequality, and the fact that, for $n \ge N_{\frac{1}{4}}$, (3.14) holds (so that, in particular, all factors are bounded by 1 from above), we get that

$$\langle \| \mathbf{d}_\omega^+(n) \|^2 \rangle \le \liminf_{N\to\infty} \sum_{k=0}^N (2k+1) \left(\frac{1}{2}\right)^k, \tag{3.17}$$

so that $\langle \| \mathbf{d}_\omega^+(n) \|^2 \rangle$ is bounded in n. Thus we are left with proving the equality (3.16), or in other words, with proving

$$\lim_{K\to\infty} \left(\sum_{j=n+1}^\infty \sum_{k=0}^\infty \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{+,k}(j) - \sum_{j=n+1}^\infty \sum_{k=0}^K \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{+,k}(j) \right) = 0$$

which is the same as

$$\lim_{K\to\infty} \sum_{j=n+1}^\infty \sum_{k=K+1}^\infty \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{+,k}(j) = 0 \tag{3.18}$$

P-almost surely. Denote

$$\mathfrak{d}_\omega^{+,K}(j) = \sum_{k=K+1}^{\infty} \mathbf{d}_\omega^{+,k}(j). \tag{3.19}$$

Then,

$$\begin{aligned}
&P\left\{\left\|\sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathfrak{d}_\omega^{+,K}(j)\right\| \geq \frac{1}{K}\right\} \\
&\leq K^2 \left\langle \left\|\sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathfrak{d}_\omega^{+,K}(j)\right\|^2 \right\rangle \\
&\leq K^2 \liminf_{N\to\infty} \sum_{j=n+1}^{N} \left\langle \left\|\tilde{\mathcal{U}}_\omega(j)\mathfrak{d}_\omega^{+,K}(j)\right\|^2 \right\rangle \\
&\leq K^2 \left(\sum_{j=n+1}^{\infty} \left\langle \left\|\tilde{\mathcal{U}}_\omega(j)\right\|^2 \right\rangle\right) \sup_{j\geq n+1} \left\langle \left\|\mathfrak{d}_\omega^{+,K}(j)\right\|^2 \right\rangle = **\,.
\end{aligned}$$

The same considerations that lead to (3.17), lead to the conclusion that $** \in \ell^1(K)$ and therefore, by Borel Cantelli (3.18) holds, P-almost surely, and we are done.

To construct $\mathbf{d}^-$ go through the same procedure, constructing $\mathbf{d}_\omega^{-,k}(n)$, with $\mathbf{d}^{-,0} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and define

$$\mathbf{d}_\omega^{-}(n) = \sum_{k=0}^{\infty} \mathbf{d}_\omega^{-,k}(n)$$

for n large enough. Everything works the same as for the construction of $\mathbf{d}_\omega^{+}(n)$.

To show (3.9), we define

$$\mathcal{X}(n) \equiv \begin{pmatrix} 1 & 0 \\ 0 & f_+(n) \end{pmatrix} \tag{3.20}$$

and note that for $m > n$,

$$\left\|\mathcal{X}(n)\mathcal{X}(m)^{-1}\right\| \leq 1. \tag{3.21}$$

Denote now

$$\mathcal{W}_\omega(n) = \mathcal{X}(n)\tilde{\mathcal{U}}_\omega(n)\mathcal{X}(n)^{-1}, \tag{3.22}$$

and

$$\tilde{\mathbf{d}}_\omega^{-,k}(n) = \mathcal{X}(n)\mathbf{d}_\omega^{-,k}(n),$$

so that we have

$$\begin{aligned}
&\tilde{\mathbf{d}}_\omega^{-,k}(n)\\
&= \mathcal{X}(n) \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{-,(k-1)}(j)\\
&= \mathcal{X}(n) \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathcal{X}(j)^{-1}\mathcal{X}(j)\mathbf{d}_\omega^{-,(k-1)}(j)\\
&= \mathcal{X}(n) \sum_{j=n+1}^{\infty} \mathcal{X}(j)^{-1}\mathcal{X}(j)\tilde{\mathcal{U}}_\omega(j)\mathcal{X}(j)^{-1}\tilde{\mathbf{d}}_\omega^{-,(k-1)}(j)\\
&= \sum_{j=n+1}^{\infty} \mathcal{X}(n)\mathcal{X}(j)^{-1}\mathcal{W}_\omega(j)\tilde{\mathbf{d}}_\omega^{-,(k-1)}(j).
\end{aligned}$$

Since, by (3.4),

$$\sum_{n=1}^{\infty} \left\langle \|\mathcal{W}_\omega(n)\|^2 \right\rangle < \infty, \tag{3.23}$$

and by (3.21)

$$\sum_{j=n+1}^{\infty} \left\langle \left\|\mathcal{X}(n)\mathcal{X}(j)^{-1}\mathcal{W}_\omega(j)\right\|^2 \right\rangle \leq \sum_{j=n+1}^{\infty} \left\langle \|\mathcal{W}_\omega(j)\|^2 \right\rangle, \tag{3.24}$$

we can repeat the argument in the first part of the proof to show that

$$\sup_j \left\langle \left\|\tilde{\mathbf{d}}_\omega^{-}(j)\right\|^2 \right\rangle \equiv \sup_j \left\langle \left\|\mathcal{X}(j)\mathbf{d}_\omega^{-}(j)\right\|^2 \right\rangle < \infty. \tag{3.25}$$

Now,

$$\begin{aligned}
\left\| \mathcal{X}(n)\mathbf{d}_\omega^{-}(n) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| &= \left\| \mathcal{X}(n)\mathbf{d}_\omega^{-}(n) - \mathcal{X}(n)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|\\
&= \left\| \mathcal{X}(n) \sum_{j=n+1}^{\infty} \tilde{\mathcal{U}}_\omega(j)\mathbf{d}_\omega^{-}(j) \right\|\\
&= \left\| \sum_{j=n+1}^{\infty} \mathcal{X}(n)\mathcal{X}(j)^{-1}\mathcal{W}_\omega(j)\mathcal{X}(j)\mathbf{d}_\omega^{-}(j) \right\|.
\end{aligned} \tag{3.26}$$

Since $\mathcal{X}(j)\mathbf{d}_\omega^{-}(j)$ is a measurable function of $\{\tilde{b}_\omega(j+1), \tilde{b}_\omega(j+2), \ldots\}$, and, since $\mathcal{X}(n)\mathcal{X}(j)^{-1}\mathcal{W}_\omega(j) = \tilde{b}_\omega(j)\hat{\mathcal{W}}(n,j)$ (where $\hat{\mathcal{W}}(n,j)$ is a deterministic matrix), one can repeat the proof of Lemma 2.2, using (3.24) and (3.25), to show that

$$\lim_{n\to\infty}\left\|\sum_{j=n+1}^{\infty}\mathcal{X}(n)\mathcal{X}(j)^{-1}\mathcal{W}_\omega(j)\mathcal{X}(j)\mathbf{d}_\omega^-(j)\right\|=0$$

P-almost surely, which, by (3.26), is exactly (3.8) and (3.9). □

As remarked earlier, this lemma is actually a 'random variation' on a deterministic stability result. This random version uses the zero mean of the random variables in order to replace an ℓ^1 summability condition (which is the natural condition in the deterministic case) with an ℓ^2 condition. It is natural to ask whether it is possible to obtain such a result for a situation in which there is a combination of terms – coefficients which are ℓ^2 with zero mean and coefficients that are ℓ^1. The following lemma is an extension of Lemma 3.1 in this direction (in the special case '$f_+\equiv 1$'), which is tailored especially for our needs in the next section.

Lemma 3.2. *Let*

$$\left\{\mathcal{U}(n)=\begin{pmatrix} u_{11}(n) & u_{12}(n)\\ u_{21}(n) & u_{22}(n)\end{pmatrix}\right\}_{n=1}^{\infty},$$
$$\left\{\mathcal{V}(n)=\begin{pmatrix} v_{11}(n) & v_{12}(n)\\ v_{21}(n) & v_{22}(n)\end{pmatrix}\right\}_{n=1}^{\infty},$$
$$\left\{\mathcal{W}(n)=\begin{pmatrix} w_{11}(n) & w_{12}(n)\\ w_{21}(n) & w_{22}(n)\end{pmatrix}\right\}_{n=1}^{\infty}$$

be three matrix-valued sequences, and let $\{\tilde{b}_{1,\omega}(n)\}$, $\{\tilde{b}_{2,\omega}(n)\}$, $\{\tilde{b}_{3,\omega}(n)\}$ *be three sequences of random variables that satisfy the following properties:*

(1) *For any* $i,j=1,2,3$ *and* $n_1\neq n_2$, $\tilde{b}_{i,\omega}(n_1)$ *and* $\tilde{b}_{j,\omega}(n_2)$ *are independent random variables.*
(2) *For any* n,

$$\left\langle\tilde{b}_{1,\omega}(n)\right\rangle=\left\langle\tilde{b}_{2,\omega}(n)\right\rangle=\left\langle\tilde{b}_{1,\omega}(n)\tilde{b}_{2,\omega}(n)\right\rangle=\left\langle\tilde{b}_{1,\omega}(n)\tilde{b}_{3,\omega}(n)\right\rangle=0 \tag{3.27}$$

(3) *For all* n *and any* $\omega\in\Omega$, $\tilde{b}_{3,\omega}(n)\geq 0$.
(4)

$$\sum_{n=1}^{\infty}\left\langle\tilde{b}_{1,\omega}(n)^2\right\rangle\|\mathcal{U}(n)\|_{\mathrm{HS}}^2<\infty \tag{3.28}$$

and

$$\sum_{n=1}^{\infty}\left\langle\tilde{b}_{2,\omega}(n)^2\right\rangle\|\mathcal{V}(n)\|_{\mathrm{HS}}^2<\infty. \tag{3.29}$$

(5)
$$\sum_{n=1}^{\infty}\left\langle\tilde{b}_{3,\omega}(n)^2\right\rangle^{1/2}\|\mathcal{W}(n)\|_{\mathrm{HS}}<\infty. \tag{3.30}$$
(6)
$$\sum_{n=1}^{\infty}\left\langle|\tilde{b}_{2,\omega}(n)|\tilde{b}_{3,\omega}(n)\right\rangle\|\mathcal{V}(n)\|_{\mathrm{HS}}\|\mathcal{W}(n)\|_{\mathrm{HS}}<\infty. \tag{3.31}$$

Then, P-almost surely,

$$\mathbf{d}_\omega(n)-\mathbf{d}_\omega(n-1)=\left(\tilde{b}_{1,\omega}(n)\mathcal{U}(n)+\tilde{b}_{2,\omega}(n)\mathcal{V}(n)+\tilde{b}_{3,\omega}(n)\mathcal{W}(n)\right)\mathbf{d}_\omega(n) \tag{3.32}$$

has solutions - $\mathbf{d}^+_\omega(n)\equiv\begin{pmatrix} d^+_{1,\omega}(n)\\ d^+_{2,\omega}(n)\end{pmatrix}$ and $\mathbf{d}^-_\omega(n)\equiv\begin{pmatrix} d^-_{1,\omega}(n)\\ d^-_{2,\omega}(n)\end{pmatrix}$, that satisfy

$$\lim_{n\to\infty} d^+_{1,\omega}(n) = 0 \tag{3.33}$$
$$\lim_{n\to\infty} d^+_{2,\omega}(n) = 1 \tag{3.34}$$
$$\lim_{n\to\infty} d^-_{1,\omega}(n) = 1 \tag{3.35}$$
$$\lim_{n\to\infty} d^-_{2,\omega}(n) = 0 \tag{3.36}$$

Proof. This proof follows the same strategy of the proof of Lemma 3.1. We shall try to avoid unnecessary repetitions. The first step is the construction of $\mathbf{d}^{+,k}$ for $k\geq 0$. As before, let

$$\mathbf{d}^{+,0}_\omega(n)\equiv\begin{pmatrix}0\\1\end{pmatrix}. \tag{3.37}$$

Now, note that, by Hölder,

$$\left\langle\tilde{b}_{3,\omega}(n)\right\rangle\leq\left\langle\tilde{b}_{3,\omega}(n)^2\right\rangle^{1/2}, \tag{3.38}$$

so

$$\sum_{n=1}^{\infty}\left\langle\tilde{b}_{3,\omega}(n)\right\rangle\|\mathcal{W}(n)\|_{\mathrm{HS}}\leq\sum_{n=1}^{\infty}\left\langle\tilde{b}_{3,\omega}(n)^2\right\rangle^{1/2}\|\mathcal{W}(n)\|_{\mathrm{HS}}<\infty. \tag{3.39}$$

As in the preceding proof, we want to show that if $\mathbf{d}^{+,k}_\omega(n)$ is defined P-a.s. as a measurable function of $\{\tilde{b}_{i,\omega}(j)\}_{i=1,2,3,\ j>n}$, for any n; and $\left\langle\|\mathbf{d}^{+,k}_\omega(n)\|^2\right\rangle$ is bounded in n, then the same holds true for

$$\mathbf{d}^{+,(k+1)}_\omega(n)=\sum_{j=n+1}^{\infty}\left(\tilde{b}_{1,\omega}(j)\mathcal{U}(j)+\tilde{b}_{2,\omega}(j)\mathcal{V}(j)+\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\right)\mathbf{d}^{+,k}_\omega(j). \tag{3.40}$$

If indeed $\left\langle\|\mathbf{d}_\omega^{+,k}(n)\|^2\right\rangle$ is bounded in n (say, by C), then, by Hölder, so is $\left\langle\|\mathbf{d}_\omega^{+,k}(n)\|\right\rangle$ so by (3.39) and from the independence we will have

$$\begin{aligned}
\sum_{n=1}^{\infty}\left\langle\left\|\tilde{b}_{3,\omega}(n)\mathcal{W}(n)\mathbf{d}_\omega^{+,k}(n)\right\|\right\rangle &\le \sum_{n=1}^{\infty}\left\langle\tilde{b}_{3,\omega}(n)\,\|\mathcal{W}(n)\|\,\left\|\mathbf{d}_\omega^{+,k}(n)\right\|\right\rangle \\
&= \sum_{n=1}^{\infty}\left\langle\tilde{b}_{3,\omega}(n)\right\rangle\|\mathcal{W}(n)\|\left\langle\left\|\mathbf{d}_\omega^{+,k}(n)\right\|\right\rangle \\
&\le C\sum_{n=1}^{\infty}\left\langle\tilde{b}_{3,\omega}(n)\right\rangle\|\mathcal{W}(n)\| < \infty.
\end{aligned} \tag{3.41}$$

Therefore, monotone convergence implies that

$$\sum_{n=1}^{\infty}\tilde{b}_{3,\omega}(n)\mathcal{W}(n)\mathbf{d}_\omega^{+,k}(n) \tag{3.42}$$

is absolutely convergent P-a.s. Theorem 2.3 implies the almost sure convergence of the first two summands in (3.40) so $\mathbf{d}_\omega^{+,(k+1)}(n)$ is defined P-a.s. as a measurable function of $\{\tilde{b}_{i,\omega}(j)\}_{i=1,2,3,\ j>n}$. Thus, we are left with showing that $\left\langle\|\mathbf{d}_\omega^{+,(k+1)}(n)\|^2\right\rangle$ is bounded in n. We proceed to estimate

$$\begin{aligned}
&\left\langle\left\|\mathbf{d}_\omega^{+,(k+1)}(n)\right\|^2\right\rangle \\
&\le \liminf_{N\to\infty}\Big\langle\Big\|\sum_{j=n+1}^{N}\tilde{b}_{1,\omega}(j)\mathcal{U}(j)\mathbf{d}_\omega^{+,k}(j) \\
&\quad+\sum_{j=n+1}^{N}\tilde{b}_{2,\omega}(j)\mathcal{V}(j)\mathbf{d}_\omega^{+,k}(j)+\sum_{j=n+1}^{N}\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\mathbf{d}_\omega^{+,k}(j)\Big\|^2\Big\rangle \\
&= \liminf_{N\to\infty}\Bigg(\Big\langle\Big|\sum_{j=n+1}^{N}\Big(\tilde{b}_{1,\omega}(j)(u_{11}(j)d_{1,\omega}^{+,k}(j)+u_{12}(j)d_{2,\omega}^{+,k}(j)) \\
&\quad+\tilde{b}_{2,\omega}(j)(v_{11}(j)d_{1,\omega}^{+,k}(j)+v_{12}(j)d_{2,\omega}^{+,k}(j)) \\
&\quad+\tilde{b}_{3,\omega}(j)(w_{11}(j)d_{1,\omega}^{+,k}(j)+w_{12}(j)d_{2,\omega}^{+,k}(j))\Big)\Big|^2\Big\rangle \\
&\quad+\Big\langle\Big|\sum_{j=n+1}^{N}\Big(\tilde{b}_{1,\omega}(j)(u_{21}(j)d_{1,\omega}^{+,k}(j)+u_{22}(j)d_{2,\omega}^{+,k}(j)) \\
&\quad+\tilde{b}_{2,\omega}(j)(v_{21}(j)d_{1,\omega}^{+,k}(j)+v_{22}(j)d_{2,\omega}^{+,k}(j))
\end{aligned}$$

$$+ \tilde{b}_{3,\omega}(j)(w_{21}(j)d_{1,\omega}^{+,k}(j) + w_{22}(j)d_{2,\omega}^{+,k}(j))\Big)\Big|^2\Big\rangle\Big)$$

$$\equiv \liminf_{N\to\infty} \Big(R_1(n,N) + R_2(n,N)\Big). \tag{3.43}$$

Now, using independence and (3.27) we see that

$$\begin{aligned}
&R_1(n,N) \\
&= \Big\langle \Big| \sum_{j=n+1}^{N} \Big(\tilde{b}_{1,\omega}(j)(u_{11}(j)d_{1,\omega}^{+,k}(j) + u_{12}(j)d_{2,\omega}^{+,k}(j)) \\
&\quad + \tilde{b}_{2,\omega}(j)(v_{11}(j)d_{1,\omega}^{+,k}(j) + v_{12}(j)d_{2,\omega}^{+,k}(j)) \\
&\quad + \tilde{b}_{3,\omega}(j)(w_{11}(j)d_{1,\omega}^{+,k}(j) + w_{12}(j)d_{2,\omega}^{+,k}(j))\Big)\Big|^2\Big\rangle \\
&= \sum_{j=n+1}^{N} \Big\langle \tilde{b}_{1,\omega}(j)^2 \Big\rangle \Big\langle \Big(u_{11}(j)d_{1,\omega}^{+,k}(j) + u_{12}(j)d_{2,\omega}^{+,k}(j)\Big)^2\Big\rangle \\
&\quad + \sum_{j=n+1}^{N} \Big\langle \tilde{b}_{2,\omega}(j)^2 \Big\rangle \Big\langle \Big(v_{11}(j)d_{1,\omega}^{+,k}(j) + v_{12}(j)d_{2,\omega}^{+,k}(j)\Big)^2\Big\rangle \\
&\quad + \sum_{j=n+1}^{N} \Big(\Big\langle \tilde{b}_{2,\omega}(j)\tilde{b}_{3,\omega}(j)\Big\rangle \\
&\quad \times \Big\langle (v_{11}(j)d_{1,\omega}^{+,k}(j) + v_{12}(j)d_{2,\omega}^{+,k}(j))(w_{11}(j)d_{1,\omega}^{+,k}(j) + w_{12}(j)d_{2,\omega}^{+,k}(j))\Big\rangle\Big) \\
&\quad + \Big\langle \Big(\sum_{j=n+1}^{N} \Big(\tilde{b}_{3,\omega}(j)\Big(w_{11}(j)d_{1,\omega}^{+,k}(j) + w_{12}(j)d_{2,\omega}^{+,k}(j)\Big)\Big)\Big)^2\Big\rangle \\
&\equiv R_1^I(n,N) + R_1^{II}(n,N) + R_1^{III}(n,N) + R_1^{IV}(n,N)
\end{aligned} \tag{3.44}$$

and a similar expression holds for $R_2(n,N)$. Thus we have:

$$\begin{aligned}
R_1(n,N) + R_2(n,N) &= \underbrace{R_1^I(n,N) + R_2^I(n,N)}_{R^I(n,N)} + \underbrace{R_1^{II}(n,N) + R_2^{II}(n,N)}_{R^{II}(n,N)} \\
&+ \underbrace{R_1^{III}(n,N) + R_2^{III}(n,N)}_{R^{III}(n,N)} + \underbrace{R_1^{IV}(n,N) + R_2^{IV}(n,N)}_{R^{IV}(n,N)}.
\end{aligned}$$

(3.28) means that

$$\begin{aligned} R^I(n,N) &\le \sum_{j=n+1}^{N} \left\langle \tilde{b}_{1,\omega}(j)^2 \right\rangle \|\mathcal{U}(j)\|^2 \left\langle \|\mathbf{d}_\omega^{+,k}(j)\|^2 \right\rangle \\ &\le C \sum_{j=1}^{\infty} \left\langle \tilde{b}_{1,\omega}(j)^2 \right\rangle \|\mathcal{U}(j)\|^2 = D^I < \infty. \end{aligned} \tag{3.45}$$

Similarly for $R^{II}(n,N)$, (3.29) says

$$\begin{aligned} R^{II}(n,N) &\le \sum_{j=n+1}^{N} \left\langle \tilde{b}_{2,\omega}(j)^2 \right\rangle \|\mathcal{V}(j)\|^2 \left\langle \|\mathbf{d}_\omega^{+,k}(j)\|^2 \right\rangle \\ &\le C \sum_{j=1}^{\infty} \left\langle \tilde{b}_{2,\omega}(j)^2 \right\rangle \|\mathcal{V}(j)\|^2 = D^{II} < \infty. \end{aligned} \tag{3.46}$$

and for $R^{III}(n,N)$, (3.31) implies

$$\begin{aligned} R^{III}(n,N) &\le \sum_{j=n+1}^{N} \left\langle |\tilde{b}_{2,\omega}(j)|\tilde{b}_{3,\omega}(j) \right\rangle \|\mathcal{V}(j)\|_{\mathrm{HS}}\|\mathcal{W}(j)\|_{\mathrm{HS}} \left\langle \|\mathbf{d}_\omega^{+,k}(j)\|^2 \right\rangle \\ &\le C \sum_{j=1}^{\infty} \left\langle |\tilde{b}_{2,\omega}(j)|\tilde{b}_{3,\omega}(j) \right\rangle \|\mathcal{V}(j)\|_{\mathrm{HS}}\|\mathcal{W}(j)\|_{\mathrm{HS}} = D^{III} < \infty. \end{aligned} \tag{3.47}$$

The procedure we apply to R^{IV} is a little more involved. Applying Hölder's inequality (in the second inequality below) and then using independence we get

$$\begin{aligned} &R^{IV}(n,N) \\ &= \left\langle \left(\sum_{j=n+1}^{N} \left(\tilde{b}_{3,\omega}(j) \left(w_{11}(j) d_{1,\omega}^{+,k}(j) + w_{12}(j) d_{2,\omega}^{+,k}(j) \right) \right) \right)^2 \right\rangle \\ &\quad + \left\langle \left(\sum_{j=n+1}^{N} \left(\tilde{b}_{3,\omega}(j) \left(w_{21}(j) d_{1,\omega}^{+,k}(j) + w_{22}(j) d_{2,\omega}^{+,k}(j) \right) \right) \right)^2 \right\rangle \\ &\le 2 \sum_{i,j=n+1}^{N} \left\langle \tilde{b}_{3,\omega}(j)\tilde{b}_{3,\omega}(i) \|\mathbf{d}_\omega^{+,k}(j)\| \|\mathbf{d}_\omega^{+,k}(i)\| \right\rangle \\ &\quad \times \|\mathcal{W}(j)\|_{\mathrm{HS}}\|\mathcal{W}(i)\|_{\mathrm{HS}} \\ &\le 2 \sum_{i,j=n+1}^{N} \left\langle \tilde{b}_{3,\omega}(j)^2 \|\mathbf{d}_\omega^{+,k}(j)\|^2 \right\rangle^{1/2} \left\langle \tilde{b}_{3,\omega}(i)^2 \|\mathbf{d}_\omega^{+,k}(i)\|^2 \right\rangle^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\times \|\mathcal{W}(j)\|_{\mathrm{HS}} \|\mathcal{W}(i)\|_{\mathrm{HS}} \\
&= 2 \sum_{i,j=n+1}^{N} \left\langle \tilde{b}_{3,\omega}(j)^2 \right\rangle^{1/2} \left\langle \|\mathbf{d}_\omega^{+,k}(j)\|^2 \right\rangle^{1/2} \left\langle \tilde{b}_{3,\omega}(i)^2 \right\rangle^{1/2} \left\langle \|\mathbf{d}_\omega^{+,k}(i)\|^2 \right\rangle^{1/2} \\
&\quad \times \|\mathcal{W}(j)\|_{\mathrm{HS}} \|\mathcal{W}(i)\|_{\mathrm{HS}} \\
&= 2 \Big(\sum_{j=n+1}^{N} \left\langle \tilde{b}_{3,\omega}(j)^2 \right\rangle^{1/2} \left\langle \|\mathbf{d}_\omega^{+,k}(j)\|^2 \right\rangle^{1/2} \|\mathcal{W}(j)\|_{\mathrm{HS}} \Big)^2 \\
&\leq 2C \Big(\sum_{j=n+1}^{N} \left\langle \tilde{b}_{3,\omega}(j)^2 \right\rangle^{1/2} \|\mathcal{W}(j)\|_{\mathrm{HS}} \Big)^2 \\
&\leq 2C \Big(\sum_{j=1}^{\infty} \left\langle \tilde{b}_{3,\omega}(j)^2 \right\rangle^{1/2} \|\mathcal{W}(j)\|_{\mathrm{HS}} \Big)^2 = D^{IV} < \infty
\end{aligned} \tag{3.48}
$$

by (3.30). We see, therefore, that

$$
\begin{aligned}
\left\langle \|\mathbf{d}_\omega^{+,(k+1)}(n)\|^2 \right\rangle \leq \liminf_{N \to \infty} \big(& R^I(n,N) + R^{II}(n,N) \\
& + R^{III}(n,N) + R^{IV}(n,N) \big)
\end{aligned}
$$

is bounded in n. It follows that $\mathbf{d}_\omega^{+,k}(n)$ is defined for all k, n and P-almost every ω.

The estimates above also imply that there exists a constant C_0 such that

$$
\begin{aligned}
\left\langle \|\mathbf{d}_\omega^{+,k}(n)\|^2 \right\rangle \leq C_0 \big(& \sup_{j>n} \left\langle \|\mathbf{d}_\omega^{+,(k-1)}(j)\|^2 \right\rangle \big) \\
& \times \Big(\sum_{j=n+1}^{\infty} \left\langle \tilde{b}_{1,\omega}(j)^2 \right\rangle \|\mathcal{U}(j)\|^2 + \sum_{j=n+1}^{\infty} \left\langle \tilde{b}_{2,\omega}(j)^2 \right\rangle \|\mathcal{V}(j)\|^2 \\
& + \sum_{j=n+1}^{\infty} \left\langle |\tilde{b}_{2,\omega}(j)| \tilde{b}_{3,\omega}(j) \right\rangle \|\mathcal{V}(j)\|_{\mathrm{HS}} \|\mathcal{W}(j)\|_{\mathrm{HS}} \\
& + \big(\sum_{j=n+1}^{\infty} \left\langle \tilde{b}_{3,\omega}(j)^2 \right\rangle^{1/2} \|\mathcal{W}(j)\|_{\mathrm{HS}} \big)^2 \Big)
\end{aligned} \tag{3.49}
$$

which, as in the proof of the previous lemma, implies in turn that for large enough n,

$$
\left\langle \|\mathbf{d}_\omega^{+,k}(n)\| \right\rangle^2 \leq \left\langle \|\mathbf{d}_\omega^{+,k}(n)\|^2 \right\rangle \leq \left(\frac{1}{4} \right)^k \tag{3.50}
$$

and therefore that for such n,

$$\mathbf{d}_\omega^+(n) = \sum_{k=0}^{\infty} \mathbf{d}_\omega^{+,k}(n)$$

converges almost surely.

The next step is to show

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{j=n+1}^{\infty}\left(\tilde{b}_{1,\omega}(j)\mathcal{U}(j)+\tilde{b}_{2,\omega}(j)\mathcal{V}(j)+\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\right)\mathbf{d}_\omega^{+,k}(j)\\ &=\sum_{j=n+1}^{\infty}\sum_{k=0}^{\infty}\left(\tilde{b}_{1,\omega}(j)\mathcal{U}(j)+\tilde{b}_{2,\omega}(j)\mathcal{V}(j)+\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\right)\mathbf{d}_\omega^{+,k}(j) \quad (3.51)\end{aligned}$$

from which the first half of the theorem ((3.33) and (3.34)), will follow. Note, first, that from (3.50) and (3.30), P-a.s.

$$\sum_{k=0}^{\infty}\sum_{j=n+1}^{\infty}\left\|\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\mathbf{d}_\omega^{+,k}(j)\right\| < \infty \quad (3.52)$$

so that we have

$$\sum_{k=0}^{\infty}\sum_{j=n+1}^{\infty}\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\mathbf{d}_\omega^{+,k}(j) = \sum_{j=n+1}^{\infty}\sum_{k=0}^{\infty}\tilde{b}_{3,\omega}(j)\mathcal{W}(j)\mathbf{d}_\omega^{+,k}(j) \quad (3.53)$$

with probability one. Thus we are only left with showing

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{j=n+1}^{\infty}\left(\tilde{b}_{1,\omega}(j)\mathcal{U}(j)+\tilde{b}_{2,\omega}(j)\mathcal{V}(j)\right)\mathbf{d}_\omega^{+,k}(j)\\ &=\sum_{j=n+1}^{\infty}\sum_{k=0}^{\infty}\left(\tilde{b}_{1,\omega}(j)\mathcal{U}(j)+\tilde{b}_{2,\omega}(j)\mathcal{V}(j)\right)\mathbf{d}_\omega^{+,k}(j). \quad (3.54)\end{aligned}$$

The proof of (3.54) is precisely the same as the corresponding step in the proof of Lemma 3.1. It is therefore omitted from the argumentation.

Obviously, to show (3.35) and (3.36) one simply follows the exact same procedure outlined above, with a different initial vector, so we are done. □

4. Proof of Theorems 1.1 and 1.3

In this section we present the proofs of Theorems 1.1 and 1.3.

Proof of Theorem 1.1. There are two cases to consider:

- case 1. Assume that there exists a subsequence $a(n_j) \to 0$ as $j \to \infty$. Then we may choose a summable subsequence $a(k_j)$. Define
$$a^1(k) = \begin{cases} a(k_j) & \text{if } k = k_j \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$$
and let $\tilde{J}_0 = J_0 - J(\{a^1(n)\}, \mathbf{0})$. Then $\tilde{J}_0$ is a direct sum of finite rank operators so that its spectrum is pure point. Since J_0 is a trace class perturbation of $\tilde{J}_0$, it follows that $\tilde{J}_0$ has no absolutely continuous spectrum. If Γ is empty as well, then we are done. Otherwise, (1.10) implies that $\tilde{a}_\omega(n) \to 0$ as $n \to \infty$ almost surely. This is because $\|T^E(n)\| \geq C \min(1, a(n)^{-(1/2)})$ for some universal constant C. Therefore, repeating the argument above for J_ω, we find that it has no absolutely continuous spectrum as well. Thus (1.9) follows for this case.
- case 2. There is a constant $c_0 > 0$ such that $a(n) > c_0$ for any n. Then it follows from (1.8), that
$$\frac{1}{a(n) + \tilde{a}_\omega(n)} < (\delta c_0)^{-1} \tag{4.1}$$
for all n and ω.

Fix $E \in \Gamma$ and let $\{N_j\}_{j=1}^\infty$ be a sequence for which
$$\lim_{j\to\infty} \left(\frac{1}{N_j} \sum_{n=1}^{N_j} \frac{1}{a(n)} \right) > 0.$$
We want to show that with probability one,
$$\liminf_{j\to\infty} \tfrac{1}{N_j} \textstyle\sum_{n=1}^{N_j} \parallel T_0^E(n) \parallel^2 < \infty$$
$$\Longleftrightarrow$$
$$\liminf_{j\to\infty} \tfrac{1}{N_j} \textstyle\sum_{n=1}^{N_j} \parallel T_\omega^E(n) \parallel^2 < \infty \tag{4.2}$$
(where we use the notation introduced in the beginning of the previous section). Then, by Fubini, it will follow that there exists a set of full P-measure of realizations of the perturbation, such that for Lebesgue-a.e. energy in Γ, the asymptotic properties of the transfer matrices (in the above sense) remain the same. By Proposition 2.1, this implies (1.9) (since $\tilde{a}_\omega(n) \to 0$ almost surely).

The one-step transfer matrices have the form
$$S_\omega^E(n) = \begin{pmatrix} \frac{E - b(n) - \tilde{b}_\omega(n)}{a(n) + \tilde{a}_\omega(n)} & \frac{-a(n-1) - \tilde{a}_\omega(n-1)}{a(n) + \tilde{a}_\omega(n)} \\ 1 & 0 \end{pmatrix}$$

where $a(0)+\tilde{a}_\omega(0) \equiv 1$. Due to the $\frac{a(n-1)+\tilde{a}_\omega(n-1)}{a(n)+\tilde{a}_\omega(n)}$ term, these matrices are not independent. A crucial ingredient in the proof of Lemma 3.2 is the independence of the matrices. We begin, therefore, with a modification to these matrices following [10]: Define

$$K_\omega(n) = \begin{pmatrix} 1 & 0 \\ 0 & a(n)+\tilde{a}_\omega(n) \end{pmatrix}. \tag{4.3}$$

Then

$$\begin{aligned} K_\omega(n) S_\omega^E(n) K_\omega(n-1)^{-1} &= \begin{pmatrix} \frac{E-b(n)-\tilde{b}_\omega(n)}{a(n)+\tilde{a}_\omega(n)} & \frac{-1}{a(n)+\tilde{a}_\omega(n)} \\ a(n)+\tilde{a}_\omega(n) & 0 \end{pmatrix} \\ &\equiv \tilde{S}_\omega^E(n). \end{aligned} \tag{4.4}$$

Note that $\tilde{S}_\omega^E(n)$ are independent and unimodular. One may now define

$$\begin{aligned} \tilde{T}_\omega^E(n) \equiv \tilde{S}_\omega^E(n) \cdot \ldots \cdot \tilde{S}_\omega^E(1) &= K_\omega(n) T_\omega^E(n) K_\omega^E(0)^{-1} \\ &= K_\omega(n) T_\omega^E(n). \end{aligned} \tag{4.5}$$

Define $\tilde{\mathcal{D}}_\omega^E(n)$ through

$$\tilde{T}_\omega^E(n) = \tilde{T}_0^E(n) \tilde{\mathcal{D}}_\omega^E(n). \tag{4.6}$$

Then

$$T_\omega^E(n) = K_\omega(n)^{-1} K_0(n) T_0^E(n) \tilde{\mathcal{D}}_\omega^E(n),$$

so, using (1.8), in order to show (4.2) almost surely, it suffices to show that $\tilde{\mathcal{D}}_\omega^E(n)$ converge to a limit with probability one.

From (4.6) it follows that

$$\tilde{\mathcal{D}}_\omega^E(n-1) = (I + \tilde{\mathcal{U}}_\omega^E(n)) \tilde{\mathcal{D}}_\omega^E(n), \tag{4.7}$$

where

$$\begin{aligned} &\tilde{\mathcal{U}}_\omega^E(n) \\ &= \tilde{T}_0^E(n)^{-1} (\tilde{S}_0^E(n) \tilde{S}_\omega^E(n)^{-1} - I) \tilde{T}_0^E(n) \\ &= \tilde{T}_0^E(n)^{-1} \begin{pmatrix} \frac{\tilde{a}_\omega(n)}{a(n)} & \frac{\tilde{b}_\omega(n)}{a(n)(a(n)+\tilde{a}_\omega(n))} \\ 0 & -\frac{\tilde{a}_\omega(n)}{a(n)+\tilde{a}_\omega(n)} \end{pmatrix} \tilde{T}_0^E(n) \\ &= \frac{\tilde{a}_\omega(n)}{a(n)} \tilde{T}_0^E(n)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{T}_0^E(n) \\ &\quad + \frac{\tilde{b}_\omega(n)}{a(n)(a(n)+\tilde{a}_\omega(n))} \tilde{T}_0^E(n)^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{T}_0^E(n) \end{aligned}$$

$$+\frac{\tilde{a}_\omega(n)}{a(n)+\tilde{a}_\omega(n)}\tilde{T}_0^E(n)^{-1}\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\tilde{T}_0^E(n). \tag{4.8}$$

Writing

$$\frac{\tilde{a}_\omega(n)}{a(n)+\tilde{a}_\omega(n)}=\frac{\tilde{a}_\omega(n)}{a(n)}-\frac{(\tilde{a}_\omega(n))^2}{a(n)(a(n)+\tilde{a}_\omega(n))},$$

and denoting

$$\mathcal{V}^E(n)=\tilde{T}_0^E(n)^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\tilde{T}_0^E(n), \tag{4.9}$$

$$\mathcal{U}^E(n)=\tilde{T}_0^E(n)^{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\tilde{T}_0^E(n), \tag{4.10}$$

$$\mathcal{W}^E(n)=\tilde{T}_0^E(n)^{-1}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\tilde{T}_0^E(n), \tag{4.11}$$

we get

$$\begin{aligned}\tilde{\mathcal{U}}_\omega^E(n)=\frac{\tilde{a}_\omega(n)}{a(n)}\mathcal{V}^E(n)+\frac{\tilde{b}_\omega(n)}{a(n)(a(n)+\tilde{a}_\omega(n))}\mathcal{U}^E(n)+\\ +\frac{(\tilde{a}_\omega(n))^2}{a(n)(a(n)+\tilde{a}_\omega(n))}\mathcal{W}^E(n).\end{aligned} \tag{4.12}$$

Since the $\tilde{T}_0^E(n)$ are unimodular (so they have norm equal to their inverses') it follows that

$$\|\mathcal{U}^E(n)\|\leq\|\tilde{T}_0^E(n)\|^2 \tag{4.13}$$

and the same holds for $\mathcal{V}^E(n)$ and $\mathcal{W}^E(n)$. Furthermore, (1.10) implies that

$$\begin{aligned}&\sum_{n=1}^\infty\left(\left\langle\tilde{a}_\omega(n)^4\right\rangle^{1/2}+\left\langle\tilde{b}_\omega(n)^2\right\rangle\right)\|\tilde{T}_0^E(n)\|^4\\ &\leq\sum_{n=1}^\infty\left(\left\langle\tilde{a}_\omega(n)^4\right\rangle^{1/2}+\left\langle\tilde{b}_\omega(n)^2\right\rangle\right)\|K_0(n)\|^4\|T_0^E(n)\|^4\\ &\leq C\sum_{n=1}^\infty\left(\left\langle\tilde{a}_\omega(n)^4\right\rangle^{1/2}+\left\langle\tilde{b}_\omega(n)^2\right\rangle\right)(a(n)+1)^4\|T_0^E(n)\|^4<\infty.\end{aligned} \tag{4.14}$$

(Here C is some universal constant.) Thus, from (4.1) and from (1.10), it follows that

$$\sum_{n=1}^\infty\left\langle\frac{\tilde{b}_\omega(n)^2}{a(n)^2(a(n)+\tilde{a}_\omega(n))^2}\right\rangle\|\mathcal{U}^E(n)\|_{\mathrm{HS}}^2$$

$$\leq C_1 \sum_{n=1}^{\infty} \left\langle \tilde{b}_\omega(n)^2 \right\rangle \|\tilde{T}_0^E(n)\|^4 < \infty \tag{4.15}$$

and

$$\sum_{n=1}^{\infty} \left\langle \frac{\tilde{a}_\omega(n)^2}{a(n)^2} \right\rangle \|\mathcal{V}^E(n)\|_{\mathrm{HS}}^2$$
$$\leq C_2 \sum_{n=1}^{\infty} \left\langle \tilde{a}_\omega(n)^4 \right\rangle^{1/2} \|\tilde{T}_0^E(n)\|^4 < \infty \tag{4.16}$$

for some constants $C_1,\ C_2 > 0$. Also, since $\|\tilde{T}_0^E(n)\| \geq 1$ (recall these are unimodular),

$$\sum_{n=1}^{\infty} \left\langle \frac{|\tilde{a}_\omega(n)|\tilde{a}_\omega(n)^2}{a(n)^2(a(n)+\tilde{a}_\omega(n))} \right\rangle \|\mathcal{V}^E(n)\|_{\mathrm{HS}} \|\mathcal{W}^E(n)\|_{\mathrm{HS}}$$
$$\leq C_3 \sum_{n=1}^{\infty} \left\langle |\tilde{a}_\omega(n)|^3 \right\rangle \|\tilde{T}_0^E(n)\|^4$$
$$\leq C_3 \sum_{n=1}^{\infty} \left\langle |\tilde{a}_\omega(n)|^4 \right\rangle^{3/4} \|\tilde{T}_0^E(n)\|^4$$
$$= C_3 \sum_{n=1}^{\infty} \left(\left\langle |\tilde{a}_\omega(n)|^4 \right\rangle^{1/2} \|\tilde{T}_0^E(n)\|^4 \right)^{3/2} < \infty \tag{4.17}$$

where $C_3 > 0$ is a constant. Finally, again using the fact that $\|\tilde{T}_0^E(n)\| \geq 1$, we get, for some constant $C_4 > 0$,

$$\sum_{n=1}^{\infty} \left\langle \frac{\tilde{a}_\omega(n)^4}{a(n)^2(a(n)+\tilde{a}_\omega(n))^2} \right\rangle^{1/2} \|\mathcal{W}^E(n)\|_{\mathrm{HS}}$$
$$\leq C_4 \sum_{n=1}^{\infty} \left\langle \tilde{a}_\omega(n)^4 \right\rangle^{1/2} \|\tilde{T}_0^E(n)\|^2$$
$$\leq C_4 \sum_{n=1}^{\infty} \left\langle \tilde{a}_\omega(n)^4 \right\rangle^{1/2} \|\tilde{T}_0^E(n)\|^4 < \infty. \tag{4.18}$$

Thus, we see that the conditions of Lemma 3.2 are satisfied, with $\tilde{b}_1 \equiv \frac{\tilde{b}}{a(a+\tilde{a})}$, $\tilde{b}_2 \equiv \frac{\tilde{a}}{a}$, $\tilde{b}_3 \equiv \frac{\tilde{a}^2}{a(a+\tilde{a})}$ and the obvious correspondence for the matrices. This implies that with probability one, the matrices $\tilde{\mathcal{D}}_\omega^E(n)$ converge to the identity matrix. As explained above, this finishes the proof of case 2 and therefore completes the proof of the theorem.

□

Remark. It is important to note that for a perturbation along the diagonal alone (that is - for the case of $\tilde{a}_\omega(n) \equiv 0$), a much shorter proof can be provided: Note that in this case, $\tilde{\mathcal{U}}^E_\omega(n)$ of (4.7) reduces to

$$\tilde{\mathcal{U}}^E(\omega, n) = \frac{\tilde{b}_\omega(n)}{a(n)^2}\mathcal{U}^E(n) \tag{4.19}$$

so that

$$\left\langle \tilde{\mathcal{U}}^E_\omega(n) \right\rangle = \frac{\left\langle \tilde{b}_\omega(n) \right\rangle}{a(n)^2}\mathcal{U}^E(n) = 0. \tag{4.20}$$

It is clear also that $\det\left(I + \tilde{\mathcal{U}}^E_\omega(n)\right) = 1$ so that we have also

$$\left\langle \left(I + \tilde{U}^E_\omega, n)\right)^{-1} \right\rangle = I. \tag{4.21}$$

Thus, from equation (4.7), and since $\tilde{\mathcal{U}}^E_\omega(n)$ is a function of the perturbing potential at the point n alone, it follows that $\tilde{\mathcal{D}}^E_\omega(n)$ is a matrix-valued martingale. Γ is precisely the set where this martingale is bounded, so the theorem follows from the martingale convergence theorem. The extra work we do (in Lemma 3.2) is due to the term $\frac{\tilde{a}_\omega(n)^2}{a(n)(a(n)+\tilde{a}_\omega(n))}$ which does not have zero mean but is ℓ^1 almost surely.

Proof of Theorem 1.3. Recall the definition of $\varphi_{1,\theta}$ and $\varphi_{2,\theta}$ ((1.14) and (1.15)) for a given potential $\{b(n)\}$. Note that for the operator H_θ the transfer matrices have the form

$$T^E(n) = \begin{pmatrix} \varphi_{1,\theta}(n+1) & \varphi_{2,\theta}(n+1) \\ \varphi_{1,\theta}(n) & \varphi_{2,\theta}(n) \end{pmatrix}. \tag{4.22}$$

For the perturbing random potential $\{\tilde{b}_\omega(n)\}$, and given the characterization (1.19) of $\Sigma_{\alpha s}$, it is obvious that in order to prove stability of the local Hausdorff dimension for a given energy - E, it suffices to show that a.s. there exist two solutions $\psi_{1,\omega}(n)$ and $\psi_{2,\omega}(n)$ of

$$\psi(n+1) + \psi(n-1) + (b(n) + \tilde{b}_\omega(n))\psi(n) = E\psi(n) \tag{4.23}$$

that satisfy

$$\lim_{L\to\infty} \frac{\| \psi_{1,\omega} \|_L}{\| \varphi_{1,\theta(E)} \|_L} = 1 \tag{4.24}$$

$$\lim_{L\to\infty} \frac{\| \psi_{2,\omega} \|_L}{\| \varphi_{2,\theta(E)} \|_L} = 1. \tag{4.25}$$

We shall prove that relations (4.24) and (4.25) hold almost surely, for every energy in the set Λ. A simple application of Fubini's theorem (as in the previous proof), then yields the inclusion in the theorem up to a set of measure zero, for any fixed measure.

Fix $E \in \Lambda$. It isn't difficult to see (see Lemma 4.3 in [16]) that for any $\varepsilon > 0$ there exist ε-dependent constants $C_1,\ C_2,\ C_3,\ C_4$, so that for large N

$$C_2 N^{1-\frac{1}{2\beta(E)}-\varepsilon} \leq \| \varphi_{1,\theta(E)} \|_N \leq C_1 N^{1/2+\varepsilon} \tag{4.26}$$

$$C_4 N^{1/2-\varepsilon} \leq \| \varphi_{2,\theta(E)} \|_N \leq C_3 N^{\frac{1}{2\beta(E)}+\varepsilon}. \tag{4.27}$$

(Recall that $\beta(E) > 0$.) Now, applying Lemma 3.1, with

$$\begin{aligned} \mathcal{U}(n) = &\begin{pmatrix} \varphi_{1,\theta(E)}(n+1) & \varphi_{2,\theta(E)}(n+1) \\ \varphi_{1,\theta(E)}(n) & \varphi_{2,\theta(E)}(n) \end{pmatrix}^{-1} \cdot \\ &\cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{1,\theta(E)}(n+1) & \varphi_{2,\theta(E)}(n+1) \\ \varphi_{1,\theta(E)}(n) & \varphi_{2,\theta(E)}(n) \end{pmatrix}, \end{aligned} \tag{4.28}$$

$\tilde{b}_\omega(n)$, and $f_+(n) = n^{\tilde{\eta}}$ (recall (1.25)), we see that with probability one, there exist sequences $\mathbf{d}^+_\omega(n) \equiv \begin{pmatrix} d^+_{1,\omega}(n) \\ d^+_{2,\omega}(n) \end{pmatrix}$ and $\mathbf{d}^-_\omega(n) \equiv \begin{pmatrix} d^-_{1,\omega}(n) \\ d^-_{2,\omega}(n) \end{pmatrix}$, that solve (3.5) and satisfy (3.6)–(3.9). Let

$$\begin{aligned} \psi_{1,\omega}(n) &= d^-_{1,\omega}(n)\varphi_{1,\theta(E)}(n) + d^-_{2,\omega}(n)\varphi_{2,\theta(E)}(n) \\ &= d^-_{1,\omega}(n-1)\varphi_{1,\theta(E)}(n) + d^-_{2,\omega}(n-1)\varphi_{2,\theta(E)}(n) \end{aligned} \tag{4.29}$$

$$\begin{aligned} \psi_{2,\omega}(n) &= d^+_{1,\omega}(n)\varphi_{1,\theta(E)}(n) + d^+_{2,\omega}(n)\varphi_{2,\theta(E)}(n) \\ &= d^+_{1,\omega}(n-1)\varphi_{1,\theta(E)}(n) + d^+_{2,\omega}(n-1)\varphi_{2,\theta(E)}(n). \end{aligned} \tag{4.30}$$

(The last equality in each equation follows from (3.5) with the above definition for $\mathcal{U}(n)$.) Using

$$\varphi_{1,\theta(E)}(n)\varphi_{2,\theta(E)}(n-1) - \varphi_{1,\theta(E)}(n-1)\varphi_{2,\theta(E)}(n) = \det T^E(n-1) = 1$$

and (3.5), we get that

$$\begin{aligned} &d^\pm_{1,\omega}(n-1)\varphi_{1,\theta(E)}(n-1) + d^\pm_{2,\omega}(n-1)\varphi_{2,\theta(E)}(n-1) \\ &= d^\pm_{1,\omega}(n)\varphi_{1,\theta(E)}(n-1) + d^\pm_{2,\omega}(n)\varphi_{2,\theta(E)}(n-1) \\ &\quad - \tilde{b}_\omega(n)(d^\pm_{1,\omega}(n)\varphi_{1,\theta(E)}(n) + d^\pm_{2,\omega}(n)\varphi_{2,\theta(E)}(n)). \end{aligned}$$

Thus we see that

$$\begin{aligned} &d^\pm_{1,\omega}(n)\varphi_{1,\theta(E)}(n+1) + d^\pm_{2,\omega}(n)\varphi_{2,\theta(E)}(n+1) \\ &\quad + d^\pm_{1,\omega}(n-1)\varphi_{1,\theta(E)}(n-1) + d^\pm_{2,\omega}(n-1)\varphi_{1,\theta(E)}(n-1) \\ &\quad + \big(b(n) + \tilde{b}_\omega(n)\big)\big(d^\pm_{1,\omega}(n)\varphi_{1,\theta(E)}(n) + d^\pm_{2,\omega}(n)\varphi_{2,\theta(E)}(n)\big) \\ &= E(d^\pm_{1,\omega}(n)\varphi_{1,\theta(E)}(n) + d^\pm_{2,\omega}(n)\varphi_{2,\theta(E)}(n)) \end{aligned}$$

which means that $\{\psi_{1,\omega}(n)\}$ and $\{\psi_{2,\omega}(n)\}$ solve (4.23).

Now, from

$$\begin{aligned}\left|\frac{\|\psi_{2,\omega}\|_L}{\|\varphi_{2,\theta(E)}\|_L}-1\right| &\leq \frac{\|\psi_{2,\omega}-\varphi_{2,\theta(E)}\|_L}{\|\varphi_{2,\theta(E)}\|_L} \\ &= \frac{\|d^+_{1,\omega}\varphi_{1,\theta(E)}+(d^+_{2,\omega}-1)\varphi_{2,\theta(E)}\|_L}{\|\varphi_{2,\theta(E)}\|_L} \\ &\leq \sqrt{2}\left(\frac{\|d^+_{1,\omega}\varphi_{1,\theta(E)}\|_L}{\|\varphi_{2,\theta(E)}\|_L}+\frac{\|(d^+_{2,\omega}-1)\varphi_{2,\theta(E)}\|_L}{\|\varphi_{2,\theta(E)}\|_L}\right)\end{aligned}$$

it follows immediately that (4.25) holds, (since $\varphi_{1,\theta(E)}$ is subordinate). For (4.24) write, similarly,

$$\begin{aligned}\left|\frac{\|\psi_{1,\omega}\|_L}{\|\varphi_{1,\theta(E)}\|_L}-1\right| &\leq \frac{\|\psi_{1,\omega}-\varphi_{1,\theta(E)}\|_L}{\|\varphi_{1,\theta(E)}\|_L} \\ &= \frac{\|(d^-_{1,\omega}-1)\varphi_{1,\theta(E)}+d^-_{2,\omega}\varphi_{2,\theta(E)}\|_L}{\|\varphi_{1,\theta(E)}\|_L} \\ &\leq \sqrt{2}\left(\frac{\|(d^-_{1,\omega}-1)\varphi_{1,\theta(E)}\|_L}{\|\varphi_{1,\theta(E)}\|_L}+\frac{\|d^-_{2,\omega}\varphi_{2,\theta(E)}\|_L}{\|\varphi_{1,\theta(E)}\|_L}\right).\end{aligned}$$

That the first term on the left hand side converges to zero is immediate (recall that $\beta(E)>0$ so that $\lim_{L\to\infty}\|\varphi_{1,\theta(E)}\|_L=\infty$). For the second term we note, first, that for any $\varepsilon>0$, there is an L_0 such that for any $n\geq L_0$ $|d^-_{2,\omega}(n)|<\frac{\varepsilon}{n^{\tilde{\eta}}}$. Second, from (4.26) and (4.27), we get that there exists a constant $D>0$ for which

$$\frac{\|\varphi_{2,\theta(E)}\|_L}{\|\varphi_{1,\theta(E)}\|_L}\leq DL^{\eta(E)}.$$

Now, using summation by parts,

$$\begin{aligned}&\frac{\left\|d^-_{2,\omega}\varphi_{2,\theta(E)}\right\|_L}{\left\|\varphi_{1,\theta(E)}\right\|_L} \\ &= \frac{\left(\sum_{n=1}^{L}d^-_{2,\omega}(n)^2|\varphi_{2,\theta(E)}(n)|^2\right)^{1/2}}{\left\|\varphi_{1,\theta(E)}\right\|_L} \\ &\leq \frac{\left\|d^-_{2,\omega}\varphi_{2,\theta(E)}\right\|_{L_0}}{\left\|\varphi_{1,\theta(E)}\right\|_L}+\frac{\left(\sum_{n=1}^{L}\frac{\varepsilon^2}{n^{2\tilde{\eta}}}|\varphi_{2,\theta(E)}(n)|^2\right)^{1/2}}{\left\|\varphi_{1,\theta(E)}\right\|_L} \\ &\leq \frac{\left\|d^-_{2,\omega}\varphi_{2,\theta(E)}\right\|_{L_0}}{\left\|\varphi_{1,\theta(E)}\right\|_L}+\frac{\varepsilon}{L^{\tilde{\eta}}}\frac{\left\|\varphi_{2,\theta(E)}\right\|_L}{\left\|\varphi_{1,\theta(E)}\right\|_L}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(\sum_{n=1}^{L}\left(\left(\frac{\varepsilon^2}{n^{2\tilde{\eta}}}-\frac{\varepsilon^2}{(n+1)^{2\tilde{\eta}}}\right)\sum_{j=1}^{n}|\varphi_{2,\theta(E)}(j)|^2\right)\right)^{1/2}}{\left\|\varphi_{1,\theta(E)}\right\|_L}\\
&\leq \frac{\left\|d_{2,\omega}^{-}\varphi_{2,\theta(E)}\right\|_{L_0}}{\left\|\varphi_{1,\theta(E)}\right\|_L}+D\cdot\frac{\varepsilon}{L^{\tilde{\eta}}}L^{\eta(E)}\\
&+D\cdot\varepsilon\frac{\left(\sum_{n=1}^{L}\frac{1}{n^{2\tilde{\eta}+1}}n^{2\eta(E)}\sum_{j=1}^{n}|\varphi_{1,\theta(E)}(j)|^2\right)^{1/2}}{\left\|\varphi_{1,\theta(E)}\right\|_L}\\
&\leq \frac{\left\|d_{2,\omega}^{-}\varphi_{2,\theta(E)}\right\|_{L_0}}{\left\|\varphi_{1,\theta(E)}\right\|_L}+D\cdot\varepsilon+D\cdot\varepsilon\left(\sum_{n=1}^{L}\frac{1}{n^{1+2(\tilde{\eta}-\eta(E))}}\right)^{1/2},
\end{aligned}
$$

so (4.24) follows from the fact that $\tilde{\eta}>\eta(E)$. This finishes the proof of the theorem. □

5. An Application of Theorem 1.3 to Sparse Potentials

In this section, we present an application of Theorem 1.3 to one-dimensional Schrödinger operators with sparse potentials, studied by Zlatoš in [25]. The family $H_\theta(v,\gamma)$ of operators constructed there has a potential of the form

$$
b^{v,\gamma}(n)=\begin{cases} v & n=n_j\equiv\gamma^j \text{ for some } j\geq 1\\ 0 & \text{otherwise.}\end{cases} \tag{5.1}
$$

for some $v\neq 0$ and $\gamma>1$ an integer.

We say that a measure μ has *fractional Hausdorff dimension* in some interval I if $\mu(I\cap\cdot)$ is α-continuous and $(1-\alpha)$-singular for some $1>\alpha>0$. Considering potentials of the form (5.1), Zlatoš proves the following:

Proposition 5.1 (Theorem 4.1 in [25]). *For any closed interval of energies $I\subseteq(-2,2)$ there are $v_0>0$ and $\gamma_0\in\mathbb{N}$ such that if $0<|v|<v_0$ and $\gamma\geq\gamma_0$ is an integer, then for Lebesgue-almost every θ, the measure μ_θ, corresponding to the operator $H_\theta(v,\gamma)$, has fractional Hausdorff dimension in I.*

An important feature of operators with sparse potentials, is that the modulus of the solutions undergoes significant changes only near the points where the potential does not vanish (this is easily seen using EFGP transform - see [15]). Thus, it is possible to obtain estimates on the pointwise behavior of the solutions looking at points in the support of the potential. The proof of Proposition 5.1 goes through such estimates.

It is actually shown there that, given γ large enough and v small enough, there exist constants $\beta_1 < \beta_2 < \frac{1}{2}$, depending only on I, γ and v, and a set $I' \subseteq I$ of full Lebesgue measure, such that for any $E \in I'$, equation (1.11) with $b(n) = b^{v,\gamma}(n)$ has two solutions - φ_1^E and φ_2^E for which the following holds for sufficiently large n and some small ε:

$$\left(|\varphi_1^E(n-1)|^2 + |\varphi_1^E(n)|^2\right)^{1/2} \leq n_j^{-\beta_1-\varepsilon} \quad n_j < n \leq n_{j+1}, \tag{5.2}$$

$$n_j^{\beta_1} \leq \left(|\varphi_2^E(n-1)|^2 + |\varphi_2^E(n)|^2\right)^{1/2} \leq n_j^{\beta_2} \quad n_j < n \leq n_{j+1}. \tag{5.3}$$

From this, using subordinacy theory [9] and the theory of rank one perturbations [23], Zlatoš shows that for almost every boundary condition, θ the spectral measure μ_θ is $(1-2\beta_2)$-continuous and $(1-2\beta_1)$-singular on I.

(5.2) and (5.3) provide us with a natural setting to apply Theorem 1.3. For the sake of simplicity and to make things explicit, we shall examine perturbing potentials of the form

$$\tilde{b}_{s,\omega}(n) = \frac{X_\omega(n)}{n^s} \tag{5.4}$$

where $\{X_\omega(n)\}$ are i.i.d. with a uniform distribution on an interval (say $[-1,1]$) and $s > 0$ to be specified later.

Theorem 5.2. *Let $I \subseteq (-2,2)$ be a closed interval and assume that $H_\theta(v,\gamma)$ is an operator satisfying the requirements of Proposition 5.1 so that for a.e. θ its spectral measure is α-continuous and $(1-\alpha)$-singular on J, for some $0 < \alpha < 1$. Let*

$$\beta_1 = \beta_1(I,\gamma,v) \tag{5.5}$$

and

$$\beta_2 = \beta_2(I,\gamma,v) \tag{5.6}$$

be as in the discussion above, and let

$$s > \frac{4\beta_2}{1-2\beta_2} - 2\beta_1 + \frac{1}{2}. \tag{5.7}$$

Then P-almost surely, the spectral measure of the random operator

$$H_\omega(v,\gamma,s) = H(v,\gamma) + \tilde{b}_{s,\omega} \tag{5.8}$$

is α-continuous and $(1-\alpha)$-singular on I.

Remark. It is not hard to see that, using the result of Kiselev-Last-Simon [16] described in the introduction, one needs to demand

$$|\tilde{b}(n)| \leq Cn^{-(s+\frac{1}{2})}, \tag{5.9}$$

in order to obtain this kind of stability.

Proof. We want to apply the perturbation only to sites $2, 3....$ at first, so denote by $\tilde{b}^0_\omega(n)$ the sequence

$$\tilde{b}^0_{s,\omega}(n) = \begin{cases} \tilde{b}_{s,\omega}(n) & \text{if } n > 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $I' \subseteq I$ be a set of full Lebesgue measure for which (5.2) and (5.3) hold. Since the spectral measure for the unperturbed operator is $(1 - 2\beta_2)$-continuous for almost every boundary condition, it follows from the theory of rank-one perturbations, from the fact that $H_\theta(v, \gamma)$ has no absolutely continuous spectrum on I' and from Theorem 1.4 in [16], that there exists a set $I'' \subseteq I'$ of full Lebesgue measure such that for any $E \in I''$

$$\beta(E) \geq \frac{1 - 2\beta_2}{2 - (1 - 2\beta_2)} = \frac{1 - 2\beta_2}{1 + 2\beta_2} > 0, \tag{5.10}$$

and therefore

$$\eta(E) = \frac{1 - \beta(E)}{\beta(E)} \leq \frac{4\beta_2}{1 - 2\beta_2}. \tag{5.11}$$

From the fact that almost every energy is regular, we may also assume that

$$\Lambda_0 \cap I'' = I''. \tag{5.12}$$

Thus, in order to get almost sure stability of the asymptotic behavior of the generalized eigenfunctions on I'', we only need to show

$$\sum_{n=1}^{\infty} r^E_{\tilde{\eta}}(n) \left\langle \tilde{b}_{s,\omega}(n)^2 \right\rangle = \sum_{n=1}^{\infty} r^E_{\tilde{\eta}}(n) \frac{1}{n^{2s}} < \infty \tag{5.13}$$

for every $E \in I''$ and some $\tilde{\eta} > \eta(E)$ (recall (1.24)). Given (5.7), (5.11) and (5.2)-(5.3), it is easy to verify that this is indeed the case. Thus, it follows from Theorem 1.3 that up to a set of Lebesgue measure zero

$$I'' \cap \Sigma_{\alpha\mathrm{c}}\left(\{b^{v,\gamma}(n)\}\right) = \Sigma_{\alpha\mathrm{c}}\left(\{b^{v,\gamma}(n) + \tilde{b}^0_{s,\omega}(n)\}\right)$$

$$I'' \cap \Sigma_{(1-\alpha)\mathrm{s}}\left(\{b^{v,\gamma}(n)\}\right) = \Sigma_{(1-\alpha)\mathrm{s}}\left(\{b^{v,\gamma}(n) + \tilde{b}^0_{s,\omega}(n)\}\right)$$

for a.e. ω.

Now, from the fact that the probability distribution of $\tilde{b}_{s,\omega}(1)$ is absolutely continuous, it follows, using the theory of rank one perturbations [23], that for almost every realization of the random perturbing potential, the spectral measure of $H_\omega(v, \gamma, s)$ is α-continuous and $(1 - \alpha)$-singular on J. $\square$

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