Convolution-Wedge Product of Fields in Anti-Symmetric Metric Regime Is Defined Through Electric-Magnetic Duality and Mirror Symmetry*

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Abstract

In this paper we use the pair of electric-magnetic (or GNO, or Langlands) duality groups $G = \mathrm{Sp}(1)$ and ${}^LG = \mathrm{SO}(3)$ and the T-transformation in mirror symmetry (or the S-duality, or the Fourier-Mukai transformation) to define the wedge product of fields: first by using gauge transformation, we reduce the fields with values in $\mathrm{Lie}\, G = \mathfrak{sp}(1)$ to the fields with values in the Lie algebra of the maximal torus $\mathfrak{t} \subset \mathrm{Lie}\, G = \mathfrak{sp}(1)$. Next we use the Fourier-Mukai transformation of fields to have the images as fields with values in the Lie algebra of the Langlands dual torus ${}^L\mathfrak{t}$ in $\mathrm{Lie}\, {}^LG = \mathfrak{so}(3)$. The desired wedge product of two fields is defined as the pre-image of the ordinary wedge product of images with values in ${}^L\mathfrak{t} \subset \mathfrak{so}(3)$.

1 Introduction

We discuss the problem of how to define some product of two fields in the anti-symmetric regime. The new idea is to use the so called GNO duality, which is

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equivalance to the Langlands duality and mirror symmetry. In our situation of symmetry groups, the Fourier Mukai can be extracted to defined product.

There are misterious analogy between events in arithmetics, in geometry and in physics. In seventies of 20th century, R. Langlands formulated a conjecture for finding some parametrization of discrete series represnetations of semi-simple Lie groups by some admissible finite-dimensional representations of some Galois type groups. This correspodence conjecture was proven in some special case and had some famous application in arithmetics, like in the final step of proving the Taniyama-Weil conjecture and then the Fermat Last Theorem.

The great ideas of arithmetics was then translated into geometry and some analog of the Langlands conjecture was known, and then proven as *Geometric Langlands Correspondence*.

By some heuristic reason, in 1977 M. F. Atiyah ask E. Witten of whether there is some analog of this event in physics; M. F. Atiyah referred to two works of P. Goddard J. Nuyts and D. Olive and F. Englert and P. Windey. It is known nowtheday that this misteriuos correspondence figured also in physics and it is a great success of many physicists to study mirror symmetry and Langlands correspondence in physics as Electric-Magnetic Duality.

Our main result is to use these results to define some convolution-wedge product of fields, which is very useful in physics and mathematics. This definition will be done in §5. For this goal we need to analyze the electric-magnetic duality(Theorem 1) and compare the Fourier-Mukai transformation with the ordinary Fourier transformation in coordinates (Theorem 3). As a corollary we deduce some property of wedge-convolution product (Theorem 6). We will discuss essential physical meaning of the wedge-convolution product in some another place.

2 Electric-magnetic Duality

It is well-known that the Dirac monopoles were associated with the internal symmetry group $U(1)$. Recently, the theory was extended to the general symmetric compact Lie groups H. Embedding $H \subseteq G$ in the general symmetry group G, we say also that it is broken down the symmetry to H. Let us remind that under a magnetic monopole we understand a solution, in which the magnetic part (space-space components) of the the gauge field tensor take the form

$$G_{ij} = \frac{\varepsilon_{ijk} r_k}{r^3} G(r), i,j = 1,2,3 (r >> 1) \tag{2.1}$$

The electric field is asymptotically radial and obeys the *generalized inverse square law* and additionally, assumed to be covariantly constant

$$\mathcal{D}_i G(r) = \partial_i G(r) - ie[W_i(r), G(r)] \equiv 0 \tag{2.2}$$

(supposed to be finite energy solution in the frame work of 't Hooft - Polyakov)

There is an important condition which should be satisfied, the so called Quantization condition: $\exp(4\pi ieG(r)) \equiv 1$. One can look at $G(r)$ as one generator of the symmetry group H.

$$J = \{\exp(ie\Omega G(r)); 0 \leq \Omega \leq 4\pi\} \tag{2.3}$$

It was the idea of Englert & Windey that up to normalization the possible values $\beta_1, \ldots, \beta_r$ are the weights of a new group $\check{H}$.

We think of $\check{H}$ as electric group, H as magnetic group. The main result of GNO duality is the fact that the double dual group is isomorphic to itself: $\check{\check{H}} \cong H$.

Let us denote by H a compact connected Lie group, $\mathfrak{h} = \mathcal{L}(H) = \mathrm{Lie}(H)$ the corresponding Lie algebra, which is decomposed into sum of two parts: $\mathcal{L}(H) = \underbrace{\mathcal{C}(\mathcal{L}(H))}_{\text{center}} + \underbrace{[\mathcal{L}(H), \mathcal{L}(H)]}_{\text{semi-simple part}}$. Let us denote the associated Cartan basis by

$$T_1, \ldots, T_\ell, T_{\ell+1}, \ldots, T_r, E_{\alpha_1}, E_{-\alpha_1}, \ldots E_{\alpha_\ell}, E_{-\alpha_\ell} \tag{2.4}$$

with the relations: $[T_i, E_\alpha] = \alpha_i E_\alpha$ $[T_i, T_j] = 0$. The corresponding Cartan matrix is introduced as

$$\begin{bmatrix} \underline{g_{ij}\,}| & & & \\ & 0 & & \\ & & \boxed{\begin{matrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{matrix}} & \\ & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix}. \tag{2.5}$$

We have $\alpha_i = \sum_j g_{ij}\alpha^j$, or $g_{ij} = \sum_\alpha \alpha_i \alpha_j$, normalized by the conditions: $g_{ij} = \delta_{ij}$. The corresponding exponentiated element in the Lie group, $\exp T_1, \ldots \exp T_r$ generalize a compact abelian subgroup, i.e. a torus $\mathbf{T} = \mathcal{N}_H(\langle T_1, \ldots, T_r \rangle)$.

It is well-known that for all $h \in H$, there exists $s \in H$ such that $h = sts^{-1}$, for some $t \in \mathbf{T}$. It is clear that $Z(H) \subset \mathbf{T}$, Therefore, there exists an element $S \in H$ such that

$$eG(P) = S \sum_{i=1}^{r} \beta_i T_i S^{-1}.$$

The coefficients $\beta_1 \ldots, \beta_r$ are the so called *magnetic weights*. Let us denote $\tilde{H}$ the universal covering group, $H = \tilde{H}/k(H)$; where $k(H) = \ker(\tilde{H} \to H)$. The *weights* of a single valued representation of H are introduced as follows.

$$\Lambda(H) = \{(w_1, \ldots, w_r) \text{ the eigenvalues of } T_1, \ldots, T_r\} \tag{2.6}$$

It is easy to check that

$$\Lambda(SU(2)) = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots\}, \tag{2.7}$$

$$\Lambda(SO(3)) = \{0, \pm 1, \pm 2, \ldots\} \tag{2.8}$$

and in general,

$$\Lambda(\tilde{H}) = \{w; \quad 2w\alpha/\alpha^2 \in \mathbb{Z}, \forall \alpha \in \Phi(H)\}. \tag{2.9}$$

It is clear that

$$\Lambda(H) \subset \Lambda(\tilde{H}) \tag{2.10}$$

Denote $\widetilde{\exp} : \mathcal{L}(H) \to \tilde{H}$ the exponential map, then $\exp(4\pi i T) \in k(H) \subseteq Z(\tilde{H})$. Any element of $Z(H)$ commutes with all the generators of $\mathcal{L}(H)$ and in particular with E_α and $\exp(4\pi i\beta.\alpha) = 1$, i.e. $2\alpha.\beta \in \mathbb{Z}, \forall \alpha \in \Phi(H)$. The set $\{\alpha/\alpha^2; \alpha \in \Phi(H)\}$ is a *root system*. We can normalize

$$\check{\Phi}(H) = \{\check{\alpha} = N^{-1}\alpha/\alpha^2; \qquad \alpha \in \Phi(H)\}. \tag{2.11}$$

If H is simple, N is the length of the largest root and if H is semi-simple N is a diagonal matrix. We have $\Phi(\check{H}) = \check{\Phi}(H)$, $\check{\check{\alpha}} = \alpha$. The most beautiful result of the GNO duality is indicated in the following table, which is equivalent to the Langlands duality.

General Langlands Duality

Magnetic Groups G	Electric Groups ${}^LG$
U(n)	U(n)
SU(n)	PSU(n) = SU(n)/$\mathbb{Z}_n$
Spin(2n)	SO(2n)/$\mathbb{Z}_2$
Sp(n)	SO(2n+1)
Spin(2n+1)	Sp(n)/$\mathbb{Z}_2$
G_2	G_2
E_8	E_8

3 Mirror Symmetry and Langlands Duality

In the Euclid 3-dimensional space we can define the wedge product of two vectors as the area of the parallelogram based on these vectors. This definition reduces to the wedge product of two fields. In the anti-symmetric metric regime, we could not use the same definition, because the area is not conserved by coordinate changes.

It is well-known that mirror symmetry plays the same role in GNO duality, or more precisely the electric-magnetic duality, as the Fourier-Mukai transformation between the pair of Langlands dual groups in the Langlands Program.

Let us consider a compact connected Lie group G, $\mathfrak{g} = \mathrm{Lie}\, G$ the Lie algebra of G, M a spin manifold with metric

$$g(x) = \sum_{\mu,\nu} g(x)_{\mu\nu} dx^\mu dx^\nu. \tag{3.1}$$

Consider also the associated Yang-Mills theory which is defined by a 1-form of connection $\mathcal{A}$ as a section of the principal bundle $P \twoheadrightarrow M$ with curvature $\mathcal{F}_\mathcal{A} = d_\mathcal{A}\mathcal{A}$. The action of the theory is of the form

$$\begin{array}{rcl} I & = & \frac{1}{8\pi}\int_M d^4x\sqrt{g}\left(\frac{4\pi}{e^2}\mathcal{F}_{mn}\mathcal{F}^{mn} - \frac{i\theta}{2\pi}\frac{1}{2}\epsilon_{mnpq}\mathcal{F}^{mn}\mathcal{F}^{pq}\right) \\ & = & -\frac{i}{8\pi}\int_M d^4x\sqrt{g}\left(\tau\mathcal{F}^+_{mn}\mathcal{F}^{+mn} - \bar{\tau}\mathcal{F}^-_{mn}\mathcal{F}^{-mn}\right), \end{array} \tag{3.2}$$

where g is the metric tensor on M,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}, \tag{3.3}$$

ϵ_{mnpq} is the Levi-Civita antisymmetric tensor, e is the so called electric charge, and $\mathcal{F}^\pm = \frac{1}{2}(\mathcal{F} \pm \star\mathcal{F})$ is the self-dual and anti-self-dual components of $\mathcal{F}$. It is well-known[KW] that this action I is invariant under transformation $T : \tau \mapsto \tau + 1$.

The most interest is that this action is invariant also under the *mirror symmetry transformation*

$$S : \tau \mapsto -\frac{1}{n_{\mathfrak{g}}\tau}, \tag{3.4}$$

where $n_{\mathfrak{g}} = 1$ for simply-laced (ABCDE) simple Lie groups, $= 2$ for F_2 and $= 3$ for G_2 in non-simply-laced case. The transformations T and S generate the so called Hecke group of symmetry $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$, which is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ in simply-laced case. An arbitrary element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ acts on τ by linear fractional transformation

$$\tau \mapsto \gamma.\tau = \frac{a\tau + b}{c\tau + d}. \tag{3.5}$$

Let us denote by $\mathbf{T}$ a fixed maximal torus of G. The group G is acting on connections by the so called *gauge transformations* of type

$$\mathcal{A} \mapsto g.\mathcal{A} := g\mathcal{A}g^{-1} + dg.g^{-1} \tag{3.6}$$

Denote by $\Omega_{\mathcal{A}}$ the orbit of $\mathcal{A}$ under the gauge action of G in the space $\Omega(M,G)$ of affine connections. It is well-known that

$$\Omega(M,G)/G \cong \Omega(M,\mathbf{T})/W, \tag{3.7}$$

where W is the Weyl group

$$W \cong W(G,\mathbf{T}) \cong \mathcal{N}_G(\mathbf{T})/\mathbf{T} \tag{3.8}$$

and $\mathcal{N}_G(\mathbf{T})$ is the normalizer of $\mathbf{T}$ in G. Hence, we reduced our consideration to the case of connection with values in the Lie algebras of commutative compact Lie groups. let us summarize the discussion as what follows.

Theorem 1 *The gauge fields are defined by their ground states in vacuum where the symmetry groups are broken down to the maximal tori of the symmetry groups, in other words the corresponding forms of connections are defined by their values in Lie algebras of tori by gauge transforms.*

4 Fourier-Mukai Transformation

Let us denote by ${}^L G$ the Langlands dual group, with a fixed maximal torus ${}^L\mathbf{T}$. The fundamental electric-magnetic duality maps G to ${}^L G$ and also the Lie algebra $\mathfrak{t}$ to $\check{\mathfrak{t}} = {}^L\mathfrak{t}$, and acted on by the same Weyl groups $W \cong {}^L W$. The vector spaces $\mathfrak{t}$ and $\check{\mathfrak{t}}$ are dual $\check{\mathfrak{t}} \cong \mathfrak{t}^*$. Chosen a Weyl-invariant metric on $\mathfrak{t}$ we can Weyl-invariantly identify them.

We remind the construction of Fourier-Mukai transformation, see [F] for more detail. Denote Loc_1 the moduli space of rank 1 local systems on a complex curve X. Recall that a local system is a pair $(\mathcal{F}, \nabla)$ consisting of a holomorphic line bundle $\mathcal{F}$ and a holomorphic connection ∇. Since $\mathcal{F}$ supports a holomorphic (hence flat) connection, the first Chern class of $\mathcal{F}$, which is the degree of $\mathcal{F}$, has to vanish. Therefore $\mathcal{F}$ defines a point of the Jacobian variety $\mathrm{Pic}_0(X) = \mathrm{Jac}(X)$, which is by definition the group of isomorphic classes of line bundles. We have therefore a natural map $p : \mathrm{Loc}_1(X) \to \mathrm{Jac}(X)$. The fibers of this map is the space of holomorphic connection on $\mathcal{F}$, i.e. the space of connection 1-forms in $\mathrm{H}^1(X, \Omega_X)$. Let $\mathcal{P}$ be the universal flat holomorphic line bundle, whose restriction to $(\tilde{\mathcal{F}}, \tilde{\nabla}) \times \mathrm{Jac}(X)$ is the line bundle $(\tilde{\mathcal{F}}$ on $\mathrm{Jac}(X)$. We have the diagram

$$\begin{tikzcd} & \mathcal{P} \arrow[d] & \\ & \mathrm{Loc}_1(X) \times \mathrm{Jac}(X) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Loc}_1(X) & & \mathrm{Jac}(X) \end{tikzcd}$$

where p_1, p_2 are the natural projections of the Cartesian product onto the first and the second factors, respectively.

One defines the pair of functors

$$F : D^b(\mathcal{D}_{\mathrm{Jac}(X)} - mod) \to D^b(\mathcal{O}_{\mathrm{Loc}_1(X)} - mod)$$

and

$$G : D^b(\mathcal{O}_{\mathrm{Loc}_1(X)} - mod) \to D^b(\mathcal{D}_{\mathrm{Jac}(X)} - mod),$$

as follows

$$(4.1) \quad F : \mathcal{M} \in D^b(\mathcal{D}_{\mathrm{Jac}(X)} - mod) \mapsto Rp_{1*}p_2^*(\mathcal{M} \otimes \mathcal{P}) \in D^b(\mathcal{O}_{\mathrm{Loc}_1(X)} - mod),$$

(4.2) $G : \mathcal{K} \in D^b(\mathcal{O}_{\mathrm{Loc}_1(X)} - mod) \mapsto Rp_{2*}p_1^*(\mathcal{K} \otimes \mathcal{P}) \in D^b(\mathcal{D}_{\mathrm{Jac}(X)} - mod).$

Let $E = (\mathcal{F}, \nabla)$ be a point of $\mathrm{Loc}_1(X)$ and consider the "skyscraper" sheaf $\mathcal{S}_E$ supported at this point. Then $G(\mathcal{S}_E) = (\tilde{\mathcal{F}}, \tilde{\nabla})$ is a $\mathcal{D}$-module on $\mathrm{Jac}(X)$. G. Laumon and M. Rothstein proved that these two functors F and G are inverse each to another.

Definition 2 *The general Fourier-Mukai transformation* means the equivalences of derived categories

$$D^b(\mathcal{O}_{\mathrm{Loc}_1(X)} - mod) \underset{F}{\overset{G}{\rightleftarrows}} D^b(\mathcal{D}_{\mathrm{Jac}(X)} - mod). \tag{4.3}$$

Let us examine in the concrete case of dimension 1, i.e. on $\mathbf{R}$. The "skyscraper" sheaves supported at a point x is the delta-type functions. The ordinary Fourier transform sends the delta-functions $\delta_x, x \in \mathbf{R}$, which are some sort of skyscraper sheaves to the exponent function $\Phi(y) = \exp(ixy)$, which is the solution of the differential equation

$$(\partial_y - ix)\Phi(y) = 0. \tag{4.4}$$

The function $\Phi(y)$ therefore corresponds to the $\mathcal{D}$-module of generator $\nabla = \partial_y - ix$. Because for an arbitrary function, one has decompositions:

$$f(x) = \int f(y)\delta_x(y)dy \tag{4.5}$$

and

$$f(x) = c \int \hat{f}(y) \exp(iyx)dy, \tag{4.6}$$

where $c = \frac{1}{2\pi}$ is a constant and

$$\hat{f}(y) = \int f(x) \exp(-iyx)dx, \tag{4.7}$$

we see that *on* $\mathbf{R}$*, the Fourier-Mukai transformaion is coincided with the ordinary Fourier transformation.*

On multidimensional torus fiber case, denote $\check{\mathfrak{t}} = \tilde{\mathfrak{t}}$ the Langlands dual torus, more precisely the local system on the torus by $\tilde{\mathbf{T}} = \mathrm{Hom}(\pi_1(\mathbf{T}), \mathbf{S}^1)$ and we have *Langlands duality* $\mathbf{T} \cong \mathrm{Hom}(\pi_1(\tilde{\mathbf{T}}), \mathbf{S}^1)$. Let us fix some local coordinates $(t^1, \ldots, t^m)$ on $\mathbf{T}$ and the corresponding local coordinates $(\tilde{t}^1, \ldots, \tilde{t}^m)$ on the dual torus $\tilde{\mathbf{T}}$. If we take a point $y \in \mathbf{T}$ with local coordinates $(t^1, \ldots, t^m)$, i.e. $y = (y^1 = \exp(it^1), \ldots, y^m = \exp(it^m))$, following the Fourier-Mukai transformation, we have some connection D_y on the dual torus with connection 1-form

$$A(\tilde{t}) = i \sum_j t^j d\tilde{t}_j = id\langle t, \tilde{t}\rangle \tag{4.8}$$

and

$$D_y = D_A = d + i \sum_j t^j d\tilde{t}_j. \tag{4.9}$$

This means that under the Fourier-Mukai transformation on torus, to any function

$$f(t) = c \int_{\tilde{\mathbf{T}}} \hat{f}(\tilde{t}) \exp(i\langle t, \tilde{t}\rangle) d\tilde{t} \tag{4.10}$$

on the torus $\mathbf{T}$ exactly corresponds the Fourier transform

$$\hat{f}(\tilde{t}) = \int_{\mathbf{T}} f(t) \exp(-i\langle t, \tilde{t}\rangle) dt. \tag{4.11}$$

We summarize this discussion in the following statement.

Theorem 3 *For the functions with values in the Lie algebras of tori, the Fourier-Mukai transformation becomes the ordinary Fourier transformation.*

5 Convolution-Wedge Product

Let us recall that under the Fourier transformation

$$\hat{f}(y) = \int f(x) \exp(-iyx) dx \tag{5.1}$$

the operator of multiplication of $f(x)$ by the variable x comes to the operator of differentiation $\frac{1}{i}\partial_y$ on the Fourier images and the convolution product of two functions

$$(f * g)(x) = \int f(y) g(x - y) dy, \tag{5.2}$$

comes to the ordinary point product of the Fourier transforms

$$\widehat{f * g} = \hat{f}.\hat{g}. \tag{5.3}$$

Definition 4 *The convolution-wedge product* of two fields in the anti-symmetric metric regime is defined as the inverse image of the ordinary wedge product of two images under the Fourier-Mukai transformation of fields in the symmetric metric regime. Denote the convolution-wedge product of fields by $\wedge_*$. If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\mathbf{a} \wedge_* \mathbf{b} := (a_2 * b_3 - a_3 * b_2, a_3 * b_1 - a_1 * b_3, a_1 * b_2 - a_2 * b_1). \tag{5.4}$$

The following proposition is an easy consequence of the definition.

Proposition 5 *The Fourier image of the convolution-wedge product is the vector product of two images,*

$$\widehat{\mathbf{a} \wedge_* \mathbf{b}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}}. \tag{5.5}$$

In our situation, the Fourier-Mukai transformation become the ordinary Fourier transformation. First we apply the Fourier transform to have ordinary product in symmetric regime in place of convolution product. After all we reduce to the ordinary vector analysis in $\mathrm{SO}(3)$ regime. Finally we use the inverse Fourier product to have the result.

Theorem 6 (Curl of Convolution-Wedge Product)

$$\mathrm{rot}_*(B_1 \wedge_* B_2) = \nabla \wedge_* (B_1 \wedge_* B_2) = \mathrm{div}_*(B_2)B_1 + \mathrm{div}_*(B_1)B_2. \tag{5.6}$$

We intend to discuss some physical applications of this convolution-wedge product of fields in a separated paper.

6 Conclusion

Convolution-wedge product of fields in anti-Symmetric metric regime is defined through electric-magnetic duality and mirror symmetry, which is important for physical applications.

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