On Properties of Impurity Spectrum in the Disordered Exactly Solvable Model*

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Abstract

The random point interaction Hamiltonian $-\Delta_{\alpha(\omega)}$ is considered on $\mathbb{L}^2(\mathbb{R}^d)$, $d = 2$ or $d = 3$. Existence and certain bounds of the non-empty pure point component and exponential decay of the corresponding eigenfunctions with probability one, within region of impurity spectrum of $-\Delta_{\alpha(\omega)}$, are rigorously established. In order to prove the localization result, the structure of the generalized eigenfunctions of $-\Delta_{\alpha(\omega)}$ is explicitly described, and the relation between its spectral properties, and the properties of spectra of finite-difference infinite-order operators on $\ell^2(\mathbb{Z}^d)$, is established. The multiscale analysis scheme is applied to investigate the point spectrum of finite-difference operators. In addition, the generalized spectral theorem and the absolute continuity of the integrated density of states of $-\Delta_{\alpha(\omega)}$ at the negative (impurity) part of the spectrum, rigorously proved. Applications of the new approximation scheme include straightforward analysis of the absolutely continuous conductivity spectrum ([G4] 1999), subject to a (possible) separate publication by the author.

Contents

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†absolutely alone, non-committee's, non-slavonic, surname adopted, no relatives,
never requested any third partie(s) neither to receive, nor to entrust any of my correspondence,
nor to communicate on my behalf. Direct meaning in a non-cyrillic.

0 Introduction

The following paper describes some important research results in Mathematical Physics that were not available in the literature previously (new in 1993).

Consider the random point interaction Hamiltonian on $\mathbb{L}^2(\mathbb{R}^d)$, $d=2$, or $d=3$, formally defined by

$$-\Delta_{\alpha(\omega)} = -\Delta - \sum_{j\in\mathbf{Z}^d} \alpha_j(\omega)\delta_d(.-j) \tag{0.1}$$

where $\Delta = \sum\limits_j \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator on $\mathbb{L}^2(\mathbb{R}^d)$, $\delta_d(.-j)$ denotes the d-dimensional point potential concentrated at $j\in\mathbb{Z}^d$, $\{\alpha_j(\omega)\}_{j\in\mathbf{Z}^d}$ are independent identically distributed random variables with probability distributions of compact support and bounded density:

$$P_\alpha\{\alpha\in d\alpha\} = g(\alpha)d\alpha, \;\; \delta^{-1} = \sup_\alpha g(\alpha) < \infty \tag{0.2}$$

(δ characterizes the strength of the disorder). The probability-space $(\Omega,\mathbf{P})$ is the space of realization of random coefficients α: $\Omega = \prod\limits_{j\in\mathbf{Z}^d}(\mathbb{R}_j, dP_{\alpha_j})$. The operator (0.1) may be rigorously defined for $d\leq 3$ as the strong resolvent limit of the selfadjoint extensions of operators $-\Delta_{|\mathbf{C}_0^\infty(\mathbf{R}^d\setminus\Lambda)}$, $(\mathbb{Z}^d\supset\Lambda\to\mathbb{Z}^d)$, e.g. [AGHH] for a precise definition. The resolvent kernel $(-\Delta_\alpha - z)^{-1}(x,y)$ is defined by

$$\begin{aligned}(-\Delta_\alpha - z)^{-1}(x,y) &= (-\Delta - z)^{-1}(x,y) \\ &+ \sum_{j,j'\in\mathbf{Z}^d}(-\Delta - z)^{-1}(x,j)\,[\Gamma_\alpha(z)]^{-1}(j,j')\,(-\Delta - z)^{-1}(j',y),\end{aligned} \tag{0.3}$$

where $z\notin\sigma(-\Delta)$, and the operator $\Gamma_\alpha(z):\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)$ $(d=2,3)$, is defined as follows :

$$\Gamma_\alpha(z;j,j') = \begin{cases}\alpha_j + \beta(z), & \text{if } j=j', \\ -(-\Delta - z)^{-1}(j,j'), & \text{otherwise,}\end{cases} \tag{0.4}$$

where $\beta(z) = \lim_{|x-j|\to 0} -((-\Delta - z)^{-1}(x,j) - G_0(x-j))$; $\quad G_0(x) = \begin{cases}-(2\pi)^{-1}\ln(|x|) & \text{if } d=2, \\ (4\pi|x|)^{-1} & \text{if } d=3;\end{cases}$
$j,j'\in\mathbb{Z}^d$, $\alpha_j\in\mathbb{R}$, $x,y\in\mathbb{R}^d$.

In the case of $d=1$, the single-site point potential corresponds to the Dirac δ-function ([DFP]), and the respective one-dimensional Hamiltonian represents the standard Schrödinger operator on $\mathbb{L}^2(\mathbb{R})$ with δ- potential (the Kronig-Penney model).
If $d=2$, or $d=3$, the operator $-\Delta_{|\mathbf{C}_0^\infty(\mathbf{R}^d\setminus\Lambda)}$ initially defined on $\Lambda\subset\mathbb{Z}^d$ (the set of infinitely differentiable functions of $\mathbb{C}_0^\infty(\Lambda)$ vanishing at the points of the d-dimensional cell), is symmetric, and its self-adjoint extensions (the point interaction Hamiltonians described by (0.1)-(0.4)) do not correspond to a formal sum of the Laplace operator and any multiplication operator on $\mathbb{L}^2(\mathbb{R}^d)$.

In what follows, we consider explicitly d=3 dimensional case, all the described proofs and results are also valid simultaneously in dimension 2, and the case $d=1$ had been studied already ([DSS]).

Notice that all our results are valid for the operator with the random point potential defined on arbitrary lattice

$$\Lambda = \{n_1e_1 + n_2e_2 + n_3e_3 \in \mathbb{R}^d | (n_1,n_2,n_3)\in\mathbb{Z}^d\},$$

where e_1, e_2, e_3 is a basis in $\mathbb{R}^d$.

In the following denote as usual $\mathbb{L}^2(\mathbb{R}^d) = \{f(x) : \mathbb{R}^d \to \mathbb{C}| \ x \in \mathbb{R}^d, f(x) \in \mathbb{C}, \|f\|_{\mathbb{L}^2} = \int_{\mathbf{R}^d} |f(x)|^2 \, \mathrm{dx} < \infty\}$, $\ell^2(\mathbb{Z}^d) = \{f(j) : \mathbb{Z}^d \to \mathbb{C}| \ j \in \mathbb{Z}^d, f(j) \in \mathbb{C}, \|f\|_{\ell^2} = \sum_{j\in\mathbf{Z}^d} \|f(j)\|^2 < \infty\}$, $\mathcal{H} = \mathbb{L}^2$ (ℓ^2) is called the Hilbert space with the scalar prodact $\langle f, g\rangle_{\mathbf{L}^2} := \int_{\mathbf{R}^d} \overline{f(x)}\, g(x)\, dx$ $(\langle f, g\rangle_{\ell^2} := \sum_{j\in\mathbf{Z}^d} \overline{f(j)}\, g(j))$. A number $\lambda \in \mathbb{C}$ is called the eigenvalue of operator $H : \mathcal{D}(H) \to \mathcal{I}(H) \subset \mathcal{H}$, if

$$H\psi = \lambda\psi \tag{0.5}$$

for some $\psi \in \mathcal{D}(H)$, where $\mathcal{D}(H)$ is the domain of H, $\mathcal{I}(H)$ is the image of H, ψ is called the eigenfunction of H. The closure of the set of all the eigenvalues of H is called the point spectrum, $\sigma_{pp}(H)$.

The set $\sigma \subset \mathbb{C}$, such that for arbitrary $\lambda \in \sigma$ there exists a sequence $f_n(\lambda) \in \mathcal{D}(H)$, $\|f_n\| = 1$, $\lim_{n\to\infty} \|(H-\lambda)\| f_n = 0$, is called the continuous spectrum of H, $\sigma_c(H)$. $\sigma(H) = \sigma_c(H)\cup\sigma_{pp}(H)$ is called the spectrum of H. $\sigma(H)$ is the closed subset of $\mathbb{C}$ and may be determined as the complement to the resolvent set of operator H, that is if $z \in \mathbb{C}\backslash\sigma(H)$ the resolvent $(H - z)^{-1}$ is a bounded operator on H (i.e. $\|(H - z)^{-1}\| = \sup\limits_{\|f\|=1} \|(H - z)^{-1}f\| = \mathrm{dist}\{z, \sigma(H)\} < \infty$, $z \notin \sigma(H)$).

Complex-valued function of several complex variables $f(z)$, $z = (z_1, z_2, .., z_n)$, $z_i \in \mathbb{C}$, is called analytic (holomorphic) function in an open $\Omega \subset \mathbb{C}^n$, $n \geq 1$, if in some vicinity $\Omega(z)$ of each $z \in \Omega$, f expands in series $f = \sum\limits_{n_1,..,n_n} a_{n_1,..,n_n}(z_1 - \omega_1)^{n_1} ...(z_n - \omega_n)^{n_n}$ which converges $\forall\omega \in \Omega(z)$.

The resolvent $(H - z)^{-1}$ is operator-valued holomorphic function of $z \in \Omega \subset \mathbb{C}$, if $\Omega \cap \sigma(H) = \emptyset$. A set $S \subset \mathbb{C}^n$ is called zero-set for holomorphic function $f(z)$, if $f(z) = 0$, $z \in S$.

Remind that operator H^*, such that $\langle g, Hf\rangle = \langle H^*g, f\rangle$ for arbitrary $f \in \mathcal{D}(H)$, $g \in \mathcal{D}(H^*)$, is called the operator adjoint to H. Operator A is called the symmetric operator, if $\overline{\mathcal{D}(A)} = \mathcal{H}$, and $Af = A^*f$, $f \in \mathcal{D}(A) \cap \mathcal{D}(A^*)$. Operator H is called the self-adjoint operator, if $\sigma(H) \subseteq \mathbb{R}$ (or if $H = H^*$, which means H is symmetric and $\mathcal{D}(H^*) = \mathcal{D}(H)$). Self-adjoint operator H is called a self-adjoint extension of a symmetric operator A, if $H = A$ on $\mathcal{D}(A) \subset \mathcal{D}(H)$.

Denote by $(\Omega, \mathcal{B}, \mathbb{P})$ the probability-space formed by σ-algebra $\mathcal{B}$ of $\mathbb{P}$-measurable subsets of Ω. $\mathbb{P}$-measurable operator-valued random variable $H(\omega)$, $\omega \in \Omega$, $\mathcal{D}(H(\omega)) \subset \mathcal{H}$, is called the random operator, self-adjoint, if $H(\omega)$, $\omega \in \Omega$ are self-adjoint operators. Ω is called the realization space of the corresponding random parameters.

The corresponding probability measure $\mathbb{P}$ is called ergodic, if 1)it is invariant with respect to the shift transformations: $\mathbb{P}(T_j(A)) = \mathbb{P}(A)$, for every $\mathbb{P}$-measurable $A \subset \Omega$, $T_j(A) = \{\omega(j' - j)| \ j' \in A\}$, 2)$\mathbb{P}(A) = 1$ or $\mathbb{P}(A) = 0$ for arbitrary shift-invariant subset $A \subset \Omega$: $A = T_j(A)$, $j \in \Omega$. For example, probability measure corresponding to the random field generated by independent identically distributed random variables $\{q_j(\omega)\}_{j\in\mathbf{Z}^d}$, defines ergodic random operator. In many important cases the ergodic random operators describe contensive and adopted to the rigorous study models.

The self-adjoint operators, and in particular the random ergodic operators, are of special interest and their rigorous study had been a problem of primary importance since the end of twenties of XX century, when the respective problem arised in context of the newly developed quantum mechanics. In quantum mechanics, the self-adjoint operators are assigned to the multiple physical variables in order to describe mathematically their bihaviour, such as coordinate and impulse, determining current position of a quantum particle in space and time (usually assigned with operators x, $-ih\frac{\partial}{\partial x}$), energy (usually called the Hamiltonian operator H), determining ability of a quantum particle to change its location in time in accordance with the corresponding driving equation ($H\psi(x,t) = ih\frac{\partial}{\partial t}\psi(x,t)$, the Schrödinger equation in the classical interpretation). The rigorous study of the mathematical properties of the corresponding operators allows to describe the properties of the respective physical objects. In particular, the knowledge of physical properties of an inhomogenuous material (such as alloy, polymer, semiconductor, etc.) are important in radio-electronics, optics, aerospace industry, etc. The Hamilton operator studied in the following paper represents in particular the energy operator of a quantum particle moving with influence of point impurities of the random strength in such a material. If, under certain conditions, a particle has energy belonging to a region of the pure point spectrum of the Hamiltonian, it is described in terms of the square summable wave function (associated with certain localized state), and should remain within finite distance from its initial position, which means localization, absence of diffusion, zero conductivity, and the corresponding material should have the insulator properties (under respective conditions).

The corresponding region of the spectrum of the random Hamilton operator (where the pure point spectrum could appear), is called the impurity spectrum, since it is actually produced by the impurity sources. The strength of the impurity influence is described in terms of the explicit expression for the corresponding "potential", or by the probability distributions of the corresponding real-valued random variables α_j. In the model considered, the independent identically distributed random variables have bounded density and are chosen to construct the random operator with ergodic properties, which permits its rigorous analysis via the spectral theory of random operators ([CFKS, CL]).

The point spectrum of the Hamiltonians with ergodic potential on $\mathbb{L}^2(\mathbb{R}^d)$ has been much studied recently (cf. [KS, CHM]). There were usually studied random potentials of the following type:

$$V_\omega(x) = \sum_{j\in\mathbf{Z}^d} \lambda_j(\omega)\, f_j(x - x_j(\omega)),$$

where the single-site potentials f_j corresponding to the neighbouring points of the lattice $\mathbb{Z}^d$ have supports of non-zero Lebesgue measure.

The detailed study of the spectral properties of operator (0.1) with the so-called "zero-range point potential" of general type (i.e. corresponding to the arbitrary bounded set of values of the coefficients $\{\alpha_j\}_{j\in\mathbf{Z}^d}$), is a subject to many recent publications. A quite complete survey of the results may be found in [AGHH].

In the following paper we study mainly the impurity component of the spectrum of the operator (0.1) -(0.4), in dimensions $d = 2$ or 3. We prove (Theorem 1) that while the disorder parameter δ increases to infinity, the pure point component fills the negative part of the spectrum of $-\Delta_{\alpha(\omega)}$, and when δ reaches a determined value $\delta_0 < \infty$, then all the negative spectrum becomes pure point. We also prove (Theorem 5) that the density of states of $-\Delta_{\alpha(\omega)}$ is absolutely continuous.

The explicit expression (0.3) for the resolvent kernel of $-\Delta_{\alpha(\omega)}$ implies that the spectral properties of this operator are related to the spectral properties of the operators $\Gamma_{\alpha(\omega)}(z)$, defined by (0.3) for the fixed $z \notin \sigma(-\Delta_{\alpha(\omega)})$, which are the bounded operators of infinite-order on $\ell^2(\mathbb{Z}^d)$ with random potential, selfadjoint if $z = \lambda < 0$.

The new approximation scheme developed to study the point spectrum of $-\Delta_{\alpha(\omega)}$ is composed by the two separate parts. At first we establish the convenient relation between the spectral properties of $-\Delta_{\alpha(\omega)}$ and $\Gamma_{\alpha(\omega)}(\lambda)$, and secondly we prove the relevant results for $\Gamma_{\alpha(\omega)}(\lambda)$.

We prove (Theorem 2) that the spectral measure of $-\Delta_\alpha$ is concentrated on a set of generalized eigenvalues which correspond to the "polynomially bounded" distributional solutions (Definition 1) of the equation (0.5). This theorem is an analogue of the well-known theorem for finite-difference operators (lemma B.1 [Sch]), and for the Schrödinger operators with potentials of the so-called K_ν-class, [Si1, KoSe]. This theorem is valid for arbitrary selfadjoint operator with the following property:

$$T_\delta^{-1}E(\Delta)T_\delta^{-1} \quad \text{is a trace class operator,} \tag{0.6}$$

where T_δ is the multiplication operator by $(1+|x|)^\delta$ on $\mathbb{L}^2(\mathbb{R}^d)$, $\delta > d/2$, $\mathbb{R} \supset \Delta$ is arbitrary bounded measurable set, $E(\Delta)$ is the corresponding spectral projection of $-\Delta_\alpha$.

This implies the integral estimates for the respective generalized solutions. The pointwise polynomial boundedness depends on the continuity properties of the spectral kernel.

Using the explicit expression (0.3) for the resolvent kernel, we prove the statement (0.6) for $-\Delta_\alpha$. The general properties of the point potential imply that the corresponding distributional solution does not represent a continuous function in the points $j \in \mathbb{Z}^d$: its local asymptotics is determined by $a_j G^0_\lambda(x-j)$ for $x \to j \in \mathbb{Z}^d$, where $G^0_\lambda(x,y)$ denotes the integral kernel of $(-\Delta-\lambda)^{-1}$ for $\lambda < 0$.

Theorem 3 proves that arbitrary distributional solution of (0.5), corresponding to $\lambda < 0$, admits the following representation:

$$\psi_\lambda(x) = \sum_{i \in \mathbf{Z}^d} a_i\, G^0_\lambda(x-i), \tag{0.7}$$

where for a.e. $\lambda < 0$ (with respect to $E(d\lambda)$) $\overline{a} = \{a_i\}_{i\in\mathbf{Z}^d}$ is a (generalized) polynomially bounded solution of the equation $\Gamma_\alpha(\lambda)\overline{a} = 0$. Thus there is a one-to-one correspondence between the generalized eigenfunctions of operators $-\Delta_\alpha$ and $\Gamma_\alpha(\lambda)$. This result is deterministic (i.e. is valid for arbitrary admissible values of α).

In order to prove the localization result for $-\Delta_\alpha$, it suffices to verify that within the range of the impurity spectrum (corresponding to the sufficiently low concentration of the random impurities), every polynomially bounded generalized eigenfunction of $\Gamma_{\alpha(\omega)}(\lambda)$ decays exponentially with probability one (Theorem 4).

Results of such a type were usually used to establish exponential localization for the finite-difference operator with the random potential (the Anderson model, [A]). These results involve the multiscale analysis in order to estimate the corresponding Green function ([FS, SW, DLS, D]). The infinite-order generalization of the multiscale analysis scheme ([G1, G2]) is applied to establish the required result. We briefly sketch the proof showing that estimates of Sect. 4 are uniform in λ.

Notice, that the new approximation scheme allows to avoid the straightforward analysis in $\mathbb{L}^2(\mathbb{R}^d)$ (however, this also remains possible: localization for $-\Delta_{\alpha(\omega)}$ was first established by the author in that way, which however turned out to be more technical and less natural than the scheme of the proof of the following paper).

We also establish that the integrated density of states of $-\Delta_{\alpha(\omega)}$ at the negative spectrum is absolutely continuous, describing the relation between the integrated density of states of operators $-\Delta_{\alpha(\omega)}$ and $\Gamma_{\alpha_{|\Lambda}(\omega)}(\lambda)$ by using monotonicity in λ of the corresponding operators.

The new approximation scheme presented in the following paper is directly applicable (i.e. the respective proofs may be repeated) to prove the localization at weak disorder (via applying suitable adaptation of the multiscale analysis) for the random point interaction hamiltonians on $\mathbb{L}^2(\mathbb{R}^d)$, as well as for the operators on $\ell^2(\mathbb{Z}^d)$ ([G8]) (established previously for operators with random potential on $\ell^2(\mathbb{Z}^d)$ ([An]) via different approximation technics).

The results presented are also applicable to study the delocalization properties and to establish existence of the pure absolutely continuous spectrum in multidimensional random models ([G4, G6]), as well as in multiple other applications ([ES]).

In Section 1 there are listed some notations and definitions used in the text, stated main results and discussed some applications.

In Section 2 is proved Theorem 2 on the polynomial boundedness of the generalized eigenfunctions.

In Section 3 is proved Theorem 3 on the structure of distributional solutions.

In Section 4 is proved Theorem 4 (localization for finite-difference infinite-order operators).

In Section 5 is proved Theorem 5 (result on the density of states).

In Section 6 is proved the generalized spectral theorem.

In the Appendix there are proved some auxiliary technical results.

1 Results

Denote by $G_z(x,y) = (-\Delta_\alpha - z)^{-1}(x,y), \quad z \notin \sigma(-\Delta_\alpha)$, and $G_z^0(x,y) = (-\Delta - z)^{-1}(x,y) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}, \quad z \notin [0,+\infty)$, the Green's functions (resolvent kernels) of the operators $-\Delta_\alpha$ and $-\Delta$; by $\mathcal{D}(-\Delta)$ and $\mathcal{D}(-\Delta_\alpha)$ the domains of the operators $-\Delta$ and $-\Delta_\alpha$ (for the fixed value of $\alpha = \alpha(\omega)$):

$$
\begin{aligned}
\mathcal{D}(-\Delta) &= \left\{ f \mid f \in \mathbb{L}^2(\mathbb{R}^3), \;\; \Delta f \in \mathbb{L}^2(\mathbb{R}^d) \;\; \text{in the distributional sense} \right\}, \\
\mathcal{D}(-\Delta_\alpha) &= \left\{ \begin{array}{l} \varphi \mid \varphi(x) = \varphi_0(x) + \sum_{i \in \mathbf{Z}^d} a_i \, G_z^0(x-i), \;\; x \in \mathbb{R}^d \setminus \mathbb{Z}^d, \\ \text{where} \;\;\; \varphi_0(x) \in \mathcal{D}(-\Delta), \;\; a_i = \sum_{j \in \mathbf{Z}^d} \Gamma_z^{-1}(i,j)\, \varphi_0(j) \end{array} \right\}.
\end{aligned} \tag{1.1}
$$

If $\varphi \in \mathcal{D}(-\Delta_\alpha)$,

$$
(-\Delta_\alpha - z)\,\varphi = (-\Delta - z)\,\varphi_0. \tag{1.2}
$$

Also denote $\overline{a} = \{a_i\}_{i \in \mathbf{Z}^d}$, $\overline{\varphi}_0 = \{\varphi_0(i)\}_{i \in \mathbf{Z}^d}$.

It is known ([AGHH]) that selfadjoint operators $\{-\Delta_{\alpha(\omega)}\}_{\omega \in \Omega}$ are ergodic, the spectrum $\Sigma = \sigma(-\Delta_{\alpha(\omega)})$, as well as the point spectrum (σ_{pp}), the absolutely continuous spectrum (σ_{ac}) and the singular continuous spectrum (σ_{sc}) are non-random subsets of $\mathbb{R}$ (i.e. they coincide with non-random sets with probability one), and the discrete spectrum is empty (with probability one).

We say that $\alpha = \{\alpha(i) \in \operatorname{supp} P_{\alpha_i}\}_{i \in \mathbf{Z}^d}$ is an admissible potential. Denote by A the set of all admissible potentials.

Proposition 1 **1** . *$\sigma\left(-\Delta_\alpha\right) \subseteq \Sigma, \;\; \alpha \in A$;*

2 . *$\Sigma = \bigcup_{\alpha \in A} \sigma\left(-\Delta_\alpha\right)$.*

Proposition 2 *Suppose* $\operatorname{supp} P_{\alpha_0} \subset [\mu, \nu]$. *Then*

$$
\sigma(-\Delta_\alpha) = \overline{\bigcup_{m \in \mathbf{N}} [a_m, b_m]},
$$

where $-\infty < a_0 < 0$, $a_m < b_m$, $b_m \leq \infty$, $m \in \mathbb{N}$ (the corresponding intervals may intersect). Furthermore, there exists $\lambda_0 < \infty$ such that if $\nu < \lambda_0$, and $\operatorname{supp} P_{\alpha_0} = [\mu, \nu]$, *then*

$$
\sigma(-\Delta_\alpha) = [a_0(\mu), b_0(\nu)] \cup [A(\mu), \infty) = \sigma(-\Delta_\mu) \cup \sigma(-\Delta_\nu),
$$

where $b_0(\nu) < 0$, $A(\mu) \geq 0$.

The proofs of Propositions 1, 2 could be found in [AGHH].

The following is the result on localization.

Theorem 1 *Given arbitrary r, $0 < r < 1$, there exist $\delta_0 = \delta_0(r, P_{\alpha_0})$ and $\delta_1 = \delta_1(r, P_{\alpha_0})$, $\delta_0 > \delta_1 > 0$, such that:*

1 . *if $\delta_1 \leq \delta < \delta_0$, there exists an increasing function $E_0 = E_0(\delta) \leq -r_0 < 0$ determining the pure point spectrum of $-\Delta_{\alpha(\omega)}$ in the interval $(-\infty, E_0)$:*

$$\sigma(-\Delta_{\alpha(\omega)}) \cap (-\infty, E_0) \subset \sigma_{pp};$$

2 . *if $\delta \geq \delta_0(r)$, then $\sigma(-\Delta_{\alpha(\omega)}) \cap (-\infty, -r) \subset \sigma_{pp}$. Moreover, if $\operatorname{supp} P_{\alpha_0} = [\mu, \nu]$, and $\nu < \lambda_0$ (λ_0 is defined by Prop.2), then δ_0 may be chosen such that for $\delta > \delta_0$:*

$$\sigma(-\Delta_{\alpha(\omega)}) \cap (-\infty, 0) \subset \sigma_{pp}.$$

3 . *The eigenfunctions of $-\Delta_\alpha$, corresponding to the eigenvalues of the point spectrum in the cases 1 and 2 decay exponentially at infinity, with probability one:*

$$\psi_{\lambda,\omega}(x) = \sum_{i \in \mathbf{Z}^d} a_{\lambda,\omega}(i)\, G^0_\lambda(x - i),$$

where

$$|a_{\lambda,\omega}(i)| \leq C(\lambda, \omega)\, e^{-m_\lambda |i|}, \quad 0 < m_\lambda < \sqrt{|\lambda|}.$$

Remark. Eigenfunction corresponding to an eigenvalue $\lambda < 0$, decays exponentially at infinity with rate at least $\sqrt{|\lambda|}$.

Remark. Theorem 1 holds for the Hölder continuous distributions P_{α_0}. In this case one considers the potential $\{\lambda\, \alpha_i\}_{i \in \mathbf{Z}^d}$ and λ measures the strength of the disorder.

Theorem 1 follows by Theorems 2,3,4. In order to prove Theorem 1, we study the properties of the spectral kernel of $-\Delta_\alpha$ and show that its spectral measure is concentrated on the set of generalized eigenvalues, corresponding to the solutions of equation (0.5) that are "polynomially bounded" (Definition 1).

Denote by $E_{\alpha(\omega)}(\Delta)$ the spectral projection of $-\Delta_{\alpha(\omega)}$ (we usually omit dependence on ω in our notation when this does not lead to contradiction).

We say that the statement $S(\lambda)$ is true for a.e. $\lambda \in I \subset \mathbb{R}$ with respect to the spectral measure of $-\Delta_\alpha$, if $E_\alpha(\Delta) = 0$ for $\Delta = \{\lambda \in I \mid S(\lambda) \text{ is not true}\}$.

Denote $\mathbb{L}^2_{-\delta} = \mathbb{L}^2(\mathbb{R}^d, d\mu_{-\delta})$, where $d\mu_{-\delta}(x) = (1 + |x|)^{-2\delta}\, \mathrm{d}x$; and $\mathbb{L}^2_\delta = \mathbb{L}^2(\mathbb{R}^d, d\mu_\delta)$, where $d\mu_\delta(x) = (1 + |x|)^{2\delta}\, \mathrm{d}x$, $\delta > d/2$.

Denote $\ell^2_{-\delta} = \ell^2(\mathbb{Z}^d, d\nu_{-\delta})$, where $\nu_{-\delta}(i) = (1 + |i|)^{-2\delta}$, $i \in \mathbb{Z}^d$, and $\ell^2_\delta = \ell^2(\mathbb{Z}^d, d\nu_{+\delta})$, where $\nu_{+\delta}(i) = (1 + |i|)^{2\delta}$, $i \in \mathbb{Z}^d$, $\delta > d/2$.

Definition 1 *A number $\lambda \in \mathbb{R}$ is called the generalized eigenvalue, and a function $\psi_\lambda \in \mathbb{L}^2_{-\delta}$ the generalized eigenfunction of $-\Delta_\alpha$, if*

$$\langle \psi_\lambda, (-\Delta_\alpha - \lambda)\varphi \rangle = 0 \tag{1.3}$$

for arbitrary φ of type (1.1) with $\varphi_0 \in \mathbb{C}_0^\infty$, and $z \notin \sigma(-\Delta_\alpha)$.

Denote

$$\mathcal{D}_0(-\Delta_\alpha) \;=\; \{\varphi \in \mathcal{D} |\; \varphi_0 \in \mathbb{C}_0^\infty,\; z \le E_0(\alpha,\lambda) < 0\}.$$

The following remarks mention that Definition 1 is correct.

Remark. Lemma A1 of Appendix implies that it is possible to choose E_0 in such a way that for every admissible $\alpha(\omega)$, $\omega \in \Omega$, $\;-\Delta_\alpha(\mathcal{D}_0(-\Delta_\alpha)) \subset \mathbb{L}^2_\delta$.

Remark. $\overline{\mathcal{D}_0(-\Delta_\alpha)} = \overline{\mathcal{D}(-\Delta_\alpha)}$, where $\overline{A}$ denotes the closure of $A \subset \mathcal{H}$ in $\mathbb{L}^2$-norm (Lemma 6.3).

The following theorem is deterministic (i.e. is valid for arbitrary admissible $\alpha \in A$).

Theorem 2 *Almost every $\lambda \in \mathbb{R}$ with respect to the spectral measure of $-\Delta_\alpha$ is a generalized eigenvalue.*

The following theorem describes explicitly the structure of the generalized eigenfunctions of $-\Delta_\alpha$.

A function $f(x)$, $x \in \mathbb{Z}^d$ is said to decay exponentially at infinity with rate $m > 0$, if $\limsup\limits_{|x|\to\infty} \frac{\ln|f(x)|}{|x|} \le -m$.

Theorem 3 *1. $f_\lambda(x)$ is a generalized eigenfunction of $-\Delta_\alpha$, relative to a negative generalized eigenvalue $\lambda < 0$, if and only if*

$$f_\lambda(x) = \sum_{i \in \mathbf{Z}^d} f_i\, G^0_\lambda(x-i), \tag{1.4}$$

where $\overline{f}_\lambda = \{f_i\}_{i\in\mathbf{Z}^d} \in \ell^2_{-\delta}$ is a generalized solution of the equation

$$\Gamma_\alpha(\lambda)\, \overline{f}_\lambda = 0. \tag{1.5}$$

2. $\lambda < 0$ is a proper eigenvalue of $-\Delta_\alpha$ if and only if 0 is an eigenvalue of $\Gamma_\alpha(\lambda)$.

Theorem 3b *$F_\lambda(x)$ is a generalized eigenfunction of $-\Delta_\alpha$, relative to a positive generalized eigenvalue $\lambda > 0$, if and only if*

$$F_\lambda(x) = \phi_\lambda(x) + \sum_{i\in\mathbf{Z}^d} f_i\, G^0_\lambda(x-i) \;\in\; \mathbb{L}^2_{-\delta}, \tag{1.6}$$

where $\phi_\lambda(x)$ is the distributional solution of the equation $(-\Delta - \lambda)\phi(x) = 0$, and $\overline{f} = \{f_j\}_{j\in\mathbf{Z}^d} \in \ell^2_{-\delta}$ satisfies

$$\sum_{j\in\mathbf{Z}^d} \Gamma_\alpha(\lambda;k,j)\, f_j \;=\; \phi_\lambda(k) \quad k \in \mathbb{Z}^d,$$

where $\Gamma_\alpha(\lambda; k, j)$ is determined by (0.4).

Theorems 2 and 3 imply that there is the direct correspondence between the spectral properties of operator $-\Delta_{\alpha(\omega)}$ and spectral properties of operators $\Gamma_\alpha(\lambda)$. Lemma A2 of Appendix implies that generalized solutions of the equations (0.5) and (1.5) exist simultaneously, and obey the same asymptotics at infinity.

The point spectrum of finite-difference operators of infinite-order, with the random potential was studied by [G1]. The respective results are formulated by the following theorem:

Theorem 4 *Given an interval $I \subset (-\infty, -r)$, $r > 0$, there exist increasing function $E_0 = E_0(\delta)$ ($E_0 \le -r \ \forall \delta > 0$), and $0 < \delta_0 = \delta_0(r, |I|) < \infty$, such that each generalized solution $\overline{f}_\lambda$ of (1.5) decays exponentially at infinity with probability 1, if λ satisfy one of the following conditions a), b):*

a) $\lambda \in I$ *if* $\delta \ge \delta_0(r, |I|)$,

or

b) $\lambda \in (-\infty, -E_0(\delta)) \cap I$ *if* $0 < \delta < \delta_0(r)$.

Theorem 4 is proved in Section 4.

Remak. Results presented by [G1, G2] permit arbitrary complex-valued off-diagonal terms

$$\Gamma_\alpha(\lambda; j, j') = \overline{\Gamma_\alpha(\lambda; j', j)}, \quad j, j' \in \mathbb{Z}^d.$$

The following is the result on the density of states.

Consider $N(\lambda)$, the integrated density of states of $-\Delta_{\alpha(\omega)}$ (the discussion is presented in Sect. 5).

Theorem 5 *Suppose P_{α_0} has a density bounded by δ^{-1}. Given $r > 0$, there exists $C = C(r)$ such that if $a < b$, $(a, b) \subset (-\infty, -r)$, then*

$$|N(b) - N(a)| \le 2C\delta^{-1} \, |b - a|.$$

Remark. If P_{α_0} is Hölder continuous of order $0 < \gamma < 1$, then $N(\lambda)$ is also Hölder continuous of the same order.

Theorem 5 follows by Lemmas 5.1 and 5.2. The following statements are some corollaries of the results above.

Corollary 1. It is not hard to develop the arguments of Theorem 2, and to prove the theorem on expansion in the generalized eigenfunctions for $-\Delta_\alpha$, as it was done by [Ti, Si1] for the wide class of selfadjoint operators, which however does not concern the point interaction Hamiltonians (Sect. 2, 6).

Corollary 2. If λ is a generalized eigenvalue of $-\Delta_\alpha$, then $\lambda \in \sigma(-\Delta_\alpha)$.

Corollary 2 follows by Theorem 3, Lemma 3.1 (Sect. 3) since the generalized eigenvalues of a finite-difference operator belong to its spectrum.

Corollary 3. $\sigma(-\Delta_\alpha)$ is the closure of the set of generalized eigenvalues.
Corollary 3 follows by Theorem 2, Corollary 2 since if $\lambda_0 \in \sigma(-\Delta_\alpha)$ and $\varepsilon > 0$, then $E_\alpha(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) > 0$.

Corollary 4.\ *(Location of the negative spectrum)*

$$\sigma\left(-\Delta_{\alpha(\omega)}\right) \cap (-\infty, 0) = \left\{\lambda < 0 \,|\, 0 \in \sigma\left(\Gamma_{\alpha(\omega)}(\lambda)\right)\right\}, \tag{1.7}$$

$$\sigma_{pp}\left(-\Delta_{\alpha(\omega)}\right) \cap (-\infty, 0) = \left\{\lambda < 0 \,|\, 0 \in \sigma_{pp}\left(\Gamma_{\alpha(\omega)}(\lambda)\right)\right\}. \tag{1.8}$$

Proof. Suppose $\lambda \in \sigma(-\Delta_{\alpha(\omega)})$. Then Proposition 1 implies $\lambda \in \sigma(-\Delta_\alpha)$, where α is some admissible potential. Hence there exists a sequence $F_n \in \mathcal{D}(-\Delta_\alpha)$, such that $\|F_n\|_{\mathbf{L}^2} = 1$, $\|(-\Delta_\alpha - \lambda)F_n\|_{\mathbf{L}^2} \to 0$. Lemma 3.1 implies that there exists a sequence $f_n \in \ell^2(\mathbb{Z}^d)$, such that $\|f_n\|_{\ell^2} = 1$, $\|\Gamma_\alpha\, f_n\|_{\ell^2} \to 0$, hence $0 \in \sigma(\Gamma_\alpha) \subset \sigma(\Gamma_{\alpha(\omega)})$ (since Proposition 1 holds for finite-difference ergodic operators). Now assume $\lambda \in \sigma_{pp}(-\Delta_{\alpha(\omega)})$. Then there exists a set $\Omega_0 \subset \Omega$, such that $\mathbf{P}\,(\Omega_0) = 1$, and $\lambda \in \sigma_{pp}(-\Delta_{\omega_0})$ for $\omega_0 \in \Omega_0$. According to Theorem 3 this implies that $0 \in \sigma_{pp}(\Gamma_{\omega_0})\ \forall \omega_0 \in \Omega_0$, and hence $0 \in \sigma_{pp}(\Gamma_{\alpha(\omega)})$. The converse is proved by the same way.

The generalized spectral theorem (Theorem 6) is proved in Section 6.

Theorem 7. (Absence of the mixed point spectrum)

$$\sigma_{pp}(-\Delta_\alpha) \cap (0, +\infty) = \emptyset.$$

The proof of Theorem 7 is analogues to the proof of absence of the mixed point spectrum in the Anderson model ([G4]).

Remark.

$$\sigma(-\Delta_{\alpha(\omega)})\ \cap\ (a, +\infty) \subset\ \sigma_{ac},$$

for some $a \geq 0$ (i.e. the conductivity spectrum is pure absolutely continuous (1999)).

The proof is a part of possible separate publication by the author.

Approximation scheme presented by the following paper is applicable to describe spectral properties of different other models important in physics.

Example. (Pure point spectrum for random point interaction Hamiltonian in the presence of magnetic field)
Consider the random point interaction Hamiltonian on $\mathbb{L}^2(\mathbb{R}^d)$, $d = 2$ or $d = 3$, in the presence of magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = \text{const} \neq 0$:

$$H_{\alpha(\omega)}\ =\ (-i\nabla - \mathbf{A})^2\ +\ \sum_{j \in \mathbf{Z}^d} \lambda\alpha_\omega(j)\delta_d(.-j),$$

where $\alpha(\omega)$ are i.i.d.v. with bounded distribution density of compact support.
Applying methods and results described by Theorems 1-7, it is not hard to prove that the spectrum of H is pure point, and corresponding eigenfunctions are exponentially localized (with probability 1) in any closed interval $I \subset (E_n, E_{n+1}) \cap \sigma(H_{\alpha(\omega)})$, where $\{E_n\}_{n \in \mathbf{N}}$ are the corresponding Landau levels, if $\lambda > \lambda_0(\mathbf{B}, |I|, d, P_\alpha)$ (at large disorder).
The rigorous poof of the latter result requires to prove the exponential decay of $((-i\nabla - A)^2 - z)^{-1}(x, y)$ as function of $|x - y|$, to verify polynomial boundedness of the generalized eigenfunctions (presented proof of Theorem 6 requires only semiboundedness of H), and to confirm validity of the initial estimates $(P1)$, $(P2)$ at high disorder (appropriate values of $\lambda, \mathbf{B}$), which according to Theorem 4, ensure exponential localization in the corresponding spectral bands.

2 On properties of the spectral kernel

Proof of Theorem 2.
Denote by E_α the spectral projection of $H = -\Delta_\alpha$, $G_E = (-\Delta_\alpha - E)^{-1}$, $E < \inf \sigma(-\Delta_\alpha)$, $T_\delta = (1+|x|)^\delta$ (multiplication operator), $\delta > d/2$.

Lemma 2.1 *If $\Delta \subset \mathbb{R}$ is a bounded measurable set, $A(\Delta) = T_\delta^{-1} E(\Delta) T_\delta^{-1}$ is a trace class operator.*

Proof. Suppose $E < \inf \sigma(-\Delta_\alpha)$ and consider the function

$$S_0(x) = \frac{1}{|x|} \chi_{\Lambda_{1/2}}(x),$$

where $\chi_{\Lambda_{1/2}}(x)$ is the characteristic function of the cube $\Lambda_{1/2}(0)$, centered at 0, with side-length 1/2. Denote

$$S(x) = \sum_{i \in \mathbf{Z}^d} S_0(x - i),$$

then $S(x) > 0$ and $|S^{-1}(x)| < 1/2$. Denote by S the corresponding multiplication operator. Operator on $\mathbb{L}^2(\mathbb{R}^d)$ with the integral kernel $a(x,y)$:

$$\sup_x \int_{\mathbf{R}^d} |a(x,y)|^2 \, dy < \infty,$$

is called a Carleman operator.

The resolvent formula (0.3)-(0.4) implies that $S^{-1} G_E$ is a Carleman operator, since

$$\sup_x \int_{\mathbf{R}^d} S^{-2}(x) \, |G_E(x,y)|^2 \, dy < \infty$$

(apply Lemma A2 of Appendix and the dominated convergence theorem). Hence $S^{-1} G_E$ is a bounded mapping from $\mathbb{L}^2$ to $\mathbb{L}^\infty$ ([Kor, Si1]). Now prove that if a real function $f(x)$ satisfies $f(x) \leq \frac{C}{1+|x|}$, $x \in \sigma(-\Delta_\alpha)$, then the operator $S^{-1} f(-\Delta_\alpha)$ maps $\mathbb{L}^2$ to $\mathbb{L}^\infty$ (i.e. is of Carleman type ([Kor])). For this purpose consider the function

$$g(x) = (x - E) f(x), \quad E < \inf \sigma(-\Delta_\alpha).$$

Then $f(-\Delta_\alpha) = (-\Delta_\alpha - E)^{-1} g(-\Delta_\alpha)$, i.e.

$$S^{-1} f(-\Delta_\alpha) = [S^{-1} \, (-\Delta_\alpha - E)^{-1}] \, g(-\Delta_\alpha).$$

Since $g(-\Delta_\alpha)$ is bounded in $\mathbb{L}^2$, and it is proved that $S^{-1}(-\Delta_\alpha - E)^{-1}$ is bounded from $\mathbb{L}^2$ to $\mathbb{L}^\infty$, then $S^{-1} f(-\Delta_\alpha)$ maps $\mathbb{L}^2$ to $\mathbb{L}^\infty$. Thus $S^{-1} E(\Delta)$ is a Carleman operator. Denote its kernel by $[S^{-1} E(\Delta)](x,y)$, then

$$\sup_x \int_{\mathbf{R}^d} [S^{-1} \, E(\Delta)]^2(x,y) \, dy = B < \infty.$$

Now prove that $T_\delta^{-1} E(\Delta)$ is a Hilbert-Schmidt operator. Indeed, $T_\delta^{-1} \, E(\Delta) = T_\delta^{-1} S \, [S^{-1} E(\Delta)]$, and

$$\int\limits_{\mathbf{R}^d\times\mathbf{R}^d} (1+|x|)^{-2\delta}\, S^2(x)\, [S^{-1}E(\Delta)]^2(x,y)\, dxdy$$

$$\leq B \int\limits_{\mathbf{R}^d} (1+|x|)^{-2\delta}\, S^2(x)\, dx < \infty.$$

Hence

$$T_\delta^{-1}E(\Delta), \quad E(\Delta)T_\delta^{-1} = (T_\delta^{-1}E(\Delta))^*$$

are the Hilbert-Schmidt operators, and $A(\Delta) = T_\delta^{-1}E(\Delta)T_\delta^{-1}$ is of trace class. Lemma 2.1 is proved.

Now follow the standard argument ([Si1]). Clearly, since $A(\Delta)$ is nonnegative, given arbitrary bounded set $\Delta = \cup_{n=1}^{\infty}\Delta_n$, such that $\Delta_i \cap \Delta_j = \emptyset$,

$$A(\Delta) = s - \lim_{m\to\infty} \sum_{n=1}^{m} A(\Delta_n).$$

It follows that the operator-valued measure $A(\Delta)$ is absolutely continuous with respect to the scalar measure

$$\rho(\Delta) = \mathrm{Tr}(T_\delta^{-1}\, E(\Delta)\, T_\delta^{-1}),$$

so, by the Radon-Nikodym theorem, for $d\rho$-a.e. λ, there exists a positive measurable trace class operator-valued function $a(\lambda)$ such that

$$A(\Delta) = \int\limits_{\Delta} a(\lambda)\, d\rho(\lambda), \quad \mathrm{Tr}(a(\lambda)) = 1.$$

It is easy to check that $d\rho$ is equivalent to $dE(\Delta)$ ($d\rho$ is the spectral measure). By $a(x,y;\lambda)$ denote the integral kernel of $a(\lambda)$. Consider the function

$$F(\lambda; x, y) = (1+|x|)^\delta\, a(x,y;\lambda)\, (1+|y|)^\delta.$$

Lemma 2.2 *There exist a spectral measure $d\rho(\lambda)$, and a function $F(\lambda;x,y)$, measurable in x,y and λ, which has the following properties (all the statements are valid for $d\rho$-almost all λ):*

(1)

$$\int\limits_{\mathbf{R}^d\times\mathbf{R}^d} |F(\lambda;x,y)|^2\, (1+|x|)^{-2\delta}\, (1+|y|)^{-2\delta}\, dxdy \leq 1. \tag{2.1}$$

(2) *Given arbitrary bounded measurable function $g(\lambda)$, $\lambda \in \sigma(-\Delta_\alpha)$, and arbitrary $\psi \in \mathcal{D}_0(-\Delta_\alpha)$, $\varphi \in \mathbb{L}^2_\delta$:*

$$\langle \varphi, g(-\Delta_\alpha)\psi\rangle = \int\limits_{\sigma(-\Delta_\alpha)} g(\lambda)\, [\int\limits_{\mathbf{R}^d\times\mathbf{R}^d} F(\lambda;x,y)\, \overline{\varphi(x)}\, \psi(y)\, dxdy]\, d\rho(\lambda). \tag{2.2}$$

(3) *$F(\lambda;x,y)$ is continuous in $(x,y) \in (\mathbb{R}^d\times\mathbb{R}^d)\backslash(\mathbb{Z}^d\times\mathbb{Z}^d)$.*

(4) *Consider* $\widetilde{H_0} = -\Delta_x - \Delta_y,\ x, y \in \mathbb{R}^d$, *then for arbitrary* $\widetilde{f_0} \in \widetilde{C_0^\infty}(\mathbb{R}^d \times \mathbb{R}^d \setminus \mathbb{Z}^d \times \mathbb{Z}^d)$ *(the set of infinitely differentiable functions on* $\mathbb{L}^2(\mathbb{R}^6)$ *with compact support containing no point of* $\mathbb{Z}^d \times \mathbb{Z}^d$*):*

$$\langle F(\lambda;\cdot,\cdot), (\widetilde{H_0} - 2\lambda)\widetilde{f_0}\rangle = 0. \tag{2.3}$$

(5) $F(\lambda;\cdot,y)$ *is a generalized eigenfunction of* $-\Delta_\alpha$.

Proof. 1. Follows since $\mathrm{Tr}(a(\lambda)) = 1$.

2. (2.1) implies that the integrand in [.] is absolutely integrable, and the integral is a bounded function of λ. Thus (2.2) follows by the definition of $E(\Delta)$.)

3. Since $-\Delta_{\alpha|\mathbf{C}_0^\infty(\mathbf{R}^d\setminus\mathbf{Z}^d)} = -\Delta_{|\mathbf{C}_0^\infty(\mathbf{R}^d\setminus\mathbf{Z}^d)}$, $\widetilde{H_0}$ is a proper Laplacian on $\widetilde{\mathbb{C}_0^\infty}(\mathbb{R}^d \times \mathbb{R}^d \setminus \mathbb{Z}^d \times \mathbb{Z}^d)$, the statement 3) follows by 4).

4. Suppose $\varphi_x, \psi_y \in \mathbb{C}_0^\infty(\mathbb{R}^d \setminus \mathbb{Z}^d)$, then

$$\begin{aligned}
[\widetilde{H_0}\,\varphi_x\,\psi_y](x,y) &= [-\Delta_x\varphi](x)\,\psi(y) + \varphi(x)\,[-\Delta_y\psi](y) \\
&= [H_x\varphi](x)\,\psi(y) + \varphi(x)\,[H_y\psi](y).
\end{aligned}$$

Hence by (2.2) for arbitrary $g \in \mathbb{C}_0^\infty(\sigma(-\Delta_\alpha))$, $\phi, \psi \in \mathbb{C}_0^\infty(\mathbb{R}^d \setminus \mathbb{Z}^d)$:

$$\begin{aligned}
&\int_{\sigma(-\Delta_\alpha)} g(\lambda) \int_{\mathbf{R}^d\times\mathbf{R}^d} F(\lambda;x,y)\,[\widetilde{H_0}\overline{\varphi_x}\,\psi_y](x,y)\,dxdy\,d\rho(\lambda) \\
=\ &\int_{\sigma(-\Delta_\alpha)} g(\lambda)\,[\int_{\mathbf{R}^d\times\mathbf{R}^d} 2\lambda\,F(\lambda;x,y)\,\overline{\varphi(x)}\,\psi(y)\,dxdy]\,d\rho(\lambda),
\end{aligned}$$

which implies (2.3).

5. Relation (2.2) and Fubini theorem imply that for arbitrary $\varphi \in \mathcal{D}_0(-\Delta_\alpha),\ \psi \in \mathbb{C}_0^\infty$

$$\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} F(\lambda;x,y)\,(-\Delta_\alpha - \lambda)\varphi(x)\,dx\right)\,\psi(y)\,dy = 0,$$

hence

$$\int_{\mathbf{R}^d} F(\lambda;x,y)\,(-\Delta_\alpha - \lambda)\varphi(x)\,dx = 0 \tag{2.4}$$

in every point y of continuity of F. Now fix arbitrary $y \notin \mathbb{Z}^d$ and consider functions $f_\lambda(x) = F(\lambda;x,y)$. Then for $d\rho$ - a.e. λ, $f_\lambda \in \mathbb{L}^2_{-\delta}$ ((2.1)), and f_λ satisfies (2.4). Lemma 2.2 is proved. The functions constructed are the generalized eigenfunctions of $-\Delta_\alpha$ for dE - almost all $\lambda \in \Delta$, Theorem 2 is proved.

3 Explicit expressions for the generalized eigenfunctions

Proof of Theorem 3.

Suppose $f_\lambda(x)$ is a generalized eigenfunction of $-\Delta_\alpha$, $\lambda < 0$, then $f \in \mathbb{L}^2_{-\delta}$, $\delta > d/2$, and

$$\langle f, (-\Delta_\alpha - \lambda)\,\varphi\rangle = 0 \tag{3.1}$$

for arbitrary φ of type:

$$\varphi(x) = \varphi_0(x) + \sum_{i \in \mathbf{Z}^d} a_i \, G_E(x-i), \tag{3.2}$$

for some $E < \inf \operatorname{supp} P_{\alpha_0}$, $a_i = \sum_{j \in \mathbf{Z}^d} \Gamma_\alpha^{-1}(E; i, j) \, \varphi(j)$, $\varphi_0 \in \mathbb{C}_0^\infty$.
(3.1) and the definition of $-\Delta_\alpha$ imply, that if

$$\phi \in H_0^{2,2}(\mathbb{R}^d \setminus \mathbb{Z}^d) := \Big\{ \varphi \in H_0^{2,2}(\mathbb{R}^d) \mid \varphi(j) = 0, \ j \in \mathbb{Z}^d, \ \operatorname{supp} \varphi \text{ is finite} \Big\},$$

then

$$\langle f, \, (-\Delta - \lambda)\varphi \rangle = 0. \tag{3.3}$$

Now prove that if $g \in \mathbb{L}^2_{-\delta}$ satisfies (3.3) for arbitrary $\varphi \in H_0^{2,2}(\mathbb{R}^d \setminus \mathbb{Z}^d)$, then

$$g \in \overline{\operatorname{Lin}\{G_\lambda^0(. - j)\}}_{j \in \mathbf{Z}^d}$$

(i.e. g belongs to the closure of the linear span of $\{G_\lambda^0(. - j)\}_{j \in \mathbf{Z}^d}$ in $\mathbb{L}^2_{-\delta}$).
First prove that there exists a sequence $\{\gamma_i\}_{i \in \mathbf{Z}^d}$, depending only on g and λ, such that for arbitrary $\psi \in \mathbb{C}_0^\infty \subset \mathcal{D}(-\Delta)$ with $\operatorname{supp} \psi \subset \Lambda \subset \mathbb{R}^d$:

$$\langle g, \, (-\Delta - \lambda)\psi \rangle = \sum_{i \in \Lambda} \gamma_i \, \psi(i).$$

For this purpose consider the function

$$\widetilde{\psi}(x) = \psi(x) - \sum_{i \in \Lambda} \psi(i) \, \theta(x-i),$$

where the function $\theta \subset \mathbb{C}_0^\infty$ is such that $\operatorname{supp} \theta \subset \Lambda_{1/2}(0)$, $\theta(0) = 1$, $|(-\Delta - \lambda)\theta(x)| \le \tau_\lambda < \infty$. Then $\widetilde{\psi} \in H_0^{2,2}(\mathbb{R}^d \setminus \mathbb{Z}^d)$, and

$$\langle g, \, (-\Delta - \lambda)\widetilde{\psi}) \rangle = 0.$$

It follows that

$$\begin{aligned} \langle g, \, (-\Delta - \lambda)\psi \rangle &= \sum_{i \in \mathbf{Z}^d} \langle g, \, (-\Delta - \lambda) \, \theta(x-i) \rangle \, \psi(i) \\ &= \sum_{i \in \mathbf{Z}^d} \gamma_i \, \psi(i) \end{aligned}$$

(the r.h.s. is finite since the sum is taken over a finite number of points $i \in \mathbb{Z}^d \cap \operatorname{supp} \psi$). Hence $g \in \mathbb{L}^2_{loc}$, and

$$\begin{aligned} \gamma_i &= \langle g, \, (-\Delta - \lambda)\theta(x-i) \rangle \\ &\le C_\lambda \left(\int_{\Lambda_{1/2}(i)} |g(x)|^2 \, (1+|x|)^{-2\delta} \, dx \right)^{1/2} \left(\int_{\Lambda_{1/2}(i)} (1+|x|)^{2\delta} \, dx \right)^{1/2} \\ &\le C_\lambda \, g_i \, (2+|i|)^\delta, \end{aligned} \tag{3.4}$$

where $\sum_{i \in \mathbf{Z}^d} g_i^2 < C < \infty$.

Notice that γ_i are uniquely defined by g and λ. Indeed, if there exist $\{\gamma_i\}_{i\in\mathbf{Z}^d}$ and $\{\beta_i\}_{i\in\mathbf{Z}^d}$ such that for arbitrary $\psi \in \mathbb{C}_0^\infty$

$$\langle g, (-\Delta-\lambda)\psi\rangle = \sum_{i\in\mathbf{Z}^d} \gamma_i\, \psi(i) = \sum_{i\in\mathbf{Z}^d} \beta_i\, \psi(i),$$

choose ψ such that $\operatorname{supp}\psi \subset \Lambda_{1/2}(i)$, then $\gamma_i = \beta_i$, $i \in \mathbb{Z}^d$. Suppose

$$h(x) = \sum_{i\in\mathbf{Z}^d} \gamma_i\, G_\lambda^0(x-i). \tag{3.5}$$

Since γ_i satisfy (3.4) then for $\lambda < 0$ the series (3.5) is absolutely convergent for arbitrary $x \notin \mathbb{Z}^d$ and $h \in \mathbb{L}^2_{-\delta}$ (Lemma A2 of Appendix); if $\lambda > 0$, define (3.5) in the distributional sense.

It follows that

$$\langle g-h, (-\Delta-\lambda)\psi\rangle = 0$$

if $\psi \in \mathbb{C}_0^\infty$, so $\phi = (g-h)$ is a distributional solution of the equation

$$(-\Delta-\lambda)\phi = 0. \tag{3.6}$$

The following paper mainly concerns with the case $\lambda < 0$. Then $\phi \in \mathbb{L}^2_{-\delta}$, and (3.6) implies that $\phi = 0$.

It is proved, that if $f \in \mathbb{L}^2_{-\delta}$ satisfies (3.3), then

$$f = \sum_{i\in\mathbf{Z}^d} f_i\, G_\lambda^0(x-i), \tag{3.7}$$

where the series converges in $\|.\|_{\mathbf{L}^2_{-\delta}}$ and $\{f_i\}_{i\in\mathbf{Z}^d}$ satisfy (3.4).
So far there had only been used the property that $-\Delta_\alpha$ is a selfadjoint extension of $-\Delta_{|\mathbf{C}_0^\infty(\mathbf{R}^d\setminus\mathbf{Z}^d)}$. Now consider the explicit expression of this extension in order to determine the coefficients of the expansion (3.7). Consider an arbitrary function φ of type (3.2), with $\varphi_0 \in \mathbb{C}_0^\infty$. Set $\overline{\varphi}_0 = \{\varphi_0(j)\}_{j\in\mathbf{Z}^d}$, $\overline{c} = \{c(j)\}_{j\in\mathbf{Z}^d}$, $\overline{c} = \Gamma_E^{-1}\overline{\varphi}_0$; $E < \inf\operatorname{supp} P_{\alpha_0}$, and $|E|$ is chosen large enough, so that $\varphi \in \mathbb{L}^2_\delta$ and $(-\Delta_\alpha-\lambda)\varphi \in \mathbb{L}^2_\delta$ (Lemma A1).

Since

$$(-\Delta_\alpha-\lambda)\varphi(x) = (-\Delta-\lambda)\varphi_0(x) + (E-\lambda)\sum_{i\in\mathbf{Z}^d} c_i\, G_E(x-i), \tag{3.8}$$

substituting (3.7) and (3.8) in (3.1), and taking into account

$$\begin{aligned}
&\langle G_\lambda^0(x-i), (-\Delta-\lambda)\varphi_0\rangle = \varphi_0(i),\\
&\langle G_\lambda^0(x-i), G_E^0(x-j)\rangle = \frac{1}{E-\lambda}(\Gamma_\lambda(i,j)-\Gamma_E(i,j)),
\end{aligned}$$

it follows

$$\sum_{i\in\operatorname{supp}\phi_0} f_i\, \varphi_0(i) \;+\; \sum_{i,j\in\mathbf{Z}^d} f_i\, c_j\, \left[\Gamma_\lambda(i,j)-\Gamma_E(i,j)\right] = 0.$$

Since $\overline{\varphi}_0 = \Gamma_E\overline{c}$, the latter equality could be written as

$$\sum_{k\in \mathbf{Z}^d} c_k \, \{ \sum_{j\in \mathbf{Z}^d} \Gamma_\lambda(k,j)\, f_j \,+\, (E-\lambda) \int_{\mathbf{R}^d} \phi_\lambda(x) G_E^0(x-k)\, dx \,\} \;=\; 0. \tag{3.9}$$

The statement of Theorem 3a 1) follows immediately from (3.9) since $\phi_\lambda = 0$, and $\Gamma_E^{-1} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is a bijection, if $\lambda < 0$ and $|E| >> 0$.
By choosing the sequence $\overline{\varphi}_{0,n} = \{\delta(i-n)\}_{i\in\mathbf{Z}^d}$ determine the sequence $\overline{c}_n = \{\Gamma_E^{-1}(j,n)\}_{j\in\mathbf{Z}^d}$, where each $\overline{c}_n$ satisfies (3.9). This implies

$$\Gamma_E^{-1} \left[\sum_{j\in\mathbf{Z}^d} \Gamma_\lambda(k,j)\, f_i \right] = 0, \qquad k \in \mathbb{Z}^d. \tag{3.10}$$

If $\lambda < 0$, then

$$\Gamma_\lambda\, \overline{f} \;=\; 0, \quad |f_j| \le (1+|j|)^\delta, \quad j\in\mathbb{Z}^d. \tag{3.11}$$

Consider the function $f(x)$ defined by (3.7). It follows from Lemma A2, that $f \in \mathbb{L}^2_{-\delta}$ and the argument above implies that f satisfies (3.1) since (3.9) is valid. This proves the statement of Theorem 3a 1).

Suppose that 0 is an eigenvalue of Γ_λ, $\lambda<0$, i.e. (3.11) is valid and $\overline{f} \in \ell^2(\mathbb{Z}^d)$. Lemma A2 implies that f, defined by (3.7), belongs to $\mathbb{L}^2(\mathbb{R}^d)$. Define

$$\varphi(x) \;=\; \varphi_0(x) \;+\; \sum_{j\in\mathbf{Z}^d} f_j\, G_E^0(x-j) \tag{3.12}$$

for some $E<0,\, E\neq\lambda$, where

$$\varphi_0(x) = (\lambda - E)\, (-\Delta - E)^{-1} \sum_{j\in\mathbf{Z}^d} f_j\, G_\lambda^0(x-j). \tag{3.13}$$

Clearly

$$\begin{aligned}
\varphi_0(x) \;&=\; (\lambda - E) \sum_{j\in\mathbf{Z}^d} f_j\, G_E^0\, G_\lambda^0(x-j) \\
&=\; \sum_{j\in\mathbf{Z}^d} f_j\, \left\{ G_\lambda^0(x-j) - G_E^0(x-j) \right\},
\end{aligned}$$

and hence

$$\varphi(x) \;=\; f(x) \;=\; \sum_{i\in\mathbf{Z}^d} f_i\, G_\lambda^0(x-i).$$

Notice that $\varphi \in \mathcal{D}(-\Delta_\alpha)$, since it follows by (3.11) and (3.13) that

$$\varphi_0(j) \;=\; \sum_{i\in\mathbf{Z}^d} f_i\, \left[\Gamma_E(j,i) - \Gamma_\lambda(j,i)\right] \;=\; \sum_{i\in\mathbf{Z}^d} f_i\, \Gamma_E(j,i).$$

It remains to check that f is the eigenfunction:

$$-\Delta_\alpha\, f \;=\; (-\Delta - E)\, f \;+\; E\, f \;=\; (\lambda - E)\, f \;+\; E\, f \;=\; \lambda\, f.$$

This proves Theorem 3a.

Proof of Theorem 3b. Notice that (3.1) is valid, by limiting agument, for abitrary $\varphi \in \mathcal{D}(-\Delta_\alpha)$ such that $(-\Delta_\alpha - \lambda)\varphi \in \mathbb{L}^2_\delta$. Consider the sequence $\varphi_n(x)$ of functions of type (3.2), $n \in \mathbb{Z}^d$, where $\varphi^0_n(x) \in \mathcal{D}(-\Delta_\alpha)$ satisfy

$$\varphi^0_n(x) = \Gamma_E(x-n),$$

$\overline{a}_n = \Gamma_E^{-1}\,\overline{\varphi}^0_n \;=\; \{\delta(n-k)\}_{k\in\mathbf{Z}^d}$, i.e.

$$\overline{a}_n(k) \;=\; \begin{cases} 1, & \text{if } k=n, \\ 0, & \text{if } k\neq n \end{cases},$$

then φ_n satisfies (3.1), and (3.9) implies

$$\sum_{j\in\mathbf{Z}^d} \Gamma_\alpha(\lambda;k,j)\, f_j \;=\; \phi_\lambda(k), \quad k\in\mathbb{Z}^d. \tag{3.14}$$

Theorem 3 is proved.

Modifying slightly the latter argument, prove the following useful statement.

Lemma 3.1 *1. Suppose that a sequence* $f_n \in \ell^2(\mathbb{Z}^d)$, $0 < \|f_n\|_{\ell^2} \le C < \infty$ *is such that for some* $\lambda < 0$,

$$\|\Gamma_\lambda\, f_n\|_{\ell^2} \to 0 \;\; \textit{for} \;\; n\to\infty, \tag{3.15}$$

then there exists the sequence $F_n \in \mathcal{D}(-\Delta_\alpha)$ *of functions which satisfy*

$$\|(-\Delta_\alpha - \lambda)\, F_n\|_{\mathbf{L}^2} \to 0 \quad \textit{for} \quad n\to\infty \quad \textit{and} \quad 0 < \|F_n\|_{\mathbf{L}^2} \le C_1 < \infty. \tag{3.16}$$

2. Suppose that a sequence $F_n \in \mathbb{L}^2(\mathbb{R}^d)$ *satisfies (3.16). Then there exists the corresponding sequence* $f_n \in \ell^2(\mathbb{Z}^d)$ *such that (3.15) is valid.*

In both cases one may choose the functions of norm 1 for the second sequence if the first one consists of functions of norm 1.

Proof. 1. Define

$$\begin{aligned} F_n(x) &= \varphi_n(x) + \sum_{i\in\mathbf{Z}^d} f_n(i)\, G^0_E(x-i), \\ \varphi_n(x) &= (\lambda - E)\,(-\Delta - E)^{-1} \sum_{i\in\mathbf{Z}^d} f_n(i)\, G^0_\lambda(x-i) \;+\; g_n(x), \end{aligned}$$

where $g_n(x) \;=\; \sum_{i\in\mathbf{Z}^d} a_{n,i}\,\theta(x-i)$, the function $\theta\in\mathbb{C}^\infty_0$ is such that $\operatorname{supp}\theta \subset \Lambda_{1/2}$, $\theta(0)=1$, $a_{n,i} = \sum_{i\in\mathbf{Z}^d} \Gamma_\lambda(i,j)\, f_n(j) \;=\; \Gamma_\lambda\, \overline{f}_n\,(i)$, $E<0$.
Then

$$F_n \;=\; \sum_{i\in\mathbf{Z}^d} f_n(i)\, G^0_\lambda(x-i) \; + g_n(x) \in \mathcal{D}(-\Delta_\alpha),$$

and

$$(-\Delta_\alpha - \lambda)\, F_n(x) = (-\Delta - \lambda)\, g_n(x).$$

It follows that $\|\overline{a}_n\|_{\ell^2} = \|\Gamma_\lambda\, \overline{f}_n\|_{\ell^2} \to 0$ as $n \to \infty$, so $\|g_n\|_{\mathbf{L}^2} \to 0$ and $\|(-\Delta - \lambda) g_n\|_{\mathbf{L}^2} \to 0$ as $n \to \infty$. The argument of Lemma A2 implies

$$C_1' \|f_n\|_{\ell^2} \le \|F_n\|_{\mathbf{L}^2} \le C_1'' \|f_n\|_{\ell^2}.$$

This implies that one may take normalized functions $\widetilde{F}_n = \frac{F_n}{\|F_n\|}$, if f_n are normalized.

2. Suppose $(-\Delta_\alpha - \lambda)\, F_n(x) = H_n(x)$. Since $F_n \in \mathcal{D}(-\Delta_\alpha)$, it has the form

$$F_n(x) = \psi_n(x) + \sum_{i \in \mathbf{Z}^d} a_{n,i}\, G^0_{E_n}(x - i), \tag{3.17}$$

for some $E_n < 0$, $\psi_n \in \mathcal{D}(-\Delta)$, $\overline{a}_n = \{a_{n,i}\}_{i \in \mathbf{Z}^d}$, $\overline{\psi}_n = \{\psi_n(i)\}_{i \in \mathbf{Z}^d}$, such that $\Gamma_{E_n} \overline{a}_n = \overline{\psi}_n$. Define $\overline{f}_n = \overline{a}_n$. It follows by (3.17):

$$(-\Delta_\alpha - \lambda) F_n(x) \;=\; (-\Delta - E_n)\psi_n(x) \;+\; (E_n - \lambda) F_n(x),$$

and hence

$$(-\Delta - \lambda)\psi_n(x) \;=\; H_n(x) \;+\; (\lambda - E_n) \sum_{i \in \mathbf{Z}^d} a_{n,i}\, G^0_{E_n}(x - i).$$

The resolvent identity implies

$$\psi_n(x) = G^0_\lambda\, H_n(x) \;+\; \sum_{i \in \mathbf{Z}^d} (G^0_\lambda - G^0_{E_n})(x - i)\, a_{n,i}. \tag{3.18}$$

Substituting $x = j$ into (3.18),

$$\psi_n(j) \;=\; G^0_\lambda\, H_n(j) \;+\; \sum_{i \in \mathbf{Z}^d} a_{n,i}\, [\Gamma_{E_n}(i - j) - \Gamma_\lambda(i - j)].$$

Since $\Gamma_{E_n} \overline{a}_n = \overline{\psi}_n$, it follows that a_n satisfies the equation:

$$(-\Delta - \lambda)^{-1}\, H_n(j) \;=\; \Gamma_\lambda\, \overline{a}_n(j), \quad j \in \mathbb{Z}^d. \tag{3.19}$$

Observe that the r.h.s. of (3.18) is a continuous function of x since it belongs to $\mathcal{D}(-\Delta)$, i.e. it is well defined in the points of $\mathbb{Z}^d$ and tends to 0 as $n \to \infty$. It follows by (3.17) and (3.18)

$$F_n(x) = \sum_{i \in \mathbf{Z}^d} a_{n,i}\, G^0_\lambda(x - i) \;+\; (-\Delta - \lambda)^{-1}\, H_n(x).$$

Since F_n are uniformly bounded in $\mathbb{L}^2(\mathbb{R}^d)$, $\|(-\Delta - \lambda)^{-1}\, H_n\|_{\mathbf{L}^2}$ tends to zero as $n \to \infty$, $(\lambda < 0)$, and the argument of Lemma A2 implies

$$C_2' \|F_n\|_{\mathbf{L}^2} \le \|a_n\|_{\ell^2} \le C_2'' \|F_n\|_{\mathbf{L}^2},$$

which implies that a_n may be chosen normalized if F_n are normalized.

Remark. This statement implies that there is the geometrical relation between the spectral sets of the corresponding operators (Sect.1, Corollary 4).

4 Localization

In this section sketch the proof of Theorem 4 ([G2]). By $x, y, z...$ denote integers of $\mathbb{Z}^d$.

Consider selfadjoint operators on $\ell^2(\mathbb{Z}^d)$ defined as follows:

$$A(\lambda) = A_0(\lambda) + Q_\lambda, \tag{4.1}$$

where $A_0(\lambda)$ is a non-random finite-difference operator of infinite order

$$(A_0(\lambda)\Psi)(x) = \sum_{|x-y|\geq 1} a_\lambda(x-y)\Psi(y), \quad x \in \mathbb{Z}^d, \;\; \Psi \in \ell^2(\mathbb{Z}^d), \tag{4.2}$$

and for each λ the sequence $\{a_\lambda(x)\}_{x\in \mathbf{Z}^d}$ satisfy the conditions

$$a_\lambda(x) = \overline{a_\lambda(|x|)}, \;\; |a_\lambda(x)| \leq Ce^{-\sqrt{|\lambda|}|x|}, \;\; C > 0. \tag{4.3}$$

The potential Q_λ is the random multiplication operator:

$$(Q_\lambda\Psi)(x) = \alpha(x)\Psi(x) + \frac{\sqrt{|\lambda|}}{4\pi}\Psi(x), \tag{4.4}$$

where $\{\alpha(x)\}_{x\in \mathbf{Z}^d}$ are independent random variables with identical distribution and bounded (by δ^{-1}) density ((0.2)).

Operators $A(\lambda)$ correspond to the single probability space, permitting use of their explicit dependence on λ to verify that the multiscale analysis scheme may be applied ([G2]). Denote by $B_L(x)$ the cube centered at $x \in \mathbb{Z}^d$, with sides of length L, and by $A_B(\lambda)$ the restriction of the operator $A(\lambda)$ to $\ell^2(B)$ (Dirichlet boundary conditions), by $R_B(z,\lambda) = (A_B(\lambda) - z)^{-1}$ the resolvent of the operator $A_B(\lambda)$ ($z \notin \sigma(A_B(\lambda))$), by $R_B(z, \lambda; x, y)$ the corresponding Green function, continued by zero to $\mathbb{Z}^d \times \mathbb{Z}^d$

$$R_B(z, \lambda; x, y) = \begin{cases} (A_B(\lambda) - z)^{-1}(x, y), & \text{if } x, y \in B; \\ 0, & \text{otherwise.} \end{cases}$$

Consider $m > 0$, $E \in J \subset \mathbb{R}$, $\lambda \in I \subset (-\infty, -r)$.

Definition 4.1 *The cube $B_L(x)$ is called (h, m, E, λ) -regular (for a fixed $\omega \in \Omega$) if $E \notin \sigma(A_{B_L(x)}(\lambda))$, and*

$$|R_{B_L(x)}(E, \lambda; x, y)| \leq e^{-m|y-x|} \;\; \forall y : h/2 \leq |y - x| \leq L/2.$$

If this inequality holds for all $E \in J \subset \mathbb{R}$, $\lambda \in I$ and for $h = 0$, say that $B_L(x)$ is $(m, E \in J, \lambda \in I)$-regular.

Definition 4.2 *The cube B_l is (θ, E, λ) non-resonant (for a fixed $\omega \in \Omega$) if*

$$\operatorname{dist}(E, \sigma(A_{B_L}(\lambda)) \geq e^{-l^\theta} \;\; \textit{for some } \theta, \;\; 0 < \theta < 1.$$

Lemma 4.1 *Given intervals $J \subset \mathbb{R}$, $I \subset (-\infty, -r)$, suppose that for $0 < L_0 < L_1 < \infty$*

(P1) $\mathbf{P}\{$ *the cube B_{L_i} is $(m_0, E \in J, \lambda \in I)$- regular* $\} \geq 1 - L_i^{-p}$, $i = 0, 1$;
for some $p > d$ and $0 < m_0 \leq \rho$;

(P2) $\mathbf{P}\{$ *the cube B_L is (θ, E, λ)non-resonant* $\} \geq 1 - L^{-q}$
for all $L \geq L_0$, $E \in J$, $\lambda \in I$, for some θ ($0 < \theta \leq 1/2$) and $q > q_0(p)$.

Then given m $(0 < m < m_0)$ there exist $\alpha = \alpha(p)$ $(1 < \alpha < \sqrt{2})$ and $B_0 = B_0(p, \theta, \alpha, q, m, m_0) < \infty$ such that if $L_0 > B_0$ and $L_n = (L_0)^{(\alpha)^n}$, then $\forall n \in \mathbb{N}$ the following inequality

$$\mathbf{P}\left\{\begin{array}{c} \text{if } (E,\lambda) \in J \times I \text{ one of two cubes} \\ B_{L_n}(x) \text{ or } B_{L_n}(y) \text{ is} \\ (L_n^{\beta}, m, E, \lambda) \text{ - regular} \end{array}\right\} \geq 1 - (L_n)^{-2p} \tag{4.5}$$

holds for $x, y \in \mathbb{Z}^d : |x - y| \geq L_n$, if $\beta = \frac{2}{1+\alpha^3}$.

Lemma 4.2 *Suppose that operators A_λ satisfy assumption (P2) of Lemma 4.1, and that (4.5) holds for every $(E, \lambda) \in J \times I$, $I \subset (-\infty, -r)$.*
Then with probability 1 every generalized solution $f \in \ell^2_{-\delta}$ of the equation

$$(A(\lambda) - E)\, f = 0$$

decays exponentially at infinity with rate $m < m_0$.

The proofs of Lemmas 4.1-4.2 do not much differ from those of [G2], where the corresponding results were established for the single operator (4.1) - (4.4) (i.e. for the fixed value of λ). In order to complete the proof of Theorem 4, it is sufficient to verify validity of assumptions (P1) and (P2) of Lemma 4.1. (P1) holds in one of the two following cases.

Lemma 4.3 *("Low energy" case) Given $0 < m < \sqrt{r}$, $L_1 > L_0 > 0$, there exists $\lambda_0 = \lambda_0(P_{\alpha_0}, m, r, p, L_0, L_1) \leq -r < 0$ such that for arbitrary interval $I \subset (-\infty, -\lambda_0)$*

$$\mathbf{P}\{ \text{ cube } B_{L_i} \text{ is } (m, E \in (-1,1), \lambda \in I) \text{ - regular}\} \geq 1 - L_i^{-p}, \quad i = 0, 1. \tag{4.6}$$

Lemma 4.4 *("High disorder" case) Given $l > 0$, $0 < m < \sqrt{r}$, $L_1 > L_0 > 0$, there exists $\delta_0 = \delta_0(P_{\alpha_0}, m, r, p, L_0, L_1, l) < \infty$ such that if $\delta > \delta_0$, then inequalities (4.6) hold for arbitrary interval $I \subset (-\infty, -r)$ of length $|I| = l$.*

The proofs of the Lemmas 4.3 - 4.4 are almost the same as in [G2]. Validity of Assumption (P2) follows from the bound on the density of states ([W]) for the finite-difference operators. Reproduce the result in the convenient form.

Consider $N_{\Gamma_B(E)}(\lambda) = \frac{1}{|B|}\, \#\left\{\lambda_i(\omega) < \lambda | \lambda_i \in \sigma(\Gamma_B(E))\right\}$, the normalized counting function of $\Gamma_B(E)$, $B \subseteq \mathbb{Z}^d$.

Lemma 4.5 *(Estimate on the density of states in $\ell^2(\mathbb{Z}^d)$)*
Given arbitrary $E < 0$, $I = (a, b) \subset \mathbb{R}$,

$$\mathbf{E}\left\{N_{\Gamma_B(E)}(I)\right\} = \mathbf{E}\left\{N_{\Gamma_B(E)}(b) - N_{\Gamma_B(E)}(a)\right\} \leq 2|b - a|\delta^{-1}.$$

Remark. Lemma 4.5 for $E < 0$ is proved in [G3]. If P_{α_0} is Hölder continuous of order γ, then $N(\lambda)$ is also Hölder continuous of the same order.
Remark. Lemmas 4.1 - 4.5 prove more than is stated by Theorem 4. They imply that all the operators A_λ (with λ in the interval corresponding to the high disorder situation) have pure point spectrum in $(-1, 1)$ with probability 1.

5 Density of states

In this section consider the integrated density of states of $-\Delta_{\alpha(\omega)}$. Since the proof of existence of the integrated density of states for the operator $-\Delta_\alpha$ did not appear in the literature as yet, at first briefly discuss a possible way to define it. As before, denote by $E_\omega(\Delta)$ the spectral projection of $-\Delta_{\alpha(\omega)}$, χ_Λ the characteristic function of the cube $\Lambda \in \mathbb{R}^d$, which sides do not cross sites of $\mathbb{Z}^d$. Define the measure

$$d\,K_{\Lambda,\omega}(\Delta) \;=\; \frac{1}{|\Lambda|}\,\mathrm{Tr}\,\left(\chi_\Lambda\, E_\omega(\Delta)\,\chi_\Lambda\right).$$

The standard argument based on the ergodic theorem (cf. [CFKS]) implies that $dK_{\Lambda,\omega}$ converges, in the distributional sense for $\mathbf{P}$-almost all $\omega \in \Omega$, to the non-random limiting measure

$$dN(\Delta) \;=\; \mathbf{E}\,\mathrm{Tr}\,\left(\chi_1\, E_\omega(\Delta)\,\chi_1\right), \tag{5.1}$$

where χ_1 is the unit cube in $\mathbb{R}^d$, and $\mathbf{E}\{.\}$ denotes the expectation value on the probability space $(\Omega, \mathbf{P})$. $N(\lambda)$ is called the integrated density of states of $-\Delta_{\alpha(\omega)}$. The following is the rigorous analysis of the continuity properties of $N(\lambda)$ for $\lambda < 0$.

Theorems 2-3 and Lemma A2 (cf. (5.1)) imply that $N(\lambda)$ is a continuous function for $\lambda < 0$. Indeed,

$$\begin{aligned} &\mathbf{P}\,\{\,\lambda < 0 \quad \text{is an eigenvalue of} \quad -\Delta_{\alpha(\omega)}\} \\ =\;& \mathbf{P}\,\{\,0 \quad \text{is an eigenvalue of} \quad \Gamma_{\alpha(\omega)}(\lambda)\} \;=\; 0. \end{aligned} \tag{5.2}$$

The first equality is a consequence of Theorem 3. The second one follows from the continuity of the integrated density of states $N_{\Gamma(\lambda)}$ (Lemma 4.5). In order to prove this, denote by $E_{\Gamma_\omega(\lambda)}\{\Delta\}(x,y)$ the kernel of the spectral projection of operator $\Gamma_\omega(\lambda)$, then

$$dN_{\Gamma(\lambda)}\{0\} \;= \mathbf{E}\,\left\{E_{\Gamma_\omega(\lambda)}\{0\}(0,0)\right\}.$$

It follows by Lemma 4.5 that $N_{\Gamma(\lambda)}(t)$ is continuous in $t = 0$, therefore $E_{\Gamma_\omega(\lambda)}\{0\}(0,0) = 0$ with probability 1, which implies (since $E_{\Gamma_\omega(\lambda)}\{0\}$ is ergodic projection) that $E_{\Gamma_\omega(\lambda)}\{0\}(x,x) = 0$ $\forall x \in \mathbb{Z}^d$ with probability 1. Hence $E_{\Gamma_\omega(\lambda)}\{0\} = 0$ with probability 1, and (5.2) follows.
Since (5.2) is zero, it follows that $E_\omega\{\lambda\} = 0$ with probability 1, and the statement $dN\{\lambda\} = 0$ is implied by (5.1). A more detailed analysis is required to prove the absolute continuity of $N(\lambda)$.

Consider a cube $B \subset \mathbb{R}^d$, $\Lambda = B \cap \mathbb{Z}^d$, the operator $-\Delta_{\overline{\alpha},\Lambda}$, where $\overline{\alpha}$ is the restriction of the function $\alpha(j) : \mathbb{Z}^d \to \mathbb{R}$ to a finite subset $\Lambda \subset \mathbb{Z}^d$ (i.e. the interactions at the points $\mathbb{Z}^d \setminus \Lambda$ are "switched off"). The resolvent kernel of this operator ([AGHH]) is given by

$$\begin{aligned} (-\Delta_{\overline{\alpha},\Lambda} - z)^{-1}(x,y) \;&=\; (-\Delta - z)^{-1}(x,y) \\ &+\; \sum_{j,j'\in\Lambda} (-\Delta - z)^{-1}(x,j)\,[\Gamma_{\overline{\alpha}(\omega)}(z)]^{-1}(j,j')\,(-\Delta - z)^{-1}(j',y), \end{aligned} \tag{5.3}$$

where $\Im z > 0$ and operator $\Gamma_{\overline{\alpha}(\omega)}(z) : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is defined as follows :

$$\Gamma_{\overline{\alpha}(\omega)}(z;j,j') = \begin{cases} \alpha_j - \frac{i\sqrt{z}}{4\pi}, & \text{if } j = j', \\ -(-\Delta - z)^{-1}(j,j') & \text{otherwise} \end{cases}, \quad j, j' \in \Lambda. \tag{5.4}$$

It is known ([AGHH]) that for arbitrary fixed ω, the negative spectrum of $\Gamma_{\overline{\alpha}(\omega)}$ is discrete and consists of not more than $N = |\Lambda|$ eigenvalues (number of points in $\Lambda \subset \mathbb{Z}^d$). It follows that for $E < 0$, the normalized counting function of $\Gamma_{\overline{\alpha}(\omega)}$ is well-defined:

$$N_{\Lambda,\omega}(E) = \frac{1}{|\Lambda|} \# \{\lambda_i(\omega) < E\}$$

If $z < 0$ and $|z|$ is sufficiently large, then the sequence of resolvents $(-\Delta_{\overline{\alpha},\Lambda} - z)^{-1}$ is monotone increasing as $\Lambda \to \mathbb{Z}^d$, and strongly converges over the filter of finite subsets $\Lambda \subset \mathbb{Z}^d$:

$$s - \lim_{\Lambda \nearrow \mathbf{Z}^d} (-\Delta_{\overline{\alpha},\Lambda} - z)^{-1} = (-\Delta_\alpha - z)^{-1}.$$

Thus the corresponding sequence of operators $-\Delta_{\overline{\alpha}}$ is monotone decreasing and converges in the strong resolvent sense to $-\Delta_\alpha$ as $\Lambda \to \mathbb{Z}^d$ (this is the way to define $-\Delta_\alpha$, [AGHH]). It follows that the corresponding sequence $N_{\Lambda,\omega}(E)$ is a superadditive stochastic process, that is given closed cubes Λ_1, Λ_2, such that $\Lambda_1 \cap \Lambda_2$ has empty interior:

$$N_{\Lambda_1 \cup \Lambda_2,\omega}(E) \geq N_{\Lambda_1,\omega}(E) + N_{\Lambda_2,\omega}(E).$$

The standard arguments ([Ki]) based on the superadditive ergodic theorem show that the sequence of distribution functions $\{N_{\Lambda_n,\omega}(E)\}$ converges in the distributional sense with probabability 1 to a non-random limiting distribution function $\widetilde{N}(E)$, if

$$\sup_n \frac{1}{|\Lambda_n|} \mathbf{E} \{N_{\Lambda_n,\omega}(E)\} < \infty. \tag{5.5}$$

In order to prove the formulated theorem, it is enough to observe the following: since Δ_α is the strong resolvent limit of a decreasing sequence $-\Delta_{\overline{\alpha},\Lambda}$, then for arbitrary interval $I_\eta \subset (-\infty, -r)$ of length $\eta > 0$, which endpoints are not atoms of $E_\omega(\Delta)$, it follows

$$s - \lim_{\Lambda \nearrow \mathbf{Z}^d} E_{\Lambda,\omega}(I_\eta) = E_\omega(I_\eta). \tag{5.6}$$

Since $-\Delta_{\alpha(\omega)}$ a.s. has no fixed (at least negative) eigenvalues ((5.2)), it follows that (5.6) is valid for arbitrary fixed interval, with probability 1. (5.1) and (5.6) imply

$$\begin{aligned} N(I_\eta) &\leq \sup_\Lambda \frac{1}{|\Lambda|} \mathbf{E} \{\mathrm{Tr}\,(E_{\Lambda,\omega}(I_\eta))\} \\ &= \sup_\Lambda \mathbf{E} \{N_{\Lambda,\omega}(I_\eta)\}\,. \end{aligned} \tag{5.7}$$

Since the set $\Lambda \subset \mathbb{Z}^d$ is finite, then for every fixed ω

$$\begin{aligned} N_\Lambda(I_\eta) &= \frac{1}{|\Lambda|} \# \{E \in I_\eta \mid 0 \in \sigma(\Gamma_\Lambda, (E))\} \\ &= \frac{1}{|\Lambda|} \# \{\lambda \mid -\eta \leq \lambda \leq 0,\ \ 0 \in \sigma(\Gamma_\Lambda(E_0 + \lambda))\}\,, \end{aligned} \tag{5.8}$$

where $E_0 \leq -r < 0$ is the right endpoint of I_η, and for simplicity denote $\Gamma_\Lambda = \Gamma_{\overline{\alpha},\Lambda}$ ((5.4)).

Lemma 5.1 *Assume that* $0 \in \sigma(\Gamma_\Lambda(E+\lambda))$, $-\eta \le \lambda < 0$, $E \le -r < 0$. *Then* $\operatorname{dist}(0, \sigma(\Gamma_\Lambda(E))) < C|\lambda|$, *and hence there exists* $x \in \sigma(\Gamma_\Lambda(E))$ *such that* $|x| < C|\lambda|$ *(with* $C = C(r)$*).*

Proof. It follows

$$\Gamma_\Lambda(E+\lambda) = \Gamma_\Lambda(E) + \Delta(E, \lambda).$$

From the resolvent identity:

$$\Gamma_\Lambda^{-1}(E+\lambda) = \Gamma_\Lambda^{-1}(E) - \Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda)\, \Gamma_\Lambda^{-1}(E+\lambda),$$

hence

$$\Gamma_\Lambda^{-1}(E+\lambda) = (1 + \Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda))^{-1}\, \Gamma_\Lambda^{-1}(E), \tag{5.9}$$

if $1 + \Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda)$ is invertible. From the Hilbert equation

$$\Delta(E,\lambda) = G_E^0 - G_{E+\lambda}^0 = -\lambda G_E^0\, G_{E+\lambda}^0,$$

i.e. if $|E| \le -r < 0$, then

$$\|\Delta(E,\lambda)\| \le \frac{|\lambda|}{r^2}.$$

Suppose that $\|\Gamma_\Lambda^{-1}(E)\| \le \frac{1}{C|\lambda|}$ (i.e. $\operatorname{dist}(0, \sigma(\Gamma_\Lambda(E))) > Ct$) then

$$\|\Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda)\| \le \frac{1}{r^2 C}, \tag{5.10}$$

from which, by choosing $C \ge \frac{2}{r^2}$, it follows:

$$\|\Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda)\| \le \frac{1}{2}.$$

This implies that the operator $1 + \Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda)$ is invertible and

$$\|(1 + \Gamma_\Lambda^{-1}(E)\, \Delta(E,\lambda))^{-1}\| \le 2. \tag{5.11}$$

It follows by (5.9) and (5.11) that

$$\|\Gamma_\Lambda^{-1}(E+\lambda)\| \le 2\|\Gamma_\Lambda^{-1}(E)\|.$$

It is proved that if $C \ge \frac{2}{r^2}$ and $\operatorname{dist}(0, \sigma(\Gamma_\Lambda(E))) \ge C|\lambda|$, then

$$\operatorname{dist}(0, \sigma(\Gamma_\Lambda(E+\lambda))) \ge \frac{C|\lambda|}{2} > 0.$$

Lemma 5.1 is proved.

Lemma 5.2 *Denote by* $\gamma_1(E), ..., \gamma_k(E)$, $1 \le k \le |\Lambda|$ *the eigenvalues of matrix* $\Gamma_\Lambda(E)$, $E < 0$. *Then* $\gamma_j(E)$ *are strictly increasing functions of* $|E|$.

Proof follows by verifying the matrix $\frac{\partial}{\partial\sqrt{|E|}}\Gamma_\Lambda = \left[\frac{1}{4\pi}\, e^{-\sqrt{|E|}|j-j\prime|}\right]_{j,j\prime=1}^{|\Lambda|}$ is positive-defined ([Si2]).

Now finish the proof of Theorem 5. Return to (5.8) and suppose that

$$0 \in \sigma(\Gamma_\Lambda(E_0+\lambda_i)), \quad -\eta \le \lambda_i < 0, \quad i = 1,..,k.$$

It follows by Lemma 5.1 that there exist $\{x_i\}_{i=1,\dots,m}$ such that $|x_i| \le C\eta$, and $x_i \in \sigma(\Gamma_\Lambda(E_0))$. Denote by $x_i(\lambda)$ the corresponding eigenvalues of $\Gamma_\Lambda(E_0+\lambda)$. According to Lemma 5.2, each $x_i(\lambda)$ increases as λ decreases to $-\eta$, and hence cannot cross change of sign more than once, i.e. $m \ge k$. It follows:

$$\mathbf{E}\left\{N_\Lambda(I_\eta)\right\} \le \mathbf{E}\left\{N_{\Gamma_\Lambda(E_0)}(E_0 - C\eta, E_0 + C\eta)\right\},$$

where $N_{\Gamma_\Lambda(E_0)}(\lambda)$ is the normalized counting function of $\Gamma_\Lambda(E_0)$. The estimate on the density of states for finite-difference operators Γ_Λ (Lemma 4.5) implies that there is the bound

$$\mathbf{E}\left\{N_{\Gamma_\Lambda(E_0)}(E_0 - C\eta, E_0 + C\eta)\right\} \le 2C\eta\delta^{-1}.$$

Theorem 5 is proved.

6 Generalized spectral theorem

Theorem 6 (Spectral Theorem) *There exist a spectral measure $d\rho(\lambda)$, an integer valued function $N(\lambda)$, $\sup_\lambda N(\lambda) \equiv N \le \infty$, and disjoint $d\rho$-measurable sets $\{\Delta_n\}_{n=1}^N$:*

$$\Delta_m \cap \Delta_n = \emptyset, \ \cup_{n=1}^N \Delta_n = \mathrm{supp}\{d\rho\},$$

such that for a.e. $\lambda \in \Delta_n$ (with respect to $d\rho$), there are n functions $\{F_j(\lambda;x)\}_{j=1}^n$ satisfying the following conditions:

- **(i)** *For λ fixed, $\{F_j(\lambda;\cdot)\}_{j=1}^n$ are linearly independent and $F_j(\lambda;\cdot) \in \mathbb{L}^2_{-\delta}$.*
- **(ii)** *$F_j(\lambda)$ are the generalized eigenfunctions of $-\Delta_\alpha$, relative to λ.*
- **(iii)** *Consider the mapping $\widetilde{U}\, \mathbb{L}^2_\delta \to \mathcal{H}$, where*

$$\mathcal{H} = \oplus_{n=1}^N \mathbb{L}^2(\mathbb{R}, \mathbb{C}^n; \chi_n\, d\rho),$$

χ_n is the characteristic function of Δ_n:

$$(\widetilde{U}\, f)_j(\lambda) \ = \int\limits_{\mathbf{R}^d} \overline{F_j(\lambda;x)}\, f(x)\, dx.$$

Then $\widetilde{U}$ is a unitary map, i.e.

$$\|\widetilde{U}\, f\|^2_{\mathcal{H}} \ = \int\limits_{\sigma(H)} \{\sum_{j=1}^N |(\widetilde{U} f)_j(\lambda)|^2\}\, d\rho(\lambda) \ = \ \|f\|^2_{\mathbf{L}^2}.$$

(iv) *$\widetilde{U}$ extends to the unitary map $U : \mathbb{L}^2(\mathbb{R}^d) \xrightarrow{\text{onto}} \mathcal{H}$ defined for arbitrary $f \in \mathbb{L}^2(\mathbb{R}^d)$ by*

$$(U\,f)_j(\lambda) \;=\; \mathcal{H} - \lim_{R\to\infty} \int\limits_{|x|\le R} F_j(\lambda;x) f(x)\,dx,$$

where lim *means the limit in $\mathcal{H}$ - norm.*

(v) *If g is a bounded measurable function on $\sigma(H)$, then for arbitrary $f \in \mathbb{L}^2(\mathbb{R}^d)$:*

$$U\,[g(H)\,f] \;=\; g(\lambda)[U\,f].$$

(vi) *If $f \in \mathcal{D}(-\Delta_\alpha)$ then*

$$U(Hf) = \lambda(Uf).$$

(vii) *If $h \in \mathcal{H}$, the inverse map U^{-1} is defined by*

$$(U^{-1}\,h)(x) \;=\; \mathbb{L}^2 - \lim_{\substack{E\to\infty \\ N\to\infty}} \int\limits_{|\lambda|\le E} \sum_{j=1}^{\min(N,N(\lambda))} h_j(\lambda)\,F_j(\lambda;x)\,d\rho(\lambda). \tag{6.1}$$

Proof. Denote by $H = -\Delta_\alpha$ the point interaction Hamiltonian on $\mathbb{L}^2(\mathbb{R}^d)$, $1 < d < 4$, $|\alpha_j| \le \sup\{\alpha_j\} < \infty$, $T_\delta = (1+|x|)^\delta$ (multiplication operator), $\delta > d/2$, $E(\Delta)$ is the spectral projection of $-\Delta_\alpha$.

Lemma 6.1 *For arbitrary bounded measurable set $\Delta \subset \mathbb{R}$, $A(\Delta) = T_\delta^{-1} E(\Delta) T_\delta^{-1}$ is a trace class operator.*

Lemma 6.1 is proved in section 2. It follows by the Radon-Nikodym theorem that there exists a positive measurable trace class operator-valued function $a(\lambda)$, $\operatorname{Tr} a(\lambda) = 1$, such that

$$A(\Delta) \;=\; \int_\Delta a(\lambda)\,d\rho(\lambda),$$

where the scalar measure (which is the spectral measure of H) is defined by

$$\rho(\Delta) \;=\; \operatorname{Tr}\,\{T_\delta^{-1}\,E(\Delta)\,T_\delta^{-1}\}.$$

Define

$$F(\lambda;x,y) \;=\; (1+|x|)^\delta\,a(\lambda;x,y)\,(1+|y|)^\delta,$$

where $a(\lambda;x,y)$ is the integral kernel of $a(\lambda)$, i.e. $F(\lambda;x,y)$ can be regarded as the integral kernel of the operator $\frac{dE(\lambda)}{d\rho(\lambda)}$ on $\mathbb{L}^2(\mathbb{R}^d)$.

The kernel of the Hilbert-Schmidt operator $a(\lambda)$:

$$a(\lambda;x,y) \;=\; \sum_{j=1}^{N(\lambda)} \overline{\varphi_j(\lambda;x)}\,\varphi_j(\lambda;y), \tag{6.2}$$

where $\varphi_j(\lambda)$ are (non-normalized) eigenfunctions of $a(\lambda)$. They are orthogonal, may be chosen jointly measurable in x, λ and their norms satisfy the relation:

$$\sum_{j=1}^{N(\lambda)} \|\varphi_j(\lambda)\|^2_{\mathbf{L}^2} \;=\; \mathrm{Tr}\, a(\lambda) \;=\; 1. \tag{6.3}$$

Define

$$F_j(\lambda; x) \;=\; (1+|x|)^{\delta}\, \varphi_j(\lambda; x), \;\; x \in \Delta_n,$$

$j = 1, .., n$, $\Delta_n \;=\; \{\lambda | N(\lambda) = n\}$.
(i) follows by definition of F_j since $\|\varphi_j\| \leq 1$.
(ii) By definition of F_j, since $\varphi_j(\lambda)$ are eigenfunctions of $a(\lambda)$:

$$\int\limits_{|x-y|\leq 1} |F_j(\lambda; x)|^2\, dx \;\leq\; C(1+|y|)^{\delta},$$

and

$$\lambda\, F_j(\lambda; x) \;=\; \int\limits_{\mathbf{R}^d} F(\lambda; x, y)\, (1+|y|)^{-2\delta}\, F_j(\lambda; y)\, dy \tag{6.4}$$

Given arbitrary $\psi_0 \in \mathcal{D}(-\Delta_\alpha)$,

$$\begin{aligned} \langle F_j(\lambda), H\psi_0\rangle \;&=\; \frac{1}{\lambda}\langle \int\limits_{\mathbf{R}^d} F(\lambda; x, y)\, (1+|y|)^{-2\delta}\, F_j(\lambda; y) dy, H\psi_0\rangle \\ &=\; \int\limits_{\mathbf{R}^d} \langle F(\cdot, y; \lambda), H\psi_0\rangle\, (1+|y|)^{-2\delta} F_j(\lambda; y)\, dy \\ &=\; \lambda\, \langle F_j(\lambda; \cdot), \psi_0(\cdot)\rangle, \end{aligned}$$

this proves (ii).

Let us prove (iii). Since $F_j(\lambda) \in \mathbb{L}^2_{-\delta}$, the mapping $\widetilde{U}\psi$ is correctly defined if $\psi \in \mathbb{L}^2_{\delta}$.

By definition of F and (6.2),

$$\begin{aligned} \int\limits_{\mathbf{R}^d\times\mathbf{R}^d} F(\lambda; x, y)\overline{\psi(x)}\psi(y)\, dxdy \;&=\; \sum_{j=1}^{N(\lambda)} \int\limits_{\mathbf{R}^d\times\mathbf{R}^d} \overline{F_j(\lambda; x)} F_j(\lambda; y)\, \overline{\psi(x)}\psi(y)\, dxdy \\ &=\; \sum_{j=1}^{N(\lambda)} |\langle F_j(\lambda), \psi\rangle|^2 \;=\; \sum_{j=1}^{N(\lambda)} |(\widetilde{U}\psi)_j(\lambda)|^2, \end{aligned} \tag{6.5}$$

Relations (2.2) and (6.5) imply that given arbitrary $\psi \in \mathbb{L}^2_{\delta}$, and bounded measurable function g on $\sigma(H)$:

$$\langle \psi, g(H)\psi\rangle \;=\; \int\limits_{\sigma(H)} g(\lambda) \sum_{j=1}^{N(\lambda)} |(\widetilde{U}\psi)_j(\lambda)|^2\, d\rho(\lambda). \tag{6.6}$$

This implies the required result (iii) by substituting $g \equiv 1$.

The statement (iv) for arbitrary $g \in \mathbb{L}^2(\mathbb{R}^d)$ follows via limiting argument by (iii), and it remains to prove that U maps $\mathbb{L}^2(\mathbb{R}^d, dx)$ onto $\mathcal{H}$.

Lemma 6.2

$$S(\lambda; x) \stackrel{\text{def}}{=} \sum_{j=1}^{N(\lambda)} |\, F_j(\lambda; x)|^2 \le C(\lambda, dist(x, \mathbb{Z}^d))\,(1+|x|)^{2\delta}, \tag{6.7}$$

for arbitrary $x \notin \mathbb{Z}^d$, $d\rho$- a.e. $\lambda \in \sigma(-\Delta_\alpha)$, $C(\lambda, t) > 0$, $t > 0$.

Proof. Relation (6.3) implies $S(\lambda) \in \mathbb{L}^2_{-\delta}$ for $d\rho$- a.e. $\lambda \in \sigma(-\Delta_\alpha)$, and

$$\int |S(\lambda; x)|\,(1+|x|)^{-2\delta}\,dx \;=\; \mathrm{Tr}\,a(\lambda) \;=\; 1. \tag{6.8}$$

The function $S(\lambda; x)$ is continuous in $A_\varepsilon = \{x \in \mathbb{R}^d |\ \mathrm{dist}(x, \mathbb{Z}^d) \ge \varepsilon\}$ if $\varepsilon > 0$. Denote $A_\varepsilon \supset B_r(x_0) = \{x \in A_\varepsilon | |x - x_0| \le r\}$, $r > 0$, $\mathrm{dist}(x_0, \mathbb{Z}^d) \ge \varepsilon > 0$. By the Harnack inequality

$$|F_j(\lambda; y)| \ge C(\lambda, r, \varepsilon)\,|F_j(\lambda; x_0)|, \quad y \in B_r(x_0).$$

Suppose $S(\lambda; x)(1+|x|)^{-2\delta}$ is unbounded in A_ε, then there exists sequence $\{x_n\}_{n\in\mathbf{N}}$, $|x_n| \to \infty$, $x_n \in A_\varepsilon$, $\mathrm{dist}(x_i, x_j) > 1$, $i \ne j$, such that $\forall x \in B_{\varepsilon/2}(x_n)$, $n \in \mathbb{Z}^d$,

$$S(\lambda; x)\ \ge C^2(1+|x|)^{2\delta},$$

which contradicts to (6.8), so (6.7) follows. Lemma 6.2 is proved.

Suppose $h \in \mathcal{H}$ is orthogonal to the range of U. Since $\sum_{j=1}^{N(\lambda)} |h_j(\lambda)|^2 < \infty$ for $d\rho$ - a.e. λ, one may define

$$K(\lambda; x) = \sum_{j=1}^{N(\lambda)} \overline{h_j(\lambda)}\,F_j(\lambda; x)\ \in\ \mathbb{L}^2_{-\delta}.$$

Then if $f_0 \in C_0^\infty(\mathbb{R}^d),\ g_0 \in C_0^\infty(\sigma(H))$,

$$\langle h, g_0(\lambda)(Uf_0)\rangle_{\mathcal{H}} \;=\; \int f_0(x)\,g_0(\lambda) \sum_{j=1}^{N(\lambda)} \overline{h_j(\lambda)}F_j(\lambda; x)\,dx d\rho(\lambda). \tag{6.9}$$

At first prove (v) for arbitrary $f_0 \in C_0^\infty(\mathbb{R}^d)$, $g_0 \in C_0^\infty(\sigma(H))$.

Lemma 2.2 (2) implies that $F(., y; \lambda)$ is the distributional solution of the equation

$$[g(H) - g(\lambda)]\psi \;=\; 0,$$

and by the argument similar to one applied above to prove (ii) it follows that $F_j(\lambda; x)$ is the distributional solution of the same equation. Notice, that if $f_0 \in \mathbb{C}_0^\infty$, then $U\,g_0(H)\,f_0$ is correctly defined since $g_0(-\Delta_\alpha)$ is the local operator simultaneously with $-\Delta_\alpha$. This implies

$$\begin{aligned}(U\,g_0(H)\,f_0)_j(\lambda) \;&=\; \int \overline{F_j(\lambda; x)}\,g_0(H)\,f_0(x)\,dx \\ &=\; g_0(\lambda) \int \overline{F_j(x; \lambda)}\,f_0(x)\,dx \;=\; g_0(\lambda)\,(Uf_0)_j\,(\lambda). \end{aligned} \tag{6.10}$$

It follows by (6.10) and Lemma 2.2 (2), that if $g_0 \in C_0^\infty(\sigma(H))$,

$$\int f_0(x)\, g_0(\lambda)\, K(\lambda;x)\, d\rho(\lambda)\, dx \;=\; \langle h, U g_0(H) f_0\rangle_{\mathcal{H}} \;=\; 0,$$

hence $K(\lambda;x) \equiv 0$ for a.e. x, $d\rho$- a.e. λ. Thus $(1+|x|)^{-2\delta} K(\lambda;x) = 0$ as an element of $\mathbb{L}^2(\mathbb{R}^d)$. Since $\varphi_j(\lambda;x) = (1+|x|)^{-\delta}\, F_j(\lambda;x)$ are orthogonal and non-zero, it follows that $K(\lambda;x) = 0$ implies $h_j(\lambda) = 0$, $j = 1,...,N(\lambda)$, for a.e. λ (with respect to $d\rho$). This proves (iv), and (v) supposing $f_0 \in \mathbb{C}_0^\infty(\mathbb{R}^d)$, $g_0 \in \mathbb{C}_0^\infty(\sigma(H))$.

The statement (v) in case $f \in \mathbb{L}^2(\mathbb{R}^d)$ follows by the limiting argument (similar to the proof of (vi) below). Now assume that g is a bounded measurable function on $\sigma(H)$. Then there exist functions $g_n \in C_0^\infty$, such that $g_n(\lambda) \to g(\lambda)$ $\forall \lambda \in \sigma(H)$, and $\|g_n\|_\infty \;\le\; \|g\|_\infty \;=\; \sup_x\, |g(x)|$. Then $g(H) \;=\; \mathrm{s} - \lim\limits_{n\to\infty} g_n(H)$, and if $f \in \mathbb{L}^2$

$$\begin{aligned} U[g(H)f] \;&=\; \mathcal{H} - \lim_{n\to\infty} U g_n(H) f \\ &=\; \mathcal{H} - \lim_{n\to\infty} g_n(\lambda) U f \;=\; g(\lambda) U f. \end{aligned}$$

This proves (v).

Lemma 6.3 *Given $f \in \mathcal{D}(-\Delta_\alpha)$, there exist $\{f_n\}_{n\in\mathbf{N}} \subset \mathcal{D}_0(-\Delta_\alpha)$, such that*

$$\begin{aligned} \mathbb{L}^2 - \lim_{n\to\infty} f_n \;&=\; f, \\ \mathbb{L}^2 - \lim_{n\to\infty} (-\Delta_\alpha - \lambda) f_n \;&=\; (-\Delta_\alpha - \lambda) f. \end{aligned} \tag{6.11}$$

Proof. Given arbitrary $f \in \mathcal{D}(H)$,

$$f(x) = \varphi(x) \;+\; \sum_{j\in\mathbf{Z}^d} a_j\, G_z^0(x-j), \tag{6.12}$$

where $\overline{a} = \{a_j\}_{j\in\mathbf{Z}^d} \;=\; \Gamma_\alpha(z)^{-1}\overline{\varphi}$, $\;\overline{\varphi} \;=\; \{\varphi(j)\}_{j\in\mathbf{Z}^d}$, $z \notin \sigma(-\Delta_\alpha)$, $\varphi \in \mathcal{D}(-\Delta)$, consider

$$f_n \;=\; \varphi_n \;+\; \sum_{j\in\mathbf{Z}^d} b_n^j\, G_z^0(x-j) \in \mathcal{D}_0(H), \tag{6.13}$$

where $\overline{b_n} = \Gamma_\alpha(z)^{-1}\overline{\varphi_n}$, and

$$\varphi_n \in C_0^\infty, \quad \mathbb{L}^2 - \lim_{n\to\infty} \varphi_n \;=\; \varphi, \quad \mathbb{L}^2 - \lim_{n\to\infty} (-\Delta - z)\varphi_n \;=\; (-\Delta - z)\varphi. \tag{6.14}$$

A sequence $\{\varphi_n\}_{n\in\mathbf{N}}$ satisfying (6.14) exists for $z \in \mathbb{C}$, $\varphi \in \mathbb{L}^2$. Given arbitrary $z \notin (0,+\infty)$, this follows since $(-\Delta - z)^{-1}$ is bounded for such z, and $(-\Delta - z)(C_0^\infty)$ is dense in $\mathbb{L}^2$. The same sequence φ_n satisfies (6.14) $\forall z \in \mathbb{C}$.

Prove that $\ell^2 - \lim\limits_{n\to\infty} \overline{\varphi}_n \;=\; \overline{\varphi}$.

If $\psi \in \mathcal{D}(-\Delta)$, then $\psi = (-\Delta - \lambda)^{-1} h$ for some $\lambda \in \mathbb{C} \setminus \sigma(-\Delta_\alpha)$, $h \in \mathbb{L}^2(\mathbb{R}^d)$. Hence

$$\overline{\psi}_j \;=\; \psi(j) \;=\; \int_{\mathbf{R}^d} G_\lambda^0(j,x)\, h(x)\, dx \;=\; \sum_{k\in\mathbf{Z}^d} \int_{\Lambda(k)} G_\lambda^0(j,x)\, h(x)\, dx, \quad j \in \mathbb{Z}^d,$$

where $\Lambda(k) = \Lambda_{1/2}(k)$ is the cube of sidelength $1/2$ centered at $k \in \mathbb{Z}^d$,

$$|\overline{\psi}_j| \;\leq\; \sum_{k \in \mathbf{Z}^d} B(j,k)\,\overline{h}_k \;<\infty, \tag{6.15}$$

$$B(j,k;x) \;=\; \Big(\int_{\Lambda(k)\subset \mathbf{Z}^d} |G^0_\lambda(j,x)|^2 dx\Big)^{1/2},$$

$$\overline{h}_k \;=\; \Big(\int_{\Lambda(k)} |h(x)|^2\,dx\Big)^{1/2},$$

i.e. $\overline{h} \in \ell^2(\mathbb{Z}^d)$, and $\{B(j,k)\}_{j,k\in \mathbf{Z}^d}$ define bounded operator B: $\ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$:

$$\|B\|_{\ell^2} \;\leq\; C(\lambda) \;<\; \infty.$$

It follows that $\overline{\psi} \in \ell^2(\mathbb{Z}^d)$:

$$\|\overline{\psi}\|_{\ell^2} \leq \|B\|_{\ell^2}\,\|h\|_{\mathbf{L}^2}. \tag{6.16}$$

Now consider functions $\{\varphi_n\}_{n\in\mathbf{N}}, \varphi \subset \mathcal{D}(-\Delta_\alpha)$, satisfying (6.14), i.e. for some $z < 0$:

$$h_n \stackrel{\text{def}}{=} (-\Delta - z)\varphi_n; \quad h \stackrel{\text{def}}{=} (-\Delta - z)\varphi;$$
$$\mathbb{L}^2 - \lim_{n\to\infty} h_n \;=\; h.$$

Relation (6.16) implies

$$\|\overline{\varphi}_n - \overline{\varphi}\|_{\ell^2} \leq \|B\|_{\ell^2}\,\|h^n - h\|_{\mathbf{L}^2},$$

i.e.

$$\ell^2 - \lim_{n\to\infty} \overline{\varphi}_n \;=\; \overline{\varphi}. \tag{6.17}$$

Since $\Gamma_\alpha(z)^{-1}$ is bounded,

$$\ell^2 - \lim_{n\to\infty} \overline{b}_n \;=\; \overline{a}, \tag{6.18}$$

where $\overline{a} \;=\; \Gamma_\alpha(z)^{-1}\,\overline{\varphi}$, $\overline{b}_n \;=\; \Gamma_\alpha(z)^{-1}\,\overline{\varphi}_n$.

Relations (6.12), (6.13) imply

$$\|f - f_n\|_{\mathbf{L}^2} \;\leq\; \|\varphi - \varphi_n\|_{\mathbf{L}^2} \;+\; C_z\|\overline{a} - \overline{b}_n\|_{\ell^2},$$

hence (6.18) imply

$$\mathbb{L}^2 - \lim_{n\to\infty} f_n \;=\; f. \tag{6.19}$$

Given arbitrary $\lambda > 0$, it follows by (6.14)

$$\|(H-\lambda)(f - f_n)\|_{\mathbf{L}^2} \;\leq\; \|(-\Delta - z)(\varphi - \varphi_n)\|_{\mathbf{L}^2} \;+\; |z-\lambda|\|f - f_n\|_{\mathbf{L}^2}. \tag{6.20}$$

Relations (6.14), (6.19), (6.20) imply

$$\mathbb{L}^2 - \lim_{n\to\infty} (H-\lambda) f_n \;=\; (H-\lambda) f \tag{6.21}$$

Lemma 6.3 is proved.

Now turn to the proof of (vi). Suppose $f_0 \in \mathcal{D}_0(H)$. Lemma A1 (Appendix A) and (ii) imply that $(H-\lambda)f_0 \in L^2_\delta$, and

$$\int \overline{F_j(\lambda;x)}\,(H-\lambda)\,f_0(x)\,dx \;=\; 0, \quad j=1,..,n;$$

which implies

$$U[(H-\lambda)f_0] \;=\; 0. \tag{6.22}$$

Now given $f \in \mathcal{D}(H)$, consider $f_n \in \mathcal{D}_0(H)$ such that

$$\begin{aligned} \mathbb{L}^2-\quad \lim_{n\to\infty} f_n \;&=\; f, \\ \mathbb{L}^2-\quad \lim_{n\to\infty} (H-\lambda) f_n \;&=\; (H-\lambda)f \end{aligned}$$

(cf. Lemma 6.3). Relations (iv), (v) and (6.22) imply

$$\begin{aligned} U[Hf] \;&=\; \mathcal{H}-\lim_{n\to\infty} U[Hf_n] \\ &=\; \mathcal{H}-\lim_{n\to\infty} \lambda[Uf_n] \;=\; \lambda\,[Uf], \end{aligned} \tag{6.23}$$

(vi) is proved. Let us prove (vii). Suppose $f \in \mathbb{L}^2_\delta(\mathbb{R}^d)$. Consider

$$\begin{aligned} \widetilde{f(x)} \;&=\; \mathbb{L}^2 - \lim_{\substack{E\to\infty \\ N\to\infty}} \int\limits_{|\lambda|\le E} \left\{ \sum_{j=1}^{\min(N,N(\lambda))} \int\limits_{\mathbf{R}^d} \overline{F_j(\lambda;y)} F_j(\lambda;x) f(y)\,dy \right\} d\rho(\lambda) \\ &=\; \lim_{E\to\infty} \int\limits_{|\lambda|\le E} \int\limits_{\mathbf{R}^d} F(\lambda;x,y)\,f(y)\,dy\,d\rho(\lambda), \end{aligned} \tag{6.24}$$

then for arbitrary $g \in L^2_\delta$:

$$\langle g, \widetilde{f}\rangle \;=\; \lim_{E\to\infty} \int\limits_{|\lambda|\le E} \int\limits_{\mathbf{R}^d\times\mathbf{R}^d} F(\lambda;x,y)\overline{g(x)} f(y)\,dx\,dy\,d\rho(\lambda) \;=\; \langle g, f\rangle. \tag{6.25}$$

It follows that $\|\widetilde{f}\| = \|f\|$, hence the limit at the r.h.s. of (6.24) exists. Since $\mathbb{L}^2_\delta$ is dense in $\mathbb{L}^2$, (6.25) implies $\widetilde{f} = f = U^{-1}h$, where U^{-1} is defined by (6.1).
Now suppose $f \in \mathbb{L}^2$, denote $f_R(x) = f(x)\chi_{(-R,R)}(x)$. Consider

$$\begin{aligned} \widetilde{f(x)} \;&=\; \mathbb{L}^2 - \lim_{\substack{E\to\infty \\ N\to\infty}} \int\limits_{|\lambda|\le E} \left\{ \sum_{j=1}^{\min(N,N(\lambda))} F_j(\lambda;x)\,\mathcal{H}-\lim_{R\to\infty} \int\limits_{|y|\le R} \overline{F_j(\lambda;y)} f(y)\,dy \right\} d\rho(\lambda) \\ &=\; \mathbb{L}^2 - \lim_{E\to\infty} \int\limits_{|\lambda|\le E} \left\{ \mathcal{H}-\lim_{R\to\infty} \int\limits_{\mathbf{R}^d} F(\lambda;x,y)\,f_R(y)\,dy \right\} d\rho(\lambda), \end{aligned} \tag{6.26}$$

The limits in (6.26) exist, since for arbitrary $g \in \mathbb{L}^2_\delta$,

$$\begin{aligned} \langle g, f\rangle &= \lim_{\mathbf{R}\to\infty} \langle g, f_R\rangle \\ &= \langle g, U^{-1}[\mathbb{L}^2 - \lim_{\mathbf{R}\to\infty} U f_R] = \langle g, \widetilde{f}\rangle, \end{aligned}$$

where the latter lim means the limit in $\mathbb{L}^2$-norm, and it follows $\widetilde{f} = f$. Theorem 6 is proved. □

Appendix A

Lemma A1. *Suppose* $\varphi \in \mathcal{D}_0(-\Delta_\alpha)$, *i.e.*

$$\varphi(x) = \varphi_0(x) + \sum_{i\in\mathbf{Z}^d} a_i\, G^0_E(x-i),$$

where $\varphi_0 \in \mathbb{C}_0^\infty$, $E \le \inf\operatorname{supp} P_{\alpha_0} - d$, $a_i = \sum_{j\in\mathbf{Z}^d} \Gamma_E^{-1}(i,j)\,\varphi_0(j)$. *Then there exists* d_0 *such that if* $d > d_0$, *then* $(-\Delta_\alpha - \lambda)\varphi \in \mathbb{L}^2_\delta$.

Proof. Denote $\Gamma^0_E = \Gamma_E - \frac{\sqrt{|E|}}{4\pi}$. Then (cf. (0.4)):

$$\|\Gamma^0_E\| \le \sum_{i\in\mathbf{Z}^d\setminus 0} \frac{1}{4\pi\,|i|}\, e^{-\sqrt{|E|}\,|i|} = \varepsilon(E) \to 0 \quad E \to -\infty.$$

It follows from the ergodicity that if $E < 0$

$$\sigma(\Gamma_E) = \operatorname{supp} P_{\alpha_0} \dot{+} \sigma(\Gamma^0_E) + \frac{\sqrt{|E|}}{4\pi}$$

(the sum above is the algebraic sum of subsets of $\mathbb{R}$), so

$$\operatorname{dist}(0, \sigma(\Gamma_E)) \ge \operatorname{dist}\left(-\frac{\sqrt{|E|}}{4\pi}, \inf\operatorname{supp} P_{\alpha_0} - \varepsilon(|E|)\right) \ge d/2,$$

if $|E|$ is large enough. The Combes-Thomas argument implies

$$|\Gamma_E^{-1}(i,j)| \le \frac{2}{d(E)}\, e^{-m(E)\,|i-j|},$$

hence if $\operatorname{supp}\varphi_0 \subset \Lambda$, then

$$|a_i| \le \sum_{j\in\mathbf{Z}^d\cap\Lambda} \Gamma_E^{-1}(i,j)\, |\varphi_0(j)| \le \|\varphi_0\|_\infty\, C_1(E, |\Lambda|)\, e^{-m(|E|)|i|}.$$

The next Lemma A2 implies that $\varphi \in \mathbb{L}^2_\delta$ and $(-\Delta_\alpha - \lambda)\varphi \in \mathbb{L}^2_\delta$.

Lemma A2. *Suppose* $\lambda < 0$, *consider*

$$F(x) = \sum_{i\in\mathbf{Z}^d} f_i\, G^0_\lambda(x-i), \qquad \overline{f} = \{f_i\}_{i\in\mathbf{Z}^d}. \tag{A.1}$$

1. If $F \in \mathbb{L}^2(\mathbb{R}^d, d\mu)$ *(i.e. the series converges in* $\|.\|_{\mathbb{L}^2(d\mu)}$*), where* $\mu(x) = g^2(x)\,dx$ *(*$g(x) = g(|x|) > 0$ *is a monotone, continuous function of* $|x|$*), and* $\varepsilon > 0$*, then*

$\overline{f} \in \ell^2(\mathbb{Z}^d, d\widetilde{\mu})$, where $\widetilde{\mu}(i) = g^2(|i| - t)$, $i \in \mathbb{Z}^d$, for some t: $|t| = \varepsilon$.

2. If $\overline{f} \in \ell^2(\mathbb{Z}^d, d\widetilde{\mu})$, where $\widetilde{\mu}(i) = \widetilde{g}^2(i)$, $i \in \mathbb{Z}^d$ $(e^{-\varepsilon\sqrt{|\lambda|}\,|i|} \leq \widetilde{g}(i) \leq e^{\varepsilon\sqrt{|\lambda|}\,|i|}$ for some $0 < \varepsilon < 1$, $\widetilde{g}$ is a monotone function of $|i|$, $i \in \mathbb{Z}^d)$, then $F \in \mathbb{L}^2(\mathbb{R}^d, d\mu)$, with $d\mu(x) = g^2(x)\,dx$, and g is monotone, continuous function of $|x|$, $g(i) = \widetilde{g}(i)$.

Proof. 1. Apply once again the argument described in the proof of Theorem 3 (Sect. 3). Given $\psi \in \mathbb{C}_0^\infty$

$$\langle F, (-\Delta - \lambda)\psi\rangle = \sum_{i \in \mathbf{Z}^3 \cap \operatorname{supp}\psi} f_i\, \psi(i).$$

Consider the function $\theta \in \mathbb{C}_0^\infty$, $\operatorname{supp}\theta \subset \Lambda_{1/2}(0)$, $\theta(0) = 1$, $|(-\Delta - \lambda)\theta(x)| \leq \tau_\lambda < \infty$, and denote $\theta_i(x) = \theta(x - i)$. Then

$$\langle F,\, (-\Delta - \lambda)\theta_i\rangle = f_i.$$

It follows that f_i obey

$$|f_i| \leq \left(\int_{\Lambda_{1/2}(i)} |F(x)|^2\, g(x)^2\, dx\right)^{1/2} \tau_\lambda \left(\int_{\Lambda_{1/2}(i)} g(x)^{-2}\, dx\right)^{1/2}. \tag{A.2}$$

Since $g(x) = g(|x|)$ is monotone and continuous, $\sup_{x \in \Lambda_{1/2}} |g(x)|^{-2} \leq |g(|i| - t)|^{-2}$, for some $|t| = 1/2$, and it follows by (A.2)

$$\sum_{i \in \mathbf{Z}^d} |f_i\, g(|i| - t)|^2 \leq \|F\|_{\mathbf{L}^2(d\mu)} \leq \infty.$$

To get the result with $0 < |t| < \varepsilon$, for arbitrary $\varepsilon > 0$ just consider the function $\theta_\varepsilon(x)$ with $\operatorname{supp}\theta \subset \Lambda_\varepsilon$. Statement 1 is proved.

2. Since $\widetilde{g}(i) \geq e^{\varepsilon\sqrt{|\lambda|}\,|i|}$, for arbitrary fixed $x \notin \mathbb{Z}^d$, series (A.1) converges absolutely. Moreover, if function $g(x) = g(|x|)$ is monotone, continuous and $g(i) = \widetilde{g}(i)$, then

$$\|F\|^2_{\mathbf{L}^2(d\mu)} = \sum_{i \in \mathbf{Z}^d} f_i \sum_{j \in \mathbf{Z}^d} f_j\, \langle G^0_\lambda(x - i), G^0_\lambda(x - j)\rangle_{\mathbf{L}^2(d\mu)},$$

where

$$\begin{aligned}\langle G^0_\lambda(x - i), G^0_\lambda(x - j)\rangle_{\mathbf{L}^2(d\mu)} &= \frac{1}{16\pi^2} \int_{\mathbf{R}^d} \frac{e^{-\sqrt{|\lambda|}\,|x-i|}\, e^{-\sqrt{|\lambda|}\,|x-j|}}{|x - i|\,|x - j|}\, g(x)^2\, dx \\ &\leq C\, e^{-m\sqrt{|\lambda|}\,|i-j|}\, g(i) g(j), \quad 0 < m < 1.\end{aligned}$$

Hence

$$\begin{aligned}\|F\|^2_{\mathbf{L}^2(d\mu)} &\leq C \sum_{i \in \mathbf{Z}^d} f_i g(i) \sum_{j \in \mathbf{Z}^d} f_j g(j) e^{-m\sqrt{|\lambda|}\,|i-j|} \\ &= C \sum_{i \in \mathbf{Z}^d} f_i g(i)\, \phi_i < C_1 \|f\|^2_{\ell^2(d\widetilde{\mu})},\end{aligned}$$

since $\{\phi_i\}_{i \in \mathbf{Z}^d} \in \ell^2(\mathbb{Z}^d)$. Lemma A2 is proved. $\square$

Appendix B

Lemma B.1.
Suppose h is arbitrary selfadjoint operator in $\ell^2(\mathbb{Z}^d)$ with simple spectrum, $\Psi(\lambda, x)$ is the generalized eigenfunction of h, $\rho(\lambda)$ is the spectral measure of h. Then for ρ - a.e. $\lambda \in \sigma(h)$:

$$\sum_{x\in\mathbf{Z}^d} |\Psi(\lambda, x)|^2(1 + |x|)^{-\nu} \le C < \infty, \tag{B.1}$$

where $\nu > d$, $C = C(\lambda) < \infty$.
Proof. Consider $I = (a, b) \subset \mathbb{R}$, $R(i) = (h - i)^{-1}$. By the spectral theorem,

$$\Im R(i; x, x) = \int_{\mathbf{R}} |\Psi(\lambda, x)|^2(\lambda^2 + 1)^{-1} d\rho(\lambda) \ge (b^2 + 1)^{-1} \int_{(a,b)} |\Psi(\lambda, x)|^2 d\rho(\lambda). \tag{B.2}$$

Multiplying both sides of (B.2) by $(1 + x)^{-\nu}$ and summing over x, for $d < \nu$, one gets:

$$\begin{aligned} \infty &\ge \sum_{x\in\mathbf{Z}^d} (1 + |x|)^{-\nu} |\Im R(i; x, x)| \\ &\ge (b^2 + 1)^{-1} \sum_{x\in\mathbf{Z}^d} (1 + |x|)^{-\nu} \int_{(a,b)} |\Psi(\lambda, x)|^2 d\rho(\lambda). \end{aligned}$$

It follows that the series (B.1) is absolutely integrable, so Lemma B.1 is proved. □.

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