WETTING OF GRADIENT FIELDS: PATHWISE ESTIMATES

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ABSTRACT. We consider the wetting transition in the framework of an effective interface model of gradient type, in dimension 2 and higher. We prove pathwise estimates showing that the interface is localized in the whole thermodynamically-defined partial wetting regime considered in earlier works. Moreover, we study how the interface delocalizes as the wetting transition is approached. Our main tool is reflection positivity in the form of the chessboard estimate.

1. INTRODUCTION AND RESULTS

1.1. **The model.** Effective interface models of gradient type have been a very active field of research in recent year. In particular, the understanding of the interaction of an interface with various types of external potentials (wall, pinning potential, etc.) has motivated numerous works, resulting in substantial progress on such issues. We refer to [12, 13, 18] for reviews of the problems investigated and references.

Among such questions, the analysis of the effect of an attractive wall on the behaviour of an interface is of particular relevance. Such a situation is commonly modelized as follows. Let $\Lambda_L = \{-\lceil L/2\rceil + 1, \dots, \lfloor L/2\rfloor\}^d$, $\overline{\Lambda}_L = \{-\lceil L/2\rceil, \dots, \lfloor L/2\rfloor + 1\}^d$ and $\partial\Lambda_L = \overline{\Lambda}_L \setminus \Lambda_L$. Interface configurations are given by $\varphi = \{\varphi_i\}_{i\in\Lambda_L} \in \mathbb{R}^{\overline{\Lambda}_L}$. Let also $V : \mathbb{R} \to \mathbb{R}$ be an even, convex function, with $V \not\equiv 0$ and $V(0) = 0$. Given $\eta, \lambda \geq 0$, we introduce the following probability measure on $\mathbb{R}^{\overline{\Lambda}_L}$.

$$\mu^0_{L;\lambda,\eta}(\mathrm{d}\varphi) = \frac{1}{\mathbf{Z}^0_{L;\lambda,\eta}} \exp\Big[-\frac{1}{8d} \sum_{\substack{i,j\in\Lambda_L\\ i\sim j}} (\varphi_i - \varphi_j)^2 - \lambda \sum_{i\in\Lambda_L} V(\varphi_i)\Big] \times \prod_{i\in\Lambda_L} \big(\mathrm{d}\varphi_i + \eta\delta_0(\mathrm{d}\varphi_i)\big) \prod_{i\in\partial\Lambda_L} \delta_0(\mathrm{d}\varphi_i),$$

where $\mathrm{d}\varphi_i$ and $\delta_0(\mathrm{d}\varphi_i)$ denote respectively Lebesgue measure and the Dirac mass at 0, and $i \sim j$ means that $\|i - j\|_1 = 1$. Writing $\Omega_+ = \{\varphi_i \geq 0, \forall i \in \Lambda_L\}$, we then introduce the probability measure

$$\mu^{+,0}_{L;\lambda,\eta}(\,\cdot\,) = \mu^0_{L;\lambda,\eta}(\,\cdot\,|\,\Omega_+).$$

This is the measure we shall be mostly interested in this work. We shall denote by

$$\mathbf{Z}^{+,0}_{L;\lambda,\eta} = \mu^0_{L;\lambda,\eta}(\Omega_+) \tag{1}$$

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the associated partition function. Before going on, let us briefly describe the physical meaning of all the pieces entering the definition of $\mu^{+,0}_{L;\lambda,\eta}$. Interpreting as usual φ_i as the height of the interface above site i, the positivity constraint Ω_+ corresponds to the presence of a hard wall at height 0, which the interface cannot cross. The term

$$\frac{1}{8d}\sum_{\substack{i,j\in\Lambda_L\\ i\sim j}}(\varphi_i-\varphi_j)^2$$

represents the internal energy associated to deformation of the interface from the horizontal plane. The term

$$\lambda\sum_{i\in\Lambda_L}V(\varphi_i)$$

represents the contribution to the energy coming from the presence of an external potential. A common choice if $V(x)=x^2$ (usually termed a *mass term*), but given the situation we want to model here a more natural choice is $V(x)=|x|$. The latter choice allows for the interpretation of the interface as separating a thermodynamically stable phase (above) from a thermodynamically unstable phase (below), the latter being stabilized locally becaused it is favoured by the wall; λ then measures the difference of free energies between the stable and unstable phases (both being stable when $\lambda=0$); see [17, 18] for a more detailed explanation. Finally, for $\eta>0$, the measure

$$\prod_{i\in\Lambda_L}\bigl(\mathrm{d}\varphi_i+\eta\delta_0(\mathrm{d}\varphi_i)\bigr)$$

modelizes the local attractivity of the interface/wall interaction, by rewarding each contact between the interface and the wall. One way to see this better (which also turns out to be technically useful later) is to realize that $\mu^{0}_{L;\lambda,\eta}$ can be seen as the weak limit of the measures

$$\mu^{0,(\epsilon)}_{L;\lambda,\eta}(\mathrm{d}\varphi)=\frac{1}{\mathbf{Z}^{0,(\epsilon)}_{L;\lambda,\eta}}\exp\bigl[-\frac{1}{8d}\sum_{\substack{i,j\in\Lambda_L\\ i\sim j}}(\varphi_i-\varphi_j)^2-\lambda\sum_{i\in\Lambda_L}V(\varphi_i)-\sum_{i\in\Lambda_L}U^{(\epsilon)}_{\eta}(\varphi_i)\bigr]\\ \times\prod_{i\in\Lambda_L}\mathrm{d}\varphi_i\prod_{i\in\partial\Lambda_L}\delta_0(\mathrm{d}\varphi_i),$$

where $e^{-U^{(\epsilon)}_{\eta}(\varphi_i)}=1+\frac{\eta}{2\epsilon}\mathbf{1}_{\{|\varphi_i|\leq\epsilon\}}$, as $\epsilon\downarrow 0$. Similarly, $\mu^{+,0}_{L;\lambda,\eta}$ is easily seen to be given by the weak limit, as $\epsilon\downarrow 0$, of

$$\mu^{+,0,(\epsilon)}_{L;\lambda,2\eta}=\mu^{0,(\epsilon)}_{L;\lambda,2\eta}(\,\cdot\,|\,\Omega_+). \tag{2}$$

1.2. **Earlier results.** Various aspects of this model have been studied in several papers. Let us briefly review earlier works relevant to the present contribution. Many of the results quoted below are valid in the more general context of gradient field with uniformly strictly convex interactions, *i.e.* those for which the term $(\varphi_i-\varphi_j)^2$ in the definition of the measure is replaced by $U(\varphi_i-\varphi_j)$ with $U:\mathbb{R}\to\mathbb{R}$ an even function with second derivative uniformly bounded away from 0 and ∞. To keep the discussion short, we shall not discuss this here (nor shall we discuss the case of non-nearest-neighbour interactions), and refer to the cited papers, and to the reviews mentionned at the beginning, for more information. Let us just remark that most of our analysis actually extends to this case as well, the gaussian character of the measure being used in an essential way only in very few places. However, most earlier results about the free energy in the wetting problem, upon which rests our whole approach, concern exclusively the gaussian setting (or Lipschitz interactions).

1.2.1. *Free interface.* We very briefly recall what is known when $\lambda = \eta = 0$, for the measure without the positivity constraint, *i.e.*, for the measure $\mu^0_{L;0,0}$. In that case, the measure is gaussian, and therefore amenable to explicit computations. Many things are known, but for our purposes here, it is enough to say that the variance of the field satisfies[1]

$$\langle \varphi_0^2 \rangle^0_{L;0,0} = \begin{cases} (g(1) + o_L(1))L & (d=1),\\ (g(2) + o_L(1)) \log L & (d=2),\\ g(d) + o_L(1) & (d \geq 3),\end{cases}$$

for explicit constants $g(d) > 0$, which shows that this measure describes a delocalized interface, with unbounded fluctuations, in dimensions 1 and 2, and a localized interface in dimension 3 and higher. In the latter case, although localized, the interface is strongly correlated,

$$\lim_{L\to\infty} \langle \varphi_i \varphi_j \rangle^0_{L;0,0} = (a(d) + o_{\|i-j\|_2}(1))\, \|i-j\|_2^{2-d},$$

avec $a(d) > 0$, $d \geq 3$.

1.2.2. *Interface and pinning potential.* Setting $\eta > 0$, keeping everything as before, changes dramatically the behaviour of the field *however small η is chosen.* More precisely, it is known that the interface is localized in any dimension [11, 2, 10], and has exponentially decaying covariances [2, 15]. Moreover, detailed informations on the critical behaviour as $\eta \downarrow 0$ is available [6], showing for example that

$$\lim_{L\to\infty} \langle \varphi_0^2 \rangle^0_{L;0,\eta} = \begin{cases} \frac12 \eta^{-2} + o(\eta^{-2}) & (d=1),\\ \frac1\pi |\log \eta| + O(\log|\log \eta|) & (d=2),\end{cases}$$

and that the rate $m(\eta)$ of exponential decay of $\lim_{L\to\infty} \langle \varphi_i \varphi_j \rangle^0_{L;0,0}$ satisfies

$$m(\eta) = \begin{cases} \frac12 \eta^2 + o(\eta^2) & (d=1),\\ O(\eta^{1/2}/|\log\eta|^{3/4}) & (d=2),\\ O(\eta^{1/2}) & (d\geq 3).\end{cases}$$

1.2.3. *Interface and hard-wall.* The measure with hard-wall constraint, but no external potentials, *i.e.* $\mu^{+,0}_{L;0,0}$ has been the subject of numerous works, focusing on the associated entropic repulsion phenomenon. Among the results that have been obtained, we highlight the two most relevant in the

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[1] We write $o_\ell(1)$ to denote a function such that $\lim_{\ell\to\infty} o_\ell(1) = 0$.

present context. Let $d \geq 3$; then [4, 8]

$$\lim_{L\to\infty} \Big| \frac{\langle\varphi_0\rangle^{+,0}_{L;0,0}}{\sqrt{\log L}} - 2\sqrt{g(d)} \Big| = 0.$$

The corresponding result in dimension 2, whose proof is substantially more intricate, is proved in [3] and takes the form

$$\lim_{L\to\infty} \Big| \frac{\langle\varphi_0\rangle^{+,0}_{L;0,0}}{\log L} - 2\sqrt{g(2)} \Big| = 0.$$

(Actually, the statement in [3] has only been proved when the positivity constraint acts on the sub-box $\Lambda_{\delta L}$, $0 < \delta < 1$, but it is clear that the previous result is true, and that it should be provable in the same way, with some additional, but minor, complications.)

The main thing to observe here is the fact that the interface is repelled by the wall, at a distance that is much larger than its typical fluctuations (which are of order $\sqrt{\log L}$ when $d = 2$, and of order 1 when $d \geq 3$). This is the phenomenon of entropic repulsion. Of course, this does not happen when $d = 1$, since the pinned random walk conditionned to be positive converges under diffusive scaling to the brownian excursion.

1.2.4. *Interface and attractive hard-wall: wetting transition.* We want to describe the behaviour of the field when both a hard-wall and a pinning potential are present, $\mu^{+,0}_{L;0,\eta}$. In this situation, there is a competition between the entropic repulsion due to the hard-wall constraint and the localizing effect of the pinning potential.

Let us introduce the finite-volume average density of pinned sites

$$\rho_L(\eta) = \langle |\Lambda_L|^{-1} \sum_{i\in\Lambda_L} \mathbf{1}_{\{\varphi_i = 0\}} \rangle^{+,0}_{L;0,\eta},$$

and its limit $\rho(\eta) = \lim_{L\to\infty} \rho_L(\eta)$. It is easy to show that ρ is non-decreasing in η, so the following critical value is well-defined,

$$\eta_c = \inf\{\eta \,:\, \rho(\eta) > 0\}.$$

This critical point can be given an equivalent definition (the equivalence is proved, *e.g.*, in [7]). Let us introduce the free energy (or surface tension, or wall free energy)

$$f_L(\lambda,\eta) = |\Lambda_L|^{-1} \frac{\mathbf{Z}^{+,0}_{L;\lambda,\eta}}{\mathbf{Z}^{+,0}_{L;\lambda,0}},$$

and $f(\lambda,\eta) = \lim_{L\to\infty} f_L(\lambda,\eta)$. Then

$$\eta_c = \inf\{\eta \,:\, f(0,\eta) > 0\}.$$

The sets $\{\eta \leq \eta_c\}$, resp. $\{\eta > \eta_c\}$, are called regimes of *complete wetting*, resp. *partial wetting.* They are supposed to correspond to regimes in which the interface is delocalized, resp. localized. The phase transition taking place at η_c is known as the *wetting transition.* It is known that $\eta_c = 0$ when $d \geq 3$ [5], while $\eta_c > 0$ when $d = 2$ [7][2]. The fact that $\eta_c > 0$ in dimension 1 is easily checked, and has been proven long ago by physicists.

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[2] Actually, it is interesting to observe that it is also proved in [7] that $\eta_c > 0$ in any dimensions if the interaction term $(\varphi_i - \varphi_j)^2$ is replaced by, say, $|\varphi_i - \varphi_j|$.

Contrarily to the results described above, there are only very few pathwise results in this setting, except in dimension 1, where specific features (in particular, a natural renewal structure) makes it possible to fully describe the process [9]. Before the present paper, the only pathwise results available in dimensions 2 and larger are those of [17], and state that

- In dimension 2 and for all η sufficiently large, the interface is localized, and covariances decay exponentially.
- In dimension 2, for all $\eta < \eta_{\mathrm{c}}$, the interface delocalizes, in the sense that $\lim_{L\to\infty} \langle \varphi_0 \rangle^{+,0}_{L;0,\eta} = \infty$.
- In dimension 2 and for η sufficiently small[3], $\langle \varphi_0 \rangle^{+,0}_{L;0,\eta} \asymp \log L$.

Notice that $\eta_c = 0$ in dimensions 3 and higher, and thus the analogue of the last statement reduces to the entropic repulsion estimate given above.

The main goal of the present paper is to provide detailed pathwise information on the localized regime in the whole partial wetting regime and in any dimensions. Moreover, we shall give some informations on the rate of divergence of the height as the wetting transition is approached (implying among other things that divergence does occur also in dimension 2).

1.2.5. *Interface, attractive hard-wall, away from coexistence: prewetting.* Finally, letting $\lambda > 0$ introduces another source of localization of the interface. Of course, under our assumptions on this potential, it is not surprising that it always yields localization of the interface, which corresponds to the impossibility of growing a large film of thermodynamically unstable phase. The main question here is to understand what happens as the system is brought close to phase coexistence, *i.e.* when $\lambda \downarrow 0$.

The situation studied in [14, 17] is the following: Fix $0 \le \eta < \eta_{\mathrm{c}}$, in dimension 1 or 2, or take $\eta = 0$ in dimension 3 and larger. Set also $\lambda > 0$. Then, as $\lambda \downarrow 0$, the system gets closer and closer to the regime of phase coexistence, and in that regime, because of the choice for η, the interface is delocalized. The problem was then to determine the rate at which this delocalization takes place. The main result of [17] can be stated as follows: For all $\lambda > 0$ sufficiently small,

$$\lim_{\lambda\downarrow 0} \lim_{L\to\infty} \langle \varphi_0 \rangle^{+,0}_{L;\lambda,\eta} \asymp \begin{cases} |\log \lambda| & (d=2), \\ |\log \lambda|^{1/2} & (d \ge 3). \end{cases}$$

This result is valid for any even, convex, not identically zero, external potential V satisfying some mild growth condition (*e.g.*, any polynomial growth is fine). In dimension 1, on the other hand, the critical behaviour does depend on the choice of V, see [14]; in this case, it has also been possible to prove exponential decay of covariances.

1.3. **New results.** We consider the measure $\mu^{+,0}_{L;\lambda,\eta}$. Let $\eta > \eta_{\mathrm{c}}$ and $\lambda > 0$. When $\lambda = 0$ (*i.e.*, at phase coexistence), the system is in the partial wetting regime, and the interface is expected to be localized. The next theorem shows that this is indeed the case, in the sense that the height of the interface remains bounded as phase coexistence is approached, $\lambda \downarrow 0$. Moreover, it

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[3] $a \asymp b$ meaning here and in the rest of this paper, that there exists a constant $c > 0$, depending on nothing except possibly the dimension, such that $ac \le b \le a/c$.

relates the rate of vanishing of the free energy with the divergence rate of the interface height, as η is sent to η_{c}.

Theorem 1.1. *There exists* $\bar{\eta} > \eta_{\mathrm{c}}$ *and* $\alpha_d > 0, C_d < \infty$ *such that, for any* $\eta \in (\eta_{\mathrm{c}}, \bar{\eta})$,

$$\mu^{+,0}_{L;\lambda,\eta}\big(\varphi_i \geq T|\log f(\lambda,\eta)|\big) \leq C_2 f(\lambda,\eta)^{\alpha_2 T^2},$$

for $d = 2$, *and*

$$\mu^{+,0}_{L;\lambda,\eta}\big(\varphi_i \geq T\sqrt{|\log f(\lambda,\eta)|}\big) \leq C_d f(\lambda,\eta)^{\alpha_d T^2},$$

for $d \geq 3$. *In particular, there exist* $c'_d < \infty$ *such that, for all* $\eta \in (\eta_{\mathrm{c}}, \bar{\eta})$,

$$\lim_{\lambda\downarrow 0}\lim_{L\to\infty} \langle\varphi_0\rangle^{+,0}_{L;\lambda,\eta} \leq c'_2 |\log f(0,\eta)|,$$

when $d = 2$, *while*

$$\lim_{\lambda\downarrow 0}\lim_{L\to\infty} \langle\varphi_0\rangle^{+,0}_{L;\lambda,\eta} \leq c'_d \sqrt{|\log f(0,\eta)|},$$

when $d \geq 3$.

Remark 1.1. *Since* $\langle\varphi_0\rangle^{+,0}_{L;\lambda,\eta}$ *is non-increasing in* η *(by FKG inequality), it follows, for example, that*

$$\lim_{\lambda\downarrow 0}\lim_{L\to\infty} \langle\varphi_0\rangle^{+,0}_{L;\lambda,\eta} < \infty,$$

for any $\eta > \eta_{\mathrm{c}}$, *and any dimension* $d \geq 2$.

Remark 1.2. *Actually, the result we prove in dimension* 3 *and higher is stronger than what is stated above, since it applies to the model with periodic boundary condition defined below (and thus, by FKG inequality, to the model with* 0*-boundary condition). In dimension* 2*, the result is only proved for* 0*-boundary condition, since we don't know how to prove the analogue of Lemma 2.1 for periodic boundary condition.*

We expect the $|\log f(0,\eta)|$ and $\sqrt{|\log f(0,\eta)|}$ upper bounds to be of the correct order. Below, we state lower bounds of this type, but only for periodic boundary condition (which, for a lower bound, is weaker than the corresponding result with 0-boundary condition).

Let us first define the corresponding model. We denote by $\mathbb{T}^d_L$ the torus $\mathbb{Z}^d/(L\mathbb{Z}^d)$. Configurations are then given by $\varphi \in \mathbb{R}^{\mathbb{T}^d_L}$. The measures $\mu^{\mathrm{per}}_{L;\lambda,\eta}$ and $\mu^{+,\mathrm{per}}_{L;\lambda,\eta}$ are defined precisely as before,

$$\mu^{\mathrm{per}}_{L;\lambda,\eta}(\mathrm{d}\varphi) = \frac{1}{\mathbf{Z}^{\mathrm{per}}_{L;\lambda,\eta}} \exp\big[-\frac{1}{8d}\sum_{\substack{i,j\in\mathbb{T}^d_L\\ i\sim j}} (\varphi_i-\varphi_j)^2 - \lambda\sum_{i\in\mathbb{T}^d_L} V(\varphi_i)\big] \times \prod_{i\in\mathbb{T}^d_L}\big(\mathrm{d}\varphi_i + \eta\delta_0(\mathrm{d}\varphi_i)\big),$$

$$\mu^{\mathrm{per}}_{L;\lambda,\eta}(\,\cdot\,) = \mu^{\mathrm{per}}_{L;\lambda,\eta}(\,\cdot\,|\,\Omega_+),$$

reinterpreting $i \sim j$ to mean that i and j are neighbouring vertices on $\mathbb{T}^d_L$. Notice that for these measures to be well-defined, it is necessary that $\lambda > 0$.

We denote by $\mathbf{Z}^{\mathrm{per}}_{L;\lambda,\eta}$ and $\mathbf{Z}^{+,\mathrm{per}}_{L;\lambda,\eta}$ the corresponding partition functions, and by

$$f^{\mathrm{per}}_L(\lambda,\eta) \overset{\text{déf}}{=} L^{-d}\log(\mathbf{Z}^{+,\mathrm{per}}_{L;\lambda,\eta}/\mathbf{Z}^{+,\mathrm{per}}_{L;\lambda,0})$$

the corresponding free energy. In the thermodynamic limit, this free energy and the one defined with 0-boundary condition agree.

Lemma 1.1. *For all $\lambda \geq 0, \eta \geq 0$, the limit $f(\lambda,\eta) = \lim_{L\to\infty} f^0_L(\lambda,\eta)$ exists and is increasing in λ. Moreover, for all $\lambda > 0, \eta \geq 0$, the limit $\lim_{L\to\infty} f^{\mathrm{per}}_L(\lambda,\eta)$ also exists and coincide with $f(\lambda,\eta)$.*

We can now state the announced lower bounds.

Theorem 1.2. *Let $\alpha > 1$. There exists $\bar\eta > \eta_{\mathrm{c}}$ and $c''_d = c''_d(\alpha) > 0$ such that, for all $\eta \in (\eta_{\mathrm{c}}, \bar\eta)$,*

$$\lim_{\lambda\downarrow 0}\lim_{L\to\infty} \langle\varphi_0\rangle^{+,\mathrm{per}}_{L;\lambda,\eta} \geq c''_2 |\log f(0,\alpha\eta)|,$$

when $d = 2$, while

$$\lim_{\lambda\downarrow 0}\lim_{L\to\infty} \langle\varphi_0\rangle^{+,\mathrm{per}}_{L;\lambda,\eta} \geq c''_d \sqrt{|\log f(0,\alpha\eta)|},$$

when $d \geq 3$.

Remark 1.3. *Although these results are interesting, it would be more informative to have estimates of the height that are expressed directly in terms of the microscopic parameter η, and not in terms of the free energy. To do this, one needs to understand the dependence of the latter in η close to the wetting transition, a task that seems too hard for the moment. It is however possible to extract a bound of this type from the proof in* [5]*, which shows that, for $d \geq 3$,*

$$f(0,\eta) \geq c_1(d) e^{-c_2/\eta}, \tag{3}$$

for some constants $0 < c_1, c_2 < \infty$. This, combined with the above estimates, implies that

$$\lim_{\lambda\downarrow 0}\lim_{L\to\infty} \langle\varphi_0\rangle^{+,0}_{L;\lambda,\eta} \leq c_3\eta^{-1},$$

for some constant $c_3 < \infty$. Observe that if, as we believe, the estimate (3) *is of the correct order, then the rate of divergence of the interface height is much faster than the logarithmic divergences seen in the results described in the previous subsection. This would of course be due to the very low density of pinned sites as η gets close to η_{c}.*

Remark 1.4. *Observe also that the introduction of the parameter α in the lower bound should be irrelevant. Indeed, if the logarithm of the free energy is behaves (as indicated by the lower bound of* [5] *for $d \geq 3$) like a polynomial function of $1/\eta$, as $\eta \downarrow \eta_{\mathrm{c}}$, our upper and lower bounds would actually differ only by a multiplicative constant.*

1.4. **Open problems.** Even though the results presented here subtantially improves the description of the wetting transition in these effective models, there remain a number of open problems.

- Prove lower and upper bounds on the interface height for the same boundary condition, when $d = 2$. To do that for periodic boundary condition, it would be enough to extend Lemma 2.1 to this boundary condition.
- It would also be desirable to extend the above result to the model with $\lambda = 0$, and not only in the limit $\lambda \downarrow 0$. This seems to imply the use of an alternative to the chessboard estimate, because it would preclude the use of periodic boundary condition.
- Prove that the covariances are exponentially decaying with the distance. It is not clear how this should be tackled. The only way of proving this type of result we are aware of is when some graphical representation is available (some random walk representation, for example). However, all the representations available for this model only apply to 2-point function, not covariances, and therefore do not seem very helpful.
- Obtain information on the behaviour of the free energy as a function of η close to the wetting transition. This seems too hard at the present time when $d = 2$, but there might be some way to prove lower bounds when $d \geq 3$. At a future stage, it would of course be extremely interesting to determine the critical exponent describing the divergence of the height.
- The present work deals only with the partial wetting regime. The situation concerning the complete wetting regime is still not as satisfactory as we would like. In particular, it would be quite desirable to prove that, in the whole complete wetting regime (or at least in the interior of this domain), the interface height diverges like $\log L$ under the measure $\mu^{+;0}_{L;0,\eta}$ in dimension 2. This is only known to hold far from the critical point. Of course, one expects even more, namely that the fields under $\mu^{+;0}_{L;0,\eta}$ and $\mu^{+;0}_{L;0,0}$ should be very close.

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2. PROOFS

The main tools used in the proofs below are FKG inequalities and the chessboard estimate. Both hold for the measures considered here, because they do for the gaussian measure, and are insensitive to perturbation of the form $\prod_i e^{U(\varphi_i)}$ (after suitably smoothing our potential – in particular the positivity constraint and the pinning potential – and taking weak limits). We refer, *e.g.*, to Appendix B of [13] for additional informations and references on the validity and use of FKG inequality in the context of gradient fields, and to [1] for a nice review on reflection positivity (and, in particular, the chessboard estimate).

2.1. **Elementary results on the free energy: proof of Lemma 1.1.** The existence of $\lim_{L\to\infty} f^0_L(\lambda,\eta)$ follows, for example, by FKG inequalities and completely standard subadditivity arguments. Its monotonicity in λ is also immediate

$$\frac{\partial}{\partial\lambda} f_L^0(\lambda,\eta) = L^{-d} \sum_{i\in\Lambda_L} \bigl\{ \langle V(\varphi_i)\rangle_{L;\lambda,0}^{+,0} - \langle V(\varphi_i)\rangle_{L;\lambda,\eta}^{+,0} \bigr\} \geq 0,$$

by FKG inequalities, since V is increasing on $\mathbb{R}^+$.

Let us denote by $\mathbf{Z}[A]$ the restriction of the partition function $\mathbf{Z}$ to configurations satisfying the condition A. With this notation, we have,

$$\tfrac{1}{2}\mathbf{Z}_{L;\lambda,\eta}^{+,\mathrm{per}} \leq \mathbf{Z}_{L;\lambda,\eta}^{+,\mathrm{per}}\bigl[\varphi_i \leq V^{-1}((2d/\lambda)\log L), \forall i \in \mathbb{T}_L^d\bigr] \leq \mathbf{Z}_{L;\lambda,\eta}^{+,\mathrm{per}}, \tag{4}$$

for all L large enough. Indeed,

$$\frac{\mathbf{Z}_{L;\lambda,\eta}^{+,\mathrm{per}}\bigl[\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L), \forall i \in \mathbb{T}_L^d\bigr]}{\mathbf{Z}_{L;\lambda,\eta}^{+,\mathrm{per}}} = \mu_{L;\lambda,\eta}^{+,\mathrm{per}}(\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L), \forall i \in \mathbb{T}_L^d),$$

which proves the second inequality, and, by FKG inequality,

$$\mu_{L;\lambda,\eta}^{+,\mathrm{per}}(\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L), \forall i \in \mathbb{T}_L^d)) \geq \prod_{i\in\mathbb{T}_L^d} \mu_{L;\lambda,\eta}^{+,\mathrm{per}}(\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L)).$$

But another application of FKG, and the chessboard estimate yield

$$\begin{aligned}
\mu_{L;\lambda,\eta}^{+,\mathrm{per}}(\varphi_i > V^{-1}(\frac{2d}{\lambda}\log L)) &\leq \mu_{L;\lambda,0}^{+,\mathrm{per}}(\varphi_i > V^{-1}(\frac{2d}{\lambda}\log L)) \\
&\leq \mu_{L;\lambda,0}^{+,\mathrm{per}}(\varphi_j > V^{-1}(\frac{2d}{\lambda}\log L), \forall j \in \mathbb{T}_L^d)^{1/|\mathbb{T}_L^d|} \\
&\leq \exp\bigl(-\lambda V(V^{-1}(\frac{2d}{\lambda}\log L))\bigr) \\
&= L^{-2d}.
\end{aligned}$$

Therefore,

$$\mu_{L;\lambda,\eta}^{+,\mathrm{per}}(\varphi_i \leq V^{-1}((2d/\lambda)\log L), \forall i \in \mathbb{T}_L^d)) \geq e^{-2L^{-d}} \geq \frac{1}{2},$$

for all L large enough. This proves (4). Notice now that the same also holds for 0-boundary condition. Indeed, the upper bound is again trivial, and for the lower bound, we can use FKG inequality to get

$$\begin{aligned}
&\mu_{L;\lambda,\eta}^{+,0}(\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L), \forall i \in \mathbb{T}_L^d)) \\
&\qquad\qquad \geq \mu_{L+1;\lambda,\eta}^{+,\mathrm{per}}(\varphi_i \leq V^{-1}((2d/\lambda)\log L), \forall i \in \mathbb{T}_L^d)).
\end{aligned}$$

It is thus sufficient to compare the partition function with periodic and 0-boundary conditions, under the constraint that all spins satisfy $\varphi_i < V^{-1}(\frac{2d}{\lambda}\log L)$. However, for a configuration φ on the torus satisfying this constraint, the change in energy resulting from setting one height to 0 is bounded above by $\frac{1}{2}(V^{-1}(\frac{2d}{\lambda}\log L))^2$ and bounded below by

$$-\frac{1}{2}(V^{-1}(\frac{2d}{\lambda}\log L))^2 - 2d\log L.$$

This shows that, for all fixed $\lambda > 0$,

$$L^{-d}\left|\log \frac{\mathbf{Z}_{L;\lambda,\eta}^{+,0}\bigl[\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L), \forall i \in \mathbb{T}_L^d)\bigr]}{\mathbf{Z}_{L+1;\lambda,\eta}^{+,\mathrm{per}}\bigl[\varphi_i \leq V^{-1}(\frac{2d}{\lambda}\log L), \forall i \in \mathbb{T}_L^d\bigr]}\right| = o_L(1),$$

implying that the limiting free energies coincide. By the above considerations, this is also true of the unrestricted partition functions and free energies.

2.2. **Upper bound on the height: proof of Theorem 1.1.**

In this section, we prove the upper tail estimate for the height at the origin, and the resulting upper bound on the height of the interface.

In the proof, it will be convenient to assume from the start that the free energy is small enough, or equivalently that η is close enough to $\eta_{\rm c}$; more precisely, we shall always assume that the $f(0,\eta)\le e^{-1}$, which ensures that $|\log f|\ge 1$ and $f^{-1/d}\ge 1$.

Let us first observe that FKG inequality implies that $\lim_{L\to\infty}\langle\varphi_0\rangle^{+,0}_{L;\lambda,\eta}$ exists in $\mathbb{R}\cup\{+\infty\}$. It is therefore sufficient for us to restrict our attention to boxes of size $L=2^N$, $N\ge 0$.

2.2.1. *The two-dimensional case.* Let us fix $\lambda>0$ and $\eta>\eta_{\rm c}$ as above, and, for $k\ge 1$, set $R(k)=\lfloor k\,f(\lambda,\eta)^{-1/2}\rfloor$. Expanding over pinned sites (see [10, 15, 6], for example), we have

$$\begin{aligned}&\mu^{+,0}_{L;\lambda,\eta}\bigl(\varphi_0\ge T|\log f(\lambda,\eta)|\bigr)\\&\quad=\sum_{k\ge1}\sum_{\substack{A\cap B_{R(k-1)}=\emptyset\\A\cap B_{R(k)}\neq\emptyset}}\zeta^{+,0}_{L;\lambda,\eta}(A)\,\mu^{+,0}_{L;\lambda,0}\bigl(\varphi_0\ge T|\log f(\lambda,\eta)|\;\big|\;\varphi_i=0,\forall i\in A\bigr),\end{aligned}$$

where we used the notation $B_R=\bigl\{i\in\mathbb{T}^d_L\,:\,\|i\|_\infty\le R\bigr\}$, $R\ge 0$.

By FKG and Lemma 2.1 below (provided T is large enough), we can find c_1 such that

$$\begin{aligned}&\mu^{+,0}_{L;\lambda,0}\bigl(\varphi_0\ge T|\log f(\lambda,\eta)|\;\big|\;\varphi_i=0,\forall i\in A\bigr)\\&\qquad\le\sup_{i\in B_{R(k)}\setminus B_{R(k-1)}}\mu^{+,0}_{L;0,0}\bigl(\varphi_0\ge T|\log f(\lambda,\eta)|\;\big|\;\varphi_i=0\bigr)\\&\qquad=\sup_{R(k)\ge R>R(k-1)}\;\sup_{\|i\|_\infty=R}\mu^{+,0}_{L;0,0}\bigl(\varphi_0\ge T|\log f(\lambda,\eta)|\;\big|\;\varphi_i=0\bigr)\\&\qquad\le\sup_{R(k)\ge R>R(k-1)}\exp\bigl(-cT^2|\log f(\lambda,\eta)|^2/\log R\bigr)\\&\qquad\le\exp\bigl(-cT^2|\log f(\lambda,\eta)|^2/\log R(k)\bigr)\\&\qquad\le f(\lambda,\eta)^{c_1T^2},\end{aligned}$$

uniformly in A such that $A\cap B_{R(k-1)}=\emptyset$ and $A\cap B_{R(k)}\neq\emptyset$, with $k<f(\lambda,\eta)^{-1/2}\equiv k_{\max}$. This implies that

$$\begin{aligned}&\mu^{+,0}_{L;\lambda,\eta}\bigl(\varphi_0\ge T|\log f(\lambda,\eta)|\bigr)\\&\qquad\le f(\lambda,\eta)^{c_1T^2}\sum_{k=1}^{k_{\max}}\sum_{A\cap B_{k-1}=\emptyset}\zeta^{+,0}_{L;\lambda,\eta}(A)+\sum_{k>k_{\max}}\sum_{A\cap B_{k-1}=\emptyset}\zeta^{+,0}_{L;\lambda,\eta}(A).\end{aligned}$$

But, for all $k\ge1$,

$$\sum_{A\cap B_{k-1}=\emptyset}\zeta^{+,0}_{L;\lambda,\eta}(A)=\frac{\mathbf{Z}^{+,0}_{L;\lambda,\eta}(B^{\rm c}_{R_k})}{\mathbf{Z}^{+,0}_{L;\lambda,\eta}},$$

where $\mathbf{Z}^{+,0}_{L;\lambda,\eta}(B^{\mathrm{c}}_{R(k)})$ is defined as in (1) but with the pinning potential acting only on $B^{\mathrm{c}}_{R(k)}$.

To estimate this last ratio, we would like to follow the idea in [2] and use reflexion positivity of the Gibbs measure (which holds, since we are considering a nearest-neighbour gradient field, with on-site potentials). Of course, the pinning potential is a bit singular, and makes the application of this inequality awkward, so we first replace it by its more regular approximation (2). The only remaining obstacle now is that we have 0-boundary condition instead of periodic boundary conditions. To remove this problem, we use once more FKG inequalities to obtain, for any $k \geq 1$,

$$\begin{aligned}\frac{\mathbf{Z}^{+,0,(\epsilon)}_{L;\lambda,\eta}}{\mathbf{Z}^{+,0,(\epsilon)}_{L;\lambda,\eta}(B^{\mathrm{c}}_{R(k)})} &= \langle \prod_{i\in B_{R(k)}} e^{U^{(\epsilon)}_\eta(\varphi_i)} \rangle^{+,0,(\epsilon)}_{L;\lambda,\eta}(B^{\mathrm{c}}_{R(k)}) \\ &\geq \langle \prod_{i\in B_{R(k)}} e^{U^{(\epsilon)}_\eta(\varphi_i)} \rangle^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta}(B^{\mathrm{c}}_{R(k)}) = \frac{\mathbf{Z}^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta}}{\mathbf{Z}^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta}(B^{\mathrm{c}}_{R(k)})}.\end{aligned}$$

Let $n \in \mathbb{N}$ be such that $2^n \leq R(k) < 2^{n+1}$, and set $\bar{R} = 2^{n-1}$. Applying the chessboard estimate starting with the block $\{-\bar{R}+1, \dots, \bar{R}\}^2$, we get

$$\begin{aligned}\frac{\mathbf{Z}^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta}(B^{\mathrm{c}}_{R(k)})}{\mathbf{Z}^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta}} &= \langle \prod_{i\in B_{R(k)}} e^{-U^{(\epsilon)}_\eta(\varphi_i)} \rangle^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta} \\ &\leq \langle \prod_{i\in B_{\bar{R}}} e^{-U^{(\epsilon)}_\eta(\varphi)} \rangle^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta} \\ &\leq \big\{ \langle \prod_{i\in \mathbb{T}^2_{L+1}} e^{-U^{(\epsilon)}_\eta(\varphi_i)} \rangle^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta} \big\}^{|B_{\bar{R}}|/|\mathbb{T}^2_{L+1}|} \\ &\leq \big\{ \frac{\mathbf{Z}^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,0}}{\mathbf{Z}^{+,\mathrm{per},(\epsilon)}_{L+1;\lambda,\eta}} \big\}^{|B_{\bar{R}}|/|\mathbb{T}^2_{L+1}|}\end{aligned}$$

We now obtain the result for our original ratio by taking the limit $\epsilon \downarrow 0$, which yields

$$\begin{aligned}\frac{\mathbf{Z}^{+,0}_{L;\lambda,\eta}(B^{\mathrm{c}}_{R(k)})}{\mathbf{Z}^{+,0}_{L;\lambda,\eta}} &\leq \big\{ \frac{\mathbf{Z}^{+,\mathrm{per}}_{L+1;\lambda,0}}{\mathbf{Z}^{+,\mathrm{per}}_{L+1;\lambda,\eta}} \big\}^{|B_{\bar{R}}|/|\mathbb{T}^2_{L+1}|} \\ &= e^{-|B_{\bar{R}}|\, f_{L+1}(\lambda,\eta)} \\ &\leq e^{-\frac{1}{4}|B_{R(k)}|\, f_{L+1}(\lambda,\eta)}.\end{aligned}$$

Collecting these estimates, and using Lemma 1.1, we obtain finally that

$$\frac{\mathbf{Z}^{+,0}_{L;\lambda,\eta}(B^{\mathrm{c}}_{R(k)})}{\mathbf{Z}^{+,0}_{L;\lambda,\eta}} \leq e^{-\frac{1}{4}|B_{R(k)}|\,(f(\lambda,\eta)-o_L(1))} \leq e^{-c_2 k^2},$$

for all L large enough. Consequently,

$$\begin{aligned}\mu^{+,0}_{L;\lambda,\eta}\big(\varphi_0 \geq T|\log f(\lambda,\eta)|\big) &\leq f(\lambda,\eta)^{c_1T^2}\sum_{k=1}^{k_{\max}} e^{-c_2k^2} + \sum_{k>k_{\max}} e^{-c_2k^2}\\ &\leq f(\lambda,\eta)^{c_1T^2}\sum_{k=1}^{\infty} e^{-c_2k^2} + e^{-c_2k_{\max}^2}\sum_{k=1}^{\infty} e^{-c_2k^2}\\ &\leq c_3 f(\lambda,\eta)^{c_1T^2}.\end{aligned}$$

We can now also easily deduce the stated upper bound on the mean height.

$$\begin{aligned}\langle\varphi_0\rangle^{+,0}_{\infty;\lambda,\eta} &\leq c_4|\log f(\lambda,\eta)|\\ &\qquad + \sum_{n\geq 1}(n+1)T|\log(f(\lambda,\eta)|\,\mu^{+,0}_{L;\lambda,\eta}\big(\varphi_0 \geq nT|\log f(\lambda,\eta)|\big)\\ &\leq c_5|\log f(\lambda,\eta)|.\end{aligned}$$

All statements have been proved now, except for the following lemma, which was used in the above argument.

Lemma 2.1. *There exists T_0 and $c>0$ such that, for all $T>T_0$ and $R\geq 1$,*

$$\sup_{\|i\|_\infty=R} \mu^{+,0}_{L;0,0}\big(\varphi_0 \geq T\log R \bigm| \varphi_i = 0\big) \leq R^{-cT^2}.$$

Proof. The proof is a variant of the one given in [13, Lemma 4.4]. Let i be such that $\|i\|_\infty = R$. By FKG inequality, the probability only increases if we replace the 0-boundary condition with $\alpha\log L$ boundary condition, $\alpha>0$,

$$\mu^{+,0}_{L;0,0}\big(\varphi_0 \geq T\log R\big) \leq \mu^{+,\alpha\log L}_{L;0,0}\big(\varphi_0 \geq T\log R \bigm| \varphi_i = 0\big).$$

Choosing α large enough, one can guarantee that $\mu^{+,\alpha\log L}_{L;0,0}\big(\Omega^+ \bigm| \varphi_i = 0\big) > \frac{1}{2}$, see [13, Lemma 4.4]. Consequently, we can remove the positivity constraint,

$$\mu^{+,\alpha\log L}_{L;0,0}\big(\varphi_0 \geq T\log R \bigm| \varphi_i = 0\big) \leq 2\mu^{\alpha\log L}_{L;0,0}\big(\varphi_0 \geq T\log R \bigm| \varphi_i = 0\big).$$

Now, under $\mu^{\alpha\log L}_{L;0,0}\big(\,\cdot \bigm| \varphi_i = 0\big)$, φ_0 is gaussian, and the random walk representation implies that its mean is at most $c'\alpha\log R$, while its variance is at least $c''\log R$. Therefore,

$$\mu^{\alpha\log L}_{L;0,0}\big(\varphi_0 \geq T\log R \bigm| \varphi_i = 0\big) \leq \exp\big(-cT^2\log R\big),$$

which proves the claim. □

2.2.2. *Dimension 3 and higher.* The argument in dimensions 3 and higher is unfortunately more involved, due to the fact that pinning a single point does not localize the interface anymore. One way to solve this problem would be to use reflection positivity to deduce that the distribution of pinned point dominates (in a suitable sense) some Bernoulli percolation-type process, and then study the entropic repulsion problem for an interface with a such a random distribution of pinned points. This looks certainly feasible, but we decided to try a different path relying more on reflection positivity, but in our opinion technically simpler.

We start in a way similar to what we did in dimension 2. As before, we fix $\lambda > 0$ and $\eta > \eta_c$, and write $R(k) = \lfloor k\, f(\lambda,\eta)^{-1/d} \rfloor$. For $k \geq 0$, let $\mathscr{E}_1(k) = \{\varphi_j > 0,\, \forall j \in B_{R(k)}\}$ and $\mathscr{E}_2(k) = \{\exists i \in B_{R(k+1)} \setminus B_{R(k)},\, \varphi_i = 0\}$. It follows from Cauchy-Schwarz inequality that

$$\begin{aligned}
\mu^{+,\mathrm{per}}_{L;\lambda,\eta}&(\varphi_0 \geq T\sqrt{|\log f|}) \\
&= \sum_{k\geq 0} \mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq T\sqrt{|\log f|}, \mathscr{E}_1(k), \mathscr{E}_2(k)) \\
&\leq \sum_{k\geq 0} \Bigl[\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq T\sqrt{|\log f|}, \mathscr{E}_2(k))\, \mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\mathscr{E}_1(k))\Bigr]^{1/2}.
\end{aligned}$$

First, observe that applying the chessboard estimate similarly as was done in the two-dimensional case, we obtain

$$\begin{aligned}
\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\mathscr{E}_1(k)) &\leq \Bigl[\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_j > 0,\, \forall j \in \mathbb{T}^d_L)\Bigr]^{c_1\left(\frac{R(k)}{L}\right)^d} \\
&\leq \left[\frac{\mathbf{Z}^{+,\mathrm{per}}_{L;\lambda,0}}{\mathbf{Z}^{+,\mathrm{per}}_{L;\lambda,\eta}}\right]^{c_1\left(\frac{R(k)}{L}\right)^d} \\
&\leq \exp\bigl(-c_2(f(\lambda,\eta) - o_L(1))R(k)^d\bigr) \\
&\leq \exp\bigl(-c_3\, k^d\bigr),
\end{aligned}$$

for all L large enough.

It remains to estimate $\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq T\sqrt{|\log f|}, \mathscr{E}_2(k))$. To lighten notations, let us write $h = T\sqrt{|\log f|}$. Let $n \in \mathbb{N}$ be such that $2^n \geq 2R(k)+1 > 2^{n-1}$, and set $\bar{R} = 2^n$. Let us fix $i \in B_{R(k+1)} \setminus B_{R(k)}$. Applying once more the chessboard estimate, starting this time with the block $i + \{-\bar{R}+1, \dots, \bar{R}\}^d$, yields

$$\begin{aligned}
\mu^{+,\mathrm{per}}_{L;\lambda,\eta}&(\varphi_0 \geq h, \varphi_i = 0) \\
&\leq \Bigl[\mu^{+,\mathrm{per}}_{L;\lambda,\eta}\bigl(\varphi_j \geq h, \forall j \in \mathscr{R}(0), \varphi_k = 0, \forall k \in \mathscr{R}(i)\bigr)\Bigr]^{|B_{\bar{R}}|/|\mathbb{T}^d_L|},
\end{aligned}$$

where, for j a site of the original block, $\mathscr{R}(j)$ is the set of sites obtained after applying all the reflections to j.

We can now get rid of the mass and the pinning potential. Using FKG inequality,

$$\begin{aligned}
\mu^{+,\mathrm{per}}_{L;\lambda,\eta}&\bigl(\varphi_j \geq h, \forall j \in \mathscr{R}(0), \varphi_k = 0, \forall k \in \mathscr{R}(i)\bigr) \\
&\leq \mu^{+,\mathrm{per}}_{L;\lambda,\eta}\bigl(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,|\, \varphi_k = 0, \forall k \in \mathscr{R}(i)\bigr) \\
&\leq \mu^{+,\mathrm{per}}_{L;0,0}\bigl(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,|\, \varphi_k = 0, \forall k \in \mathscr{R}(i)\bigr).
\end{aligned}$$

For technical reasons, we shall need the distance between the points of $\mathscr{R}(i)$ to be larger than R_0, for some constant $R_0 = R_0(d)$ to be fixed later. In order to do that, let us introduce $\Delta(d) = \min\bigl\{n \geq 1 \,:\, 2n\bar{R} \geq R_0\bigr\}$ and let us denote by $\mathscr{R}_\Delta(i)$ the subset of $\mathscr{R}(i)$ obtained by replacing $\bar{R}$ by $\Delta\bar{R}$ in the construction. Of course, the distance between points in $\mathscr{R}_\Delta(i)$ is at least

R_0. By FKG inequality,

$$\begin{aligned}&\mu^{+,\mathrm{per}}_{L;0,0}\big(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}(i)\big)\\ &\qquad\qquad \leq \mu^{+,\mathrm{per}}_{L;0,0}\big(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big).\end{aligned}$$

We want now to deal with the positivity constraint.

$$\begin{aligned}&\mu^{+,\mathrm{per}}_{L;0,0}\big(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big)\\ &\qquad\qquad \leq \frac{\mu^{\mathrm{per}}_{L;0,0}\big(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big)}{\mu^{\mathrm{per}}_{L;0,0}\big(\Omega_+ \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big)}.\end{aligned}$$

The denominator in the last expression has already been bounded below in [5] in the case of 0-boundary condition (it is here that we need $2\Delta\bar{R} \geq R_0$). Using FKG inequality to change correspondingly the boundary condition, this yields

$$\begin{aligned}\mu^{\mathrm{per}}_{L;0,0}\big(\Omega_+ \,|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big) &\geq \mu^{0}_{L-1;0,0}\big(\Omega_+ \,|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big)\\ &\geq \exp(-c_4\,(L/(\Delta\bar{R}))^d \log(\Delta\bar{R}))\\ &\geq \exp(-c_5\,(L/R(k))^d \log R(k)).\end{aligned}$$

Obviously,

$$\begin{aligned}&\mu^{\mathrm{per}}_{L;0,0}\big(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big)\\ &\qquad\qquad \leq \mu^{\mathrm{per}}_{L;0,0}\Big(\sum_{j\in\mathscr{R}(0)} \varphi_j \geq h|\mathscr{R}(0)| \,\Big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\Big).\end{aligned}$$

Under $\mu^{\mathrm{per}}_{L;0,0}\big(\cdot \,\big|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big)$, $\sum_{j\in\mathscr{R}(0)}\varphi_j$ is a gaussian random variable of mean 0, and variance

$$\sum_{j,k\in\mathscr{R}(0)} \sum_{n\geq 0} \mathbb{P}_j(X_n = k, \tau_{\mathscr{R}_\Delta(i)} > n),$$

where $\mathbb{P}_j$ is the law of the simple random walk $(X_n)_{n\geq 0}$ on $\mathbb{T}^d_L$ with $X_0 = j$, and $\tau_{\mathscr{R}(i)} = \min\{n \geq 0 \,:\, X_n \in \mathscr{R}_\Delta(i)\}$. Lemma 2.2 below implies that this variance is bounded above by $c_6|\mathscr{R}(0)|$, and thus

$$\mu^{\mathrm{per}}_{L;0,0}\big(\varphi_j \geq h, \forall j \in \mathscr{R}(0) \,|\, \varphi_k = 0, \forall k \in \mathscr{R}_\Delta(i)\big) \leq \exp(-c_7\,(L/R(k))^d h^2),$$

Putting all together, we have shown that

$$\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq h, \varphi_i = 0) \leq \exp(-\tfrac{1}{2}c_7\,h^2) \equiv \exp(-c_8\,h^2),$$

provided $h^2 > (2c_5/c_7)\log R(k)$, which is satisfied if, say, $k < f(\lambda,\eta)^{-1} \equiv k_{\max}$ and $T^2 \geq (2c_5/c_7)(d+1)/d \equiv T_1$. Thus,

$$\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq T\sqrt{|\log f(\lambda,\eta)|}, \mathscr{E}_2(k)) \leq f(\lambda,\eta)^{c_8\,T^2},$$

for all $k < k_{\max}$.

We have proved that

$$\begin{aligned}\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq T\sqrt{|\log f(\lambda,\eta)|}) &\leq f(\lambda,\eta)^{c_8\,T^2} \sum_{k=1}^{k_{\max}} e^{-c_3 k^d} + \sum_{k>k_{\max}} e^{-c_3 k^d}\\ &\leq c_{11} f(\lambda,\eta)^{c_8\,T^2},\end{aligned}$$

provided that $T \geq T_1$.

One finally obtains

$$\begin{aligned}\langle\varphi_0\rangle^{+,\mathrm{per}}_{L;\lambda,\eta} &\leq T\sqrt{|\log f(\lambda,\eta)|}\\ &\qquad + \sum_{n\geq 1}(n+1)T\sqrt{|\log f(\lambda,\eta)|}\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\varphi_0 \geq nT\sqrt{|\log f(\lambda,\eta)|})\\ &\leq c_{12}\sqrt{|\log f(\lambda,\eta)|}.\end{aligned}$$

To complete the proof, it only remains to prove the following lemma.

Lemma 2.2. *For $\mathscr{R}(0)$ and $\mathscr{R}_\Delta(i)$ defined as above,*

$$\sum_{j,k\in\mathscr{R}(0)}\sum_{n\geq 0}\mathbb{P}_j(X_n = k, \tau_{\mathscr{R}_\Delta(i)} > n) \leq c_6|\mathscr{R}(0)|.$$

Proof. Of course,

$$\sum_{j,k\in\mathscr{R}(0)}\sum_{n\geq 0}\mathbb{P}_j(X_n = k, \tau_{\mathscr{R}_\Delta(i)} > n) = \sum_{j\in\mathscr{R}(0)}\sum_{n\geq 0}\mathbb{P}_j(X_n \in \mathscr{R}(0), \tau_{\mathscr{R}_\Delta(i)} > n).$$

We start by periodizing the sets $\mathscr{R}_\Delta(i)$ and $\mathscr{R}(0)$.

$\mathscr{R}_\Delta(i)$ contains exactly one image of i in each of the blocks used during its construction; it can thus be partitioned into 2^d disjoint periodic subsets of equal sizes. We denote by S_i the subset containg i. If we only kill the random walk once it enters S_i, then the sum we want to control is only made bigger.

Similarly, the set $\mathscr{R}(0)$ contains a fixed number of sites in each of these blocks (their number depending on the value of Δ). As before, we can decompose $\mathscr{R}_0$ as a finite union of disjoint periodic arrays of sites, with the same period as S_i. We can of course restrict our attention to one of these subsets only, because if we show separately for each of these subsets that the average number of times they are visited by the walk, before it dies, is bounded above uniformly in everything but the dimension, then the same will be true for their union. Let us therefore consider one of these subset, which we call S_0.

To prove the lemma, we are going to show that after each visit of the random walk to the set S_0, the walk has a positive probability, depending only on the dimension of the lattice, of hitting S_i before reentering S_0. This will show that the number of visits to $\mathscr{R}(0)$ before entering S_i is stochastically dominated by a geometric random variable of positive parameter (uniformly in everything, but the dimension), which immediately implies the claim.

Now, the periodicity of R_0 and R_i allows us to reinterpret the problem as being on a torus of size the common period of these two sets, with two distinguished sites, s_0 and s_i, coming from S_0 and S_i, respectively. In these terms, the problem can be reformulated as follows: prove that starting from s_0, the random walk has a positive probability of hitting s_1 before returning to s_0, and this probability is uniform in the size of the torus.

But this is easy. Indeed, by symmetry, at least one half of the sites of the torus satisfy

$$\mathbb{P}_x(\tau_{s_0} > \tau_{s_i}) \geq 1/2,$$

where $\tau_y = \min\{n \ge 1 \,:\, X_n = y\}$. Let us call this set $\mathscr{G}$. It is therefore possible to find r such that at least half of the sites in

$$\{x \,:\, \|x\|_\infty = r\}$$

belong to $\mathscr{G}$ (otherwise it would be impossible for $\mathscr{G}$ to contain at least half of the sites of the torus). But the probability that the random walk starting at 0 exits the box of radius r at one of the sites belonging to $\mathscr{G}$ before returning to 0, is bounded away from 0, uniformly in everything but the dimension (since (i) the random walk is transient, and (ii) the probabilities that the random walk exits the box through any given site are comparable (see, *e.g.*, Lemma 1.7.4 in [16])). The conclusion follows, for once the walk has reached a site of $\mathscr{G}$, it hits s_i first with probability at least 1/2. □

2.3. **Lower bound on the height: proof of Theorem 1.2.**

Let $\alpha > 1$ and set $R = \lfloor \big((Kf(\lambda,\alpha\eta)\big)^{-1/d}\rfloor$, where K will be chosen (large enough) later (depending on α). Let $\mathscr{B}_R$ be the event that there are no pinned sites at distance less than R from the origin, $\mathscr{B}_R = \{\varphi_i > 0,\, \forall i \in B_R\}$. We first write

$$\langle\varphi_0\rangle^{+,\mathrm{per}}_{L;\lambda,\eta} \ge \langle\varphi_0 \,|\, \mathscr{B}_R\rangle^{+,\mathrm{per}}_{L;\lambda,\eta}\; \mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\mathscr{B}_R).$$

It follows from FKG inequality that pinning all the sites outside B_R only reduces the expectation. Therefore

$$\langle\varphi_0 \,|\, \mathscr{B}_R\rangle^{+,\mathrm{per}}_{L;\lambda,\eta} \ge \langle\varphi_0\rangle^{+,\mathrm{per}}_{2R+1;\lambda,0}.$$

Now, standard entropic repulsion estimate imply that

$$\lim_{\lambda\downarrow 0}\langle\varphi_0\rangle^{+,\mathrm{per}}_{2R+1;\lambda,0} = \langle\varphi_0\rangle^{+,\mathrm{per}}_{2R+1;0,0} \ge \begin{cases}\hat{c}_2\log R & (d=2),\\ \hat{c}_d\sqrt{\log R} & (d\ge 3),\end{cases}$$

for suitable constants $\hat{c}_d > 0$. Of course, $\log R \asymp |\log f(\lambda,\alpha\eta)|$, and therefore the result will be proved once we show that the probability of $\mathscr{B}_R$ is bounded away from zero uniformly in everything but the dimension (provided α is chosen large enough).

To see that, let us decompose the torus into a grid whose cells are disjoint translates of B_{2R}; we call these cells the *big boxes*. Let us place a translate of B_R at the middle of each big box; we call them *small boxes*. On the event $\mathscr{E} = \{$at least one half of the big boxes are entirely free of pinned sites$\}$, at least $2^{-d}\frac{1}{2}|\mathbb{T}^d_L|$ sites of the torus are at a distance at least R from all pinned sites (consider the sites in the small boxes at the center of the big boxes which are entirely free of pinned sites). By translation invariance, this implies that

$$\mu^{+,\mathrm{per}}_{L;\lambda,\eta}(\mathscr{B}_R \,|\, \mathscr{E}) \ge 2^{-d-1}.$$

Let us bound the probability of $\mathscr{E}^{\mathrm{c}}$ from above. On $\mathscr{E}^{\mathrm{c}}$, the density of pinned sites in the torus is at least $\frac{1}{2}|\mathbb{T}^d_L|/|B_{2R}| \ge 2^{-d-1}Kf(\lambda,\alpha\eta)^{-1}$. Let us denote by $\mathscr{A}$ the set of pinned sites and introduce the density of pinned sites,

$$\rho^{\mathrm{per}}_L(\lambda,\eta) = |\mathbb{T}^d_L|^{-1}\langle|\mathscr{A}|\rangle^{+,\mathrm{per}}_{L;\lambda,\eta}.$$

This density can easily be bounded above (just use the standard integration-differentiation trick, or see [7]):

$$f^{+,\mathrm{per}}_L(\lambda,\alpha\eta) = \int_0^{\alpha\eta}\frac{1}{t}\rho^{\mathrm{per}}_L(\lambda,t)\,\mathrm{d}t \ge \int_\eta^{\alpha\eta}\frac{1}{t}\rho^{\mathrm{per}}_L(\lambda,t)\,\mathrm{d}t \ge \log(\alpha)\,\rho^{\mathrm{per}}_L(\lambda,\eta),$$

where we used the fact that $\rho_L^{\rm per}(\lambda,\eta)$ is an increasing function of η. We thus have that

$$\mu^{+,\rm per}_{L;\lambda,\eta}(\mathscr{E}^{\rm c}) \leq \mu^{+,\rm per}_{L;\lambda,\eta}\big(|\mathscr{A}| > 2^{-d-1}K\log\alpha\rho_L^{\rm per}(\lambda,\eta)\big) \leq \frac{1}{2},$$

provided that $K > 2^{d+2}/\log\alpha$.

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