# Large deviations for currents in the spin-boson model

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**Abstract:** We consider a finite quantum system coupled to quasifree thermal reservoirs at different temperatures. Under the assumptions of small coupling and exponential decay of the reservoir correlation function, the large deviation generating function is shown to be analytic on a compact set. Our method is different from the spectral deformation technique which was introduced recently in the study of spin-boson-like models. As a corollary, we derive the Gallavotti-Cohen fluctuation relation for the entropy production.

**KEY WORDS:** weak coupling limit, large deviations, nonequilibrium statistical mechanics, cluster expansions, spin-boson model

# 1 Introduction

## 1.1 Large deviations

The theory of large deviations lies at the heart of classical equilibrium statistical mechanics. Indeed thermodynamical potentials like e.g. the free energy can be viewed as large deviation generating functions. In the quest for a general theory of nonequilibrium statistical mechanics, it is hence natural to take large deviations as a starting point. Recently, some remarkable results have been obtained with that strategy, see e.g. [31] and [3]. A nonequilibrium large deviation result which has received huge attention in the last decade is the Gallavotti-Cohen fluctuation theorem [17, 18, 15], which states a symmetry in the fluctuations of entropy production.

What are large deviations? Assume that we have a family of measures $\mathbb{P}_{t\in\mathbb{R}^+}$ and corresponding random variables $A_t$ taking values in $\mathbb{R}^d$. Heuristically, the family $A_t$ satisfies a LDP (large deviation principle) iff. ,

$$\mathbb{P}_t(A_t \approx \alpha) \sim \mathrm{e}^{-tI(\alpha)}, \qquad \alpha \in \mathbb{R}^d, t \uparrow \infty \tag{1.1}$$

for some function $I(\alpha)$ which is called the *rate function*. A precise definition (see [6] for further details) states that there exists a lower-semicontinuous convex function $I : \mathbb{R}^d \to \mathbb{R}^+$

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satisfying

$$\limsup_{t\nearrow\infty}\frac{1}{t}\log\mathbb{P}_t[A_t\in E] \;\leq\; -\inf_{\alpha\in E}I(\alpha), \qquad E \quad \text{closed} \tag{1.2}$$

$$\liminf_{t\nearrow\infty}\frac{1}{t}\log\mathbb{P}_t[A_t\in E] \;\geq\; -\inf_{\alpha\in E}I(\alpha), \qquad E \quad \text{open} \tag{1.3}$$

Define the large deviation generating function on $\mathbb{R}^d$

$$F(\kappa) := \lim_{t\uparrow\infty}\frac{1}{t}\log\int \mathrm{d}\mathbb{P}_t(\alpha)\mathrm{e}^{-t(\kappa|\alpha)} \tag{1.4}$$

where $(\cdot|\cdot)$ is the canonical scalar product on $\mathbb{R}^d$.

The useful Gartner-Ellis theorem states that if $F(\kappa)$ exists and is differentiable, then (1.2, 1.3) hold with $I$ being the Legendre transform of $F$. By a theorem of Bryc [4], analyticity of $F$ in a neighbourhood of $\kappa=0$ implies that the variables $A_t$ satisfy a central limit theorem, with covariance matrix given by the Hessian of $F$ in $\kappa=0$.

We now specialize the setup to nonequilibrium physics. We consider the spin-boson model with several quasifree boson reservoirs (indexed by $k\in\mathcal{K}$) at inverse temperatures $\beta_k$, interacting through a small system (the 'spin'). Let $t$ be time and let $A_{k,t}$ be the time-averaged integrated heat flow up to time $t$ into reservoir $k$. By $F_A$ we denote the large deviation generating function corresponding to $A_{k,t}$ as outlined above. Let $E_t := \sum_{k\in\mathcal{K}}\beta_k A_{k,t}$, which is interpreted as the time-averaged entropy production up to time $t$ and let $I_E, F_E$ be the rate function/generating function associated to $E_t$. The Gallavotti-Cohen fluctuation theorem states

$$I_E(\alpha)-I_E(-\alpha)=-\alpha, \qquad \text{or, equivalently} \qquad F_E(\kappa)=F_E(1-\kappa) \tag{1.5}$$

under very general conditions. It is easy to derive heuristically, see e.g. [24] for a quantum version. The technical work lies in establishing the LDP.

In the present paper, we establish analyticity of the large deviation generating function $F_A$ (corresponding to the variables $A_{k,t}$) on a compact set (Our method does not extend to the full $\mathbb{R}^{|\mathcal{K}|}$). This is described in Theorem 2.4. As a consequence, one has also analyticity of $F_E$ and we reproduce the Gallavotti-Cohen fluctuation theorem (Corollary 2.12). For completeness, we state also the central limit theorem (Corollary 2.13).

In sharp contrast to classical lattice systems, where the existence of the generating function follows from quite general considerations, see [35], there is no standard quantum theory of large deviations yet, not even in equilibrium. However, some results have been obtained in [26, 20, 33, 28] and recently in [1].

## 1.2 Technique: classical polymer model

A large part of the interest of the present paper resides in the technique of the proof. In the past years, quite some work has been devoted to spin-boson and spin-fermion models.

We mention early work [19],[2], the series of papers by Jakšić and Pillet, e.g. [23, 22] and work on the Green-Kubo relations [21]. The elegant technique which was developed for these models consists in identifying an operator whose spectrum contains information on the approach to equilibrium. This spectrum is then studied by complex deformations. A drawback is that the conditions imposed on the coupling between the small system and the quasi-free bath are typically very restrictive, e.g. in [22] the coupling function is required to be analytic in a strip, see however [9], [10] and [32] for an approach which requires much weaker conditions.

We propose a different technique which reduces the time-evolution of the coupled system to a classical polymer model with complex polymer weights. This can be done by exploiting the weak coupling limit in which the evolution of the small system becomes Markovian and dissipative (see [5] for the original rigorous paper and [27] or [8] for a review.) The polymers represent excitations around the Markovian dynamics, an idea which is inspired by [30].

(On the other hand, the above-mentioned spectral approach also exploits the weak coupling limit, but on the level of the resolvent. Hence, one could call our technique the dynamic counterpart to the spectral approach.)

The drawback of our technique is its focus on the small system, and, most importantly, that it is much less elegant than the spectral approach. A possible advantage of our technique is its conceptual simplicity and its robustness. For example, it is straightforward to prove and quantify the approach to a stationary state and decay of correlations in the small system under conditions comparable to those in [10]. This will be done in a future paper.

In this paper, we however assume quite strong conditions, namely exponential decay of the reservoir correlation functions (see Assumption 2.3). These strong conditions are necessary to control the cluster expansion which proves analyticity of the large deviation generating function. This means however that our conditions are almost identical to those in [22].

## 1.3 What are current fluctuations ?

The question what quantum current fluctuations are, and how one should generalize the Gallavotti-Cohen theorem, has received quite some attention in the statistical physics community, see [13] for a discussion of our viewpoint and a list of references. For the sake of consistency, we outline our approach and motivation.

We consider finite system where all questions can be answered and then we take the thermodynamical limit of interesting quantities, *in casu* the Laplace transform of the probability distribution of energy transport.

Fix a finite-dimensional Hilbert space $\mathcal{H}_{\mathrm{S}}$ with self-adjoint Hamiltonian $H_{\mathrm{S}}$ and let $\mathcal{K}$ be a finite set which indexes the heat reservoirs at inverse temperatures $\beta_{k\in\mathcal{K}} > 0$. To each $k \in \mathcal{K}$, we associate (the superscript $n \in \mathbb{N}$ indicates that the thermodynamic limit has not yet been taken)

1) The self-adjoint coupling operator $V_k \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$.
2) A finite-dimensional one-particle Hilbert space $\mathfrak{h}_k^n$ and its bosonical second quantization $\Gamma(\mathfrak{h}_k^n)$. For $g \in \mathfrak{h}_k^n$, let $a^*(g)/a(g)$ be creation/annihilation operators on $\Gamma(\mathfrak{h}_k^n)$, satisfying
$$[a(g), a^*(g')] = \langle g, g' \rangle_{\mathfrak{h}_k^n}, \qquad g, g' \in \mathfrak{h}_k^n \tag{1.6}$$
3) A self-adjoint one-particle Hamiltonian $h_k^n \in \mathcal{B}(\mathfrak{h}_k^n)$ with corresponding second quantization $H^n_{\mathrm{R}_k} := \mathrm{d}\Gamma(h_k^n)$ and a coupling function $f_k^n \in \mathfrak{h}_k^n$.
4) A Gibbs state $\rho^n_{k,\beta_k}$ on $\mathcal{B}(\Gamma(\mathfrak{h}_k^n))$ at inverse temperature $\beta_k$
$$\rho^n_{k,\beta_k}[\cdot] = \frac{1}{Z_k^n(\beta_k)} \operatorname{Tr}\left[\mathrm{e}^{-\beta_k H^n_{\mathrm{R}_k}} \cdot\right], \qquad Z_k^n(\beta_k) = \operatorname{Tr}\left[\mathrm{e}^{-\beta_k H^n_{\mathrm{R}_k}}\right] \tag{1.7}$$

We define the total interacting Hamiltonian on $\mathcal{H}_{\mathrm{S}} \otimes_{k\in\mathcal{K}} \Gamma(\mathfrak{h}_k^n)$ as
$$H^n_\lambda \;=\; H_{\mathrm{S}} + \sum_{k\in\mathcal{K}} H^n_{\mathrm{R}_k} + \lambda \sum_{k\in\mathcal{K}} V_k \otimes (a^*(f_k^n) + a(f_k^n)) \tag{1.8}$$
from which we construct the unitary time-evolution
$$U_t^{\lambda,n} := \mathrm{e}^{-\mathrm{i}tH^n_\lambda} \tag{1.9}$$
We take as initial state
$$\rho_{\mathrm{S}} \otimes \rho^n_{\mathrm{R}} := \rho_{\mathrm{S}} \otimes [\otimes_{k\in\mathcal{K}} \rho^n_{k,\beta_k}] \tag{1.10}$$
Note that the $H^n_{\mathrm{R}_k}$ mutually commute and that they have discrete spectrum. Denote by $x \in \mathbb{R}^{|\mathcal{K}|}$ (with components $x_{k\in\mathcal{K}}$) elements of $\times_{k\in\mathcal{K}} \mathrm{sp} H^n_{\mathrm{R}_k}$ and let $P^n_x$ be the spectral projection on $(x_k)_{k\in\mathcal{K}}$ corresponding to $(H^n_{\mathrm{R}_k})_{k\in\mathcal{K}}$.

Inspired by physics, we define the probability to measure an energy increase of resp. $y_k$ in reservoir $k$ after time $t$ and starting from the state (1.10), as (we group $y_{k\in\mathcal{K}}$ into $y \in \mathbb{R}^{|\mathcal{K}|}$)
$$\mathbb{P}^n_{\rho_{\mathrm{S}},t,\lambda}(y) := \sum_{\substack{x, x' \in \times_{k\in\mathcal{K}} \mathrm{sp} H^n_{\mathrm{R}_k} \\ x'_k - x_k = y_k}} \rho_{\mathrm{S}} \otimes \rho^n_{\mathrm{R}} \left[P^n_x U^{\lambda,n}_{-t} P^n_{x'} U^{\lambda,n}_t P^n_x\right] \tag{1.11}$$
where $\mathbb{P}^n_{\lambda,t,\rho_{\mathrm{S}}}(y)$ is set to zero when the sum on the RHS is empty. The physical idea behind this formula is clear: measure the energies $H^n_{\mathrm{R}_k}$ in the state $\rho_{\mathrm{S}} \otimes \rho^n_{\mathrm{R}}$ (thereby reducing the reservoirs to $P^n_x$), then switch on the time evolution $U^{\lambda,n}_t$, finally measure again (reducing the reservoirs to $P^n_{x'}$).

We use that the initial state commutes with the projections $P_x$, i.e.
$$\rho_{\mathrm{S}} \otimes \rho^n_{\mathrm{R}}[P_x \,\cdot\,] = \rho_{\mathrm{S}} \otimes \rho^n_{\mathrm{R}}[\,\cdot\, P_x], \tag{1.12}$$

to rewrite the Laplace transform of the measure (1.11) as

$$\sum_{y\in\mathbb{R}^{|\mathcal{K}|}} \mathrm{e}^{-\sum_{k\in\mathcal{K}}\kappa_k y_k}\mathbb{P}^n_{\rho_{\mathrm{S}},t,\lambda}(y) = \rho_{\mathrm{S}}\otimes\rho^n_{\mathrm{R}}\left[Y^n_{-\kappa}U^{\lambda,n}_{-t}Y^n_{\kappa}U^{\lambda,n}_t\right]. \tag{1.13}$$

where $\kappa\in\mathbb{R}^{|\mathcal{K}|}$ and $Y^n_\kappa = \mathrm{e}^{-\sum_{k\in\mathcal{K}}\kappa_k H^n_{\mathrm{R}_k}}$.

The infinite-volume analogue of (1.13), to be defined through Lemma (2.1), is the subject of our main result: Theorem 2.4.

## 1.4 Notation

For an indexed family of operators $J^m_{i=1}$, we assume the ordering

$$\prod_{i=1}^m J_i := J_m J_{m-1}\dots J_2 J_1 \tag{1.14}$$

For a Hilbert space $\mathfrak{h}$, we denote its symmetric (bosonical) second quantization by

$$\Gamma(\mathfrak{h}) := \bigoplus_{n\in\mathbb{N}} \otimes^n_{\mathrm{s}}\mathfrak{h} \tag{1.15}$$

where $\otimes^n_{\mathrm{s}}\mathfrak{h}$ is the symmetrized $n$'th tensor power of $\mathfrak{h}$. We define the 'vacuum vector' $\Psi\in\Gamma(\mathfrak{h})$ as the unit vector in $\mathbb{C}\sim\otimes^0_{\mathrm{s}}\mathfrak{h}$.

When dealing with tensor products of Hilbert spaces, we will often abbreviate $A\otimes 1$ and $1\otimes B$ by respectively $A$ and $B$.

If $\rho$ is a state (positive normalized functional) on a $\mathcal{C}^*$-algebra $\mathcal{A}$, we use the notation $\rho[R]$ with $R\in\mathcal{A}$ for the value of the functional $\rho$ in $R$. If $\mathcal{A}$ is finite-dimensional, we write $\tilde{\rho}\in\mathcal{A}$ for the associated density matrix, i.e. $\rho(R)=\mathrm{Tr}[\tilde{\rho}R]$.

If $A$ is a measurable subset of a measure space, we write $\mathrm{Ind}[A]$ for its characteristic function. ('Ind' from 'Indicator').

For $\kappa\in\mathbb{C}^d$, we let $\mathrm{Re}\,\kappa, \mathrm{Im}\,\kappa\in\mathbb{R}^d$ be defined by $(\mathrm{Re}\,\kappa)_i = \mathrm{Re}\,\kappa_i, (\mathrm{Im}\,\kappa)_i = \mathrm{Im}\,\kappa_i$ for $i=1,\dots,d$.

## 1.5 Outline

In Sections 2.1 and 2.2, we state the model, which is the infinite-volume limit of the setup presented in Section 1.3. The main result, Theorem 2.4, comes in Section 2.3, some corollaries are given in Section 2.4. An alternative way of reading the paper is by starting with Proposition 2.14, which builds on Section 1.3 and allows to go immediately to the main result.

The proof is spread over Sections 3 and 4. In Section 3, the general idea of the proof is given, necessary lemma's are stated and the connections between those lemma's are indicated. In the technical Section 4, all necessary lemma's from Section 3 are proven. Proofs of the corollaries are also contained in Section 4.

# 2 Model and results

## 2.1 Zero-temperature Hamiltonian

Let $\mathcal{H}_{\mathrm{S}}$ be a finite-dimensional Hilbert space with a self-adjoint Hamiltonian $H_{\mathrm{S}}$. To each reservoir $k \in \mathcal{K}$, we associate

1) The self-adjoint coupling operator $V_k \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$ and an inverse temperature $\beta_k > 0$.
2) The one-particle Hilbert space $\mathfrak{h}_k := L^2(\mathbb{R}^+, \mathfrak{l}_k)$ for some separable Hilbert space $\mathfrak{l}_k$ and its bosonical second quantization $\Gamma(\mathfrak{h}_k)$. For $g \in \mathfrak{h}_k$, let $a^*(g)/a(g)$ be creation/annihilation operators on $\Gamma(\mathfrak{h}_k)$ (see e.g. [7]), satisfying the canonical commutation relations.
$$[a(g), a^*(g')] = \langle g, g' \rangle_{\mathfrak{h}_k}, \qquad g, g' \in \mathfrak{h}_k \tag{2.1}$$
3) The coupling function $f_k \in \mathfrak{h}_k$ and the self-adjoint one-particle Hamiltonian $h_k$, acting on $\mathfrak{h}_k = L^2(\mathbb{R}^+, \mathfrak{l}_k)$ as multiplication by the variable $x \in \mathbb{R}^+$, i.e. $(h_k g)(x) = xg(x)$.

The total Hamiltonian with coupling constant $\lambda \in \mathbb{R}$,

$$H_\lambda = H_{\mathrm{S}} + \sum_{k \in \mathcal{K}} \mathrm{d}\Gamma(h_k) + \lambda \sum_{k \in \mathcal{K}} V_k \otimes (a^*(f_k) + a(f_k)) \tag{2.2}$$

is formally defined on a dense subspace of $\mathcal{H}_{\mathrm{S}} \otimes_{k \in \mathcal{K}} \Gamma(\mathfrak{h}_k)$ (see Section 3.2.1).

## 2.2 Finite-temperature dynamics

Define the Hilbert spaces

$$\begin{aligned} \mathfrak{h} \quad &:= \quad \underset{k \in \mathcal{K}}{\oplus} (\mathfrak{h}_k \oplus \mathfrak{h}_k) && (2.3) \\ \mathcal{H} : \quad &= \quad \mathcal{H}_{\mathrm{S}} \otimes \Gamma(\mathfrak{h}) && (2.4) \end{aligned}$$

and self-adjoint operators

$$\begin{aligned} l_k := &\quad (-h_k) \oplus h_k && \text{on} \quad \mathfrak{h}_k \oplus \mathfrak{h}_k && (2.5) \\ l := &\quad \underset{k \in \mathcal{K}}{\oplus} l_k && \text{on} \quad \mathfrak{h} && (2.6) \end{aligned}$$

Identify

$$\mathfrak{h}_k \oplus \mathfrak{h}_k \sim L^2(\mathbb{R}^-, \mathfrak{l}_k) \oplus L^2(\mathbb{R}^+, \mathfrak{l}_k) \sim L^2(\mathbb{R}, \mathfrak{l}_k) \tag{2.7}$$

Remark that $l_k$, as defined by (2.5), acts on $L^2(\mathbb{R}, \mathfrak{l}_k)$ as $(l_k g)(x) = xg(x)$. Define the effective coupling functions $\breve{f}_k : \mathbb{R} \to \mathfrak{l}_k$ by

$$\breve{f}_k(x) := \begin{cases} \frac{1}{\sqrt{1-e^{-\beta_k x}}} f_k(x) & x > 0 \\ \frac{1}{\sqrt{e^{-\beta_k x}-1}} \overline{f_k(-x)} & x < 0 \end{cases} \tag{2.8}$$

where we equipped the Hilbert spaces $\mathfrak{l}_k$ with a complex conjugation $\mathfrak{l}_k \ni v \to \overline{v}$. By the forthcoming Assumption 2.3, we have $\breve{f}_k \in L^2(\mathbb{R}, \mathfrak{l}_k)$.

By the natural embedding of $\mathfrak{h}_k \oplus \mathfrak{h}_k$ into $\mathfrak{h}$, we will sometimes consider $\breve{f}_k$ as an element of $\mathfrak{h}$. We are ready to write the formal *semistandard Liouvillian* on $\mathcal{H}$ with coupling constant $\lambda \in \mathbb{R}$

$$L_\lambda = H_{\mathrm{S}} + \mathrm{d}\Gamma(l) + \lambda \sum_{k \in \mathcal{K}} V_k \otimes (a^*(\breve{f}_k) + a(\breve{f}_k)) \tag{2.9}$$

This operator can easily be constructed rigorously as a self-adjoint operator on a dense subspace of $\mathcal{H}$, see Section 3.2.1. Let $\rho_{\mathrm{S}}$ be an arbitrary state (see Section 1.4) on $\mathcal{B}(\mathcal{H}_{\mathrm{S}})$ and let $\rho_{\mathrm{R}}$ be the vacuum state on $\mathcal{B}(\Gamma(\mathfrak{h}))$, i.e.

$$\rho_{\mathrm{R}}[\cdot] = \langle \Psi, \cdot \Psi \rangle \tag{2.10}$$

where $\Psi$ is the vacuum vector in $\Gamma(\mathfrak{h})$. Our initial state on $\mathcal{B}(\mathcal{H})$ will be $\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}$.

The relevance of the functions $\breve{f}_k$ and the Liouvillian (2.9) in combination with the vacuum state $\rho_{\mathrm{R}}$ is that (2.9) generates a $W^*$-dynamics on an appropriate subalgebra (the Araki-Woods $W^*$-algebra) of $\mathcal{B}(\mathcal{H})$, on which $\rho_{\mathrm{R}}$ represents a product of thermal states (at least on the reservoir part). This elegant theory of Araki-Woods representations is extensively discussed in the literature, see [7]. We don't pursue it here since it is not necessary to state the result. In fact, we could even completely ignore the above setup and state our result starting from finite-volume quantum systems. This is illustrated in Proposition 2.14.

Define the following unitaries for $t \in \mathbb{R}, \kappa \in (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$,

$$U_t^\lambda = \mathrm{e}^{-\mathrm{i}tL_\lambda}, \qquad Y_\kappa = \mathrm{e}^{-\sum_{k\in\mathcal{K}} \kappa_k \mathrm{d}\Gamma(l_k)} \tag{2.11}$$

We are ready to define our object of interest.

**Lemma 2.1.** *Assume there is a $D \subset \mathbb{R}^{|\mathcal{K}|}$ with $0 \in D$ such that for $\kappa \in D$ and $k \in \mathcal{K}$,*

$$\langle \breve{f}_k, \mathrm{e}^{-\kappa_k l_k} \breve{f}_k \rangle_{\mathfrak{h}} < \infty \tag{2.12}$$

*Than the functions*

$$\kappa \longrightarrow \begin{cases} \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}} \left[ Y_{-\kappa} U_{-t}^\lambda Y_\kappa U_t^\lambda \right] \\ \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}} \left[ U_{-s}^\lambda Y_{-\kappa} U_{-t}^\lambda Y_\kappa U_t^\lambda U_s^\lambda \right] \end{cases} \tag{2.13}$$

*have an analytical continuation from $(\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$ into $D^\circ + (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$.*

## 2.3 Result

To state our assumptions, we need to introduce some terminology. For $e \in \mathrm{sp} H_{\mathrm{S}}$, let $1_e(H_{\mathrm{S}})$ be the spectral projection on $e$, corresponding to $H_{\mathrm{S}}$. Let $\mathcal{F} := \mathrm{sp} H_{\mathrm{S}} - \mathrm{sp} H_{\mathrm{S}}$ and define for $\omega \in \mathcal{F}$ and $V_k$ as in Section 2.1,

$$V_{\omega,k} := \sum_{e-e'=\omega} 1_e(H_{\mathrm{S}}) V_k 1_{e'}(H_{\mathrm{S}}) \tag{2.14}$$

Denote for a subset $\mathcal{C} \subset \mathcal{B}(\mathcal{H}_{\rm S})$, the commutant $\mathcal{C}'$ by

$$\mathcal{C}' = \left\{ B \in \mathcal{B}(\mathcal{H}_{\rm S}) \,\middle|\, \forall C \in \mathcal{C} :\ [C, B] = 0 \right\} \tag{2.15}$$

The first assumption expresses that the coupling is sufficiently effective.

**Assumption 2.2.** *The functions $\breve{f}_k$ as defined in (2.8) are continuous in $\mathcal{F}$ and*

$$\left\{ V_{\omega,k} \,\middle|\, k \in \mathcal{K}, \omega \in \mathcal{F},\ \|\breve{f}_k(\omega)\| \neq 0 \right\}' = \mathbb{C}1 \tag{2.16}$$

The second assumption requires the reservoir correlation functions to decay exponentially.

**Assumption 2.3.** *The bound (2.12) holds for a compact $D \subset \mathbb{R}^{|\mathcal{K}|}$ with $0 \in D$. There are constants $\alpha, c > 0$ such that for $\kappa \in D$, $k \in \mathcal{K}$ and $t \in \mathbb{R}$,*

$$\left| \langle \breve{f}_k, \mathrm{e}^{(-\kappa_k - \mathrm{i}t) l_k} \breve{f}_k \rangle_{\mathfrak{h}} \right| < c\mathrm{e}^{-\alpha|t|} \tag{2.17}$$

Our main result establishes existence and analyticity of the large-deviation generating function:

**Theorem 2.4.** *Assume Assumptions 2.2 and 2.3. There is a $\lambda_0 > 0$ such that for $|\lambda| \leq |\lambda_0|$ and $\kappa \in D$,*

$$F(\kappa, \lambda) := \lim_{t \uparrow \infty} \frac{1}{t} \log \rho_{\rm S} \otimes \rho_{\rm R} \left[ Y_{-\kappa} U^{\lambda}_{-t} Y_{\kappa} U^{\lambda}_{t} \right] \tag{2.18}$$

*exists and is independent of $\rho_{\rm S}$. The function $D^{\circ} \ni \kappa \mapsto F(\kappa, \lambda)$ is real-analytic.*

The previous result assumes a specific (decoupled) initial state $\rho_{\rm S} \otimes \rho_{\rm R}$. The result is however still valid, when starting from an initial state, which has evolved from the decoupled state $\rho_{\rm S} \otimes \rho_{\rm R}$.

**Theorem 2.5.** *Assume the conditions of Theorem 2.4, then for all $s \leq 0$,*

$$F(\kappa, \lambda) = \lim_{t \uparrow \infty} \frac{1}{t} \log \rho_{\rm S} \otimes \rho_{\rm R} \left[ U^{\lambda}_{-s} Y_{-\kappa} U^{\lambda}_{-t} Y_{\kappa} U^{\lambda}_{t} U^{\lambda}_{s} \right] \tag{2.19}$$

***Remark* 2.6.** From the proof, it follows that

$$F(\kappa, \lambda) = \lambda^2 e_{\kappa} + o(\lambda^2), \qquad \lambda \to 0 \tag{2.20}$$

where $e_{\kappa}$ is the generating function obtained by starting from the weak coupling limit

$$e_{\kappa} := \lim_{t \uparrow \infty} \lim_{\lambda \downarrow 0} \frac{1}{t} \log \rho_{\rm S} \otimes \rho_{\rm R} \left[ Y_{-\kappa} U^{\lambda}_{-\lambda^{-2} t} Y_{\kappa} U^{\lambda}_{\lambda^{-2} t} \right] \tag{2.21}$$

(see [34, 13] for a discussion). Theorem 2.4 hence implies that one can interchange the limits $\lambda \searrow 0$ and $t \nearrow \infty$.

***Remark* 2.7.** Theorem 2.5, although mathematically more appealing than Theorem 2.4, does not have a clear physical meaning. This is because states like

$$\rho_{\rm S} \otimes \rho_{\rm R} \left[ U^{\lambda}_{-s} \cdot U^{\lambda}_{s} \right] \tag{2.22}$$

in general do not commute[2] with functions of $\mathrm{d}\Gamma(l_k)$, and hence one cannot repeat the construction of Section 1.3. For the same reason, we do not attempt to redo Theorem 2.4, starting from the NESS (nonequilibrium steady state), which is not considered in the present article.

***Remark* 2.8.** By Bochner's theorem, there is a nonnegative Borel measure $\mathbb{P}_{\rho_{\rm S},t,\lambda}$ on $\mathbb{R}^{|\mathcal{K}|}$ such that

$$\rho_{\rm S} \otimes \rho_{\rm R}[Y_{-\mathrm{i}\kappa} U^{\lambda}_{-t} Y_{\mathrm{i}\kappa} U^{\lambda}_{t}] = \int_{\mathbb{R}^{|\mathcal{K}|}} \mathrm{d}\mathbb{P}_{\rho_{\rm S},t,\lambda}(y) \mathrm{e}^{-\mathrm{i}\sum_{k\in\mathcal{K}} \kappa_k y_k} \tag{2.23}$$

for $\kappa \in \mathbb{R}^{|\mathcal{K}|}$. Plugging in $\kappa = 0$, one sees that $\mathbb{P}_{\rho_{\rm S},t,\lambda}$ is a probability measure. The variables $y_{k\in\mathcal{K}}$ represent the total heat flow into the $k$'th reservoir, rather than the time-averaged heat flow, as in Section 1.1. However, by the factor $t$ in the exponent of (1.4), the generating function $F(\kappa,\lambda)$ in Theorem 2.4 is the exact analogue of that in Section 1.1.

***Remark* 2.9.** If $\breve{f}_k \in L^2(\mathbb{R}, \mathfrak{l}_k)$, then one checks straightforwardly that the bound (2.12) is automatically satisfied for $0 \leq \kappa_k \leq \beta_k$. Alternatively, one could construct the functions (2.13) for these values of $\kappa$ by using the KMS-condition.

***Remark* 2.10.** Assumption 2.3 cannot hold for $D = \mathbb{R}^{|\mathcal{K}|}$, for that would imply that

$$\mathbb{C} \ni \kappa \to \langle \breve{f}_k, \mathrm{e}^{-\kappa_k l_k} \breve{f}_k \rangle_{\mathfrak{h}_k \oplus \mathfrak{h}_k} \tag{2.24}$$

is analytic and bounded, hence constant.

## 2.4 Corollaries

We describe some straightforward, though physically relevant corollaries to our main result.

It is convenient to introduce an additional assumption, which expresses that the model has an additional symmetry (traditionally this symmetry is time-reversal invariance)

**Assumption 2.11.** There is an antiunitary operator $\Theta$ on $\mathcal{H}$, satisfying

$$\Theta^2 = 1, \qquad \Theta\Psi = \Psi, \qquad \Theta L_\lambda \Theta = L_\lambda, \qquad \Theta \mathrm{d}\Gamma(l_k)\Theta = \mathrm{d}\Gamma(l_k) \tag{2.25}$$

One derives the analog of the Gallavotti-Cohen fluctuation theorem

[2] in the sense of (1.12)

**Corollary 2.12.** *Assume the conditions of Theorem 2.4 and Assumption 2.11. Define for $\zeta \in \mathbb{R}$, the vector $\kappa(\zeta) \in \mathbb{R}^{|\mathcal{K}|}$ by*

$$\kappa_k(\zeta) := \zeta\beta_k \tag{2.26}$$

*Then, for $\zeta$ such that $\kappa(\zeta), \kappa(1-\zeta) \in D$,*

$$F(\kappa(\zeta), \lambda) = F(\kappa(1-\zeta), \lambda) \tag{2.27}$$

By standard methods, the large deviation principle with analytic rate function yields a central limit theorem.

**Corollary 2.13.** *Assume the conditions of Theorem 2.4 for a domain $D \subset \mathbb{R}^{|\mathcal{K}|}$ such that $0 \in D^\circ$. Let the symmetric $|\mathcal{K}| \times |\mathcal{K}|$- matrix $M$ (covariance matrix) be defined by*

$$M_{k,k'} = \frac{\partial^2}{\partial\kappa_k \partial\kappa_{k'}} F(\kappa, \lambda)\Big|_{\kappa=0}, \tag{2.28}$$

*Recall the measure $\mathbb{P}_{\rho_\mathrm{S},t,\lambda}$ from Remark 2.8 and let $\mathbb{E}_{\rho_\mathrm{S},t,\lambda}$ be the associated expectation. Define the random variables*

$$Y_k^t := \frac{y_k - \mathbb{E}_{\rho_\mathrm{S},t,\lambda}[y_k]}{\sqrt{t}} \tag{2.29}$$

*Then for $\kappa \in \mathbb{R}^{|\mathcal{K}|}$,*

$$\lim_{t\uparrow\infty} \mathbb{E}_{\rho_\mathrm{S},t,\lambda}\left[\exp \mathrm{i} \sum_{k\in\mathcal{K}} \kappa_k Y_k^t\right] = \exp -\frac{1}{2}\left(\sum_{k,k'\in\mathcal{K}} \kappa_k M_{k,k'} \kappa_{k'}\right) \tag{2.30}$$

The next proposition relates $\rho_\mathrm{S} \otimes \rho_\mathrm{R}\left[Y_{-\kappa} U_{-t}^\lambda Y_\kappa U_t^\lambda\right]$ to finite-volume quantities. It refers to notation introduced in Section 1.3. Identify the $V_{k\in\mathcal{K}}, H_\mathrm{S} \in \mathcal{B}(\mathcal{H}_\mathrm{S})$ and $\beta_k > 0$ from Sections 1.3 and 2.1.

**Proposition 2.14.** *Assume there is a $D \subset \mathbb{R}^{|\mathcal{K}|}$ as in Lemma 2.1. Assume that for all $\kappa \in D$, $t \in \mathbb{R}$, $k \in \mathcal{K}$ and for all choices $\zeta, \zeta' \in \{0, \kappa_k\}$,*

$$\rho_{k,\beta_k}^n\left[a^*(\mathrm{e}^{(-\mathrm{i}t+\zeta)h_k^n} f_k^n) a(\mathrm{e}^{-\overline{\zeta'} h_k^n} f_k^n)\right] \underset{n\uparrow\infty}{\longrightarrow} \langle f_k, (\mathrm{e}^{\beta_k h_k} - 1)^{-1} \mathrm{e}^{(-\mathrm{i}t+\zeta-\zeta')h_k} f_k\rangle_{\mathfrak{h}_k} \tag{2.31}$$

$$\rho_{k,\beta_k}^n\left[a(\mathrm{e}^{(-\mathrm{i}t-\overline{\zeta})h_k^n} f_k^n) a^*(\mathrm{e}^{\zeta' h_k^n} f_k^n)\right] \underset{n\uparrow\infty}{\longrightarrow} \langle f_k, (1 - \mathrm{e}^{-\beta_k h_k})^{-1} \mathrm{e}^{(\mathrm{i}t-\zeta+\zeta')h_k} f_k\rangle_{\mathfrak{h}_k}. \tag{2.32}$$

*Then,*

$$\lim_{n\uparrow\infty} \rho_\mathrm{S} \otimes \rho_\mathrm{R}^n\left[Y_{-\kappa}^n U_{-t}^{\lambda,n} Y_\kappa^n U_t^{\lambda,n}\right] = \rho_\mathrm{S} \otimes \rho_\mathrm{R}\left[Y_{-\kappa} U_{-t}^\lambda Y_\kappa U_t^\lambda\right] \tag{2.33}$$

# 3 Proof of main theorem

## 3.1 The "path space"

For a (possibly unbounded) interval $I$, we write $I_2 := \{\eta \subset I \,||\eta| = 2\}$ for the set of unordered pairs in $I$. ($|\eta|$ is the number of elements in the set $\eta$.) Introduce the set $\Omega_I$ as

$$\Omega_I := \underset{n\in\mathbb{N}}{\cup} \Omega_{I,n}, \qquad \Omega_{I,n} := \big\{\sigma \subset (I_2 \times \{\mathrm{a}, \mathrm{b}\}) \,\big|\, |\sigma| = n\big\} \tag{3.1}$$

We will write $\Omega$ to denote $\Omega_{(-\infty,\infty)}$. For an interval $I$ and $\sigma \in \Omega$ define $\sigma_I \in \Omega_I$ as

$$\sigma_I := \sigma \cap (I_2 \times \{\mathrm{a}, \mathrm{b}\}) \tag{3.2}$$

For intervals $I_1, I_2$ such that $I_1 \cup I_2$ is a disconnected set, we define

$$\sigma_{I_1\cup I_2} := \sigma_{I_1} \cup \sigma_{I_2}, \qquad \Omega_{I_1\cup I_2} := \{\sigma \cup \sigma' \,| \sigma \in \Omega_{I_1}, \sigma' \in \Omega_{I_2}\} \tag{3.3}$$

Note however that if $I := I_1 \cup I_2$ is connected (hence an interval), $\{\sigma\cup\sigma' \,|\sigma \in \Omega_{I_1}, \sigma' \in \Omega_{I_2}\}$ is in general a strict subset of $\Omega_I$. We use a special notation for the singletons in $\Omega_I$,

$$\tilde{\Omega}_I := \Omega_{I,1} \qquad \text{with elements} \qquad \tilde{\sigma} \in \tilde{\Omega}_I \tag{3.4}$$

and we slightly abuse notation by writing $\tilde{\sigma} \in \sigma$ whenever $\tilde{\sigma} \in \tilde{\Omega}$ and $\sigma \cap \tilde{\sigma} = \tilde{\sigma}$. (The 'abuse' lies in confusing a singleton set with its only element)

Define $\varphi_1 : \mathbb{R} \times \mathbb{R} \times \{\mathrm{a}, \mathrm{b}\} \mapsto \tilde{\Omega}$ by putting $\varphi_1(s_1, t_i, \ell_1) := \{(\{s_1, t_1\}, \ell_1)\}$ for $s_1, t_1 \in \mathbb{R}$ and $\ell_1 \in \{\mathrm{a}, \mathrm{b}\}$. More generally, for

$$(s_1, t_1, \ell_1; \ldots ; s_n, t_n, \ell_n) \in (\mathbb{R} \times \mathbb{R} \times \{\mathrm{a}, \mathrm{b}\})^n, \tag{3.5}$$

we define

$$\varphi_n(s_1, t_1, \ell_1; \ldots ; s_n, t_n, \ell_n) := \overset{n}{\underset{i=1}{\cup}} \varphi_1(s_i, t_i, \ell_i) \in \Omega_n. \tag{3.6}$$

We now make $\Omega$ into a measure space. Let $\mu$ be the (Lesbegue $\times$ Lesbegue $\times$ counting)-measure on $\mathbb{R} \times \mathbb{R} \times \{\mathrm{a}, \mathrm{b}\}$ and let $\mu_n$ be its $n$-fold product on $(\mathbb{R} \times \mathbb{R} \times \{\mathrm{a}, \mathrm{b}\})^n$. Let $\mathcal{A}$ be the $\sigma$-algebra on $\Omega$ generated by the sets

$$\bigcup_{n\geq 1} \{\varphi_n(A_n) \,|\, A_n \text{ is a Borel subset of } (\mathbb{R} \times \mathbb{R} \times \{\mathrm{a}, \mathrm{b}\})^n\} \tag{3.7}$$

and finally we define the measure space $(\Omega, \mathcal{A}, \mu_\sigma)$ by putting, with $A_n$ as above,

$$\mu_\sigma(\emptyset) = 1 \tag{3.8}$$

$$\mu_\sigma(\varphi_n(A_n)) = \frac{1}{2^n n!}\mu_n \left(\varphi_n^{-1}\varphi_n(A_n)\right) \tag{3.9}$$

For convenience, we choose one element of $\varphi_{|\sigma|}^{-1}(\sigma)$ to paramatrize $\sigma$, namely the element satisfying $s_i \leq t_i$ for $i = 1, \dots, |\sigma|$ and $s_i \leq s_{i+1}$ for all $i = 1, \dots, |\sigma| - 1$. The $s_i, t_i, \ell_i$ thus determined will be indicated as $s_i(\sigma), t_i(\sigma), \ell_i(\sigma)$ and simply $s(\tilde{\sigma}), t(\tilde{\sigma}), \ell(\tilde{\sigma})$ for $\tilde{\sigma} \in \tilde{\Omega}$. We will abbreviate $\mathrm{d}\mu_\sigma(\sigma)$ as $\mathrm{d}\sigma$. Note that

$$\mathrm{d}\sigma = \prod_{i=1}^{|\sigma|} \mathrm{d}s_i(\sigma) \mathrm{d}t_i(\sigma) \mathrm{d}\ell_i(\sigma) \tag{3.10}$$

where $\mathrm{d}\ell_i(\sigma)$ refers to the counting measure on $\{\mathrm{a}, \mathrm{b}\}$.

We define the *support* of $\sigma \in \Omega$ as

$$\mathrm{Supp}(\sigma \in \Omega) = \underset{\tilde{\sigma} \in \sigma}{\cup} \mathrm{Supp}(\tilde{\sigma}), \qquad \mathrm{Supp}(\tilde{\sigma} \in \tilde{\Omega}) = [s(\tilde{\sigma}), t(\tilde{\sigma})], \tag{3.11}$$

## 3.2 Perturbation expansion: the Duhamel series

### 3.2.1 Existence of dynamics

Let $P_n$ be the projector on $\mathcal{H}_\mathrm{S} \otimes \otimes_\mathrm{s}^n \mathfrak{h}$, the $n$-particle subspace of $\mathcal{H}_\mathrm{S} \otimes \Gamma(\mathfrak{h}) \subset \mathcal{H}$. Define the subspace $\mathcal{D} \subset \mathcal{H}_\mathrm{S} \otimes \Gamma(\mathfrak{h})$ as

$$\psi \in \mathcal{D} \Leftrightarrow \exists c > 0: \ \|P_n \psi\| \leq \frac{c^n}{\sqrt{n!}}, \ n \in \mathbb{N} \tag{3.12}$$

Fix $\kappa \in \mathbb{C}^{|\mathcal{K}|}$, let $L_0$ be as in (2.9) with $\lambda = 0$, and define

$$A_\kappa(t) := \mathrm{e}^{\mathrm{i}tL_0} \left( \sum_{k \in \mathcal{K}} V_k \otimes (a(\mathrm{e}^{-\overline{\kappa_k} l_k} \check{f}_k) + a^*(\mathrm{e}^{\kappa_k l_k} \check{f}_k)) \right) \mathrm{e}^{-\mathrm{i}tL_0} \tag{3.13}$$

The following lemma is Theorem 6.1 in [11].

**Lemma 3.1.** *For $\psi \in \mathcal{D}$ and $\kappa \in (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$, the series*

$$\mathrm{e}^{-\mathrm{i}t(L_0 + \lambda A_\kappa(0))} \psi := \mathrm{e}^{-\mathrm{i}tL_0} \sum_{n \geq 0} (-\mathrm{i}\lambda)^n \int_{0 \leq t_1 \leq \dots \leq t_n \leq t} \mathrm{d}t_1 \dots \mathrm{d}t_n A_\kappa(t_n) \dots A_\kappa(t_1) \psi \tag{3.14}$$

*is absolutely convergent and $\mathrm{e}^{-\mathrm{i}t(L_0 + \lambda A_\kappa(0))} \psi \in \mathcal{D}$. The one-parameter family $t \to \mathrm{e}^{-\mathrm{i}t(L_0 + \lambda A_\kappa(0))}$ extends to a strongly continuous unitary group on $\mathcal{H}$.*

We recognize $L_\lambda = L_0 + \lambda A_0(0)$ with $L_\lambda$ as in (2.9). This proves the existence of the dynamics $U_t^\lambda$.

The pull-through formula yields for $\kappa \in (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$,

$$Y_{-\kappa} U_{-t}^\lambda Y_\kappa = \mathrm{e}^{\mathrm{i}t(L_0 + \lambda A_\kappa(0))}. \tag{3.15}$$

Recall the vacuum vector $\Psi$ and remark that obviously $\mathcal{H}_{\mathrm{S}} \otimes \Psi \in \mathcal{D}$. Hence one can expand the functions (2.13) for $\kappa \in (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$ in a series involving $A_0(u)$ and $A_\kappa(u)$ for $s \le u \le t$. Splitting the interaction by $V_k = \sum_{\omega \in \mathcal{F}} V_{\omega,k}$ and using Wick's theorem, one arrives at the equality (3.17).

Note that Lemma 3.1 can be extended to some values of $\kappa \in \mathbb{C}^{|\mathcal{K}|} \setminus (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$ (sacrificing of course boundedness and unitarity of (3.14)) and hence one could construct the functions (2.13) in a more direct way. However, analytical continuation works on a larger domain.

### 3.2.2 The Duhamel series as $\mathcal{B}(\mathcal{B}(\mathcal{H}_{\mathrm{S}}))$-valued integral over $\Omega$

Let, for $n \in \mathbb{N}$, $Pair(2n)$ be the set of all partitions, (denoted by $\pi$), of $\{1, \ldots, 2n\}$ into $n$ unordered pairs $\{p, q\}$ with $p < q$. That is, $\pi \subset 2^{\{1,\ldots,2n\}}$ belongs to Pair$(2n)$ iff.

$$\pi = \{\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_n, q_n\}\}, \qquad \bigcup_{i=1}^{n} \{p_i, q_i\} = \{1, \ldots, 2n\} \tag{3.16}$$

with the convention that $p_i < q_i$ for $i = 1, \ldots, n$.

Fix $t \ge 0 \ge s$ and let $\mathcal{X} := [s, t] \times \mathcal{F} \times \mathcal{K}$.

We denote by '$x$' elements of $\mathcal{X}^n$ for any $n \in \mathbb{N}_0$ with components $(u_i, \omega_i, k_i) \in [s, t] \times \mathcal{F} \times \mathcal{K}$, $i = 1, \ldots, n$. The measure $\mathrm{d}^n x$ abbreviates $\prod_{i=1}^n \mathrm{d}u_i \mathrm{d}\omega_i \mathrm{d}k_i$ where the last two differentials $\mathrm{d}\omega_i \mathrm{d}k_i$ refer to the counting measure on resp. $\mathcal{F}, \mathcal{K}$.

We have now the necessary notation to write an expansion for the functions (2.13). For an arbitrary state $\rho_{\mathrm{S}}$ and $\lambda \in \mathbb{R}$;

$$\begin{aligned} &\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}} \left[ U^{\lambda}_{-\lambda^{-2}s} Y_{-\kappa} U^{\lambda}_{-\lambda^{-2}t} Y_\kappa U^{\lambda}_{\lambda^{-2}t} U^{\lambda}_{\lambda^{-2}s} \right] = \\ &\sum_{n \in 2\mathbb{N}} \lambda^{-n} \sum_{m=0}^{n} \mathrm{i}^{n-m} \int_{\mathcal{X}^n} \mathrm{d}^n x \, \mathrm{Ind}[u_1 < \ldots < u_m] \, \mathrm{Ind}[u_{m+1} > \ldots > u_n] \\ &\mathrm{Tr} \left[ \left( \prod_{i=1}^m V_{\omega_i, k_i} \right) \; \tilde{\rho}_{\mathrm{S}} \; \left( \prod_{i=1}^{n-m} V_{\omega_{n+1-i}, k_{n+1-i}} \right) \right] \\ &\sum_{\pi \in Pair(n)} \prod_{\{p,q\} \in \pi} \quad Q_{\kappa,\lambda}(x, \{p, q\}, m) \end{aligned} \tag{3.17}$$

where we abbreviated[3]

$$\begin{aligned} &Q_{\kappa,\lambda}(x, \{p, q\}, m) = \delta_{k_p, k_q} \\ &\left\langle \mathrm{e}^{-\overline{\zeta(x,q,m)} l} \mathrm{e}^{-\mathrm{i}\lambda^{-2} u_q (l + \omega_q)} \breve{f}_{k_q}, \quad \mathrm{e}^{\zeta(x,p,m) l} \mathrm{e}^{-\mathrm{i}\lambda^{-2} u_p (l - \omega_p)} \breve{f}_{k_p} \right\rangle_{\mathfrak{h}} \end{aligned} \tag{3.18}$$

with, for $x \in \mathcal{X}^n$ and $j, m \in \{1, \ldots, n\}$,

$$\zeta(x, j, m) := \begin{cases} \kappa_{k_j} & \text{if} \quad (0 < u_j < t \qquad \text{and} \quad j > m) \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

[3]Remark that we have written $l$ instead of $l_k$, which is justified since $f_k$ is embedded into $\mathfrak{h}$.

For $\kappa \in (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$, the equality (3.17) follows from Section 3.2.1. In Section 4.1, we check that it can be analytically continued and hence, from now on, we assume (3.17) for $\kappa \in D + (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$.

We now connect the concepts and notation of Section 3.1 with those used in (3.17). First we remark that a pair $(x,\pi), x \in \mathcal{X}^{2n}, \pi \in Pair(2n)$ is associated to $2^n$ elements of $\Omega_n$, we write

$$\sigma \sim (x,\pi) \Leftrightarrow \forall \{p,q\} \in \pi, \exists \tilde\sigma \in \sigma : \{u_p,u_q\} = \{s(\tilde\sigma), t(\tilde\sigma)\} \tag{3.20}$$

Let $(\delta_\epsilon)_{\epsilon>0}$ be a family of $C^\infty$ functions with compact support on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \mathrm{d}x \delta_\epsilon(x) = 2, \qquad \delta_\epsilon(x) = \delta_\epsilon(-x) \geq 0 \tag{3.21}$$

and for all bounded, measurable functions $f$ on $\mathbb{R}$, which are right-continuous at 0,

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^+} \mathrm{d}x f(x) \delta_\epsilon(x) = \lim_{x\downarrow 0} f(x) \tag{3.22}$$

Define, analogously to (3.18),

$$\begin{gathered} Q^{\mathrm{a}}_{\kappa,\lambda}(x,\{p,q\},m) = \delta_{k_p,k_q}\delta_{\omega_p,-\omega_q} \int_{\mathbb{R}^+} \mathrm{d}u \\ \left\langle \mathrm{e}^{-\overline{\zeta(x,q,m)}l} \breve{f}_{k_q},\, \mathrm{e}^{-\mathrm{i}\lambda^{-2}u(l-\omega_p)\mathrm{sign}(u_p-u_q)} \mathrm{e}^{\zeta(x,p,m)l}\, \breve{f}_{k_p} \right\rangle_{\mathfrak{h}} \end{gathered} \tag{3.23}$$

which exists by Assumption 2.3 (since $\|h\|_1 < \infty$ in the notation of Section 4.1). For $\sigma \in \Omega_{[s,t]}$ and $S \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$, and writing $n := 2|\sigma|$, we put

$$\begin{aligned} &W^\epsilon_{\kappa,\lambda}(\sigma)S := \\ &\sum_{\substack{x \in \mathcal{X}^n, \pi \in Pair(n) \\ \sigma \sim (x,\pi)}} \lambda^{-n} \sum_{m=0}^{n} \mathrm{i}^{n-m}\, \mathrm{Ind}[u_1 < \ldots < u_m]\, \mathrm{Ind}[u_{m+1} > \ldots > u_n] \\ &\mathrm{Tr}\left[ \left(\prod_{i=1}^m V_{\omega_i,k_i}\right)\ S\ \left(\prod_{i=1}^{n-m} V_{\omega_{n+1-i},k_{n+1-i}}\right)\right] \\ &\prod_{\{p,q\}\in\pi} \Big( \delta_{\ell(\tilde\sigma),\mathrm{a}} \left[ \delta_\epsilon\!\left(\tfrac{t(\tilde\sigma)-s(\tilde\sigma)}{\lambda^2}\right) Q^{\mathrm{a}}_{\kappa,\lambda}(x,\{p,q\},m) \right] \\ &\qquad + \delta_{\ell(\tilde\sigma),\mathrm{b}} \left[ Q_{\kappa,\lambda}(x,\{p,q\},m) - \delta_\epsilon\!\left(\tfrac{t(\tilde\sigma)-s(\tilde\sigma)}{\lambda^2}\right) Q^{\mathrm{a}}_{\kappa,\lambda}(x,\{p,q\},m) \right] \Big) \end{aligned} \tag{3.24}$$

where the $\tilde\sigma$ in the last two lines is the unique element $\tilde\sigma \in \sigma$ such that $\{u_p,u_q\} = \{s(\tilde\sigma), t(\tilde\sigma)\}$.

We state two important properties of the maps $W^\epsilon_{\kappa,\lambda}(\sigma)$,

$$\max \mathrm{Supp}(\sigma) \leq \min \mathrm{Supp}(\sigma') \Longrightarrow W^\epsilon_{\kappa,\lambda}(\sigma \cup \sigma') = W^\epsilon_{\kappa,\lambda}(\sigma') W^\epsilon_{\kappa,\lambda}(\sigma) \tag{3.25}$$

$$\text{LHS of (3.17)} = \int_{\Omega_{[s,t]}} \mathrm{d}\sigma W^{\epsilon}_{\kappa,\lambda}(\sigma)\tilde{\rho}_{\mathrm{S}} \tag{3.26}$$

which follow from (3.17) and (3.24). (Note that (3.26) holds for all $\epsilon > 0$.)

One can see in (3.24), the purpose of the coordinates $\ell(\tilde{\sigma}) = \mathrm{a}, \mathrm{b}$. Each $\tilde{\sigma}$ determines a pair of time-instants (and hence interaction terms): $s(\tilde{\sigma}), t(\tilde{\sigma})$ between which a pairing occurs. If $\ell(\tilde{\sigma}) = \mathrm{a}$, these time-instants are constrained to coincide (as $\epsilon \searrow 0$) and to have opposite values of $\omega \in \mathcal{F}$. In $\ell(\tilde{\sigma}) = \mathrm{b}$, we collect the rest of the term corresponding to this pairing. The reader who is familiar with the *weak coupling limit* from [5], recognizes the a-terms as the ones contributing in the limit $\lambda \searrow 0$.

## 3.3 Time-discretized polymers

The following construction depends on a time scale $\tau$. In order to keep the notation manageable, we will drop this time-scale from most of the notation.

### 3.3.1 Primary polymers

Let $j \in \mathbb{Z}$ and $\zeta \subset \mathbb{Z}$ and put

$$I_j := [\tau j, \tau(j+1)], \qquad I_\zeta := \cup_{j \in \zeta} I_j \tag{3.27}$$

Define a partial ordering on $\mathbb{Z}$ by

$$j \prec j' \Leftrightarrow j \leq j' + 2 \tag{3.28}$$

The same symbol will be used for subsets of $\zeta, \zeta' \subset \mathbb{Z}$;

$$\zeta \prec \zeta' \Leftrightarrow (\forall j \in \zeta, \forall j' \in \zeta' : \ j \prec j') \tag{3.29}$$

Introduce also the compatibility relation $\sim$ by

$$\zeta \sim \zeta' \Leftrightarrow (\zeta \prec \zeta' \quad \text{or} \quad \zeta' \prec \zeta) \tag{3.30}$$

We define the set of *primary polymers* $\mathfrak{U} \subset 2^{\mathbb{Z}}$. A $\gamma \in 2^{\mathbb{Z}}$ belongs to $\mathfrak{U}$ iff.

1) For all $\gamma', \gamma'' \in 2^{\mathbb{Z}}$ such that $\gamma = \gamma' \cup \gamma''$:

$$\gamma' \sim \gamma'' \Longrightarrow (\gamma' = \emptyset \quad \text{or} \quad \gamma'' = \emptyset.) \tag{3.31}$$

2) If $j \in \gamma$ and $j < -1$, then $j + 1 \in \gamma$.

We now construct a mapping $K : \Omega \to 2^{\mathbb{Z}}$.

**Definition 3.1.** *Let $M(\sigma) \subset \mathbb{Z}$ be the union of all sets $\gamma \subset \mathbb{Z}$ satisfying*

$$\sigma_{I_\gamma} = \sigma \setminus \sigma_{\mathbb{R}\setminus I_\gamma} \quad \textit{and } \forall \tilde{\sigma} \in \sigma_{I_\gamma} : \ \ell(\tilde{\sigma}) = \mathrm{a} \tag{3.32}$$

*Define $K(\sigma) := \mathbb{Z} \setminus M(\sigma)$*

We also construct operators on $\mathcal{B}(\mathcal{H}_{\mathrm{S}})$ indexed by $\gamma \in 2^{\mathbb{Z}}$ ;

$$W_{\kappa,\lambda}(\gamma) = \lim_{\epsilon\downarrow 0} \int_{K(\sigma)=\gamma} \mathrm{d}\sigma W^{\epsilon}_{\kappa,\lambda}(\sigma) \tag{3.33}$$

See Section 4.1 for well-definedness of (3.33). The following lemma is our main technical tool

**Lemma 3.2.** *Assume assumption 2.3. There is a $c(\lambda)$ such that for $\kappa \in \{z \in \mathbb{C} | \operatorname{Re} z \in D, |\operatorname{Im} z| \leq 1\}$ and for $\gamma \in \mathfrak{U}$,*

$$\|W_{\kappa,\lambda}(\gamma)\| \leq c(\lambda)^{|I_\gamma|}, \qquad c(\lambda) \underset{\lambda\searrow 0}{\longrightarrow} 0 \tag{3.34}$$

### 3.3.2 The reference dynamics

Recall the notation of Section 3.2, assume $\|h\|_1 < \infty$ (see Section 4.1) and put

$$c_\kappa(\omega, k) := \int_{\mathbb{R}^+} \mathrm{d}u \, \langle \breve{f}_k, \mathrm{e}^{-\mathrm{i}u(l_k-\omega)} \mathrm{e}^{-\kappa_k l_k} \breve{f}_k \rangle_{\mathfrak{h}_k \oplus \mathfrak{h}_k} \tag{3.35}$$

Remark that, if $\breve{f}_k$ is continuous in $\mathcal{F}$,

$$\operatorname{Re} c_\kappa(\omega, k) = \|\breve{f}_k(\omega)\|^2_{\mathfrak{l}_k} \mathrm{e}^{-\kappa_k \omega} \geq 0 \tag{3.36}$$

and define for $S \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$:

$$\mathcal{L}_\kappa(S) := \sum_{\omega \in \mathcal{F}, k \in \mathcal{K}} \Big( 2\operatorname{Re} c_\kappa(\omega, k) V_{\omega,k} S V^*_{\omega,k} - c_0(\omega, k) S V^*_{\omega,k} V_{\omega,k} - \overline{c_0(\omega, k)} V^*_{\omega,k} V_{\omega,k} S \Big) \tag{3.37}$$

For $\kappa \in \mathbb{R}^{|\mathcal{K}|}$, the family $\mathrm{e}^{t\mathcal{L}_\kappa}, t \geq 0$ is a semigroup of completely positive maps on $\mathcal{B}(\mathcal{H}_{\mathrm{S}})$, see e.g. [29], or Theorem 3.4 in [12]. For $\kappa = 0$, it is also trace-preserving. Its relevance lies in the following lemma's.

**Lemma 3.3.** *Assume the bound (2.12), continuity of $\breve{f}_k$ in $\mathcal{F}$ and $\|h\|_1 < \infty$, then for $I \subset \mathbb{R}^+$*

$$\lim_{\epsilon\searrow 0} \int_{\substack{\sigma \in \Omega_I \\ \ell_{1,\ldots,|\sigma|}(\sigma) = \mathrm{a}}} \mathrm{d}\sigma W^{\epsilon}_{\kappa,\lambda}(\sigma) = \mathrm{e}^{|I|\mathcal{L}_\kappa} \tag{3.38}$$

**Lemma 3.4.** *Assume Assumption 2.2 and the conditions of Lemma 3.3. For a compact $D \in \mathbb{R}^{|\mathcal{K}|}$, there are $\delta, g > 0$ such that for $\kappa \in \{z \in \mathbb{C} | \operatorname{Re} z \in D, |\operatorname{Im} z| \leq \delta\}$, $\mathcal{L}_\kappa$ has a simple eigenvalue $e_\kappa$ and*

$$\inf \left\{ \operatorname{Re}(e_\kappa - x) \,\middle|\, x \in \operatorname{sp}\mathcal{L}_\kappa \setminus \{e_\kappa\} \right\} \geq g \tag{3.39}$$

*The eigenvector $Z_\kappa$ corresponding to $e_\kappa$ can be chosen strictly positive.*

We adopt the normalization $\operatorname{Tr}[Z_\kappa^* Z_\kappa] = 1$. Let $P_\kappa$ be the projection on $Z_\kappa$ w.r.t. the scalar product given by $\operatorname{Tr}[\cdot]$, i.e.

$$P_\kappa S = Z_\kappa \operatorname{Tr}[Z_\kappa^* S], \qquad S \in \mathcal{B}(\mathcal{H}_{\mathrm{S}}) \tag{3.40}$$

and let $R_\kappa := 1 - P_\kappa$. Ordinary perturbation theory yields analyticity of $\kappa \to e_\kappa, P_\kappa$.

### 3.3.3 Secondary polymers

It is good to pause now and justify some concepts from the previous sections.

We make some preliminary definitions, let for $\gamma \in \mathfrak{U}$,

$$\tilde{W}_{\kappa,\lambda}(\gamma) = \mathrm{e}^{-|I_\gamma|\mathcal{L}_\kappa} W_{\kappa,\lambda}(\gamma) \tag{3.41}$$

and for $s \leq u < 0$,

$$\tilde{\rho}_{\mathrm{S},u} := \int_{\Omega_{[s,u]}} \mathrm{d}\sigma W^\epsilon_{\kappa,\lambda}(\sigma) \tilde{\rho}_{\mathrm{S}} \tag{3.42}$$

Remark that, by the same reasoning as the one leading to (3.26),

$$\operatorname{Tr}[\tilde{\rho}_{\mathrm{S},u} S] = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}} \left[ U^\lambda_{\lambda^{-2}(s-u)} (S \otimes 1) U^\lambda_{\lambda^{-2}(u-s)} \right] \tag{3.43}$$

In particular, $\operatorname{Tr}[\tilde{\rho}_{\mathrm{S},u}] = 1$ and $\tilde{\rho}_{\mathrm{S},u}$ is a density matrix corresponding to a state $\rho_{\mathrm{S},u}$, which is independent of $\kappa$ and $\epsilon$.

Let $s \leq 0 \leq t$ be such that $\tau^{-1}s, \tau^{-1}t \in \mathbb{Z}$ (this is a restriction on which we will comment later). We write $\Lambda := \mathbb{Z} \cap [\tau^{-1}s, \tau^{-1}t - 1]$. For an ordered sequence of primary polymers $\gamma_1 \prec \ldots \prec \gamma_n$ in $\mathfrak{U} \cap 2^\Lambda$, we introduce

$$u_i = \begin{cases} \min\{\inf I_{\gamma_1}, 0\} & \text{if} \quad i = 0 \\ \inf I_{\gamma_i} & \text{if} \quad 1 \leq i \leq n \\ t & \text{if} \quad i = n+1 \end{cases} \tag{3.44}$$

Starting from (3.26) and (3.33), realizing that for every $\gamma \in 2^{\mathbb{Z}}$, there is a unique sequence $\gamma_1 \prec \ldots \prec \gamma_n$ such that $\cup_{i=1}^n \gamma_i = \gamma$, and using Lemma 3.3 and (3.43), one derives

$$\begin{aligned} \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}} \left[ U^\lambda_{-\lambda^{-2}s} Y_{-\kappa} U^\lambda_{-\lambda^{-2}t} Y_\kappa U^\lambda_{\lambda^{-2}t} U^\lambda_{\lambda^{-2}s} \right] &= \operatorname{Tr}[\mathrm{e}^{t\mathcal{L}_\kappa} \tilde{\rho}_{\mathrm{S},0}] \\ + \sum_{n \geq 1} \sum_{\substack{\gamma_1 \prec \ldots \prec \gamma_n \\ \gamma_{1,\ldots,n} \in \mathfrak{U} \cap 2^\Lambda}} \operatorname{Tr} &\left[ \prod_{i=1}^n \mathrm{e}^{(u_{i+1}-u_i)\mathcal{L}_\kappa} \tilde{W}_{\kappa,\lambda}(\gamma_i) \times \mathrm{e}^{(u_1-u_0)\mathcal{L}_\kappa} \tilde{\rho}_{\mathrm{S},u} \right] \end{aligned} \tag{3.45}$$

The trick is now to use the ergodicity of $\mathrm{e}^{t\mathcal{L}_\kappa}$, as established by Lemma 3.4, to write the expansion (3.45) as a product over independent *secondary polymers*, which will be collections of $\gamma \in \mathfrak{U}$. To achieve this, we insert $1 = P_\kappa + R_\kappa$ at times $u_{i=0,\ldots,n+1}$. Consecutive primary polymers separated by $R_\kappa$ belong to the same secondary polymer. This is formalized in the remainder of this Section.

We add two abstract elements, $-\varsigma$ and $\varsigma$ to $\mathbb{Z}$ and we extend the partial ordering $\prec$ to $\mathbb{Z} \cup \{-\varsigma, \varsigma\}$ by putting

$$-\varsigma \not\prec -\varsigma, \qquad \varsigma \not\prec \varsigma, \qquad \forall j \in \mathbb{Z}: \; -\varsigma \prec j \prec \varsigma \tag{3.46}$$

The extension to subsets of $\mathbb{Z}$ is done as in (3.29)

Recall $\Lambda$ a defined above. We define the set of secondary polymers $\mathfrak{W}_\Lambda := 2^{\Lambda \cup \{-\varsigma,\varsigma\}} \setminus \{\emptyset\}$. Remark that every $\zeta \in \mathfrak{W}_\Lambda$ has a unique decomposition

$$\zeta = \epsilon_- \cup \bigcup_{i=1}^{n} \gamma_i \cup \epsilon_+ \tag{3.47}$$

where $\gamma_{1,\ldots,n} \in \mathfrak{U}$, $\gamma_1 \prec \ldots \prec \gamma_n$ and $\epsilon_- = \emptyset, \{-\varsigma\}$ and $\epsilon_+ = \emptyset, \{\varsigma\}$. We construct *polymer weights* $w^\Lambda_{\kappa,\lambda}(\zeta)$ for $\zeta \in \mathfrak{W}_\Lambda$ using the parametrization (3.47)

$$\begin{aligned} w^\Lambda_{\kappa,\lambda}(\zeta = \epsilon_- \cup \bigcup_{i=1}^{n} \gamma_i \cup \epsilon_+) = \mathrm{Tr}\Big[ & W^+_\kappa(\epsilon_+) \times \mathrm{e}^{(u_{n+1}-u_n)(\mathcal{L}_\kappa - e_\kappa)} \times \\ & \prod_{i=2}^{n} \tilde{W}_{\kappa,\lambda}(\gamma_i) R_\kappa \mathrm{e}^{(u_i - u_{i-1})(\mathcal{L}_\kappa - e_\kappa)} \times \tilde{W}_{\kappa,\lambda}(\gamma_1) \mathrm{e}^{(u_1-u_0)(\mathcal{L}_\kappa - e_\kappa)} \times W^-_\kappa(\epsilon_-) \rho_{\mathrm{S},u_0} \Big] \end{aligned} \tag{3.48}$$

where

$$W^+_\kappa(\epsilon_+) = \frac{1}{\mathrm{Tr}[Z_\kappa]} \begin{cases} P_\kappa & \text{if} \quad \epsilon_+ = \emptyset \\ R_\kappa & \text{if} \quad \epsilon_+ = \{\varsigma\} \end{cases}, \qquad W^-_\kappa(\epsilon_-) = \frac{1}{\mathrm{Tr}[Z^*_\kappa \tilde{\rho}_{\mathrm{S},0}]} \begin{cases} P_\kappa & \text{if} \quad \epsilon_- = \emptyset \\ R_\kappa & \text{if} \quad \epsilon_- = \{-\varsigma\} \end{cases} \tag{3.49}$$

Remark that both $\mathrm{Tr}[Z_\kappa], \mathrm{Tr}[Z^*_\kappa \tilde{\rho}_{\mathrm{S},0}] > 0$ by Lemma 3.4. Call

$$b_\kappa := \mathrm{Tr}[Z_\kappa] \, \mathrm{Tr}[Z^*_\kappa \tilde{\rho}_{\mathrm{S},0}] \tag{3.50}$$

For $n = 0$, the definition (3.48) reduces to

$$\begin{aligned} w^\Lambda_{\kappa,\lambda}(\epsilon_- \cup \epsilon_+) &= b_\kappa^{-1} \, \mathrm{Tr}[W^+_\kappa(\epsilon_+) \mathrm{e}^{t(\mathcal{L}_\kappa - e_\kappa)} W^-_\kappa(\epsilon_-) \tilde{\rho}_{\mathrm{S},0}] \\ &= \delta_{\epsilon_-,\emptyset} \delta_{\epsilon_+,\emptyset} + \delta_{\epsilon_-,\{-\varsigma\}} \delta_{\epsilon_+,\{\varsigma\}} b_\kappa^{-1} \, \mathrm{Tr}[R_\kappa \mathrm{e}^{t(\mathcal{L}_\kappa - e_\kappa)} R_\kappa \tilde{\rho}_{\mathrm{S},0}] \end{aligned} \tag{3.51}$$

which follows from $P_\kappa R_\kappa = R_\kappa P_\kappa = 0$.

Note that the weights $w^\Lambda_{\kappa,\lambda}(\zeta)$ depend weakly on $\rho_{\mathrm{S}}$ (also via $\rho_{\mathrm{S},0}$). However, for $\zeta$ not containing $-\varsigma$ and $j<0$, this dependence vanishes since $(\mathrm{Tr}[Z^*_\kappa\tilde\rho_{\mathrm{S},0}])^{-1}P_\kappa\rho_{\mathrm{S},0}=Z_\kappa$. We finally write the polymer model starting from (3.45) and using the polymer weights (3.48),

$$\text{LHS of (3.45)} = \mathrm{e}^{te_\kappa}b_\kappa\Big(1+\sum_{n\geq 1}\sum_{\substack{\zeta_1\prec\ldots\prec\zeta_n\\ \zeta_{1,\ldots,n}\in\mathfrak{W}_\Lambda}}\prod_{i=1}^n w^\Lambda_{\kappa,\lambda}(\zeta_i)\Big) \tag{3.52}$$

Note that the '1' corresponds to $\zeta=\emptyset$, which has been excluded from $\mathfrak{W}_\Lambda$ but has weight 1 by (3.51).

## 3.4 Cluster expansion

The expression (3.52) can be used to set up a cluster expansion for the log of the LHS. We follow closely the presentation of cluster expansions in [36].

Let $\mathcal{C}_n$ be the set of all (unoriented) connected graphs with $n$ vertices. For a finite sequence $\zeta_1,\ldots,\zeta_n$ in $\mathfrak{W}_\Lambda$, introduce the function

$$\phi(\zeta_1,\ldots,\zeta_n)=\frac{1}{n!}\begin{cases}1 & \text{if}\quad n=1\\ \sum_{G\in\mathcal{C}_n}\prod_{(i,j)\in G}\mathrm{Ind}[\zeta_i\nsim\zeta_j] & \text{if}\quad n>1\end{cases} \tag{3.53}$$

where $\prod_{(i,j)\in G}$ denotes the product over all edges $(i,j)$ of the graph $G$.

By standard combinatorics [36], (3.52) yields

$$\log(\text{LHS of (3.45)}) = te_\kappa+\log b_\kappa+\sum_{n\geq 1}\sum_{\zeta_1,\ldots,\zeta_n\in\mathfrak{W}_\Lambda}\phi(\zeta_1,\ldots,\zeta_n)\prod_{i=1}^n w^\Lambda_{\kappa,\lambda}(\zeta_i) \tag{3.54}$$

The sum on the RHS runs over sequences in $\mathfrak{W}_\Lambda$ with $n$ elements. As $\mathfrak{W}_\Lambda$ is finite, there is no convergence issue.

To control the expansion (3.54) as $t\nearrow\infty$ (recall that $\Lambda$ also depends on $t$), we apply a standard Theorem which relies on the Kotecky-Preiss criterion [25].

**Lemma 3.5** (Kotecky-Preiss criterion)**.** *Assume Assumptions 2.2 and 2.3, and let $\delta>0$ be as in Lemma 3.4. There is a $\lambda_0$ such that for $|\lambda|\leq\lambda_0$, $\kappa\in\{z\in\mathbb{C}^{|\mathcal{K}|}\,|\,\mathrm{Re}\,\kappa\in D, |\mathrm{Im}\,\kappa|\leq\delta\}$, and $t$ (and hence $\Lambda$) large enough, there is a nonnegative function $a$ on $\mathfrak{W}_\Lambda$, such that*

1)

$$\sum_{\zeta'\in\mathfrak{W}_\Lambda,\,\zeta'\nsim\zeta}|w^\Lambda_{\kappa,\lambda}(\zeta')|\mathrm{e}^{a(\zeta')}<a(\zeta) \tag{3.55}$$

2)
$$\sum_{\zeta\,\in\,\mathfrak{W}_\Lambda} |w^\Lambda_{\kappa,\lambda}(\zeta)|\mathrm{e}^{a(\zeta)} < \infty \tag{3.56}$$

*Moreover, for each* $\zeta \in \mathfrak{W}_\Lambda$*,* $w^\Lambda_{\kappa,\lambda}(\zeta)$ *is analytic in* $\kappa \in \{z \in \mathbb{C}^{|\mathcal{K}|} \,|\, \mathrm{Re}\,\kappa \in D, |\mathrm{Im}\,\kappa| \le \delta\}$.

By applying Theorem 1 in [36], we conclude from (3.52) and Lemma 3.5, that for every $\zeta \in \mathfrak{W}_\Lambda$, the following bound holds

$$v^\Lambda_{\kappa,\lambda}(\zeta) := \sum_{n\ge 1} \sum_{\substack{\zeta_1,\dots,\zeta_n\,\in\,\mathfrak{W}_\Lambda \\ \exists i\in\{1,\dots,n\}:\,\zeta_i=\zeta}} \phi(\zeta_1,\dots,\zeta_n) \prod_{i=1}^n |w^\Lambda_{\kappa,\lambda}(\zeta_i)| \le \mathrm{e}^{a(\zeta)} w(\zeta) \tag{3.57}$$

Using first (3.57) and then (3.55),

$$\sum_{n\ge1}\sum_{\zeta_1,\dots,\zeta_n\in\mathfrak{W}_\Lambda} \phi(\zeta_1,\dots,\zeta_n)\prod_{i=1}^n w^\Lambda_{\kappa,\lambda}(\zeta_i) \le \sum_{j\in\Lambda\cup\{-\varsigma,\varsigma\}}\ \sum_{\zeta\in\mathfrak{W}_\Lambda,\zeta\nsim\{j\}} v^\Lambda_{\kappa,\lambda}(\zeta) \tag{3.58}$$

$$\le \sum_{j\in\Lambda\cup\{-\varsigma,\varsigma\}}\ \sum_{\zeta\in\mathfrak{W}_\Lambda,\zeta\nsim\{j\}} w^\Lambda_{\kappa,\lambda}(\zeta)\mathrm{e}^{a(\zeta)} \le \sum_{j\in\Lambda\cup\{-\varsigma,\varsigma\}} a_\Lambda(\{j\}) \tag{3.59}$$

By the proof of Lemma 3.5, we see that $a(\zeta)$ on $\zeta = \{j\}$ with $j \in \Lambda\cup\{-\varsigma,\varsigma\}$ is constant and independent of $\Lambda$. Hence, upon dividing by $t$, (3.59) is bounded.

We define the set of *bulk polymers* $\mathfrak{W}_{\mathrm{bu.}} \subset \mathfrak{W}_{\mathbb{Z}}$ as $\mathfrak{W}_{\mathrm{bu.}} := 2^{\mathbb{N}}$. Remark that for each $\zeta \in \mathfrak{W}_{\mathrm{bu.}}$, $\zeta + m \in \mathfrak{W}_{\mathrm{bu.}}$ for $m \in \mathbb{N}$. For $\zeta \in \mathfrak{W}_{\mathrm{bu.}} \cap \mathfrak{W}_\Lambda$, we write simply $w_{\kappa,\lambda}(\zeta) := w^\Lambda_{\kappa,\lambda}(\zeta)$ (which is independent of $\rho_{\mathrm{S}}$, see above (3.52)). Define for a sequence of polymers $\zeta_1,\dots,\zeta_n$ in $\mathfrak{W}_{\mathrm{bu.}}$.

$$\eta_N(\zeta_1,\dots,\zeta_n) := \sum_{m=0}^{N-1} \phi(\zeta_1,\dots,\zeta_n)\prod_{i=1}^n w_{\kappa,\lambda}(\zeta_i+m) \tag{3.60}$$

and notice that one could as well replace $\phi(\zeta_1,\dots,\zeta_n)$ by $\phi(\zeta_1+m,\dots,\zeta_n+m)$.

We parameterize $t = N\tau$ (and hence $\Lambda$) with $N \in \mathbb{N}$. Analogous to (3.59),

$$\left| \sum_{n\ge1}\sum_{\zeta_1,\dots,\zeta_n\in\mathfrak{W}_\Lambda} \phi(\zeta_1,\dots,\zeta_n)\prod_{i=1}^n w^\Lambda_{\kappa,\lambda}(\zeta_i) - \sum_{n\ge1} \sum_{\substack{\zeta_1,\dots,\zeta_n\,\in\,\mathfrak{W}_{\mathrm{bu.}} \\ 0\,\in\,\cup_{i=1}^n\zeta_i}} \eta_N(\zeta_1,\dots,\zeta_n) \right|$$

$$\le \sum_{j=-\varsigma,\varsigma,0,\max\Lambda}\ \sum_{\zeta\in\mathfrak{W}_\Lambda,\zeta\nsim\{j\}} v^\Lambda_{\kappa,\lambda}(\zeta) \le \sum_{j=-\varsigma,\varsigma,0,\max\Lambda} a_\Lambda(\{j\}) \tag{3.61}$$

and (3.61) vanishes upon dividing by $t = \tau N$ as $N \nearrow \infty$.

Theorems 2.4 and 2.5 contain a limit $t \uparrow \infty$, whereas, as already mentioned, we restrict ourselves to $t = N\tau, N \in \mathbb{N}$ and hence we take the limit along a subsequence. It is easily seen that the general results can be obtained by modifying slightly the intervals $I_j$ for $j=\tau^{-1}s, \tau^{-1}t-1$ (such that $\cup_{j\in\Lambda} I_j=[s,t]$ still holds) and hence also modifying slightly the polymer weights of polymers containing $j=\tau^{-1}s$ or $j=\tau^{-1}t-1$. Indeed, by (3.61), these 'boundary polymers' do not contribute to $F(\kappa,\lambda)$.

That taken into account, and using (3.54) and (3.61), Theorems 2.4 and 2.5 will follow once we prove that

$$F(\kappa,\lambda)=\lambda^2 e_\kappa+\lambda^2 \lim_{N\nearrow+\infty} \sum_{\substack{\zeta_1,\dots,\zeta_n\in\mathfrak{W}_{\mathrm{bu.}}\\ 0\in\cup_{i=1}^n\zeta_i}} \frac{1}{\tau N}\eta_N(\zeta_1,\dots,\zeta_n) \tag{3.62}$$

exists. Indeed, by the bound (3.59), analyticity follows from the analyticity of $w^{\Lambda}_{\kappa,\lambda}(\zeta), \zeta\in\mathfrak{W}_\Lambda$ and the Vitali convergence theorem.

Again by the bound (3.59), existence of (3.62) will follow once we prove that for each sequence $\zeta_1,\dots,\zeta_n$, the limit $\lim_{N\nearrow+\infty}\frac{1}{N}\eta_N(\zeta_1,\dots,\zeta_n)$ exists. For simplicity, we restrict to sequences with one element $\zeta_1\in\mathfrak{W}_\Lambda$ and to $\zeta_1=\emptyset\cup\gamma\cup\emptyset$ ($\gamma\in\mathfrak{U}$) in the parametrization (3.47). the general case is treated analogously).

For $\sigma\in\Omega$ and $q\in\mathbb{R}$, we denote by $\sigma+q$ the element in $\Omega$ with coordinates $s_i(\sigma)+q, t_i(\sigma)+q, \ell_i(\sigma), i=1,\dots,|\sigma|$. Since $K(\sigma)=\gamma+m \Leftrightarrow K(\sigma-\tau m)=\gamma$ for $m\in\mathbb{N}$,

$$\frac{1}{N}\eta_N(\zeta_1):=\frac{1}{N}b_\kappa^{-1}\sum_{m=0}^{N-1}\lim_{\epsilon\searrow 0}\int_{K^{-1}(\gamma)}\mathrm{d}\sigma\ \mathrm{Tr}[P_\kappa W^{\epsilon}_{\kappa,\lambda}(\sigma+\tau m)P_\kappa\rho_{\mathrm{S},0}] \tag{3.63}$$

By inspecting (3.24), one sees that for $\mathrm{Supp}\sigma\subset\mathbb{R}^+$, $W^\epsilon_{\kappa,\lambda}(\sigma)$ differs from $W^\epsilon_{\kappa,\lambda}(\sigma+q)$ only by (a product of) periodic factors $\mathrm{e}^{\mathrm{i}\lambda^{-2}\omega q}, \omega\in\mathcal{F}$ and from this it easily follows that

$$\lim_{N\uparrow+\infty}\frac{1}{N}\sum_{m=0}^{N-1}\mathrm{Tr}[P_\kappa W^\epsilon_{\kappa,\lambda}(\sigma+\tau m)P_\kappa\rho_{\mathrm{S},0}] \tag{3.64}$$

exists for all $\sigma$. Existence of $\lim_{N\uparrow+\infty}\frac{1}{N}\eta_N(\zeta_1)$ follows hence by dominated convergence from some $m$-independent bound, for example (see (4.7))

$$\|\int_{K^{-1}(\gamma)}\mathrm{d}\sigma W^\epsilon_{\kappa,\lambda}(\sigma+\tau m)\|\leq \mathrm{e}^{\|G\|_1|I_\gamma|} \tag{3.65}$$

# 4 Proofs of various lemma's

## 4.1 Proof of Lemma 2.1, Proposition 2.14 and well-definedness of (3.33)

Remark that for $g\in L^1(\mathbb{R}^+)$, the function

$$z\to\int_{\mathbb{R}^+} g(x)\mathrm{e}^{zx} \tag{4.1}$$

has an analytical continuation from $\mathrm{Re}\, z = 0$ to $\mathrm{Re}\, z < 0$. Hence, the bound (2.12) implies that $\kappa \to Q_{\kappa,\lambda}(x, \{p,q\}, m)$ (with the notation of Section 3.2.2) is analytic on $D^\circ + (\mathrm{i}\mathbb{R})^{|\mathcal{K}|}$ and

$$h(t) := \sup_{\omega \in \mathcal{F}, k \in \mathcal{K}} \sup_{\mathrm{Re}\,\kappa \in D, |\mathrm{Im}\,\kappa| \leq 1} |\langle \breve{f}_k, \mathrm{e}^{-\kappa_k l - \mathrm{i}t(l+\omega)} \breve{f}_k \rangle| < \infty \tag{4.2}$$

Also,

$$|Q_{\kappa,\lambda}(x, \{p,q\}, m)| \;\; \leq \;\; h(|u_q - u_p|) \tag{4.3}$$

Using (3.24), (3.26),the estimates (4.3), $\|\tilde{\rho}_{\mathrm{S}}\| \leq 1$ and putting $\|V\| := \max_{k \in \mathcal{K}} \|V_k\|$, we get

$$\begin{aligned}
\| \int_{\Omega_I} \mathrm{d}\sigma W^{\epsilon}_{\kappa,\lambda}(\sigma) \tilde{\rho}_{\mathrm{S}} \| \;\; &\leq \;\; \int_{\Omega_I} \mathrm{d}\sigma \lambda^{-2|\sigma|} \|V\|^{2|\sigma|} \prod_{\tilde{\sigma} \in \sigma} (1/2)\, 4|\mathcal{K}||\mathcal{F}|^2 h(\lambda^{-2} t(\tilde{\sigma}) - \lambda^{-2} s(\tilde{\sigma})) \\
&\leq \;\; \sum_{n \in \mathbb{N}} \left( 4|\mathcal{K}||\mathcal{F}|^2 \|V\|^2 \int_I \lambda^{-2} \mathrm{d}u h(\lambda^{-2} u) \right)^n \int_{s_n \geq \ldots \geq s_1} \prod_{i=1}^{n} \mathrm{d}s_i \\
&\leq \;\; \exp\left( 4|\mathcal{K}||\mathcal{F}|^2 \|V\|^2 |I| \int_{\lambda^{-2} I} \mathrm{d}u h(u) \right)
\end{aligned} \tag{4.4}$$

The prefactors $4|\mathcal{K}||\mathcal{F}|^2$ in the first inequality account for the fact that for each $\sigma$, there are $(4|\mathcal{K}|^2|\mathcal{F}|^2)^{|\sigma|}$ pairs $x \in \mathcal{X}^{2|\sigma|}, \pi \in Pair(2|\sigma|)$ such that $\sigma \sim (x, \pi)$ (The $|\mathcal{K}|^2$ turns into $|\mathcal{K}|$ because of the factors $\delta_{k_p,k_q}$). The factor $1/2$ appears to avoid double counting, since $4|\mathcal{K}||\mathcal{F}|^2 h(\lambda^{-2} t(\tilde{\sigma}) - \lambda^{-2} s(\tilde{\sigma}))$ bounds the sum over $\ell(\tilde{\sigma}) = \mathrm{a}, \mathrm{b}$.

Since the bound (4.4) doesnot depend on $\kappa$, analyticity of (3.26) and Lemma 2.1 follow from the Vitali convergence theorem.

To show Proposition 2.14, one expands the LHS of (2.33) just as in (3.17). Note that one does not need a bound like (2.12) since $\mathfrak{h}^n_k$ is finite-dimensional. By (2.31, 2.32),the expansion converges term by term to the infinite-volume expansion (3.17). Remark that for large enough $n$ and fixed $\kappa$, the expressions on the LHS of (2.31,2.32) are bounded by some $C$, uniformly in $t \in \mathbb{R}$. The total series is hence dominated similarly to (4.4) with $C$ replacing $h(\lambda^{-2} t(\tilde{\sigma}) - \lambda^{-2} s(\tilde{\sigma}))$.

We now turn to expression (3.33), establishing some notation that will be used in further proofs. By Assumption 2.3, we have $\|h\|_1 = \int_{\mathbb{R}^+} \mathrm{d}u h(u) < \infty$. Let for $u \geq 0$,

$$G_\epsilon(u) := 4|\mathcal{K}||\mathcal{F}|^2 \|V\|^2 \left( 2h(u) + \|h\|_1 \delta_\epsilon(u) \right) \geq 0 \tag{4.5}$$

and remark $\int_{\mathbb{R}^+} \mathrm{d}u G_\epsilon(u) =: \|G\|_1 < \infty$ (independent of $\epsilon, \lambda$). Put

$$H^\epsilon_\lambda(\sigma) = \prod_{\tilde{\sigma} \in \sigma} \lambda^{-2} G_\epsilon(\lambda^{-2} t(\tilde{\sigma}) - \lambda^{-2} s(\tilde{\sigma})) \tag{4.6}$$

Let $A \subset \Omega_I$, then by (4.6) and analogous to (4.4),

$$\| \int_A \mathrm{d}\sigma W^\epsilon_{\kappa,\lambda}(\sigma) \| \leq \int_A \mathrm{d}\sigma H^\epsilon_\lambda(\sigma) \leq \mathrm{e}^{I\|G\|_1} \tag{4.7}$$

Call

$$\mathcal{V} := \{\sigma \in \Omega | \exists \tilde{\sigma} \in \sigma : s(\tilde{\sigma}) = t(\tilde{\sigma})\} \tag{4.8}$$

the set on which (products of) $\delta_\epsilon$ becomes singular and let $\mathrm{d}_{\mathcal{V}}\sigma$ stand for the measure induced by $\mathrm{d}\sigma$ on (the submanifold) $\mathcal{V}$. By the bound (4.7), one sees that $\int_A \mathrm{d}\sigma W^\epsilon_{\kappa,\lambda}(\sigma)$ has a well-defined limit as $\epsilon \searrow 0$ whenever

$$\int_{\mathcal{V}} \mathrm{d}_{\mathcal{V}}\sigma \,\mathrm{Ind}[\sigma \in \partial A] = 0 \tag{4.9}$$

Since $A := K^{-1}(\gamma)$ for $\gamma \in \mathfrak{U}$ satisfies this condition, the expression (3.33) is well-defined. Its analyticity in $\kappa$ follows analogously to analyticity of (3.26), using the bound (4.7) and analyticity of $Q^{\mathrm{a}}_{\kappa,\lambda}(x,\{p,q\},m)$.

## 4.2 Estimates to be used in Section 4.3

Assume that $\int_{\mathbb{R}^+} \mathrm{d}u\, uh(u) < \infty$. Let $I$ be a union of intervals and $a \in \mathbb{R}$. For (4.13), we assume that $0 \notin I^\circ$. Let for $j \in \mathbb{Z}$ and $\sigma \in \Omega$,

$$A_j(\sigma) := \mathrm{Ind}\left[ \begin{array}{c} \exists \tilde{\sigma}, \tilde{\sigma}' \in \sigma : \mathrm{Supp}\tilde{\sigma} \cap \mathrm{Supp}\tilde{\sigma}' \neq \emptyset, \\ (\mathrm{Supp}\tilde{\sigma} \cup \mathrm{Supp}\tilde{\sigma}') \cap I_j \neq \emptyset \end{array} \right], \qquad B_j(\sigma) := \mathrm{Ind}\left[\ \exists \tilde{\sigma} \in \sigma : \mathrm{Supp}\tilde{\sigma} \cap \partial I_j \neq \emptyset\ \right] \tag{4.10}$$

There are $c_1, c_2, c_3 > 0$, such that

$$\int_{\Omega_I} \mathrm{d}\sigma H^\epsilon_\lambda(\sigma) A_j(\sigma) \leq \lambda^2 c_1 \tau \exp\left(|I| \|G\|_1\right) \tag{4.11}$$

$$\int_{\Omega_I} \mathrm{d}\sigma H^\epsilon_\lambda(\sigma) B_j(\sigma) \leq \lambda^2 c_2 \exp\left(|I| \|G\|_1\right) \tag{4.12}$$

$$\lim_{\epsilon \downarrow 0} \left\| \int_{\substack{\sigma \in \Omega_{I,n}, \ell(\tilde{\sigma}) = b \\ \forall \tilde{\sigma}, \tilde{\sigma}' \in \sigma : \mathrm{Supp}\tilde{\sigma} \cap \mathrm{Supp}\tilde{\sigma}' = \emptyset}} \mathrm{d}\sigma W^\epsilon_{\kappa,\lambda}(\sigma) \right\| \leq (\lambda^2 c_3)^n \tag{4.13}$$

*Proof.* We start with (4.13). Recall the notation of Section 3.2.2. By the substitution $u_q \to v, u_p - u_q \to u$,

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \left| \int_{u_p > u_q \in I} \mathrm{d}u_p \mathrm{d}u_q Q_{\kappa,\lambda}(x,\{p,q\},m) - Q^{\mathrm{a}}_{\kappa,\lambda}(x,\{p,q\},m) \delta_\epsilon\left(\frac{|u_p - u_q|}{\lambda^2}\right) \right| \\
\leq\ & \delta_{\omega_p,-\omega_q} \int_I \mathrm{d}v \int_{|I|-v}^\infty \mathrm{d}u h(\lambda^{-2}u) + (1 - \delta_{\omega_p,-\omega_q}) \lambda^2 \left| \int_I \mathrm{d}v \int_0^{|I|-v} \mathrm{d}u h(\lambda^{-2}u) \mathrm{e}^{-\mathrm{i}\lambda^{-2} v(\omega_p + \omega_q)} \right| \\
\leq\ & \delta_{\omega_p,-\omega_q} \left( \int_I \mathrm{d}u\, uh(\lambda^{-2}u) + |I| \int_{|I|}^\infty \mathrm{d}u h(\lambda^{-2}u) \right) + \lambda^2 (1 - \delta_{\omega_p,-\omega_q}) \int_0^{|I|} \mathrm{d}u h(\lambda^{-2}u) \frac{2\lambda^2}{\omega_p + \omega_q}
\end{aligned}$$

(The case $u_q > u_p$ is bounded identically.) An overall factor $\lambda^4$ can be extracted and we conclude

$$\lim_{\epsilon\downarrow 0}\| \int\limits_{\tilde\sigma\in\tilde\Omega_I,\ell(\tilde\sigma)=b} \mathrm{d}\tilde\sigma W^\epsilon_{\kappa,\lambda}(\tilde\sigma)\| \leq \lambda^2 4|\mathcal{K}|\|V\|^2\left(|\mathcal{F}|\int_{\mathbb{R}^+} du\, uh(u) + 2\|h\|_1 \sum_{\omega\neq\omega'\in\mathcal{F}} |\omega-\omega'|^{-1}\right) =: \lambda^2 c_3$$

which yields (4.13) by using the property (3.25).

To show (4.11), we have

$$\int\limits_{\Omega_I} \mathrm{d}\sigma H^\epsilon_\lambda(\sigma)A_j(\sigma) \leq \int\limits_{\tilde\sigma,\tilde\sigma'\in\tilde\Omega_I} \mathrm{d}\tilde\sigma\mathrm{d}\tilde\sigma' H^\epsilon_\lambda(\tilde\sigma\cup\tilde\sigma')A_j(\tilde\sigma\cup\tilde\sigma')\int\limits_{\Omega_I}\mathrm{d}\sigma H^\epsilon_\lambda(\sigma) \tag{4.14}$$

and the second integral on the RHS is smaller than $\mathrm{e}^{|I|\|G\|_1}$ by (4.7). For the first integral in (4.14), we assume without loss of generality that $s(\tilde\sigma) < s(\tilde\sigma')$. We assume that $A_j = 1$ which implies that at least one of $s(\tilde\sigma), t(\tilde\sigma), s(\tilde\sigma'), t(\tilde\sigma')$ lie in $I_j$. We assume that $s(\tilde\sigma)\in I_j$, the other possibilities give an identical bound (hence the factor $1/4$ on the last line).

$$\int\limits_{\tilde\sigma,\tilde\sigma'\in\tilde\Omega,s(\tilde\sigma)\in I_j,s(\tilde\sigma)<s(\tilde\sigma')<t(\tilde\sigma)} \mathrm{d}\tilde\sigma'\mathrm{d}\tilde\sigma H^\epsilon_\lambda(\tilde\sigma\cup\tilde\sigma') \tag{4.15}$$

$$\leq \|G\|_1 \int\limits_{\tilde\sigma\in\tilde\Omega,s(\tilde\sigma)\in I_j,s(\tilde\sigma)<s(\tilde\sigma')<t(\tilde\sigma)} \mathrm{d}\tilde\sigma\mathrm{d}s(\tilde\sigma')\lambda^{-2}G^\epsilon(\lambda^{-2}(t(\tilde\sigma)-s(\tilde\sigma))) \tag{4.16}$$

$$\leq \|G\|_1 \int\limits_{\tilde\sigma\in\tilde\Omega,s(\tilde\sigma)\in I_j} \mathrm{d}\tilde\sigma(t(\tilde\sigma)-s(\tilde\sigma))\lambda^{-2}G^\epsilon(\lambda^{-2}(t(\tilde\sigma)-s(\tilde\sigma))) \tag{4.17}$$

$$\leq \lambda^2\|G\|_1|I_j|\int_{\mathbb{R}^+}\mathrm{d}u\, uG^\epsilon(u) =: (1/4)\lambda^2 c_1\tau \tag{4.18}$$

where we used $|I_j| = \tau$. The integral on the LHS of (4.12) is evaluated analogously (though simpler), yielding the bound $\lambda^2\int_{\mathbb{R}^+}\mathrm{d}u\, uG^\epsilon(u) =: \lambda^2 c_2$.

□

## 4.3 Proof of Lemma 3.2

For each $\gamma\in\mathfrak{U}$ and $\sigma\in K^{-1}(\gamma)$ we define a partition $K_a(\sigma)\cup K_b(\sigma) = \gamma$

**Definition 4.1.** *A $j\in\mathbb{Z}$ belongs to $K_a(\sigma)\subset 2^{\mathbb{Z}}$ iff. there is a $J\subset I_j$ such that*

$$\sigma_J = \sigma\setminus\sigma_{\mathbb{R}\setminus J}, \qquad \forall\tilde\sigma\in\sigma_J : \ell(\tilde\sigma) = b, \qquad \forall\tilde\sigma,\tilde\sigma'\in\sigma_J, \mathrm{Supp}\tilde\sigma\cap\mathrm{Supp}\tilde\sigma' = \emptyset \tag{4.19}$$

*Let for $j \in K_a(\sigma)$, $J_j \subset I_j$ be the union of all intervals with the above property and put*

$$\sigma_a(\sigma) := \cup_{j \in K_a(\sigma)} \sigma_{J_j} \tag{4.20}$$

*Let $K_b(\sigma) := K(\sigma) \setminus K_a(\sigma)$ and define $\sigma_l(\sigma) \subset \sigma$*

$$\sigma_l(\sigma) := \left\{ \tilde{\sigma} \in \sigma \setminus \sigma_a(\sigma) \,\middle|\, |\mathrm{Supp}\tilde{\sigma}| > \tau \right\} \tag{4.21}$$

We now estimate

$$\| \int_{K_{a,b}(\sigma) = \gamma_{a,b}} \mathrm{d}\sigma W^{\epsilon}_{\kappa,\lambda}(\sigma) \| \leq \int_{\sigma_l(\sigma) \,=\, \sigma} \mathrm{d}\sigma H^{\epsilon}_{\lambda}(\sigma) \tag{4.22}$$

$$\int_{\substack{K_a(\sigma' \cup \sigma) = \emptyset, \gamma_b \subset K_b(\sigma' \cup \sigma) \subset \gamma \\ \sigma_l(\sigma' \cup \sigma) = \sigma}} \mathrm{d}\sigma' H^{\epsilon}_{\lambda}(\sigma') \tag{4.23}$$

$$\Big\| \int_{\substack{\sigma_a(\sigma \cup \sigma' \cup \sigma'') = \sigma'' \\ K_a(\sigma \cup \sigma' \cup \sigma'') = \gamma_a}} \mathrm{d}\sigma'' W^{\epsilon}_{\kappa,\lambda}(\sigma'') \Big\| \tag{4.24}$$

By (4.13) and for $\lambda$ small enough and $c_a = 2c_3$,

$$\lim_{\epsilon \searrow 0} \Big\| \int_{\substack{\sigma_a(\sigma \cup \sigma' \cup \sigma'') = \sigma'' \\ K_a(\sigma \cup \sigma' \cup \sigma'') = \gamma_a}} \mathrm{d}\sigma'' W^{\epsilon}_{\kappa,\lambda}(\sigma'') \Big\| \leq \Big(\sum_{n=1}^{\infty} (\lambda^2 c_3)^n\Big)^{|\gamma_a|} \leq (c_a \lambda^2)^{|\gamma_a|} \tag{4.25}$$

In (4.23), all $j \in \gamma_b$ such that $I_j \cap \mathrm{Supp}\sigma = \emptyset$ necessarily satisfy $C_j(\sigma') := A_j(\sigma') \vee B_j(\sigma') = 1$ where $x \vee y := \max(x, y)$, hence

$$\int_{\substack{K_a(\sigma' \cup \sigma) = \emptyset, \gamma_b \subset K_b(\sigma' \cup \sigma) \subset \gamma \\ \sigma_l(\sigma' \cup \sigma) = \sigma}} \mathrm{d}\sigma' H^{\epsilon}_{\lambda}(\sigma') \leq \int_{\Omega_{I_\gamma}} \mathrm{d}_\tau \sigma' H^{\epsilon}_{\lambda}(\sigma') \prod_{\substack{j \in \gamma_b \\ \mathrm{Supp}(\sigma) \cap I_j = \emptyset}} C_j(\sigma') \tag{4.26}$$

where we have exploited $\sigma_l(\sigma') = \emptyset$ to modify the measure as

$$\mathrm{d}_\tau \sigma' := \mathrm{Ind}[\forall \tilde{\sigma}' \in \sigma' : |\mathrm{Supp}\tilde{\sigma}'| < \tau] \, \mathrm{d}\sigma'. \tag{4.27}$$

Remark that $\sigma$ in (4.26) intersects at most $(3/\tau)|\mathrm{Supp}(\sigma)|$ elements $j \in \gamma_b$ and hence the product in (4.26) contains at least $|\gamma_b| - \min\{(3/\tau)|\mathrm{Supp}(\sigma)|, |\gamma_b|\}$ elements. For $n \geq 3$, one easily establishes

$$\int_{\Omega} \mathrm{d}_\tau \sigma' C_j(\sigma') C_{j+n}(\sigma') = \int_{\Omega} \mathrm{d}_\tau \sigma' C_j(\sigma') \int_{\Omega} \mathrm{d}_\tau \sigma' C_{j+n}(\sigma') \tag{4.28}$$

By keeping hence only each third element in the product in (4.26) ,using the estimates (4.11) and (4.12) and the factorization (4.28),

$$\text{RHS } (4.26) \leq (c'(\lambda))^{|\gamma_b|-\min\{(3/\tau)|\text{Supp}(\sigma)|,|\gamma_b|\}} \left(\mathrm{e}^{|I_\gamma|\|G\|_1}\right) \tag{4.29}$$

with $c'(\lambda) = \sqrt[3]{(c_1\tau + c_2)\lambda^2}$.

We replace (4.23) by the RHS of (4.29) (omitting for the moment the number $\mathrm{e}^{|I_\gamma|\|G\|_1}$) and we perform the first integral in (4.22): By Assumption 2.3 , there is a $c'' > 0$ such that

$$|h(t)| \leq c'' \mathrm{e}^{-\alpha|t|} \tag{4.30}$$

Introduce $q_i := t_i(\sigma) - s_i(\sigma)$ and $q := \sum_{i=1}^{|\sigma|} q_i \geq |\text{Supp}\sigma|$,

$$\int_{\sigma_l(\sigma)=\sigma,\, K_b(\sigma)\subset\gamma_b} \mathrm{d}\sigma H^\epsilon_\lambda(\sigma) c'(\lambda)^{|\gamma_b|-\min\{(3/\tau)|\text{Supp}(\sigma)|,|\gamma_b|\}} \tag{4.31}$$

$$\leq c'(\lambda)^{|\gamma_b|} \sum_{n\geq 0} \frac{(c''|I_{\gamma_b}|)^n}{n!} \int_{(I_{\gamma_b})^n} \mathrm{d}q_1 \ldots \mathrm{d}q_n \mathrm{e}^{-\alpha\lambda^{-2}q} c'(\lambda)^{-\min\{|\gamma_b|,(3/\tau)q\}} \tag{4.32}$$

$$\leq c'(\lambda)^{|\gamma_b|} \sum_{n\geq 0} \frac{(c''|I_{\gamma_b}|)^n}{n!} \int_{\mathbb{R}^+} \mathrm{d}q \frac{q^{n-1}}{(n-1)!} \mathrm{e}^{-\alpha\lambda^{-2}q} c'(\lambda)^{-\min\{|\gamma_b|,(3/\tau)q\}} \tag{4.33}$$

The first inequality follows by integrating over $s_i(\sigma)$, the second by integrating over $q_{1,\ldots,n}$ with the constraint $q := \sum_{i=1}^n q_i$. By a straightforward computation, for small enough $\lambda$, the expression (4.33) is bounded by some function $c_b(\lambda)^{|\gamma_b|}$ with $c_b(\lambda) \searrow 0$ as $\lambda \searrow 0$. Combining this with the bounds (4.25) and (4.7), we get

$$\lim_{\epsilon\searrow 0} \sum_{\gamma_a\cup\gamma_b=\gamma} \| \int_{K_{a,b}(\sigma)=\gamma_{a,b}} W^\epsilon_{\kappa,\lambda}(\sigma) \| \tag{4.34}$$

$$\leq \sum_{\gamma_a\cup\gamma_b=\gamma} \mathrm{e}^{\|G\|_1|I_\gamma|} c_a(\lambda)^{|\gamma_a|} c_b(\lambda)^{|\gamma_b|} \leq \left(e^{\|G\|_1}(c_a(\lambda)+c_b(\lambda))^{\tau^{-1}}\right)^{|I_\gamma|} \tag{4.35}$$

where the sum is over partitions of $\gamma$ into $\gamma = \gamma_a \cup \gamma_b$ and we have used the binomial theorem. This proves Lemma 3.2.

## 4.4 Proof of Lemma 3.3 and Lemma 3.4

For $\tilde\sigma \in \tilde\Omega_I$ such that $\ell(\tilde\sigma) = \text{a}$.

$$\lim_{\epsilon\downarrow 0} W^\epsilon_{\kappa,\lambda}(\tilde\sigma) = \begin{cases} \mathcal{L}_\kappa & \text{if} \quad s(\tilde\sigma) > 0 \\ \mathcal{L}_0 & \text{if} \quad s(\tilde\sigma) < 0 \end{cases} \tag{4.36}$$

which follows by comparing (3.23) and (3.35). Similarly, choosing now $\mathrm{Supp}\sigma \subset \mathbb{R}^+$,

$$\ell(\sigma)_{1,\ldots,|\sigma|} = \mathrm{a} \Rightarrow \lim_{\epsilon \downarrow 0} W^{\epsilon}_{\kappa,\lambda}(\sigma) = (\mathcal{L}_\kappa)^{|\sigma|} \tag{4.37}$$

After integrating over $s_{1,\ldots,|\sigma|}$ (which yields a factor $\frac{I^{|\sigma|}}{|\sigma|!}$) and summing over $|\sigma|$, Lemma 3.3 follows by dominated convergence, using the bounds (4.7) for the LHS of (3.38).

Lemma 3.4 follows from a Perron-Frobenius theorem for completely positive maps: From [14], one easily derives (see the Appendix of [34] for more precise references)

**Theorem 4.1.** *Let $Q$ be a completely positive map on a finite dimensional $C^*-$ algebra $\mathcal{A}$ and assume that for all strictly positive (i.e. positive and invertible) $S, S' \in \mathcal{A}$,*

$$\mathrm{Tr}\,[SQS'] > 0 \tag{4.38}$$

*Then, $Q$ has a strictly positive simple eigenvalue $e$ such that, for all eigenvalues $e' \neq e$, one has $|e'| < e$. The eigenvector $Z$, corresponding to $e$ can be chosen strictly positive.*

By a well-known theorem of Frigerio [16], Assumption 2.2 implies the property (4.38).

## 4.5 Proof of Lemma 3.5

For $\zeta \in \mathfrak{W}_\Lambda$ and with the conventions $\max \emptyset = 0, \min \emptyset = \max \Lambda$, we put

$$\min \zeta \;:=\; \begin{cases} \min\{\min(\zeta \setminus \{-\varsigma, \varsigma\}), 0\} & \text{if} \quad -\varsigma \in \zeta \\ \min(\zeta \setminus \{\varsigma\}) & \text{otherwise} \end{cases} \tag{4.39}$$

$$\max \zeta \;:=\; \begin{cases} \max \Lambda = \tau^{-1} t - 1 & \text{if} \quad \varsigma \in \zeta \\ \max(\zeta \setminus \{-\varsigma\}) & \text{otherwise} \end{cases} \tag{4.40}$$

$$d(\zeta) \;:=\; \max \zeta - \min \zeta + 1 \tag{4.41}$$

Starting from (3.48), using the estimates $\|W^-_\kappa(\epsilon_-)\| \times \|W^+_\kappa(\epsilon_+)\| \leq b_\kappa^{-1}$ and Lemma 3.4, we get

$$w^\Lambda_{\kappa,\lambda}(\zeta) \leq \mathrm{e}^{-\tau d(\zeta) g} \frac{\dim \mathcal{H}_{\mathrm{S}}}{b_\kappa} \prod_{i=1}^n \|\tilde{W}_{\kappa,\lambda}(\gamma_i)\| \tag{4.42}$$

Remark that by Lemma 3.2, definition (3.41) and the simple bound $\|\mathrm{e}^{-t|\mathcal{L}_\kappa|}\| \leq \mathrm{e}^{t\|\mathcal{L}_\kappa\|}$,

$$A(\lambda) := \sup_{j \in \mathbb{Z}} \sum_{\gamma \in \mathfrak{U}, \min \gamma = j} \|\tilde{W}_{\kappa,\lambda}(\gamma)\| \to 0, \qquad \lambda \searrow 0 \tag{4.43}$$

Using that $\zeta \in \mathfrak{W}_\Lambda \setminus \{\{-\varsigma,\varsigma\}\}$ either has $n \geq 1$ (in the parametrization (3.47)) or $w^\Lambda_{\kappa,\lambda}(\zeta) = 0$ (as is clear in (3.51)), one calculates

$$
\begin{aligned}
\sup_{j\in\Lambda} \sum_{\substack{d(\zeta)=d,\min\zeta=j\\ \zeta\in\mathfrak{W}_\Lambda\setminus\{\{-\varsigma,\varsigma\}\}}} w^\Lambda_{\kappa,\lambda}(\zeta) &\leq \frac{\dim\mathcal{H}_{\mathrm{S}}}{b_\kappa}\mathrm{e}^{-\tau dg} \sup_{j\in\Lambda} \sum_{\substack{d(\zeta)=d,\min\zeta=j\\ \zeta\in\mathfrak{W}_\Lambda\setminus\{\{-\varsigma,\varsigma\}\}}} \prod_{i=1}^{n} \|\tilde{W}_{\kappa,\lambda}(\gamma_i)\| \\
&\leq \frac{4\dim\mathcal{H}_{\mathrm{S}}}{b_\kappa}\mathrm{e}^{-\tau dg}\sum_{n\in\mathbb{N}_0}\frac{(d)^n}{n!}A(\lambda)^n \\
&\leq \frac{4\dim\mathcal{H}_{\mathrm{S}}}{b_\kappa}(\mathrm{e}^{dA(\lambda)}-1)\mathrm{e}^{-\tau dg} \qquad (4.44)
\end{aligned}
$$

To obtain the second inequality, rewrite the sum over $\zeta$ as a sum over the minima of $\gamma_{i=1,\dots,n}$ (which lie necessarily between $\min\zeta$ and $\min\zeta + d$) and a sum over all possible $\gamma_i$ with fixed minima (as in (4.43)). The factor 4 counts the possibilities for $\epsilon_-, \epsilon_+$.

Choose $a(\zeta) := ad(\zeta)$ with $a > 0$.

$$
\begin{aligned}
&\sum_{\zeta'\in\mathfrak{W}_\Lambda,\zeta'\nsim\zeta} |w^\Lambda_{\kappa,\lambda}(\zeta')|\mathrm{e}^{ad(\zeta')} \qquad (4.45)\\
&\leq \sum_{d=1}^{\infty}(d+d(\zeta))\mathrm{e}^{ad}\sup_{j\in\Lambda}\sum_{\substack{d(\zeta')=d,\min\zeta'=j\\ \zeta\in\mathfrak{W}_\Lambda\setminus\{\{-\varsigma,\varsigma\}\}}} w^\Lambda_{\kappa,\lambda}(\zeta') + \frac{\dim\mathcal{H}_{\mathrm{S}}}{b_\kappa}\mathrm{e}^{-(\tau g-a)d(\{-\varsigma,\varsigma\})} \qquad (4.46)\\
&\leq \frac{\dim\mathcal{H}_{\mathrm{S}}}{b_\kappa}\sum_{d=1}^{\infty}(d+d(\zeta))(\mathrm{e}^{dA(g,\lambda)}-1)\mathrm{e}^{-d(\tau g-a)} + \frac{\dim\mathcal{H}_{\mathrm{S}}}{b_\kappa}\mathrm{e}^{-(\tau g-a)t} \qquad (4.47)
\end{aligned}
$$

To get the first inequality, isolate $\zeta' = \{-\varsigma,\varsigma\}$ in the sum over $\mathfrak{W}_\Lambda$. The factor $d(\zeta)+d$ in (4.46) is the number of possible values for $\min\zeta'$. The second inequality follows by (4.44).

To get Statement (1) of Lemma 3.5, it suffices e.g. to take $a = (1/2)g\tau$. Indeed, the last term in (4.47) is made arbitrarily small by choosing $t$ large enough, the first term is controlled since for e.g. $A(\lambda) < (1/4)\tau g$, the exponent $\tau g - a - A(\lambda) \geq (1/4)\tau g > 0$ and hence terms with large $d$ decay exponentially. The terms with small $d$ are controlled by $\mathrm{e}^{dA(\lambda)} - 1$ which is made arbitrarily small by reducing $\lambda$.

Statement (2) of Lemma 3.5 is obvious since $\mathfrak{W}_\Lambda$ is a finite set. The analyticity (in $\kappa$) of the weights $w^\Lambda_{\kappa,\lambda}(\zeta)$ is a straightforward consequence of the analyticity of (3.33), which is proven in Section 4.1, and analyticity of $\kappa \to \mathcal{L}_\kappa, \mathrm{e}_\kappa$.

## 4.6 Proof of Corollaries 2.12 and 2.13

We prove Corollary 2.12. Let $\rho_{\mathrm{S}}$ be the trace state on $\mathcal{H}_{\mathrm{S}}$, i.e. $\rho_{\mathrm{S}}[S] = (1/\dim\mathcal{H}_{\mathrm{S}})\,\mathrm{Tr}[S]$. Let $\kappa(\zeta)$ be as in (2.26) for $\zeta \in \mathbb{C}$, and let $F, F'$ be finite products of $A_0(u), u \in \mathbb{R}$, as introduced in Section 3.2.1. One easily checks (or consults [7]) that

$$\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[Y_{-\kappa(\mathrm{i}t)} F Y_{\kappa(\mathrm{i}t)} F'] = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[F' Y_{-\kappa(\mathrm{i}t-1)} F Y_{\kappa(\mathrm{i}t-1)}] \tag{4.48}$$

which expresses that the state $\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}$ is a KMS-state at temperature 1 for the dynamics given by $Y_{-\kappa(\mathrm{i}t)} \cdot Y_{\kappa(\mathrm{i}t)}$ on the appropriate Araki-Woods $W^*$-algebra.

Since on the domain $\mathcal{D}$, $U_t^\lambda$ is a sum of such terms $F$ (see (3.14)), one gets

$$\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[Y_{-\kappa(\mathrm{i}t)} U_{-t}^\lambda Y_{\kappa(\mathrm{i}t)} U_t^\lambda] = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[U_t^\lambda Y_{-\kappa(\mathrm{i}t-1)} U_{-t}^\lambda Y_{\kappa(\mathrm{i}t-1)}] \tag{4.49}$$

Upon inserting $\Theta\Theta = 1$ and using $\Theta Y_\kappa \Theta = Y_{\overline{\kappa}}$, $\Theta U_t^\lambda \Theta = U_{-t}^\lambda$ and

$$\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[\cdot] = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[\Theta \cdot \Theta], \qquad \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[\cdot] = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[Y_{-\kappa} \cdot Y_\kappa], \tag{4.50}$$

one obtains by analytic continuation

$$\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[Y_{-\kappa(\zeta)} U_{-t}^\lambda Y_{\kappa(\zeta)} U_t^\lambda] = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}[Y_{-\kappa(1-\zeta)} U_{-t}^\lambda Y_{\kappa(1-\zeta)} U_t^\lambda] \tag{4.51}$$

from which the Corollary 2.12 follows by Theorem 2.4.

Corollary 2.12 follows immediately from a theorem by Bryc [4].

# Acknowledgments

The author has benefited from good discussions with H.-T. Yau, C. Maes, J. Bricmont, J. Dereziński and C-A. Pillet. The financial support of the FWO-Flanders is greatly acknowledged.

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