A NEW CLASS OF BANACH SPACES

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Abstract. In this paper, we construct a new class of separable Banach spaces $\mathbb{KS}^p$, which parallel the standard L^p spaces, but contains each of these spaces, as well as the space of finitely additive measures, as compact dense embeddings. Equally important is the fact that these spaces contain all Henstock-Kurzweil integrable functions and, in particular, the Feynman kernel and the Dirac measure, as norm bounded elements.

1. Introduction

The standard university analysis courses tend to produce a natural bias and unease concerning the use of finitely additive set functions as a basis for the general theory of integration (despite the efforts of Alexkandrov [AX], Bochner [BO], Dubins [DU], Dunford and Schwartz [DS], de Finetti [DFN] and Yosida and Hewitt [YH]). (We should remember that the concept of measure was, and is, important for geometry and some, but not all, parts of analysis. In other parts, the concept of integral tends to dominate.)

Without denying an important place for countable additivity, Dubins ([DU], [BD], and [DUK]) argues forcefully for the intrinsic advantages in using finite additivity in the basic axioms of probability theory. (The penetrating analysis of the foundations of probability theory by de Finetti [DFN] also supports this position.)

1991 *Mathematics Subject Classification.* Primary (45) Secondary(46) .

Key words and phrases. Banach space, Henstock-Kurzweil integral, Feynman path integral.

In a very interesting paper [DU], Dubins shows that the Wiener process has a number of cousins, related processes with the same finite dimensional distributions as the Wiener process. For example, there is one cousin with polynomial paths and another with piecewise linear paths. Since the Wiener measure is unique, these cousins must necessarily have finitely additive limiting distributions.

The most important of the finitely additive measures is the one generated by the Henstock-Kurzweil integral (HK-integral), which generalizes the Lebesgue, Bochner and Pettis integrals. (It was discovered independently by Henstock [HS] and Kurzweil [KW].) The HK-integral is equivalent to the the wide-sense Denjoy integral. However, it is much easier to understand (and learn) compared to the Denjoy and Lebesgue integrals, and provides useful variants of the same theorems that have made the Lebesgue integral so important. Furthermore, it arises from a simple (transparent) generalization of the Riemann integral that was taught in elementary calculus. Loosely speaking, one uses a version of the Riemann integral with the interior points chosen first, while the size of the base rectangle around any interior point is determined by an arbitrary positive function defined at that point (see Henstock [HS], Kurzweil [KW], or Pfeffer,[PF]).

1.1. **Purpose.** Clearly, the most important factor preventing the widespread use of the HK-integral in physics and engineering has been the lack of a natural Banach space structure for this class of functions (as is the case for the Lebesgue integral). The purpose of this paper is to introduce a new class of Banach spaces $\mathbb{KS}^p(\Omega),\ 1 \le p \le \infty$, with $\Omega \subset \mathbb{R}^n,\ (n = 1, 2, \cdots)$. These are all separable spaces that contain the corresponding $\mathbf{L}^p$ spaces as dense, continuous, compact embeddings. Our original interest was in the fact that each of these spaces contains the wide-sense Denjoy integrable functions, as well as all the finitely additive measures. These spaces are perfect for the highly oscillatory functions that occur in quantum theory and nonlinear analysis.

2. $\mathbb{KS}^p$ Spaces

In order to construct the spaces, first recall that the HK-integral is equivalent to the Denjoy integral (see Henstock [HS] or Pfeffer [PF]). In the one-dimensional case, Alexiewicz [AL] has shown that the class $D(\mathbf{R})$, of Denjoy integrable functions, can be normed in the following manner: for $f \in D(\mathbf{R})$, define $\|f\|_D$ by

$$\|f\|_D = \sup_s \left| \int_{-\infty}^{s} f(r)dr \right|.$$

It is clear that this is a norm, and it is known that $D(\mathbf{R})$ is not complete (see Alexiewicz [AL]). Replacing $\mathbf{R}$ by $\mathbf{R}^n$, for $f \in D(\mathbf{R}^n)$ we have the following generalization:

$$\|f\|_D = \sup_{r>0} \left| \int_{\mathbf{B}_r} f(\mathbf{x})d\mathbf{x} \right| = \sup_{r>0} \left| \int_{\mathbf{R}^n} \mathcal{E}_{\mathbf{B}_r}(\mathbf{x}) f(\mathbf{x})d\mathbf{x} \right| < \infty, \tag{2.1}$$

where $\mathbf{B}_r$ is any closed cube of diagonal r centered at the origin in $\mathbf{R}^n$, with sides parallel to the coordinate axes, and $\mathcal{E}_{\mathbf{B}_r}(\mathbf{x})$ is the characteristic function of $\mathbf{B}_r$.

Now, fix n, and let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbf{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbf{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed cube centered at $\mathbf{x}_i$, with sides parallel to the coordinate axes and diagonal $r_l = 2^{-l}, l \in \mathbb{N}$. Now choose an order for $\mathbb{N} \times \mathbb{N}$ so that the set $\{\mathbf{B}_k,\ k \in \mathbb{N}\}$ contains all closed cubes $\{\mathbf{B}_l(\mathbf{x}_i)\ |(l, i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$. Let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$, so that $\mathcal{E}_k(\mathbf{x})$ is in $\mathbf{L}^p[\mathbf{R}^n] \cap \mathbf{L}^\infty[\mathbf{R}^n]$ for $1 \le p < \infty$. Define $F_k(\,\cdot\,)$

on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$F_k(f) = \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}. \tag{2.2}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $\mathbf{L}^p[\mathbf{R}^n]$ for each k, $\|F_k\|_\infty \le 1$ and, if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $\mathbf{L}^p[\mathbf{R}^n]$ for $1 \le p \le \infty$. Fix $t_k > 0$ such that $\sum_{k=1}^\infty t_k = 1$ and define a measure $d\mathbf{P}(\mathbf{x}, \mathbf{y})$ on $\mathbf{R}^n \times \mathbf{R}^n$ by:

$$d\mathbf{P}(\mathbf{x}, \mathbf{y}) = \left[\sum\nolimits_{k=1}^\infty t_k \mathcal{E}_k(\mathbf{x}) \mathcal{E}_k(\mathbf{y})\right] d\mathbf{x} d\mathbf{y}.$$

We first construct our Hilbert space. Define an inner product $(\,\cdot\,)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$\begin{aligned} (f, g) &= \int_{\mathbf{R}^n \times \mathbf{R}^n} f(\mathbf{x}) g(\mathbf{y})^* d\mathbf{P}(\mathbf{x}, \mathbf{y}) \\ &= \sum\nolimits_{k=1}^\infty t_k \left[\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right] \left[\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{y}) g(\mathbf{y}) d\mathbf{y}\right]^*. \end{aligned} \tag{2.3}$$

We use a different choice of t_k in Gill and Zachary [GZ], which is suggested by physical analysis in another context. We call the completion of $\mathbf{L}^1[\mathbf{R}^n]$, with the above inner product, the Kuelbs-Steadman space, $\mathbb{KS}^2[\mathbf{R}^n]$. Following suggestions of Gill and Zachary, Steadman [ST] constructed this space by adapting an approach developed by Kuelbs [KB] for other purposes. Her interest was in showing that $\mathbf{L}^1[\mathbf{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the HK-integrable functions. To see that this is the case, let $f \in D[\mathbf{R}^n]$, then:

$$\|f\|_{\mathbf{KS}^2}^2 = \sum\nolimits_{k=1}^\infty t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^2 \leqslant \sup_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^2 \leqslant \|f\|_D^2 ,$$

so $f \in \mathbb{KS}^2[\mathbf{R}^n]$.

Theorem 1. *For each p, $1 \leqslant p \leqslant \infty$, $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^p[\mathbf{R}^n]$ as a dense subspace.*

Proof. By construction, $\mathbb{KS}^2[\mathbf{R}^n]$ contains $\mathbf{L}^1[\mathbf{R}^n]$ densely, so we need only show that $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ for $q \neq 1$. If $f \in \mathbf{L}^q[\mathbf{R}^n]$ and $q < \infty$, we have

$$\begin{aligned}
\|f\|_{\mathbf{KS}^2} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^{\frac{2q}{q}}\right]^{1/2} \\
&\leqslant \left[\sum\nolimits_{k=1}^{\infty} t_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{2}{q}}\right]^{1/2} \\
&\leqslant \sup_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{1}{q}} \leqslant \|f\|_q .
\end{aligned}$$

Hence, $f \in \mathbb{KS}^2[\mathbf{R}^n]$. For $q = \infty$, we have

$$\begin{aligned}
\|f\|_{\mathbf{KS}^2} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^2\right]^{1/2} \\
&\leqslant \left[\left[\sum\nolimits_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^2\right] [ess\sup|f|]^2\right]^{1/2} \leqslant M \|f\|_\infty .
\end{aligned}$$

Thus $f \in \mathbb{KS}^2[\mathbf{R}^n]$, and $\mathbf{L}^\infty[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$. ◻

The fact that $\mathbf{L}^\infty[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$, while $\mathbb{KS}^2[\mathbf{R}^n]$ is separable makes it clear in a very forceful manner that whether a space is separable or not depends on the topology. Before proceeding to additional study, we need to construct $\mathbb{KS}^p[\mathbf{R}^n]$.

To construct $\mathbb{KS}^p[\mathbf{R}^n]$ for all p and for $f \in \mathbf{L}^p$, define:

$$\|f\|_{\mathbf{KS}^p} = \begin{cases} \left\{\sum_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^p\right\}^{1/p}, 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|, p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{\mathbf{KS}^p}$ defines a norm on $\mathbf{L}^p$. If $\mathbf{KS}^p$ is the completion of $\mathbf{L}^p$ with respect to this norm, we have:

Theorem 2. *For each q, $1 \leqslant q \leqslant \infty$, $\mathbb{KS}^p[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ as a dense continuous embedding.*

Proof. As in the previous theorem, by construction $\mathbb{KS}^p[\mathbf{R}^n]$ contains $\mathbf{L}^p[\mathbf{R}^n]$ densely, so we need only show that $\mathbb{KS}^p[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ for $q \neq p$. First, suppose that $p < \infty$. If $f \in \mathbf{L}^q[\mathbf{R}^n]$ and $q < \infty$, we have

$$\begin{aligned}
\|f\|_{\mathbf{KS}^p} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^{\frac{qp}{q}}\right]^{1/p} \\
&\leqslant \left[\sum\nolimits_{k=1}^{\infty} t_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{p}{q}}\right]^{1/p} \\
&\leqslant \sup_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{1}{q}} \leqslant \|f\|_q .
\end{aligned}$$

Hence, $f \in \mathbb{KS}^p[\mathbf{R}^n]$. For $q = \infty$, we have

$$\begin{aligned}
\|f\|_{\mathbf{KS}^p} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^{p}\right]^{1/p} \\
&\leqslant \left[\left[\sum\nolimits_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^p\right] \left[ess\sup |f|\right]^p\right]^{1/p} \leqslant M \|f\|_{\infty} .
\end{aligned}$$

Thus $f \in \mathbb{KS}^p[\mathbf{R}^n]$, and $\mathbf{L}^\infty[\mathbf{R}^n] \subset \mathbb{KS}^p[\mathbf{R}^n]$. The case $p = \infty$ is obvious. □

Theorem 3. *For $\mathbb{KS}^p$, we have:*

(1) *If $1 \leq p \leq \infty$ and $f, g \in \mathbb{KS}^p$, then*

$$\|f+g\|_{\mathbf{KS^p}} \leqslant \|f\|_{\mathbf{KS^p}} + \|g\|_{\mathbf{KS^p}} \text{ (Minkowski inequality).}$$

(2) *If, with $1 \leq p \leq \infty$, K is a weakly sequentially compact subset of $\mathbf{L}^p$, it is a compact subset of $\mathbb{KS}^p$.*

(3) *If $1 < p < \infty$, then $\mathbb{KS}^p$ is uniformly convex.*

(4) *If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then the dual space of $\mathbb{KS}^p$ is $\mathbb{KS}^q$.*

Proof. The proof of (1) follows from the classical case for sums. The proof of (2) follows from the fact that, if $\{f_n\}$ is any weakly convergent sequence in K with limit f, then

$$\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left[f_n(\mathbf{x}) - f(\mathbf{x}) \right] d\mathbf{x} \to 0$$

for each k. It follows that $\{f_n\}$ converges strongly to f in $\mathbb{KS}^p$.

The proof of (3) follows from a modification of the proof of the Clarkson inequalities for l^p norms.

In order to prove (4), observe that the linear functional

$$L_g(f) = \|g\|_{\mathbf{KS}^p}^{2-p} \sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \right|^{p-2} \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) f(\mathbf{y})^* d\mathbf{y}$$

is a unique duality map on $\mathbb{KS}^q$ for each $g \in \mathbb{KS}^p$ and that $\mathbb{KS}^p$ is reflexive from (3). $\square$

In many applications, it is convenient to formulate problems on one of the standard Sobolev spaces $\mathbf{W}_p^m(\mathbb{R}^n)$. We close this section with the corresponding extension of $\mathbb{KS}^p$. First recall that the space:

$$\mathbf{X}_p^m(\mathbb{R}^n) = \{B_\alpha * g = (I - \triangle)^{-\alpha/2} g : g \in \mathbf{L}^p(\mathbb{R}^n),\ 0 < \alpha < n,\ 0 < \alpha < m\}$$

coincides with $\mathbf{W}_p^m(\mathbb{R}^n)$, where B_α is the Bessel potential of order α, $\triangle$ is the Laplacian and $*$ is the convolution operator.

Theorem 4. *The completion of $\mathbf{X}_p^m(\mathbb{R}^n)$ with the $\mathbb{KS}^p(\mathbb{R}^n)$ norm defines the space $\mathbb{KS}_p^m(\mathbb{R}^n)$, which contains $\mathbf{W}_p^m(\mathbb{R}^n)$ as a continuous dense and compact embedding.*

Proof. Since $B_\alpha \in \mathbf{L}^1(\mathbb{R}^n)$, we see from Young's inequality for convolutions that $B_\alpha * g \in \mathbf{L}^p(\mathbb{R}^n)$ if $g \in \mathbf{L}^p(\mathbb{R}^n)$ and $0 < \alpha < n$. $\square$

2.1. **Extension of Fourier and Convolution Operators.** Let $L[\mathcal{B}], L[\mathcal{H}]$ denote the bounded linear operators on a separable Banach or Hilbert space, $\mathcal{B}, \mathcal{H}$, respectively. We assume that $\mathcal{B}$ is a continuous and dense embedding in $\mathcal{H}$. A proof of the next theorem may be found in Gill et al [GBZS], and is a generalization of a well-known theorem of Lax [LX].

Theorem 5. *Let A be a bounded linear operator on $\mathcal{B}$. Then A has a bounded extension to $L[\mathcal{H}]$, with $\|A\|_{\mathcal{H}} \leq k \|A\|_{\mathcal{B}}$ with k constant;*

We now use Theorem 5 to prove that $\mathfrak{F}$ and $\mathfrak{C}$, the Fourier (transform) and the convolution operator respectively, defined on $\mathbf{L}^1[\mathbf{R}^n]$, have bounded extensions to $\mathbb{KS}^p[\mathbf{R}^n]$. It should be noted that this theorem also implies that both operators have bounded extensions to all $\mathbf{L}^p[\mathbf{R}^n]$ spaces for $1 \leq p < \infty$. This is the first proof based on functional analysis, while the traditional proof is obtained via rather deep methods of (advanced) real analysis.

Theorem 6. *Both $\mathfrak{F}$ and $\mathfrak{C}$ extend to bounded linear operators on $\mathbb{KS}^p[\mathbf{R}^n]$.*

Proof. To prove our result, first use Theorem 5 to show that, since $\mathfrak{F}$ is a bounded linear operator on $\mathbf{L}^1[\mathbf{R}^n]$ and $\mathbf{L}^1[\mathbf{R}^n] \cap \mathbf{L}^p[\mathbf{R}^n]$ is dense in $\mathbb{KS}^p[\mathbf{R}^n]$, it extends to a bounded linear operator on $\mathbb{KS}^p[\mathbf{R}^n]$, $1 \leq p < \infty$.

To prove that $\mathfrak{C}$ has a bounded extension, fix g in $\mathbf{L}^1[\mathbf{R}^n]$ and define $\mathfrak{C}_g$ on $\mathbf{L}^1[\mathbf{R}^n]$ by:

$$\mathfrak{C}_g(f)(\mathbf{y}) = \int g(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

Once again, since $\mathfrak{C}_g$ is bounded on $\mathbf{L}^1[\mathbf{R}^n]$ and $\mathbf{L}^1[\mathbf{R}^n] \cap \mathbf{L}^p[\mathbf{R}^n]$ is dense in $\mathbb{KS}^p[\mathbf{R}^n]$, by Theorem 5 it extends to a bounded linear operator on $\mathbb{KS}^p[\mathbf{R}^n]$. Now use the fact that convolution is commutative to get that $\mathfrak{C}_f$ is a bounded linear operator on $\mathbf{L}^1[\mathbf{R}^n]$ for all $f \in \mathbb{KS}^p[\mathbf{R}^n]$. Another application of Theorem 5 completes the proof. $\square$

Note that, since $\mathbf{L}^1(\mathbf{R}^n) \subset \mathbb{KS}^p(\mathbf{R}^n)$ and $\mathbb{KS}^p(\mathbf{R}^n)$ is reflexive for $1 < p < \infty$, we see that the second dual $\left\{\mathbf{L}^1(\mathbf{R}^n)\right\}^{**} = \mathfrak{M}(\mathbf{R}^n) \subset \mathbb{KS}^p(\mathbf{R}^n)$. Recall that $\mathfrak{M}(\mathbf{R}^n)$ is the space of bounded finitely additive set functions defined on the Borel sets $\mathfrak{B}(\mathbf{R}^n)$. This space contains the Dirac delta function and the free-particle Green's function for the Feynman integral.

Definition 7. *A sequence $\{\mu_k\} \subset \mathfrak{M}[\mathbf{R}^n]$ is said to converge weakly to μ, ($\mu_n \xrightarrow{w} \mu$) if, for every bounded uniformly continuous function $h(\mathbf{x})$,*

$$\int_{\mathbf{R}^n} h(\mathbf{x}) d\mu_n \to \int_{\mathbf{R}^n} h(\mathbf{x}) d\mu.$$

Theorem 8. *If $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbf{R}^n]$, then $\mu_n \xrightarrow{s} \mu$ (strongly) in $\mathbb{KS}^p[\mathbf{R}^n]$.*

Proof. Since the characteristic function of a closed cube is a bounded uniformly continuous function, $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbf{R}^n]$ implies that

$$\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) d\mu_n \to \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) d\mu$$

for each k, so that $\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathbf{KS}^p} = 0$. $\square$

A little reflection gives:

Theorem 9. *The space $\mathbb{KS}^p[\mathbf{R}^n]$ is a commutative Banach algebra with unit, for $1 \le p \le \infty$.*

2.2. **Application: Feynman Path Integral.**

The properties of $\mathbb{KS}^p[\mathbf{R}^n]$ derived earlier suggests that $\mathbb{KS}^2[\mathbf{R}^n]$ may be a better Hilbert space than $\mathbf{L}^2[\mathbf{R}^n]$ for the study of the path integral formulation of quantum theory developed by Feynman. Note that it is easy to prove that both the position and momentum operators have closed densely defined extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. Furthermore, the extensions of $\mathfrak{F}$ and $\mathfrak{C}$ insure that all of the Schrödinger and Heisenberg theory has a faithful representation on $\mathbb{KS}^2[\mathbf{R}^n]$. These issues will be discussed more fully in another venue.

Since $\mathbb{KS}^p[\mathbf{R}^n]$ contains the space of measures, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $\mathbb{KS}^p[\mathbf{R}^n]$ topology. (For example, $[\sin(\lambda \cdot \mathbf{x})/(\lambda \cdot \mathbf{x})] \in \mathbb{KS}^p[\mathbf{R}^n]$ and converges strongly to $\delta(\mathbf{x})$.) Thus, the finitely additive set function defined on the Borel sets (Feynman kernel):

$$\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B] = \int_B (2\pi i(t-s))^{-n/2} \exp\{i|\mathbf{x}-\mathbf{y}|^2 \big/ 2(t-s)\} d\mathbf{y}$$

is in $\mathbb{KS}^p[\mathbf{R}^n]$ and $\|\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B]\|_{\mathrm{KS}} \leqslant 1$, while $\|\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B]\|_{\mathfrak{M}} = \infty$ (the variation norm) and

$$\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B] = \int_{\mathbf{R}^n} \mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, \tau, d\mathbf{z}]\mathbb{K}_{\mathbf{f}}[\tau, \mathbf{z}\,;\, s, B], \text{ (HK-integral).}$$

Definition 10. *Let* $\mathbf{P}_n = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$ *be a HK-partition of the interval* $[0, t]$ *for each* n*, with* $\lim_{n\to\infty} \Delta\mu_n = 0$ *(mesh). Set* $\Delta t_j = t_j - t_{j-1}, \tau_0 = 0$ *and, for* $\psi \in \mathbb{KS}^p[\mathbf{R}^n]$*, define*

$$\int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\, \mathbf{x}(0)] = e^{-\lambda t} \sum_{k=0}^{[\lambda t]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbf{R}^n} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(\tau_j)\,;\, t_{j-1}, d\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$(2.4)\ \int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(0)]\psi[\mathbf{x}(0)] = \lim_{\lambda\to\infty} \int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_{\lambda}\mathbf{x}(\tau)\ ;\ \mathbf{x}(0)]\psi[\mathbf{x}(0)]$$

whenever the limit exists.

The next result is now elementary. A more general (sum over paths) result, that covers almost all application areas, will appear later.

Theorem 11. *The function* $\psi(\mathbf{x}) \equiv 1 \in \mathbb{KS}^p[\mathbf{R}^n]$ *and*

$$\int_{\mathbf{R}^{n[s,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(s)] = \mathbb{K}_{\mathbf{f}}[t, \mathbf{x};\ s, \mathbf{y}] = \tfrac{1}{\sqrt{[2\pi i(t-s)]^n}} \exp\{i\left|\mathbf{x}-\mathbf{y}\right|^2 \big/ 2(t-s)\}.$$

The above result is what Feynman was trying to obtain without the appropriate space. These spaces have been used to provide a generalization of the constructive representation theory for the Feynman operator calculus (see [GZ1]) to the Banach space setting.

Conclusion

In this paper we have shown how to construct a natural class of separable Banach spaces $\mathbb{KS}^p$, which parallels the standard L^p spaces but contains them as dense compact embeddings. These spaces are of particular interest because they contain the Henstock-Kurzweil integrable functions. We have also constructed the corresponding spaces $\mathbb{KS}^m_p$, of Sobolev type. As a elementary application, we have used $\mathbb{KS}^2$ to construct the free particle path integral in the manner originally intended by Feynman.

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