The spectrum of Schrödinger operators with random δ magnetic fields

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Abstract. We shall consider the Schrödinger operators on $\mathbf{R}^2$ with the magnetic field given by a nonnegative constant field plus random δ magnetic fields of the Anderson type or of the Poisson-Anderson type. We shall investigate the spectrum of these operators by the method of the admissible potentials by Kirsch–Martinelli [15]. Moreover, we shall prove the lower Landau levels are infinitely degenerated eigenvalues when the constant field is sufficiently large, by estimating the growth order of the eigenfunctions using the entire function theory by Levin [17].

1 Introduction

The δ magnetic fields are sometimes called the Aharonov-Bohm fields, after the celebrated paper by Aharonov–Bohm [1]. There are numerous works which study the Aharonov-Bohm fields; see e.g. Ruijsenaars [25], Helffer [14] or Nambu [21]. Especially, Geyler–Grishanov [12] and Geyler–Šťovíček [13] studied the infinite degeneracy of the zero modes of the 2-dimensional Pauli operator with δ magnetic fields; Rozenblum–Shirokov [24] also studied the same subject in the case the magnetic field is a signed Borel measure. One of the authors [19] studied the structure of the whole spectrum of the Schrödinger operators with a constant magnetic field plus δ magnetic fields, in the case the number of δ fields is finite, or in the case δ fields are well-separated; the authors [20] also studied the same subject in the case δ fields vary periodically. In the present paper, we consider the case there is some randomness in the positions of δ magnetic fields or in their intensities, and study some fundamental spectral properties of the Schrödinger operators with random δ magnetic fields.

Define a differential operator $\mathcal{L}_\omega$ on $\mathbf{R}^2$ by

$$\mathcal{L}_\omega = \left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right)^2,$$

where ω is an element of a probability space Ω, and $\mathbf{a}_\omega$ is the magnetic vector potential. The magnetic field corresponding to a vector potential $\mathbf{a} = (a_x, a_y)$ is defined by $\operatorname{curl}\mathbf{a} = \partial_x a_y - \partial_y a_x$ in the distribution sense. We assume the magnetic field $\operatorname{curl}\mathbf{a}_\omega$ is given by

$$\operatorname{curl}\mathbf{a}_\omega(z) = B + \sum_{\gamma\in\Gamma_\omega} 2\pi\alpha_\gamma(\omega)\delta(z-\gamma), \tag{1}$$

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where B is a nonnegative constant, Γ_ω is a discrete subset of $\mathbf{R}^2$, $\alpha_\gamma(\omega)$ is a constant belonging to $[0,1)$, and δ is the Dirac δ measure concentrated on the origin. The assumption $\alpha_\gamma(\omega) \in [0,1)$ loses no generality, since the integral differences of $\alpha_\gamma(\omega)$'s can be gauged away; see [13, section 6].

We shall work on the following two cases in the present paper:

(i) **The Anderson type random δ magnetic fields.** The set Γ_ω is a lattice Γ of rank 2 independent of ω, that is, there exist linearly independent vectors e_1, e_2 such that $\Gamma_\omega = \Gamma = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$. The random variables $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ are independently, identically distributed (abbr. i.i.d.). We denote the common distribution measure for $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ by $\mu = \mathbf{P}\circ\alpha_\gamma^{-1}$ (independent of γ; $\mathbf{P}$ is the probability measure on Ω). We assume

$$\operatorname{supp}\mu \neq \{0\}, \tag{2}$$

since the case $\operatorname{supp}\mu = \{0\}$ is the trivial case. We denote

$$\bar\alpha = \mathbf{E}[\alpha_\gamma], \quad p = \mathbf{P}\{\alpha_\gamma \neq 0\}, \tag{3}$$

where $\mathbf{E}[X]$ denotes the expectation of the random variable X. The values $\bar\alpha$ and p are independent of γ, since $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ are i.i.d.

(ii) **The Poisson-Anderson type random δ magnetic fields.** The set Γ_ω is the Poisson configuration (the support of the Poisson point process) with intensity measure $\rho dxdy$, where ρ is a positive constant (for the definition of the Poisson point process, see e.g. Reiss [23] or Ando–Iwatsuka–Kaminaga–Nakano [3]). The random variables $\{\alpha_\gamma\}_{\gamma\in\Gamma_\omega}$ are i.i.d. with common distribution measure μ satisfying (2), which are independent of the Poisson configuration Γ_ω [3]. We use the same notation as (3).

It is known that there exists a vector potential $\mathbf{a}_\omega$ satisfying (1) (see [13, section 4]; see also section 2 below). We denote

$$L_\omega u = \mathcal{L}_\omega u, \quad D(L_\omega) = C_0^\infty(\mathbf{R}^2 \setminus \Gamma_\omega),$$

where $C_0^\infty(U)$ denotes the space of the compactly supported smooth functions in U, and $D(X)$ the operator domain of the operator X. We denote the Friedrichs extension of L_ω by H_ω. By a usual ergodicity argument, we can prove that there exists a closed set Σ in $\mathbf{R}$ independent of ω such that

$$\sigma(H_\omega) = \Sigma \tag{4}$$

almost surely (see also Proposition 3.3 below). Moreover, the inequality $H_\omega \geq B$ (see [19, Proposition 3.3 (iii)]) implies

$$\Sigma \subset [B,\infty). \tag{5}$$

[3] More precisely, we construct the random variables $\{\alpha_\gamma\}_{\gamma\in\Gamma_\omega}$ as follows. Let Ω_1 be the probability space on which the Poisson configuration Γ_ω is defined, and number the elements $\{\gamma_j\}_{j=1}^\infty$ of Γ_ω as $0 < |\gamma_1| < |\gamma_2| < \cdots$ (the probability that there exist two points of Γ_ω with the same absolute value is zero). Let Ω_2 be the probability space on which i.i.d. random variables $\{\alpha_j\}_{j=1}^\infty$ are defined. Put $\Omega = \Omega_1 \times \Omega_2$, and denote $\alpha_{\gamma_j}(\omega) = \alpha_j(\omega)$ $(j = 1, 2, \ldots)$.

We denote the free operator by H_0, that is, H_0 is the Friedrichs extension of the operator L_0 given by

$$L_0 = \left(\frac{1}{i}\nabla + \mathbf{a}_0\right)^2, \ \mathbf{a}_0 = \left(-\frac{By}{2}, \frac{Bx}{2}\right), \ D(L_0) = C_0^\infty(\mathbf{R}^2).$$

The spectrum of H_0 is well-known:

$$\sigma(H_0) = \begin{cases} [0,\infty) & (B=0), \\ \bigcup_{n=1}^\infty \{E_n\} & (B>0), \end{cases}$$

where $E_n = (2n-1)B$ is called the n-th Landau level. When $B > 0$, all the Landau levels are infinitely degenerated eigenvalues of H_0.

First we exhibit our result for the Anderson type. In the sequel, we denote

$$\mathrm{mult}(\lambda; H) = \dim \mathrm{Ker}(H - \lambda)$$

for a self-adjoint operator H.

Theorem 1.1 *Let $\mathbf{a}_\omega$ be the Anderson type. Then, we have the following:*

(i) Assume

$$\operatorname{supp}\mu \cap (\{0\} \cup \{1\}) \neq \emptyset. \tag{6}$$

Then, we have $\Sigma \supset \sigma(H_0)$. In particular, if $B = 0$ and (6) holds, then $\Sigma = [0,\infty)$.

(ii) Assume

$$\operatorname{supp}\mu \cap (\{0\} \cup \{1\}) = \emptyset \tag{7}$$

and $B = 0$. Then, we have $\inf \Sigma > 0$.

(iii) For $n \in \mathbf{N} = \{1, 2, \ldots\}$, we have

$$\mathrm{mult}(E_n; H_\omega) = \infty \qquad \text{if } \frac{B|\mathcal{D}|}{2\pi} + \bar{\alpha} > np,$$
$$\mathrm{mult}(E_1; H_\omega) = 0 \qquad \text{if } \frac{B|\mathcal{D}|}{2\pi} + \bar{\alpha} < p$$

almost surely, where $\mathcal{D}$ is the fundamental domain of Γ given by

$$\mathcal{D} = \left\{ se_1 + te_2 \mid -\frac{1}{2} \leq s < \frac{1}{2}, \ -\frac{1}{2} \leq t < \frac{1}{2} \right\},$$

and $|\mathcal{D}|$ is the area of $\mathcal{D}$.

(iv) Assume (7) and $B > 0$. Put $R = \min_{\gamma\in\Gamma,\gamma\neq 0} |\gamma|$. Put $\alpha_- = \inf \operatorname{supp}\mu$ and $\alpha_+ = \sup \operatorname{supp}\mu$ ($0 < \alpha_- \leq \alpha_+ < 1$ by (7)). Then, for any $n_0 \in \mathbf{N}$, there exist constants $C > 0$ and $c > 0$ dependent only on n_0, α_-, α_+ satisfying

the following: if $BR^2 \geq C$, then the first n_0 Landau levels $E_1, \ldots, E_{n_0}$ are the isolated, infinitely degenerated eigenvalues of H_ω almost surely, and

$$\Sigma \cap [B, E_{n_0+1}) = \bigcup_{n=1}^{n_0} \{E_n\} \cup S_n,$$

where S_n is a closed subset of $\mathbf{R}$ satisfying

$$S_n \subset \bigcup_{\alpha \in \operatorname{supp} \mu} [E_n + (2\alpha - e^{-cBR^2})B, E_n + (2\alpha + e^{-cBR^2})B].$$

The assertion (iii) is an extension of [20, Theorem 1.1], which is in the periodic case. The assertions (iii) and (iv) roughly mean the lower Landau levels tend to be stable under the perturbation by δ magnetic fields, even if it is random. Similar results are obtained in the case of (scalar) point interactions by Geĭler [11], Avishai–Redheffer–Band [4], Avishai–Redheffer [5], Avishai–Azbel–Gredeskul [6] and Dorlas–Macris–Pulé [8], or in the case of δ magnetic fields [12, 13, 24, 19]. It may be interesting to compare the above results with those in the case of regular potentials (see Zak [26], Dinaburg–Sinai–Soshnikov [9]); in that case, it is widely believed that the Landau levels are broadened and there exist some extended states corresponding to the center of the Landau level.

Next we shall exhibit the result for the Poisson-Anderson case. In the sequel, $[x]$ denotes the integer part of a real number x (the maximal integer which does not exceed x), and $\operatorname{frac}(x)$ denotes the fractional part of x (i.e. $\operatorname{frac}(x) = x - [x]$).

Theorem 1.2 *Let $\mathbf{a}_\omega$ be the Poisson-Anderson type. Then, we have the following:*

(i) $\Sigma \supset \sigma(H_0)$. *In particular, if $B = 0$, then $\Sigma = [0, \infty)$.*

(ii) Assume $B > 0$. Put

$$F = \{\operatorname{frac}(\alpha_1 + \cdots + \alpha_m) \mid \alpha_1, \ldots, \alpha_m \in \operatorname{supp} \mu,\ m \in \mathbf{N}\}. \tag{8}$$

Then, we have

$$\Sigma \supset \bigcup_{n=1}^{\infty} (E_n + 2BF), \tag{9}$$

where $E_n + 2BF = \{E_n + 2B\alpha \mid \alpha \in F\}$. In particular, if F is dense in $[0, 1)$, then we have

$$\Sigma = [B, \infty). \tag{10}$$

(iii) We have

$$\operatorname{mult}(E_n; H_\omega) = \infty \qquad \textit{if } \frac{B}{2\pi\rho} + \bar{\alpha} > np,$$
$$\operatorname{mult}(E_1; H_\omega) = 0 \qquad \textit{if } \frac{B}{2\pi\rho} + \bar{\alpha} < p$$

almost surely.

The assumption 'F is dense in $[0,1)$' is satisfied if $\operatorname{supp}\mu$ contains an irrational number or $\operatorname{supp}\mu$ is an infinite set. So the equation (10) tells $\sigma(H_\omega)$ generically fills the whole possible energy range $[B,\infty)$; similar results are found in the case of the Schrödinger operators with the Poisson-Anderson type random scalar potentials [3]. This phenomenon is caused by the maximal randomness of the Poisson point process; for any distribution of finite points, we can find an infinite number of its imitations in the Poisson configuration, so the system contains all the energy levels of all the finite point systems (see Proposition 3.3 below). However, we could not prove (10) when $\operatorname{supp}\mu$ is a finite set of rationals. The difficulty is the treatment of the perturbation of the singular magnetic fields; we could only prove the strong resolvent continuity when the intensities of δ fields and their positions are perturbed (see section 7). The assertion (iii) corresponds to (iii) of Theorem 1.1, since $1/\rho$ is the area of 'the fundamental domain' of the Poisson configuration with intensity $\rho dxdy$ (i.e. $\mathbf{E}[\#(\Gamma.\cap\mathcal{D})]=1$ if $|\mathcal{D}|=1/\rho$).

The rest of the present paper is organized as follows. In section 2, we shall prepare some notations used in the present paper. In section 3, we shall define the admissible potentials for our models. In section 4, we shall construct explicit eigenfunctions for Landau levels. In section 5, we shall prove Theorem 1.1. In section 6, we shall prove Theorem 1.2. In section 7, we shall prove the strong resolvent continuity of our operators when the intensities of δ fields and their positions are perturbed. In section 8, we shall outline a proof of a generalization of the entire function theory by Levin [17], and give a characterization of the domain of the Friedrichs extension.

2 Preliminaries

In the sequel, we identify a vector $z=(x,y)\in\mathbf{R}^2$ with a complex number $z=x+iy\in\mathbf{C}$. So $L^2(\mathbf{R}^2)=L^2(\mathbf{R}^2;dxdy)$ is identified with $L^2(\mathbf{C})=L^2(\mathbf{C};dxdy)$, etc.

For a nonnegative constant B and a meromorphic function ψ on $\mathbf{C}$ having at most 1-st order poles and real residues, put

$$\phi(z)=\frac{B\bar{z}}{2}+\psi(z). \tag{11}$$

We denote

$$\mathcal{L}_\phi=\left(\frac{1}{i}\nabla+\mathbf{a}_\phi\right)^2,$$

where

$$\mathbf{a}_\phi(z)=(\operatorname{Im}\phi(z),\operatorname{Re}\phi(z)).$$

Let Γ be the set of the (1-st order) poles of ψ. Let α_γ be the (real) residue of ψ at $z=\gamma$, and put $\alpha=(\alpha_\gamma)_{\gamma\in\Gamma}$. Then we have

$$\operatorname{curl}\mathbf{a}_\phi(z)=B+\sum_{\gamma\in\Gamma}2\pi\alpha_\gamma\delta(z-\gamma)$$

(see [13, section 4]). Define a linear operator L_ϕ by

$$L_\phi u = \mathcal{L}_\phi u, \quad D(L_\phi) = C_0^\infty(\mathbf{C} \setminus \Gamma).$$

We denote the Friedrichs extension of L_ϕ by H_ϕ.

For $r > 0$ and $z \in \mathbf{C}$, we denote

$$B_r(z) = \{w \in \mathbf{C} \mid |w - z| \le r\}.$$

When $\#(\Gamma \cap B_r(0))$ grows not extremely fast, we can prove

$$\begin{aligned} D(H_\phi) &= \{u \in L^2(\mathbf{C}) \mid \mathcal{L}u \in L^2(\mathbf{C}), \\ &\quad \limsup_{z \to \gamma} |u(z)| < \infty \text{ for any } \gamma \in \Gamma\} \end{aligned} \tag{12}$$

(see Proposition 8.7).

Define

$$D_0(H_\phi) = \{u \in D(H_\phi) \mid \operatorname{supp} u \text{ is bounded}\}.$$

We see that $D_0(H_\phi)$ is an operator core for H_ϕ by cut-off argument.

Define differential operators $\mathcal{A}_\phi$ and $\mathcal{A}_\phi^\dagger$ by

$$\mathcal{A}_\phi = 2\partial_z + \phi(z), \quad \mathcal{A}_\phi^\dagger = -2\partial_{\bar{z}} + \overline{\phi(z)}, \tag{13}$$

where $\partial z = (\partial_x - i\partial_y)/2$, $\partial \bar{z} = (\partial_x + i\partial_y)/2$. These operators satisfy the canonical commutation relation

$$\mathcal{L}_\phi = \mathcal{A}_\phi^\dagger \mathcal{A}_\phi + B = \mathcal{A}_\phi \mathcal{A}_\phi^\dagger - B$$

as an operator on $\mathcal{D}'(\mathbf{C} \setminus \Gamma)$.

Although the results are independent of the choice of the gauge, we choose the following gauge in the sequel, to clarify the argument. Put

$$\phi_\omega(z) = \frac{B\bar{z}}{2} + \frac{\alpha_0(\omega)}{z} + \sum_{\gamma \in \Gamma_\omega \setminus \{0\}} \alpha_\gamma(\omega) \left(\frac{1}{z - \gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right), \tag{14}$$

where $\alpha_0(\omega) = 0$ if $0 \notin \Gamma_\omega$. We can verify the convergence of the right hand side of (14) both in the Anderson case and in the Poisson-Anderson case (see Proposition 4.1 and (ii) of Lemma (4.4)). We denote $H_\omega = H_{\phi_\omega}$.

3 Admissible potentials

The spectra of the Schrödinger operators with random potentials are determined by those with some special potentials, called the admissible potentials. The method of this type is introduced by Kirsch–Martinelli [15], and also used by [3] et al. In our cases, their method can be formulated as follows.

Definition 3.1 (Admissible sequences for the Anderson type fields) *Let Γ be the period lattice in the definition of the Anderson type δ magnetic fields. Let $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ be a $[0,1)$-valued sequence. We say α is periodic if there exists a rank 2-sublattice Γ' of Γ such that $\alpha_{\gamma+\gamma'} = \alpha_\gamma$ for every $\gamma \in \Gamma$, $\gamma' \in \Gamma'$. We say a sequence α is admissible for the Anderson type fields if α is a* $\mathrm{supp}\,\mu$*-valued periodic sequence. We denote the set of all the admissible sequences by $\mathcal{P}_A$.*

For a periodic sequence α, take a complete system of representatives $\{\gamma_1, \dots, \gamma_K\}$ of Γ/Γ' ($K = \#(\Gamma/\Gamma')$), and define

$$\phi_\alpha(z) = \frac{B\bar{z}}{2} + \sum_{k=1}^{K} \alpha_{\gamma_k}\zeta_{\Gamma'}(z-\gamma_k),$$

where $\zeta_{\Gamma'}$ is the Weierstrass ζ-function corresponding to the lattice Γ', that is,

$$\zeta_{\Gamma'}(z) = \frac{1}{z} + \sum_{\gamma'\in\Gamma'\setminus\{0\}} \left(\frac{1}{z-\gamma'} + \frac{1}{\gamma'} + \frac{z}{\gamma'^2} \right).$$

We denote $H_\alpha = H_{\phi_\alpha}$.

Definition 3.2 (Admissible pairs for the Poisson-Anderson type fields) *We say a pair (Γ, α) is admissible for the Poisson-Anderson type fields if Γ is a finite subset of* $\mathbf{C}$ *(maybe the empty set) and $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ is a* $\mathrm{supp}\,\mu$*-valued sequence. We denote the set of all the admissible pairs by $\mathcal{F}_A$. For an admissible pair (Γ, α), we define*

$$\phi_{\Gamma,\alpha}(z) = \frac{B\bar{z}}{2} + \sum_{\gamma\in\Gamma} \frac{\alpha_\gamma}{z-\gamma}.$$

We denote $H_{\Gamma,\alpha} = H_{\phi_{\Gamma,\alpha}}$.

Proposition 3.3 *(i) Let $\mathbf{a}_\omega$ be the Anderson type. Then, we have*

$$\sigma(H_\omega) = \sigma_{\mathrm{ess}}(H_\omega) = \overline{\bigcup_{\alpha\in\mathcal{P}_A} \sigma(H_\alpha)}$$

almost surely.

(ii) Let $\mathbf{a}_\omega$ be the Poisson-Anderson type. Then, we have

$$\sigma(H_\omega) = \sigma_{\mathrm{ess}}(H_\omega) = \overline{\bigcup_{(\Gamma,\alpha)\in\mathcal{F}_A} \sigma(H_{\Gamma,\alpha})}$$

almost surely.

Though the proof is similar to those of known results [15, 3], we shall give it here to show the singularity of $\mathbf{a}_\omega$ does not violate the argument.

Proof. We prove only the assertion (ii). The proof of (i) is similar.

Take a countable dense subset X of $\mathbf{C}$, a countable dense subset S of $\operatorname{supp}\mu$ and put

$$\widetilde{\mathcal{F}_A} = \{(\Gamma, \alpha) \in \mathcal{F}_A \mid \gamma \in X \text{ and } \alpha_\gamma \in S \text{ for every } \gamma \in \Gamma\}.$$

Notice that the set $\widetilde{\mathcal{F}_A}$ is countable. We put

$$\Sigma = \overline{\bigcup_{(\Gamma,\alpha)\in\mathcal{F}_A} \sigma(H_{\Gamma,\alpha})}, \quad \widetilde{\Sigma} = \overline{\bigcup_{(\Gamma,\alpha)\in\widetilde{\mathcal{F}_A}} \sigma(H_{\Gamma,\alpha})}.$$

We shall divide the proof into 4 steps.

Step 1. $\sigma(H_\omega) \supset \sigma_{\mathrm{ess}}(H_\omega)$ clearly holds.

Step 2. $\Sigma \supset \sigma(H_\omega)$.

Proof. Let $\lambda \in \sigma(H_\omega)$. Then, for any $\epsilon > 0$, we can take an ϵ-approximating normalized eigenfunction u_ϵ of H_ω for λ (i.e. $\|u_\epsilon\| = 1$, $\|(H_\omega - \lambda)u_\epsilon\| \le \epsilon$) from $D_0(H_\omega)$. Then, using a gauge transform in an open neighborhood of $\operatorname{supp} u_\epsilon$, we can construct an ϵ-approximating normalized eigenfunction of $H_{\Gamma,\alpha}$ for λ for some $(\Gamma, \alpha) \in \mathcal{F}_A$. This implies $\operatorname{dist}(\lambda, \Sigma) \le \epsilon$, so the conclusion holds. $\square$

Step 3. $\widetilde{\Sigma} \supset \Sigma$ immediately follows from Corollary 7.5.

Step 4. $\sigma_{\mathrm{ess}}(H_\omega) \supset \widetilde{\Sigma}$ almost surely.

Proof. Since $\widetilde{\mathcal{F}_A}$ is countable and $\sigma_{\mathrm{ess}}(H_\omega)$ is closed, it suffices to show

$$\sigma_{\mathrm{ess}}(H_\omega) \supset \sigma(H_{\Gamma,\alpha})$$

almost surely, for every $(\Gamma, \alpha) \in \widetilde{\mathcal{F}_A}$. Moreover, since $\sigma_{\mathrm{ess}}(H_\omega)$ is closed, it suffices to show that, if

$$(r, s) \cap \sigma(H_{\Gamma,\alpha}) \neq \emptyset, \quad r, s \in \mathbf{Q}, \quad r < s, \quad (\Gamma, \alpha) \in \widetilde{\mathcal{F}_A}, \tag{15}$$

then we have

$$(r, s) \cap \sigma_{\mathrm{ess}}(H_\omega) \neq \emptyset \tag{16}$$

almost surely. Take $r, s, (\Gamma, \alpha)$ satisfying (15) and take $\lambda \in (r, s) \cap \sigma(H_{\Gamma,\alpha})$. Take $\epsilon > 0$ so that $(\lambda - 2\epsilon, \lambda + 2\epsilon) \subset (r, s)$. In the sequel, we use the notation in section 7.3 below. Take a bounded open set $O \supset \Gamma$, and let the subspaces $\{D_{\Gamma',\alpha'}\}$ and the operators $\{T_{\Gamma',\alpha'}\}$ as in Lemma 7.3. Take an ϵ-approximating normalized eigenfunction u of $H_{\Gamma,\alpha}$ for λ from $D_{\Gamma,\alpha}$, and put $u_{\Gamma',\alpha'} = T_{\Gamma',\alpha'}u/\|T_{\Gamma',\alpha'}u\|$. Take a bounded open set $O' \supset O \cup \operatorname{supp} u$. By Lemma 7.3, there exists a constant $\delta > 0$ such that

$$\|(H_{\Gamma',\alpha'} - \lambda)u_{\Gamma',\alpha'}\| \le \epsilon, \quad \operatorname{supp} u_{\Gamma',\alpha'} \subset O' \tag{17}$$

for any $(\Gamma', \alpha') \in \mathcal{F}$ with $d((\Gamma', \alpha'), (\Gamma, \alpha)) \le \delta$.

Take a sequence $\{z_n\}_{n=1}^\infty \subset \mathbf{C}$ such that $\{O' + z_n\}_{n=1}^\infty$ are disjoint. Put $K = \#\Gamma$, $\Gamma = (\gamma_k)_{k=1}^K$ and $\alpha = (\alpha_k)_{k=1}^K$. For $n \in \mathbf{N}$, consider the event A_n which consists of all ω satisfying

$$\Gamma_\omega \cap B_{\delta/\sqrt{2K}}(\gamma_k + z_n) = \{\gamma_{kn}(\omega)\} \quad \text{(1 point set)}, \tag{18}$$

$$|\alpha_{\gamma_{kn}(\omega)}(\omega) - \alpha_k| \le \frac{\delta}{\sqrt{2K}} \tag{19}$$

for $k = 1, \ldots, K$, and

$$\Gamma_\omega \cap \left((O' + z_n) \setminus \bigcup_{k=1}^K B_{\delta/\sqrt{2K}}(\gamma_k + z_n) \right) = \emptyset. \tag{20}$$

The events $\{A_n\}_{n\in\mathbf{N}}$ are independent and have the same positive probability. Thus, for almost sure ω, we can take a subsequence $\{n_l\}_{l=1}^\infty$ such that (18), (19) and (20) hold for $n = n_l$. By (17), (18), (19), (20) and a gauge transform, we can construct a sequence $\{v_l\}_{l=1}^\infty \subset D_0(H_\omega)$ satisfying

$$\|(H_\omega - \lambda)v_l\| \le \epsilon, \quad \|v_l\| = 1$$

and $\operatorname{supp} v_l \subset O' + z_{n_l}$, almost surely. This implies $\operatorname{dist}(\lambda, \sigma_{\rm ess}(H_\omega)) \le \epsilon$, so (16) holds. □

Thus the proof of (ii) of Proposition 3.3 is completed. □

4 Eigenfunctions for Landau levels

In this section, we assume $B > 0$ and construct eigenfunctions for Landau Levels. Similar solutions are found in [12, 13, 24, 20].

4.1 Multi-valued canonical product

There is a beautiful theory by B. Ja. Levin about the relation between the growth order of the canonical product and the distribution of its zeros [17]. His theory also holds for the multi-valued canonical product, with the modification as follows.

Let Γ be a discrete subset of $\mathbf{C}$ and $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ be a sequence of non-negative real numbers. For $r > 0$ and $\theta_1, \theta_2 \in \mathbf{R}$ with $0 \le \theta_2 - \theta_1 \le 2\pi$, put

$$n(r, \theta_1, \theta_2) = \sum_{0<|\gamma|\le r, \theta_1 \le \arg\gamma < \theta_2} \alpha_\gamma \tag{21}$$

(we abbreviate '$\gamma \in \Gamma$' in the sum, as in the sequel). Put $n(r) = n(r, 0, 2\pi)$. We assume

$$n(r) = O(r^2) \quad \text{as } r \to \infty. \tag{22}$$

Define a sum $\zeta_{\Gamma,\alpha}$ and a product $\sigma_{\Gamma,\alpha}$ by

$$\zeta_{\Gamma,\alpha}(z) = \frac{\alpha_0}{z} + \sum_{\gamma\neq 0} \alpha_\gamma \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right), \tag{23}$$

$$\sigma_{\Gamma,\alpha}(z) = z^{\alpha_0} \prod_{\gamma\neq 0} \left(1 - \frac{z}{\gamma}\right)^{\alpha_\gamma} e^{\alpha_\gamma \left(\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}\right)} \tag{24}$$

(we put $\alpha_0 = 0$ when $0 \notin \Gamma$). When Γ is a lattice of rank 2 and $\alpha_\gamma \equiv 1$, then $\zeta_{\Gamma,\alpha}$ is the Weierstrass ζ function, and $\sigma_{\Gamma,\alpha}$ is the Weierstrass σ function.

Let $\mathcal{C} = \{C_j\}_{j=1}^{\infty}$ be a system of disks, where $C_j = B_{r_j}(z_j)$. We say $\mathcal{C}$ has the upper linear density $\bar{\rho}^*(\mathcal{C})$ if

$$\bar{\rho}^*(\mathcal{C}) = \limsup_{r\to\infty} \frac{1}{r} \sum_{|z_j|\leq r} r_j.$$

We say $\mathcal{C}$ is a C^0-set if $\bar{\rho}^*(\mathcal{C}) = 0$. We often identify $\mathcal{C}$ with the union set of the disks belonging to $\mathcal{C}$.

Proposition 4.1 *Assume (22) holds. Then the following holds:*

(i) The sum (23) converges uniformly in a compact subset of $\mathbf{C} \setminus \Gamma$. If we take the branches of the functions $\{(1 - \frac{z}{\gamma})^{\alpha_\gamma}\}_{\gamma\in\Gamma\setminus\{0\}}$ appropriately, then the right hand side of (24) converges uniformly in a simply connected compact subset of $\mathbf{C} \setminus \Gamma$. For $k = 0, 1, 2, \ldots$, the function $|\left(\frac{d}{dz}\right)^k \sigma_{\Gamma,\alpha}(z)|$ is independent of the choice of the branches. Moreover, we have

$$\frac{d}{dz}\sigma_{\Gamma,\alpha}(z) = \sigma_{\Gamma,\alpha}(z)\zeta_{\Gamma,\alpha}(z). \tag{25}$$

(ii) Assume additionally that

(a) there exists $I_0 \subset [0, 2\pi)$ such that $[0, 2\pi) \setminus I_0$ is countable and the limit

$$\Delta(\theta_1, \theta_2) = \lim_{r\to\infty} \frac{n(r, \theta_1, \theta_2)}{r^2} \tag{26}$$

exists for any $\theta_1, \theta_2 \in I_0 + 2\pi\mathbf{Z}$ with $0 \leq \theta_2 - \theta_1 \leq 2\pi$, and

(b) the limit

$$\delta_{\Gamma,\alpha} = \frac{1}{2} \lim_{r\to\infty} \sum_{0<|\gamma|\leq r} \frac{\alpha_\gamma}{\gamma^2} \tag{27}$$

exists and finite.

Let $d\Delta$ be the Lebesgue-Stieltjes measure given by the relation $\int_{[\theta_1,\theta_2)} d\Delta(\psi) = \Delta(\theta_1, \theta_2)$. Then, there exists a C^0-set $\mathcal{C}$ such that

$$\lim_{r\to\infty, re^{i\theta}\notin\mathcal{C}} \frac{\log|\sigma_{\Gamma,\alpha}(re^{i\theta})|}{r^2} = H(\theta), \tag{28}$$

where the function $H(\theta)$ is defined by the Stieltjes integral

$$H(\theta) = -\int_{\theta-2\pi}^{\theta} (\psi - \theta)\sin 2(\psi - \theta)\ d\Delta(\psi) + \mathrm{Re}(e^{2i\theta}\delta_{\Gamma,\alpha}).$$

The convergence (28) is uniform with respect to $\theta \in [0, 2\pi)$.

Remark. There is a misprint in the first edition of [17]; there must be the minus sign before the integral in [17, (2.06)].

The second assertion of the above lemma is a generalization of [17, Theorem 2 in Chap. II, Sec. 1], and its proof is also similar. We shall outline a proof in the appendix.

Corollary 4.2 *In addition to the assumption of (ii) of Proposition 4.1, assume that*

$$\Delta(\theta_1, \theta_2) = c(\theta_2 - \theta_1)$$

for some positive constant c. Put

$$\widetilde{\sigma}_{\Gamma,\alpha}(z) = e^{-\delta_{\Gamma,\alpha} z^2} \sigma_{\Gamma,\alpha}(z).$$

Then, there exists some C^0-set $\mathcal{C}$ satisfying the following; for any $\epsilon > 0$, we have

$$|\widetilde{\sigma}_{\Gamma,\alpha}(z)| \leq e^{(c\pi+\epsilon)|z|^2} \tag{29}$$

for sufficiently large z, and

$$|\widetilde{\sigma}_{\Gamma,\alpha}(z)| \geq e^{(c\pi-\epsilon)|z|^2} \tag{30}$$

for sufficiently large z outside $\mathcal{C}$.

Proof. By Proposition 4.1 and the equality

$$-c\int_{\theta-2\pi}^{\theta} (\psi - \theta)\sin 2(\psi - \theta) d\psi = c\pi,$$

we see that there exists some C^0-set $\mathcal{C}$ such that both (29) and (30) hold for sufficiently large z outside $\mathcal{C}$. Since $\mathcal{C}$ is a C^0-set, the limitation $z \in \mathbf{C}\backslash\mathcal{C}$ on (29) can be eliminated by using the maximum modulus principle (see the argument after the proof of [17, Lemma 5 in Chap. II, Sec. 3]). $\square$

For an entire function f, it is well-known that f and its derivatives $\frac{d^k f}{dz^k}$ have the same exponential growth order (see [17, Chap 1., Sec. 2]). For a multi-valued holomorphic function f, we have the following.

Lemma 4.3 *Let f be a multi-valued holomorphic function on $\mathbf{C}$ and n_0 a nonnegative integer. Let Γ be the set of the branch points of f. Assume the following conditions hold:*

(i) In a neighborhood U_γ of each $\gamma \in \Gamma$, f is written as

$$f(z) = (z-\gamma)^{\alpha_\gamma} g_\gamma(z),$$

where $\alpha_\gamma > n_0$ and g_γ is a function holomorphic in U_γ.

(ii) $\#\{\gamma \in \Gamma \mid |\gamma| \leq r\} = O(r^2)$ as $r \to \infty$.

(iii) There exists a constant $a > 0$ such that

$$|f(z)| \leq e^{a|z|^2}$$

for sufficiently large z.

Then, for any $\epsilon > 0$, we have for any $k = 0, 1, \ldots, n_0$

$$\left| \frac{d^k f}{dz^k}(z) \right| \leq e^{(a+\epsilon)|z|^2} \tag{31}$$

for sufficiently large $z \in \mathbf{C} \setminus \Gamma$.

Remark. By (i), the function $|\frac{d^k f}{dz^k}(z)|$ is single-valued.

Proof. By (i), we have

$$\lim_{z \to \gamma} \left| \frac{d^k f}{dz^k}(z) \right| = 0$$

for $k = 0, \ldots, n_0$. Thus the function $M_k(r) = \max_{|z|=r} \left| \frac{d^k f}{dz^k}(z) \right|$ is monotone nondecreasing, by the maximum modulus principle. By (ii), we can take $A \in \mathbf{N}$ such that

$$\#\{\gamma \in \Gamma \mid |\gamma| \leq r\} \leq Ar^2 - 1.$$

Take $l \in \mathbf{N}$. Dividing the ring $\{l-1 < |z| \leq l\}$ into Al^2 subrings, we find a subring $\{r_l - \frac{1}{2Al^2} < |z| \leq r_l + \frac{1}{2Al^2}\}$ which contains no point of Γ. Then, for $|z| = r_l$, we have by the Cauchy integral formula

$$\frac{d^k f}{dz^k}(z) = \frac{k!}{2\pi i} \int_{|w-z| = \frac{1}{3Al^2}} \frac{f(w)}{(w-z)^{k+1}} dw.$$

Thus we have

$$M_k(l-1) \leq (3Al^2)^k k! M_0\,(l)\,.$$

Therefore (31) follows from the assumption (iii). $\Box$

4.2 Explicit solution

Let us construct the eigenfunctions for Landau levels for our random models and estimate them using the results in the previous subsection. Note that a similar argument is found in Chistyakov–Lyubarskii–Pastur [7] (they give an estimate for entire functions with randomly distributed zeros).

Put $\alpha(\omega) = (\alpha_\gamma(\omega))_{\gamma\in\Gamma_\omega}$, $\zeta_\omega = \zeta_{\Gamma_\omega,\alpha(\omega)}$ and $\sigma_\omega = \sigma_{\Gamma_\omega,\alpha(\omega)}$. Then, the operators $\mathcal{A}_\omega = \mathcal{A}_{\phi_\omega}$, $\mathcal{A}_\omega^\dagger = \mathcal{A}_{\phi_\omega}^\dagger$ defined in (13) are written as

$$\mathcal{A}_\omega = 2\partial_z + \frac{B\bar{z}}{2} + \zeta_\omega(z), \quad \mathcal{A}_\omega^\dagger = -2\partial_{\bar{z}} + \frac{Bz}{2} + \overline{\zeta_\omega(z)}. \tag{32}$$

Put

$$\widetilde{\alpha}_\gamma(\omega) = \begin{cases} 1 & (0 < \alpha_\gamma(\omega) < 1), \\ 0 & (\alpha_\gamma(\omega) = 0), \end{cases} \qquad \widetilde{\sigma}_\omega = \sigma_{\Gamma_\omega,\widetilde{\alpha}(\omega)},$$

where $\widetilde{\alpha}(\omega) = (\widetilde{\alpha}_\gamma(\omega))_{\gamma\in\Gamma_\omega}$. Notice that $\widetilde{\sigma}_\omega$ is an entire function.

Lemma 4.4 *Let $\mathbf{a}_\omega$ be the Anderson type or the Poisson-Anderson type, and n a positive integer.*

(i) Let f be an entire function. Put

$$u(z) = {\mathcal{A}_\omega^\dagger}^{\,n-1}\left(e^{-\frac{B}{4}|z|^2}|\sigma_\omega(z)|^{-1}\overline{\widetilde{\sigma}_\omega(z)^n f(z)}\right). \tag{33}$$

If $u \in L^2(\mathbf{C})$, then $u \in D(H_\omega)$ and $H_\omega u = E_n u$. Moreover, if $u \in D(H_\omega)$ satisfies $H_\omega u = Bu$, then there exists an entire function f such that (33) holds with $n = 1$.

(ii) For almost sure ω, the assumptions (a) and (b) in (ii) of Proposition 4.1 are satisfied with $\Gamma = \Gamma_\omega$, $\alpha = \beta(\omega) = (n\widetilde{\alpha}_\gamma(\omega) - \alpha_\gamma(\omega))_{\gamma\in\Gamma_\omega}$ and

$$\Delta(\theta_1,\theta_2) = \begin{cases} (\theta_2-\theta_1)(np-\bar{\alpha})/2|\mathcal{D}| & \textit{(Anderson type)}, \\ \rho(\theta_2-\theta_1)(np-\bar{\alpha})/2 & \textit{(Poisson-Anderson type)}. \end{cases}$$

(iii) Let $\omega \in \Omega$ satisfy the conclusion of (ii). Let $\delta_\omega = \delta_{\Gamma_\omega,\beta(\omega)}$ be the constant defined by (27) for $\Gamma = \Gamma_\omega$ and $\alpha = \beta(\omega)$. For a polynomial $g \not\equiv 0$, let $u_{n,g}$ be the function u defined by (33) with $f(z) = e^{-\delta_\omega z^2}g(z)$. Then, there exists a C^0-set $\mathcal{C}$ such that for any $\epsilon > 0$

$$\begin{cases} |u_{n,g}(z)| \le \exp\left(\left(-\frac{B}{4} + \frac{\pi(np-\bar{\alpha})}{2|\mathcal{D}|} + \epsilon\right)|z|^2\right) & \textit{(Anderson type)}, \\ |u_{n,g}(z)| \le \exp\left(\left(-\frac{B}{4} + \frac{\pi\rho(np-\bar{\alpha})}{2} + \epsilon\right)|z|^2\right) & \textit{(Poisson-Anderson type)} \end{cases} \tag{34}$$

for sufficiently large z, and

$$\begin{cases} |u_{1,g}(z)| \ge \exp\left(\left(-\frac{B}{4} + \frac{\pi(p-\bar{\alpha})}{2|\mathcal{D}|} - \epsilon\right)|z|^2\right) & \textit{(Anderson type)}, \\ |u_{1,g}(z)| \ge \exp\left(\left(-\frac{B}{4} + \frac{\pi\rho(p-\bar{\alpha})}{2} - \epsilon\right)|z|^2\right) & \textit{(Poisson-Anderson type)} \end{cases} \tag{35}$$

for sufficiently large z outside $\mathcal{C}$.

Proof. (i) The proof is almost the same as [20, Lemma 4.1]. The only difference is that the function u does not necessarily vanish at the point γ with $\alpha_\gamma(\omega) = 0$. So we change the σ^n in [20, (4.1)] into $\widetilde{\sigma}^n_\omega$.

(ii) We prove the statement only for the Poisson-Anderson type (the Anderson type can be treated similarly).

First we prove the assumption (a) is satisfied. For $N = m + ni \in \mathbf{Z} \oplus \mathbf{Z}i$, define a square Q_N by

$$Q_N = \left\{ s + ti \mid m - \frac{1}{2} \le s < m + \frac{1}{2},\ n - \frac{1}{2} \le t < n + \frac{1}{2} \right\}$$

and put

$$X_N(\omega) = \sum_{\gamma \in \Gamma_\omega \cap Q_N} \beta_\gamma(\omega).$$

Then the random variables $\{X_N\}_{N \in \mathbf{Z} \oplus \mathbf{Z}i}$ are independent and

$$\mathbf{E}[X_N] = \mathbf{E}_{\Omega_1}[\#(\Gamma_\cdot \cap Q_N)] \mathbf{E}_{\Omega_2}[n\widetilde{\alpha}_\gamma - \alpha_\gamma] = \rho(np - \bar{\alpha}),$$

where we used $\mathbf{E}[\#(\Gamma_\cdot \cap U)] = \rho|U|$ (for the probability spaces Ω_1 and Ω_2, see the footnote about the definition of the Poisson-Anderson fields). For $r > 0$ and $\theta_1, \theta_2 \in \mathbf{R}$ with $0 \le \theta_2 - \theta_1 \le 2\pi$, put

$$\begin{aligned} S(r, \theta_1, \theta_2) &= \{ s e^{i\theta} \mid 0 < s \le r,\ \theta_1 \le \theta < \theta_2 \}, \\ N(r, \theta_1, \theta_2) &= \{ N \in \mathbf{Z} \oplus \mathbf{Z}i \mid Q_N \subset S(r, \theta_1, \theta_2) \}, \\ \widetilde{n}(r, \theta_1, \theta_2) &= \sum_{N \in N(r, \theta_1, \theta_2)} X_N. \end{aligned}$$

Then we have

$$\frac{\widetilde{n}(r, \theta_1, \theta_2)}{r^2} = \frac{\sum_{N \in N(r, \theta_1, \theta_2)} X_N}{\# N(r, \theta_1, \theta_2)} \frac{\# N(r, \theta_1, \theta_2)}{r^2} \to \frac{\rho(np - \bar{\alpha})(\theta_2 - \theta_1)}{2}$$

almost surely, by the law of large numbers. Moreover, we readily have

$$\lim_{r \to \infty} \frac{\widetilde{n}(r, \theta_1, \theta_2) - n(r, \theta_1, \theta_2)}{r^2} = 0$$

almost surely. Thus we have

$$\lim_{r \to \infty} \frac{n(r, \theta_1, \theta_2)}{r^2} = \frac{\rho(np - \bar{\alpha})(\theta_2 - \theta_1)}{2} \tag{36}$$

almost surely, for each $\theta_1, \theta_2 \in \mathbf{Q}$ with $0 \le \theta_2 - \theta_1 \le 2\pi$. By the monotonicity of the function $n(r, \theta_1, \theta_2)$ with respect to θ_1 or θ_2, we see that (36) holds for every $\theta_1, \theta_2 \in \mathbf{R}$, almost surely.

Next we show the assumption (b) holds. Put

$$\delta(r) = \sum_{1 < |\gamma| \le r} \frac{\beta_\gamma}{\gamma^2}.$$

We shall prove $\delta(r)$ converges as $r \to \infty$, almost surely.

For $m = 1, 2, \dots$ and $k = 0, \dots, 4m-1$, put

$$U_{m,k} = \left\{ re^{i\theta} \;\middle|\; m^2 < r \le (m+1)^2, \frac{k\pi}{2m} \le \theta < \frac{(k+1)\pi}{2m} \right\},$$

$$c_{m,k} = m^2 e^{i\frac{k\pi}{2m}}, \quad \Gamma_{m,k} = \Gamma \cap U_{m,k}, \quad \delta_{m,k} = \sum_{\Gamma_{m,k}} \frac{\beta_\gamma}{\gamma^2}.$$

In the sequel, we denote the general constants independent of m, k, ω by C. For $\gamma \in U_{m,k}$, we have

$$\left| \frac{1}{\gamma^2} - \frac{1}{c_{m,k}^2} \right| = \left| \frac{(\gamma + c_{m,k})(\gamma - c_{m,k})}{\gamma^2 c_{m,k}^2} \right| \le C m^{-5}. \tag{37}$$

Put $\bar{\beta} = \mathbf{E}[\beta_\gamma] = np - \bar{\alpha}$. Then we have

$$\begin{aligned}
&|\delta_{m,k} + \delta_{m,k+m}| \\
\le\ & m^{-4} \left| \sum_{\Gamma_{m,k}} \beta_\gamma - \sum_{\Gamma_{m,k+m}} \beta_\gamma \right| + C m^{-5} \left(\#\Gamma_{m,k} + \#\Gamma_{m,k+m} \right) \\
\le\ & m^{-4} \left(\left| \sum_{\Gamma_{m,k}} (\beta_\gamma - \bar{\beta}) \right| + \left| \sum_{\Gamma_{m,k+m}} (\beta_\gamma - \bar{\beta}) \right| + |\#\Gamma_{m,k} - \#\Gamma_{m,k+m}| \bar{\beta} \right) \\
& + C m^{-5} \left(\#\Gamma_{m,k} + \#\Gamma_{m,k+m} \right),
\end{aligned} \tag{38}$$

where we used (37) and $c_{m,k+m}^2 = -c_{m,k}^2$ in the first inequality. By the Schwarz inequality and the independentness of $\{\beta_\gamma\}$, we have

$$\begin{aligned}
& \mathbf{E}\left[\left| \sum_{\Gamma_{m,k}} (\beta_\gamma - \bar{\beta}) \right| \right] = \mathbf{E}_{\Omega_1}\left[\mathbf{E}_{\Omega_2}\left[\left| \sum_{\Gamma_{m,k}} (\beta_\gamma - \bar{\beta}) \right| \right] \right] \\
\le\ & \mathbf{E}_{\Omega_1}\left[\left(\mathbf{V}_{\Omega_2}\left[\sum_{\Gamma_{m,k}} \beta_\gamma \right] \right)^{1/2} \right] = \mathbf{E}_{\Omega_1}\left[\left(\#\Gamma_{m,k} \mathbf{V}_{\Omega_2}[\beta_\gamma] \right)^{1/2} \right] \\
\le\ & \left(\mathbf{E}_{\Omega_1}[\#\Gamma_{m,k}] \right)^{1/2} \left(\mathbf{V}_{\Omega_2}[\beta_\gamma] \right)^{1/2} \le Cm,
\end{aligned} \tag{39}$$

where $V[X]$ denotes the variance of a random variable X. $\mathbf{E}\left[\left| \sum_{\Gamma_{m,k+m}} (\beta_\gamma - \bar{\beta}) \right| \right]$ is estimated in the same way. Moreover, we have

$$\begin{aligned}
\mathbf{E}[|\#\Gamma_{m,k} - \#\Gamma_{m,k+m}|] &\le 2\mathbf{E}[|\#\Gamma_{m,k} - \rho |U_{m,k}||] \\
&\le 2\mathbf{V}[\#\Gamma_{m,k}]^{1/2} \le Cm, \qquad (40) \\
\mathbf{E}[\#\Gamma_{m,k} + \#\Gamma_{m,k+m}] &= 2\rho |U_{m,k}| \le Cm^2, \qquad (41)
\end{aligned}$$

where we used $\mathbf{V}[\#\Gamma_{m,k}] = \rho|\Gamma_{m,k}| \leq Cm^2$. By (38), (39), (40) and (41), we have

$$\mathbf{E}\left[|\delta_{m,k} + \delta_{m,k+m}|\right] \leq Cm^{-3},$$

so

$$\sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \mathbf{E}\left[|\delta_{m,k} + \delta_{m,k+m} + \delta_{m,k+2m} + \delta_{m,k+3m}|\right] < \infty.$$

Therefore we conclude the sequence $\{\delta(m^2)\}_{m=1}^{\infty}$ converges almost surely.

Now it is sufficient to show that

$$\sup_{m^2 < r < (m+1)^2} |\delta(r) - \delta(m^2)| \to 0 \quad \text{as } m \to \infty \tag{42}$$

almost surely. As in the proof of (a), we can prove

$$\frac{\#\{\gamma \in \Gamma_\omega \mid m^2 < |\gamma| < (m+1)^2\}}{\pi(m+1)^4 - \pi m^4} \to \rho$$

almost surely. This implies

$$\#\{\gamma \in \Gamma_\omega \mid m^2 < |\gamma| < (m+1)^2\} \leq Cm^3$$

almost surely. Thus we have

$$|\delta(r) - \delta(m^2)| \leq \#\{\gamma \in \Gamma_\omega \mid m^2 < |\gamma| < (m+1)^2\} m^{-4} \leq Cm^{-1}$$

for $m^2 < r < (m+1)^2$, which implies (42).

(iii) By (25) and (32), we have

$$\mathcal{A}_\omega^\dagger = \operatorname{sgn}\sigma_\omega(z)^{-1}\left(-2\partial_{\bar{z}} + \frac{Bz}{2}\right)\operatorname{sgn}\sigma_\omega(z),$$

where $\operatorname{sgn}(z) = \frac{z}{|z|} = \frac{|z|}{\bar{z}}$. Thus we have

$$\begin{aligned} & u_{n,g}(z) \\ = \; & e^{-\frac{B}{4}|z|^2}\operatorname{sgn}\sigma_\omega(z)^{-1}\left(-2\partial_{\bar{z}} + Bz\right)^{n-1}\overline{\sigma_\omega(z)^{-1}\widetilde{\sigma}_\omega(z)^n e^{-\delta_\omega z^2} g(z)} \\ = \; & e^{-\frac{B}{4}|z|^2}\operatorname{sgn}\sigma_\omega(z)^{-1}\left(-2\partial_{\bar{z}} + Bz\right)^{n-1}\overline{\widetilde{\sigma}_{\Gamma_\omega,\beta(\omega)}(z) g(z)}. \end{aligned}$$

Since

$$\frac{\#(\Gamma_\omega \cap B_r(0))}{r^2} \to \pi\rho$$

almost surely, we have

$$\#(\Gamma_\omega \cap B_r(0)) = O(r^2) \quad \text{as } r \to \infty \tag{43}$$

almost surely. So the conclusion follows from (ii) of this lemma, Corollary 4.2, Lemma 4.3 and the Leibniz rule. $\square$

5 Proof of Theorem 1.1

Let us once postpone the argument about the domains of self-adjoint extensions, and begin the proof of main theorems.

Proof of Theorem 1.1 (i) Let us identify a real value λ with the constant sequence $(\lambda)_{\gamma\in\Gamma}$. When $\operatorname{supp}\mu \ni 0$ or 1, the constant sequence 0 or 1 is admissible. Thus the assertion follows from (i) of Proposition 3.3, since H_0 is the free operator and H_1 is unitarily equivalent to H_0 by [13, section 6].

(ii) The same statement in the case α is periodic is already proved by Melgaard–Ouhabaz–Rozenblum [18], and the proof of this assertion is also similar. We use the Hardy type inequality (see Laptev–Weidl [16], [18]). Similarly to [18, Proposition 7.7], we see that there exists a constant $C > 0$ dependent only on Γ such that

$$\int_{\mathbf{R}^2} |(\nabla + i\mathbf{a}_\omega)u|^2 \, dxdy \geq C\rho(\alpha)^2 \int_{\mathbf{R}^2} W(z)|u(z)|^2 dxdy$$

for every $u \in C_0^\infty(\mathbf{C}\backslash\Gamma)$, where $\rho(\alpha) = \min(\alpha_-, 1-\alpha_+)$ and $W(z) = \operatorname{dist}(z, \Gamma)^{-2}$. Since $\inf W(z) > 0$, we have the conclusion.

(iii) Suppose $B|\mathcal{D}|/2\pi + \bar{\alpha} > np$ holds. Then, there exists $\epsilon > 0$ such that $-\frac{B}{4} + \pi(np - \bar{\alpha})/2|\mathcal{D}| + \epsilon < 0$. For any polynomial g, the function $u_{n,g}$ is an eigenfunction of H_ω corresponding to the eigenvalue E_n, by (34). Thus we have $\operatorname{mult}(E_n; H_\omega) = \infty$.

Next, suppose $B|\mathcal{D}|/2\pi + \bar{\alpha} < p$ holds. Then, there exists $\epsilon > 0$ such that $-B/4 + \pi(p - \bar{\alpha})/2|\mathcal{D}| - \epsilon > 0$. By (35), we have

$$|u_{1,1}(z)| \geq 1 \tag{44}$$

for sufficiently large z outside some C^0-set $\mathcal{C}$. Adding some disk centered at the origin to $\mathcal{C}$, we may assume (44) holds for every $z \in \mathbf{C} \setminus \mathcal{C}$. Let

$$S_0 = \{r > 0 \mid \{|z| = r\} \cap \mathcal{C} = \emptyset\}.$$

Suppose some $u \in D(H)$ satisfies $Hu = Eu$. By (i) of Lemma 4.4, u is written as $u = u_{1,1}\bar{f}$ for some entire function $f = \sum_{n=0}^\infty a_n z^n$. Then we have

$$\int_{\mathbf{C}} |u|^2 dxdy \geq 2\pi \sum_{n=0}^\infty \int_{S_0} |a_n|^2 r^{2n+1} dr. \tag{45}$$

Since $\mathcal{C}$ is a C^0-set, we have

$$\int_{(0,R)\cap S_0} r^{2n+1} dr \geq |(1, R) \cap S_0| \to \infty$$

as $R \to \infty$, where $|S|$ denotes the Lebesgue measure of S. Thus the right hand side of (45) diverges if some a_n is not zero. This implies $u = 0$, so we have $\operatorname{mult}(E_1; H_\omega) = 0$.

(iv) By the scaling $z' = \sqrt{B}z$, we can reduce the proof into the case $B = 1$. Then, the assertion is an immediate corollary of [20, Theorem 1.2 (ii)] and (i) of Proposition 3.3 (notice that the constants R_0 and c in [20, Theorem 1.2 (ii)] depend only on n_0, B, α_-, α_+). $\square$

6 Proof of Theorem 1.2

Proof of Theorem 1.2 (i) This is an immediate corollary of (ii) of Proposition 3.3, since the empty set $\emptyset$ is admissible and $H_{\emptyset,\emptyset} = H_0$.

(ii) By (ii) of Proposition 3.3 and Lemma 7.6 (proved later), we have for any $n \in \mathbf{N}$ and any admissible pair (Γ, α)

$$\Sigma \supset \overline{\bigcup_{\epsilon>0} \sigma(H_{\epsilon\Gamma,\alpha})} \ni E_n + 2B\,\mathrm{frac}(\alpha_1 + \cdots + \alpha_K),$$

where $K = \#\Gamma$ and $\alpha = (\alpha_k)_{k=1}^K$. Thus we have $\Sigma \supset E_n + 2BF$.

(iii) Similar to the proof of (iii) of Theorem 1.1. $\square$

7 Perturbation of δ magnetic fields

In this section, we prove the strong resolvent continuity of $H_{\Gamma,\alpha}$ with respect to (Γ, α) (we have already used it in the proof of Proposition 3.3). Since our magnetic potential has strong singularity, a careful analysis of the domain is necessary.

7.1 Self-adjoint extensions of minimal operators

In this subsection, we review some properties about the domain of the self-adjoint extension of $D(L_\phi)$.

We prepare some notation for the case $\#\Gamma = 1$. Let $B \geq 0$, $0 \leq \alpha \leq 1$, and $\gamma \in \mathbf{C}$ (the case α=1 is contained for convenience). Put

$$\phi_\alpha^{\gamma,1}(z) = \frac{B\overline{(z-\gamma)}}{2} + \frac{\alpha}{z-\gamma}.$$

We denote $\mathcal{L}_\alpha^{\gamma,1} = \mathcal{L}_{\phi_\alpha^{\gamma,1}}$, $L_\alpha^{\gamma,1} = L_{\phi_\alpha^{\gamma,1}}$ and $H_\alpha^{\gamma,1} = H_{\phi_\alpha^{\gamma,1}}$.

Let $R > 0$. Let $\chi \in C_0^\infty(\mathbf{C})$ such that $0 \leq \chi \leq 1$ and

$$\chi(z) = \begin{cases} 0 & (|z| \geq \frac{R}{2}), \\ 1 & (|z| \leq \frac{R}{3}). \end{cases}$$

For $\gamma \in \Gamma$, put $\chi_\gamma(z) = \chi(z-\gamma)$, $r_\gamma = |z-\gamma|$, $\theta_\gamma = \arg(z-\gamma)$ and

$$\begin{aligned} f_\alpha^{\gamma,1}(z) &= \chi_\gamma(z) {r_\gamma}^{\alpha_\gamma}, & g_\alpha^{\gamma,1}(z) &= \chi_\gamma(z) e^{-i\theta_\gamma} {r_\gamma}^{1-\alpha_\gamma}, \\ h_\alpha^{\gamma,1}(z) &= \chi_\gamma(z) {r_\gamma}^{-\alpha_\gamma}, & i_\alpha^{\gamma,1}(z) &= \chi_\gamma(z) e^{-i\theta_\gamma} {r_\gamma}^{\alpha_\gamma - 1}, \\ j_\alpha^{\gamma,1}(z) &= \chi_\gamma(z) \log r_\gamma, & k_\alpha^{\gamma,1}(z) &= \chi_\gamma(z) e^{-i\theta_\gamma} \log r_\gamma. \end{aligned}$$

Lemma 7.1 *Let ϕ be a function given by (11). Assume $0 \le \alpha_\gamma \le 1$ for every $\gamma \in \Gamma$. Then, the following holds:*

(i) We regard $\mathcal{L}_\phi$ as an operator on $\mathcal{D}'(\mathbf{C} \setminus \Gamma)$. Then,

$$\begin{aligned} D(L_\phi^*) &= \{u \in L^2(\mathbf{C}) \mid \mathcal{L}_\phi u \in L^2(\mathbf{C})\}, \\ D(H_\phi) &= D(L_\phi^*) \cap \overline{D(L_\phi)}^Q, \end{aligned}$$

where $\overline{}^Q$ denotes the closure with respect to the form norm. In particular, for $u \in D(H_\phi)$ we have

$$\left(\frac{1}{i}\nabla + \mathbf{a}_\phi\right) u \in L^2(\mathbf{C})^2.$$

(ii) Let $u \in D(L_\phi^*)$ and $\chi \in C_0^\infty(\mathbf{C})$ with $\chi(z) = 1$ in a neighborhood of γ for every $\gamma \in \Gamma \cap \operatorname{supp} \chi$. Then, $\chi u \in D(L_\phi^*)$.

(iii) Let $u \in D(H_\phi)$. Then, for any $\chi \in C_0^\infty(\mathbf{C})$, we have $\chi u \in D(H_\phi)$.

(iv) Suppose that there exists a constant R satisfying

$$0 < R \le \inf_{\gamma \neq \gamma'} |\gamma - \gamma'|.$$

Suppose also that there exist functions $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ satisfying $\Phi^\gamma \in C^\infty(B_{R/2}(\gamma))$, $|\Phi^\gamma(z)| = 1$ and

$$\mathcal{L}_\phi \Phi^\gamma = \Phi^\gamma \mathcal{L}_{\alpha_\gamma}^{\gamma,1}$$

in $B_{R/2}(\gamma)$. Put

$$f_\alpha^\gamma(z) = \Phi^\gamma f_{\alpha_\gamma}^{\gamma,1}, \quad g_\alpha^\gamma(z) = \Phi^\gamma g_{\alpha_\gamma}^{\gamma,1}, \quad \ldots,$$

where the R in the definition of $f_{\alpha_\gamma}^{\gamma,1}, \ldots$ is the present one. Put

$$\begin{aligned} \Gamma_0 &= \{\gamma \in \Gamma \mid \alpha_\gamma = 0\}, \\ \Gamma_1 &= \{\gamma \in \Gamma \mid \alpha_\gamma = 1\}, \\ \Gamma_2 &= \{\gamma \in \Gamma \mid 0 < \alpha_\gamma < 1\}. \end{aligned}$$

Then, when Γ is a finite set, we have

$$\begin{aligned} D(L_\phi^*) &= D(\overline{L_\phi}) \oplus \mathrm{L.h.}\Bigg(\bigcup_{\gamma \in \Gamma_0} \{f_\alpha^\gamma, j_\alpha^\gamma\} \cup \bigcup_{\gamma \in \Gamma_1} \{g_\alpha^\gamma, k_\alpha^\gamma\} \\ &\qquad \cup \bigcup_{\gamma \in \Gamma_2} \{f_\alpha^\gamma, g_\alpha^\gamma, h_\alpha^\gamma, i_\alpha^\gamma\} \Bigg), \qquad (46) \\ D(H_\phi) &= D(\overline{L_\phi}) \oplus \mathrm{L.h.}\Bigg(\bigcup_{\gamma \in \Gamma_0} \{f_\alpha^\gamma\} \cup \bigcup_{\gamma \in \Gamma_1} \{g_\alpha^\gamma\} \cup \bigcup_{\gamma \in \Gamma_2} \{f_\alpha^\gamma, g_\alpha^\gamma\} \Bigg) \qquad (47) \end{aligned}$$

where L.h.X *denotes the finite linear combinations of the vectors in X. When Γ is an infinite set, the right hand side of (46) (or (47)) is dense in the left hand side with respect to the graph norm.*

The proof of the above proposition is similar to those of [19, Proposition 5.4, Corollary 5.10, Lemma 5.15], so we shall omit it. To prove (iv), roughly speaking, we construct an isomorphism from $D(L_\phi^*)/D(\overline{L_\phi})$ to $\bigoplus_{\gamma\in\Gamma} D(L_{\alpha_\gamma}^{\gamma,1*}/\overline{L_{\alpha_\gamma}^{\gamma,1}})$ by cut-off and gauge transform, and use the results in Albeverio–Gesztesy–Høegh-Krohn–Holden [2] and Exner–Šťovíček–Vytřas [10].

7.2 Criterion for the strong resolvent convergence

We shall prove the strong resolvent continuity via the following elementary lemma.

Lemma 7.2 *Let $\mathcal{H}$ be a Hilbert space. Let A_n ($n = 1, 2, \ldots$), A be self-adjoint operators on $\mathcal{H}$. Then, the following two conditions are equivalent:*

(i) $A_n \to A$ in the strong resolvent sense.

(ii) There exist linear operators T_n and an operator core D for A such that $T_n D \subset D(A_n)$ and for every $u \in D$

$$T_n u \to u, \quad A_n T_n u \to Au$$

in $\mathcal{H}$.

Remark. The assertion (ii)⇒(i) in the case $T_n = Id$ is well-known (see Reed-Simon [22, Theorem VIII.25]).

Proof. (i) ⇒ (ii) Put $T_n = (A_n + i)^{-1}(A + i)$, $D = D(A)$.
(ii) ⇒ (i) Let $u \in D$ and put $v = (A + i)u$. Then,

$$\begin{aligned} & (A_n + i)^{-1}v - (A + i)^{-1}v \\ = \;& (A_n + i)^{-1}\left((A + i)u - (A_n + i)T_n u\right) + T_n u - u \to 0. \end{aligned}$$

Since $(A_n + i)^{-1} - (A + i)^{-1}$ is uniformly bounded and $(A + i)D$ is dense in $\mathcal{H}$, we have the assertion. □

7.3 Strong resolvent continuity for the perturbation of δ magnetic fields

Let K be a positive integer. Let $\mathcal{F}$ be the space of all the pairs (Γ, α), where $\Gamma = (\gamma_k)_{k=1}^K$ is a sequence of K distinct complex numbers, and $\alpha = (\alpha_k)_{k=1}^K$ is a $[0, 1]$-valued sequence. Define a metric d on $\mathcal{F}$ by

$$d((\Gamma, \alpha), (\Gamma', \alpha')) = \left(\sum_{k=1}^K (|\gamma_k - \gamma_k'|^2 + |\alpha_k - \alpha_k'|^2)\right)^{1/2}$$

for $(\Gamma, \alpha), (\Gamma', \alpha') \in \mathcal{F}$, where $\Gamma' = (\gamma_k')_{k=1}^K$, $\alpha' = (\alpha_k')_{k=1}^K$. We often regard a sequence Γ as a K-point subset of $\mathbf{C}$. We denote $\mathcal{L}_{\Gamma,\alpha} = \mathcal{L}_{\phi_{\Gamma,\alpha}}$, $L_{\Gamma,\alpha} = L_{\phi_{\Gamma,\alpha}}$ and $H_{\Gamma,\alpha} = H_{\phi_{\Gamma,\alpha}}$ for $(\Gamma, \alpha) \in \mathcal{F}$, where

$$\phi_{\Gamma,\alpha}(z) = \frac{B\bar{z}}{2} + \sum_{k=1}^K \frac{\alpha_k}{z - \gamma_k}.$$

Lemma 7.3 *Let O be an open set and $(\Gamma,\alpha)\in\mathcal{F}$ with $\Gamma\subset O$. Then, there exist an open neighborhood $\mathcal{F}'$ of (Γ,α), subspaces $\{D_{\Gamma',\alpha'}\}_{(\Gamma',\alpha')\in\mathcal{F}'}$ of $L^2(\mathbf{C})$ and linear operators $\{T_{\Gamma',\alpha'}\}_{(\Gamma',\alpha')\in\mathcal{F}'}$ satisfying the following conditions:*

(i) $D_{\Gamma',\alpha'}$ *is an operator core of* $H_{\Gamma',\alpha'}$ *and* $D_{\Gamma',\alpha'}\subset D_0(H_{\Gamma',\alpha'})$*, for any* $(\Gamma',\alpha')\in\mathcal{F}'$*.*

(ii) $T_{\Gamma',\alpha'}$ *is a linear operator from* $D_{\Gamma,\alpha}$ *to* $D_{\Gamma',\alpha'}$ *for any* $(\Gamma',\alpha')\in\mathcal{F}'$*. Moreover,* $\operatorname{supp} T_{\Gamma',\alpha'}u\subset O\cup\operatorname{supp} u$*.*

(iii) When $(\Gamma',\alpha')\to(\Gamma,\alpha)$ *in* $\mathcal{F}'$*, we have*

$$T_{\Gamma',\alpha'}u\to u,\quad H_{\Gamma',\alpha'}T_{\Gamma',\alpha'}u\to H_{\Gamma,\alpha}u \tag{48}$$

in $L^2(\mathbf{C})$*, for any* $u\in D_{\Gamma,\alpha}$*.*

Proof. Take $(\Gamma,\alpha)\in\mathcal{F}$. Put

$$\begin{aligned}\mathcal{K}_0 &= \{k=1,\dots,K\mid \alpha_k=0\},\\ \mathcal{K}_1 &= \{k=1,\dots,K\mid \alpha_k=1\},\\ \mathcal{K}_2 &= \{k=1,\dots,K\mid 0<\alpha_k<1\}.\end{aligned}$$

If $\mathcal{K}_2\neq\emptyset$, put

$$A=\min_{k\in\Gamma_2}\min(\alpha_k,1-\alpha_k).$$

If $\mathcal{K}_2=\emptyset$, put $A=1$. Take $R'>0$ so that $B_{3R'}(\gamma_k)\subset O$ for $k=1,\dots,K$ and $\{B_{3R'}(\gamma_k)\}_{k=1}^K$ are disjoint. Take a small positive number $R<\min(A,R')$ (determined later) and put $\mathcal{F}'=\{(\Gamma',\alpha')\in\mathcal{F}\mid d((\Gamma',\alpha'),(\Gamma,\alpha))<R\}$.

We shall construct diffeomorphisms $\{F_{\Gamma',\alpha'}\}_{(\Gamma',\alpha')\in\mathcal{F}'}$ on $\mathbf{C}$ satisfying

$$F_{\Gamma',\alpha'}(z)=z\quad\text{for } z\in\mathbf{C}\setminus\bigcup_{k=1}^K B_{3R'}(\gamma_k), \tag{49}$$

$$F_{\Gamma',\alpha'}(z)=z-\gamma_k+\gamma_k'\quad\text{for } z\in B_{R'}(\gamma_k), \tag{50}$$

$$\|\partial_x^l\partial_y^m(F_{\Gamma',\alpha'}(z)-z)\|_\infty\le C_{lm}d((\Gamma,\alpha),(\Gamma',\alpha')) \tag{51}$$

for any $(\Gamma',\alpha')\in\mathcal{F}'$ and $l,m=0,1,2,\dots$, where C_{lm} is a positive constant independent of (Γ',α'). For this purpose, take a smooth function η on $\mathbf{R}^1$ such that $0\le\eta(r)\le1$ and

$$\eta(r)=\begin{cases}1 & (r\le R'),\\ 0 & (r\ge 2R').\end{cases}$$

Define $F_{\Gamma',\alpha'}$ by (49) and

$$F_{\Gamma',\alpha'}(z)=z+\eta(|z-\gamma_k|)(\gamma_k'-\gamma_k)$$

for $z\in B_{3R'}(\gamma_k)$, $k=1,\dots,K$. Then, $F_{\Gamma',\alpha'}$ clearly satisfies (50) and (51). If we take R sufficiently small, we see that $F_{\Gamma',\alpha'}$ is a diffeomorphism by the Hadamard inverse function theorem.

For $(\Gamma', \alpha') \in \mathcal{F}'$ and $k = 1, \ldots, K$, define

$$\Phi^k_{\Gamma',\alpha'}(z) \quad = \quad \exp i \operatorname{Im} \int_{\gamma'_k}^z \left(-\frac{B\bar{\gamma}'_k}{2} - \sum_{l \neq k} \frac{\alpha'_l}{w - \gamma'_l} \right) dw.$$

The function $\Phi^k_{\Gamma',\alpha'}$ is single-valued, smooth and satisfies

$$\mathcal{L}_{\Gamma',\alpha'} \Phi^k_{\Gamma',\alpha'} = \Phi^k_{\Gamma',\alpha'} \mathcal{L}^{\gamma'_k,1}_{\alpha'_k} \tag{52}$$

in $B_{3R'}(\gamma'_k)$.

Let us apply Lemma 7.1 with the above R and $\{\Phi^k_{\Gamma',\alpha'}\}_{k=1}^K$. Define a core $D_{\Gamma',\alpha'}$ for $H_{\Gamma',\alpha'}$ by

$$D_{\Gamma',\alpha'} = \text{L.h.} \left(\bigcup_{k \in \mathcal{K}_0} \{f^{\gamma'_k}_{\alpha'_k}\} \cup \bigcup_{k \in \mathcal{K}_1} \{g^{\gamma'_k}_{\alpha'_k}\} \cup \bigcup_{k \in \mathcal{K}_2} \{f^{\gamma'_k}_{\alpha'_k}\} \cup \{g^{\gamma'_k}_{\alpha'_k}\} \right) \oplus C_0^\infty(\mathbf{C} \setminus \Gamma').$$

For $(\Gamma', \alpha') \in \mathcal{F}'$, define a linear operator $T_{\Gamma',\alpha'}$ from $D_{\Gamma,\alpha}$ to $D_{\Gamma',\alpha'}$ by

$$\begin{aligned} & T_{\Gamma',\alpha'} \left(\sum_{k \in \Gamma_0} c_k f^{\gamma_k}_{\alpha_k} + \sum_{k \in \Gamma_1} d_k g^{\gamma_k}_{\alpha_k} + \sum_{k \in \Gamma_2} (c_k f^{\gamma_k}_{\alpha_k} + d_k g^{\gamma_k}_{\alpha_k}) + \xi \right) \\ = \quad & \sum_{k \in \Gamma_0} c_k f^{\gamma'_k}_{\alpha'_k} + \sum_{k \in \Gamma_1} d_k g^{\gamma'_k}_{\alpha'_k} + \sum_{k \in \Gamma_2} (c_k f^{\gamma'_k}_{\alpha'_k} + d_k g^{\gamma'_k}_{\alpha'_k}) + \xi \circ F^{-1}_{\Gamma',\alpha'}, \end{aligned}$$

where c_k and d_k are constants and $\xi \in C_0^\infty(\mathbf{C} \setminus \Gamma)$.

We shall check that $T_{\Gamma',\alpha'}$ satisfies the desired properties. The inclusion $\operatorname{supp} T_{\Gamma',\alpha'} u \subset O \cup \operatorname{supp} u$ holds by definition. Since the functions $f^{\gamma'_k}_{\alpha'_k}$, $g^{\gamma'_k}_{\alpha'_k}$ and $F_{\Gamma',\alpha'}$ are smooth with respect to (Γ', α') except $\gamma'_k \neq z$, the convergence (48) clearly holds pointwise almost everywhere. By the dominated convergence theorem, it suffices to show that the functions $T_{\Gamma',\alpha'} u$ and $H_{\Gamma',\alpha'} T_{\Gamma',\alpha'} u$ are bounded uniformly with respect to (Γ', α'). By (52), it suffices to show $f^{\gamma'_k,1}_{\alpha'_k}$, $g^{\gamma'_k,1}_{\alpha'_k}$, $H^{\gamma'_k,1}_{\alpha'_k} f^{\gamma'_k,1}_{\alpha'_k}$ and $H^{\gamma'_k,1}_{\alpha'_k} g^{\gamma'_k,1}_{\alpha'_k}$ are uniformly bounded. This can be done by a straightforward calculation using the polar coordinate. Put $r = |z - \gamma'_k|$, $\theta = \arg(z - \gamma'_k)$, $a = \alpha_{\gamma'_k}$, $\chi = \chi_{\gamma'_k}$. According to [10, section II], we have

$$H^{\gamma'_k,1}_{\alpha'_k} = -\frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \left(\frac{1}{i} \partial_\theta + a + \frac{Br^2}{2} \right)^2. \tag{53}$$

Thus we have

$$\begin{aligned} |f^{\gamma'_k,1}_{\alpha'_k}| \quad &= \quad \chi r^a \leq M, \\ \left| H^{\gamma'_k,1}_{\alpha'_k} f^{\gamma'_k,1}_{\alpha'_k} \right| \quad &\leq \quad (2a+1)|\partial_r \chi| r^{a-1} + (|\partial_r^2 \chi| + Ba) r^a + \frac{B^2}{4} \chi r^{a+2} \\ &\leq \quad 3m^{-1} C_1 + (C_2 + B) M + \frac{B^2}{4} M^3, \end{aligned}$$

where $M = \max\left(\frac{R}{2}, 1\right)$, $m = \min\left(\frac{R}{3}, 1\right)$, $C_1 = \sup|\partial_r \chi|$, $C_2 = \sup|\partial_r^2 \chi|$. The estimates for $g_{\alpha'_k}^{\gamma'_k,1}$ and $H_{\alpha'_k}^{\gamma'_k,1} g_{\alpha'_k}^{\gamma'_k,1}$ can be done similarly. $\square$

Corollary 7.4 *$H_{\Gamma',\alpha'} \to H_{\Gamma,\alpha}$ in the strong resolvent sense as $(\Gamma', \alpha') \to (\Gamma, \alpha)$ in $\mathcal{F}$.*

Corollary 7.5 *For any $\lambda \in \sigma(H_{\Gamma,\alpha})$, $\lim_{(\Gamma',\alpha')\to(\Gamma,\alpha)} \mathrm{dist}(\sigma(H_{\Gamma',\alpha'}), \lambda) = 0$.*

Proof. This can be easily proved by constructing approximating eigenfunctions using the operator $T_{\Gamma',\alpha'}$. $\square$

7.4 Gathering to a point

Next we consider the case some points are gathering to a point. For a sequence $\Gamma = (\gamma_k)_{k=1}^K$ and $\epsilon > 0$, we denote

$$\epsilon\Gamma = (\epsilon\gamma_k)_{k=1}^K.$$

Lemma 7.6 *Let $(\Gamma, \alpha) \in \mathcal{F}$. Then, we have*

$$\lim_{\epsilon \downarrow 0} \mathrm{dist}(\sigma(H_{\epsilon\Gamma,\alpha}), \{E_n + 2B\,\mathrm{frac}(\alpha_1 + \cdots + \alpha_K)\}) = 0 \tag{54}$$

for any $n = 1, 2, \ldots$.

Proof. Let $R = \max_{k=1,\ldots,K} |\gamma_k|$ and $\beta = \mathrm{frac}(\alpha_1 + \cdots + \alpha_K)$. If $\beta = 0$, then the assertion is trivial since $\sigma(H_{\epsilon\Gamma,\alpha})$ contains all the Landau levels by [19, Theorem 1.1 (i)].

Assume $0 < \beta < 1$. For $\epsilon > 0$ and $|z| > \epsilon R$, put

$$\Phi_\epsilon(z) = \exp i\,\mathrm{Im} \int_{\epsilon(R+1)}^{z} \left(\frac{\beta}{w} - \sum_{k=1}^{K} \frac{\alpha_k}{w - \epsilon\gamma_k} \right) dw,$$

where the integral is done along a smooth curve from $\epsilon(R+1)$ to z contained in the region $\{|z| > \epsilon R\}$. The function Φ_ϵ is single-valued, smooth and satisfies

$$\mathcal{L}_{\epsilon\Gamma,\alpha}\Phi_\epsilon = \Phi_\epsilon \mathcal{L}_\beta^{0,1} \tag{55}$$

in $\{|z| > \epsilon R\}$.

Put

$$u_n(z) = |z|^\beta z^{n-1} e^{-\frac{B}{4}|z|^2}.$$

The function u_n is an eigenfunction of $H_\beta^{0,1}$ for the eigenvalue $E_n + 2\beta B$. Take a smooth function $\chi = \chi(r)$ on $\mathbf{R}$ satisfying $0 \le \chi(z) \le 1$ and

$$\chi(r) = \begin{cases} 0 & (0 \le r \le R), \\ 1 & (2R \le r). \end{cases}$$

Put $\chi_\epsilon(z) = \chi(\frac{|z|}{\epsilon})$ and put $u_\epsilon = \Phi_\epsilon \chi_\epsilon u_n / \|\Phi_\epsilon \chi_\epsilon u_n\|$. Using (55) and the polar coordinate expression (53), we can show

$$\|(H_{\epsilon\Gamma,\alpha} - (E_n + 2\beta B))u_\epsilon\|^2 \leq C\epsilon^{2\beta+2n-4}, \tag{56}$$

where C is a positive constant independent of ϵ. When $n \geq 2$, the inequality (56) implies (54).

To treat the case $n = 1$, we introduce an auxiliary operator $H^-_{\epsilon\Gamma,\alpha}$ as in [19, Proposition 3.3] (notice that the Friedrichs extension is denoted by H_N^{AB} in [19]). The operator $H^-_{\epsilon\Gamma,\alpha}$ is a self-adjoint extension of $L_{\epsilon\Gamma,\alpha}$ satisfying

$$H_{\epsilon\Gamma,\alpha} + 2B \simeq H^-_{\epsilon\Gamma,\alpha}|_{\mathrm{Ker}(H^-_{\epsilon\Gamma,\alpha} - B)^\perp}$$

(see [19, (8)]). Thus we have

$$\mathrm{dist}(\sigma(H_{\epsilon\Gamma,\alpha}), E_1 + 2\beta B) = \mathrm{dist}(\sigma(H^-_{\epsilon\Gamma,\alpha}), E_2 + 2\beta B). \tag{57}$$

Since $\mathrm{supp}\, u_\epsilon \subset \{|z| \geq \epsilon R\}$, we have $u_\epsilon \in D(H^-_{\epsilon\Gamma,\alpha})$. Thus (56) for $n = 2$ holds even if we replace $H_{\epsilon\Gamma,\alpha}$ by $H^-_{\epsilon\Gamma,\alpha}$. Combining this fact with (57), we conclude (54) also holds for $n = 1$. $\square$

8 Appendix

8.1 Proof of Proposition 4.1

Proof of Proposition 4.1. (i) Since $|\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2}| = O(|\gamma|^{-3})$ locally uniformly with respect to z in $\mathbf{C} \setminus \Gamma$, we have

$$\begin{aligned} & \sum_{\gamma \neq 0} \alpha_\gamma \left| \frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right| \leq C \int_0^\infty r^{-3} dn(r) \\ = \;& C \left(\left[r^{-3} n(r) \right]_0^\infty + 3 \int_0^\infty r^{-4} n(r) dr \right) < \infty, \end{aligned}$$

where we used (22). Thus the sum (23) converges. Then we can define $\sigma_{\Gamma,\alpha}$ via the following formula:

$$\sigma_{\Gamma,\alpha}(z) = z^{\alpha_0} \exp\left(\int_0^z \left(\zeta_{\Gamma,\alpha}(w) - \frac{\alpha_0}{w} \right) dw \right). \tag{58}$$

The right hand side of (58) can be rewritten in the form (24), and then the product converges. The formula (25) follows from (58). If we change the path of integration from 0 to z, then $\sigma_{\Gamma,\alpha}$ is multiplied by some $e^{2\pi i \alpha_\gamma}$'s. Thus $|(\frac{d}{dz})^k \sigma_{\Gamma,\alpha}|$ is independent of the choice of the branches.

(ii) (Outline) This assertion can be proved in a similar way to the proof of [17, Theorem 2 in Chap. II, Sec. 1]. Only we have to do is to replace the definition of the function $n(r, \theta_1, \theta_2)$ by (21). Below we shall exhibit the outline of the proof, and show how the lemmas in [17] should be modified by this change. Without loss of generality, we assume $0 \notin \Gamma$, so $\alpha_0 = 0$.

Lemma 8.1 *For any positive number H, any finite set $\Gamma \subset \mathbf{C}$ and any sequence $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ of positive numbers, there is a system of disks in $\mathbf{C}$, with the sum of the radii equal to $2H$, such that for each point z outside these disks we have*

$$\prod_{\gamma\in\Gamma} |z-\gamma|^{\alpha_\gamma} > \left(\frac{H}{e}\right)^n,$$

where $n = \sum_{\gamma\in\Gamma} \alpha_\gamma$.

Outline of Proof. This is a generalization of the Cartan estimate [17, Theorem 10 in Chap. 1, Sec. 7]. For $X \subset \mathbf{C}$, put

$$n(X) = \sum_{\gamma\in\Gamma\cap X} \alpha_\gamma.$$

Put $\Gamma_0 = \Gamma$, $C_0 = \emptyset$. For $j = 1, 2, \ldots$, define disks $C_j = B_{r_j}(z_j)$ by the following inductive procedure: Put $\Gamma_j = \Gamma_{j-1} \setminus C_{j-1}$. If $\Gamma_j = \emptyset$, the procedure finishes. If $\Gamma_j \neq \emptyset$, let C_j be a disk having the largest radius among the closed disks $B_r(z)$ satisfying

$$r = \frac{H}{n} n\left(B_r(z) \cap \Gamma_j\right).$$

Since Γ is a finite set, this procedure must finish within finite steps, and we obtain disks $\{C_j\}_{j=1}^J$. Put $D_j = B_{2r_j}(z_j)$. Then the disks $\{D_j\}_{j=1}^J$ have the desired properties. The equality $\sum_{j=1}^J 2r_j = 2H$ holds by the construction of $\{C_j\}$. For $z \in \left(\bigcup_{j=1}^J D_j\right)^c$, number the elements of Γ and α as $|z-\gamma_1| \leq \cdots \leq |z-\gamma_K|$ and $\alpha_k = \alpha_{\gamma_k}$. By a similar argument as in [17] [4], we have

$$|z-\gamma_k| \geq \frac{H}{n} \sum_{j=1}^k \alpha_j.$$

Thus we have

$$\begin{aligned}
\sum_{k=1}^K \alpha_k \log|z-\gamma_k| &\geq \sum_{k=1}^K \alpha_k \left(\log H - \log n + \log \sum_{j=1}^k \alpha_j\right) \\
&> n(\log H - \log n) + \int_0^n \log x \; dx \\
&= n \log \frac{H}{e},
\end{aligned}$$

where we used the concavity of $\log x$ in the second inequality. □

[4] We show 'every disk $C = B_r(z)$ with $r \geq r_j$ satisfies $r \geq \frac{H}{n} n(C \cap \Gamma_j)$', and apply this fact to the disk $B_{|z-\gamma_k|}(z)$.

We introduce the Weierstrass primary factors

$$G(u;1) = (1-u)e^u, \quad G(u;2) = (1-u)e^{u+\frac{u^2}{2}}.$$

When we consider the function $\log G(u;p)$ $(p = 1, 2)$ in the sequel, we make a cut $[1, \infty)$ in the complex u-plane, and take the branch $\log G(0;p) = 0$. So when we consider the function $\log \sigma_{\Gamma,\alpha}(z) = \sum_{\gamma\in\Gamma} \alpha_\gamma \log G(\frac{z}{\gamma}, 2)$, the variable z belongs to the star region

$$\mathbf{C} \setminus \bigcup_{\gamma\in\Gamma} \{t\gamma \mid t \geq 1\}.$$

We denote $r = |z|$.

Lemma 8.2 *Assume (22) holds. For $0 < s < 1$, put*

$$f_s(z) = \prod_{|\gamma|<sr} G\left(\frac{z}{\gamma};1\right)^{\alpha_\gamma}.$$

Then, there exist $C_1 > 0$ and $r_1 = r_1(s) > 0$ such that

$$|\log f_s(z)| \leq C_1 s r^2$$

for $r \geq r_1$, where C_1 is independent of s, r.

The proof is similar to that of [17, Lemma 7 in Chap. 1, Sec. 17], in the case $\rho(r) = \rho = 2$, $p = 1$.

Lemma 8.3 *Assume (22) holds. For $t > 2$, put*

$$_t f(z) = \prod_{|\gamma|>tr} G\left(\frac{z}{\gamma};2\right)^{\alpha_\gamma}.$$

Then, there exist $C_2 > 0$ and $r_2 = r_2(t) > 0$ such that

$$|\log {}_t f(z)| \leq C_2 t^{-1} r^2$$

for $r \geq r_2$, where C_2 is independent of t, r.

The proof is similar to that of [17, Lemma 8 in Chap. 1, Sec. 17], in the case $\rho(r) = \rho = 2$, $p = 2$.

Lemma 8.4 *Assume that $\Gamma \subset (0, \infty)$ and the limit $\Delta = \Delta(0, 2\pi)$ (defined by (26)) exists. Put*

$$V_r(z) = \prod_{|\gamma|\leq r} G\left(\frac{z}{\gamma};1\right)^{\alpha_\gamma} \prod_{|\gamma|>r} G\left(\frac{z}{\gamma};2\right)^{\alpha_\gamma}. \tag{59}$$

Then, for $0 < \theta < 2\pi$, we have

$$\lim_{r\to\infty} \frac{\log V_r(re^{i\theta})}{r^2} = -\Delta\left(\frac{1}{2} - i(\theta - \pi)\right) e^{2i\theta}.$$

The limit is uniform with respect to $\theta \in [\eta, 2\pi - \eta]$, for any $0 < \eta < \pi$.

The proof is similar to that of [17, Lemma 9 in Chap. 1, Sec. 17], in the case $\rho(r) = \rho = 2$.

Lemma 8.5 *Suppose a discrete set* $\Gamma = \{\gamma_k\}_{k=1}^\infty$ *and a sequence* $\alpha = (\alpha_{\gamma_k})_{k=1}^\infty$ *satisfy (22). Assume* $\widetilde{\Gamma} = \{\widetilde{\gamma}_k\}_{k=1}^\infty$ *satisfies*

$$|\gamma_k| = |\widetilde{\gamma}_k|, \quad |\arg\gamma_k - \arg\widetilde{\gamma}_k| < \delta$$

for some $\delta > 0$ *independent of* k*. Let* $V_r(z)$ *as in (59), and* $\widetilde{V}_r(z)$ *is (59) with* Γ *replaced by* $\widetilde{\Gamma}$ *and* $\alpha_{\widetilde{\gamma}_k} = \alpha_{\gamma_k}$*. Then, for any* $\eta > 0$ *and* $\epsilon > 0$*, there exists* $\delta_0 > 0$ *dependent only on* η*,* ϵ *such that if* $\delta < \delta_0$ *we have*

$$\left| \log|V_r(z)| - \log|\widetilde{V}_r(z)| \right| < \epsilon r^2$$

for all z *not in the union of some disks with upper linear density less than* η*.*

The proof is similar to that of [17, Lemma 4 in Chap. 2, Sec. 3], in the case $\rho(r) = \rho = 2$. In the proof, we use Lemma 8.1, Lemma 8.2 and Lemma 8.3.

Lemma 8.6 *Let* Γ*,* α *satisfying the assumption (a) in (ii) of Proposition 4.1. Let* V_r *as in (59). Then, there exists a* C^0*-set* $\mathcal{C}$ *such that we have*

$$\lim_{r\to\infty, re^{i\theta}\notin\mathcal{C}} \frac{\log|V_r(re^{i\theta})|}{r^2} = -\int_{\theta-2\pi}^{\theta} (\psi-\theta)\sin 2(\psi-\theta)\, d\Delta(\psi). \tag{60}$$

The convergence in (60) is uniform with respect to $\theta \in [0, 2\pi)$*.*

The proof is similar to that of [17, Lemma 5 in Chap.2, Sec. 3] and the subsequent argument, with $\rho(r) = \rho = 2$. Roughly speaking, we approximate Γ by another set $\widetilde{\Gamma}$ contained in a finite number of semi-infinite lines. The asymptotics of the function $\widetilde{V}_r$ (the function V_r corresponding to $\widetilde{\Gamma}$) is obtained by Lemma 8.4, which leads to the conclusion combined with the approximating argument using Lemma 8.5.

Using the above lemmas, we shall prove (ii) of Proposition 4.1. Notice that

$$\frac{\log|\sigma_{\Gamma,\alpha}|}{r^2} = \mathrm{Re}\left(\sum_{|\gamma|\le r} \frac{\alpha_\gamma}{2\gamma^2} e^{2i\theta}\right) + \frac{\log|V_r(z)|}{r^2}. \tag{61}$$

The first term in the right hand side of (61) converges to $\mathrm{Re}(\delta_{\Gamma,\alpha}e^{2i\theta})$ by assumption (b). So the conclusion follows from Lemma 8.6. $\square$

8.2 Domain of the Friedrichs extension

Using (47), we can verify (12) if $\#\Gamma < \infty$ (see the next proposition). However, it is not trivial in the case $\#\Gamma = \infty$, especially in the case Γ is not uniformly separated (in fact, the Poisson configuration Γ_ω is not uniformly separated almost surely). Here we shall prove (12) under some rather mild condition. By gauge transform, we can assume $0 \le \alpha_\gamma < 1$ without loss of generality.

Proposition 8.7 *Put $n(r) = \#(\Gamma \cap B_r(0))$ for $r > 0$. Assume*

$$\liminf_{r\to\infty} \frac{\log\log n(r)}{r} < \log 2. \tag{62}$$

Then, (12) holds.

Of course, we regard the case $\#\Gamma \le 1$ satisfies (62). Not only the lattice Γ, but also the Poisson configuration Γ_ω also satisfies (62), by (43). The difficulty of the proof is that $u \in D(L_\phi^*)$ does not imply $(\nabla + i\mathbf{a}_\phi)u \in (L^2(\mathbf{C}))^2$ in general.

Proof. In the sequel, we omit the subscript ϕ. We denote the right hand side of (12) by $\widetilde{D}$, that is, $\widetilde{D}$ consists of the elements of $D(L^*)$ satisfying the boundary conditions

$$\limsup_{z\to\gamma} |u(z)| < \infty \tag{63}$$

for every $\gamma \in \Gamma$. We shall divide the proof into 2 steps.

Step 1. The case $\#\Gamma < \infty$.

Proof. First we show $D(H) \subset \widetilde{D}$. Since $\#\Gamma < \infty$, $D(H)$ is given by (47). The functions f_α^γ, g_α^γ satisfy (63). We shall show $u \in D(\overline{L})$ also satisfies (63). Take a sequence $u_n \in D(L)$ which approximates u in the graph norm. Then, $v_n = (H+1)u_n \to v = (H+1)u$ in L^2. By the diamagnetic inequality [18], we have

$$|(u_n - u)(z)| = |(H+1)^{-1}(v_n - v)|(z) \le (-\Delta + 1)^{-1}|v_n - v|(z) \quad \text{a.e.} \tag{64}$$

The function $(-\Delta+1)^{-1}|v_n - v| \to 0$ in the Sobolev space H^2. By the Sobolev inequality, the right hand side of (64) converges to 0 uniformly in $z \in \mathbf{C}$, so is the left hand side. Thus u is continuous and $u(\gamma) = 0$.

Conversely, by (46), (47), the function in $D(L^*) \setminus D(H)$ must have a singularity at some γ. So $\widetilde{D} \subset D(H)$. □

Step 2. The case $\#\Gamma = \infty$.

Proof. For constants a, b with $1 \le a < b$, we can find $\chi_{a,b} \in C_0^\infty(\mathbf{C})$ so that $0 \le \chi_{a,b} \le 1$ and

$$\chi_{a,b}(z) = \begin{cases} 1 & |z| \le a, \\ 0 & |z| \ge b, \end{cases}$$

$$\|\nabla\chi_{a,b}\|_\infty \le C(b-a)^{-1}, \quad \|\Delta\chi_{a,b}\|_\infty \le C(b-a)^{-2},$$

where C is a constant independent of a, b and z.

First we show $D(H) \subset \widetilde{D}$. Let $u \in D(H)$. For $n \in \mathbf{N}$, take a, b with $n \le a < b$ and put $\chi = \chi_{a,b}$. Then, we have $\chi u \in D(H)$ by (iii) of Lemma 7.1. By reducing into the finite point case using a gauge transform, we see that (63) holds for any $\gamma \in B_n(0) \cap \Gamma$. Taking $n \to \infty$, we have $u \in \widetilde{D}$.

Next we show $D(H) \supset \widetilde{D}$. Let $u \in D(L^*)$ satisfying (63). Take $\eta = \chi_{a,b}$ so that $\operatorname{supp} \nabla\eta \cap \Gamma = \emptyset$. By (ii) of Lemma 7.1, we have $\eta u \in D(L^*)$, and ηu

also satisfies (63). Thus, by reducing into the finite point case using a gauge transform, we have $\eta u \in D(H)$. By (iii) of Lemma 7.1, we have $\chi u \in D(H)$ for any $\chi = \chi_{a,b}$. Thus $(\nabla + i\mathbf{a})\chi u \in L^2$ and

$$\|(\nabla + i\mathbf{a})\chi u\|^2 = (H\chi u, \chi u).$$

By the Leibniz rule, we have

$$\|(\nabla + i\mathbf{a})\chi u\|^2 \le \|\chi H u\|\|u\| + 2\|\nabla\chi(\nabla + i\mathbf{a})u\|\|u\| + \|(\Delta\chi)u\|\|u\|.$$

Thus we have

$$\begin{aligned} & \int_{|z|\le a} |(\nabla + i\mathbf{a})u|^2\, dxdy \\ \le\ & C\left(1 + (b-a)^{-2}\right) + C(b-a)^{-1}\left(\int_{a\le|z|\le b} |(\nabla + i\mathbf{a})u|^2\, dxdy\right)^{1/2} \end{aligned} \tag{65}$$

for any $1 \le a < b$, where C is a constant independent of a, b.

We shall prove $(\nabla + i\mathbf{a})u \in L^2$. Put

$$c_k = \int_{k-1\le|z|\le k} |(\nabla + i\mathbf{a})u|^2\, dxdy.$$

Substituting $a = k$, $b = k+1$ into (65), we have

$$c_1 + \ldots + c_k \le C(1 + c_{k+1}^{1/2}), \tag{66}$$

where C is independent of k. Thus, if c_k is bounded, then we have $(\nabla + i\mathbf{a})u \in L^2$. Suppose c_k is unbounded. By (66), we see that there exist $k_0 \in \mathbf{N}$ and $h > 0$ such that $d_k = c_{k_0+k}/h$ satisfies $d_1 \ge 1$ and

$$d_1 + \cdots + d_k \le d_{k+1}^{1/2}$$

for any $k = 1, 2, \ldots$. This inequality implies $d_3 \ge 4 > e$ and $d_k^2 \le d_{k+1}$, so we have

$$\log\log d_k \ge (k-3)\log 2$$

for $k \ge 3$, therefore

$$\liminf_{k\to\infty} \frac{\log\log c_k}{k} \ge \log 2. \tag{67}$$

Dividing the ring $\{k-1 < |z| \le k\}$ into $2n(k)$ subrings, we find a subring $S_k = \{r_k - l_k < |z| \le r_k + l_k\}$ with $l_k = 1/(2n(k))$ and $S_k \cap \Gamma = \emptyset$. Put $S_k' = \{r_k - l_k/4 < |z| \le r_k + l_k/4\}$. Take a covering of S_k' by open balls $\{B_j'\}_{j=1}^J$, where $B_j' = B_{l_k/2}(z_j)$, $|z_j| = r_j$. Put $B_j = B_{l_k}(z_j)$. We may assume

$$\max_{z\in\mathbf{C}} \#\{j \mid B_j \ni z\} \le C,$$

where C is independent of k, j. Take $\Phi_j \in C^\infty(B_j)$ such that $|\Phi_j(z)| = 1$ and

$$\mathcal{L}\Phi_j = \Phi_j \mathcal{L}^{0,1}_{z_j}$$

in B_j. By this equality and the elliptic estimate, we have

$$\begin{aligned}
&\int_{B'_j} |(\nabla + i\mathbf{a})u|^2 dxdy = \int_{B'_j} |(\nabla + i\mathbf{a}^{0,1}_{z_j})\Phi_j^{-1}u|^2 dxdy \\
= \quad & \int_{B_{1/2}(0)} \left|(\nabla_{z'} + il_k^2\mathbf{a}_0(z'))\Phi_j^{-1}u(l_kz' + z_j)\right|^2 dx'dy' \\
\leq \quad & C\int_{B_1(0)} \left(|(\nabla_{z'} + il_k^2\mathbf{a}_0(z'))^2\Phi_j^{-1}u(l_kz' + z_j)|^2 + |\Phi_j^{-1}u(l_kz' + z_j)|^2\right) dx'dy' \\
\leq \quad & Cl_k^{-2}\int_{B_j} (|Hu|^2 + |u|^2)dxdy,
\end{aligned}$$

where C denotes general constants independent of k, j and $z' = x' + iy'$. Summing up this inequality with respect to j, we have

$$\int_{S'_k} |(\nabla + i\mathbf{a})u|^2 dxdy \leq Cl_k^{-2}.$$

Thus, substituting $a = r_k - l_k/4$, $b = r_k + l_k/4$ into (65), we have

$$c_1 + \cdots + c_{k-1} \leq C(1 + n(k)^2).$$

So we have

$$\liminf_{k\to\infty} \frac{\log\log c_k}{k} \leq \liminf_{k\to\infty} \frac{\log\log n(k)}{k} < \log 2.$$

This contradicts (67). Thus we have $(\nabla + i\mathbf{a})u \in L^2(\mathbf{C})$.

Put $\chi_n = \chi_{n,2n}$. Then, by the Leibniz formula and the fact u, $(\nabla + i\mathbf{a})u$, Hu are square integrable, we see that $\chi_n u \to u$ in $D(L^*)$. Since $\chi_n u \in D(H)$ and $D(H)$ is a closed subspace of $D(L^*)$, we have $u \in D(H)$. □

Thus the proposition is proved. □

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