SMOOTH DEPENDENCE OF LINEARIZATION ON PERTURBATIONS

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Abstract. We show that a linearizing homeomorphism has the C^1-dependence on perturbations of nonlinear terms of an underlying diffeomorphism. We also present an expression for the derivative of the linearization with respect to the perturbations of the nonlinear terms.

1. Introduction

The well-known theorem of Grobman and Hartman provides an essential method for local analysis of dynamical systems with hyperbolic fixed point.

Theorem 1 (Hartman, Grobman [G, H]). *Let $U \subset \mathbb{R}^n$ be a neighborhood of 0, $\mathfrak{f}\colon U \to \mathbb{R}^n$ continuously differentiable with 0 as a hyperbolic fixed point. Then there is a homeomorphism h of a neighborhood of 0 with $H \circ \mathfrak{f} = D\mathfrak{f}_0 \circ H$ near 0.*

Thus, the dynamics near the hyperbolic fixed point is topologically the same as that of the linear part of the map. Yet, the linearizing homeomorphism H depends on the higher degree terms, $\mathfrak{f} - D\mathfrak{f}_0$. (We call these higher degree terms nonlinearities of $\mathfrak{f}$.)

It is clear, that H varies continuously with small changes of nonlinearities. There arises a question: "Can the linearization have a higher regularity, than $C^0(\mathbb{R}^n, \mathbb{R}^n)$?" Addressing this question, we study differentiability of the linearization H with respect to the perturbations of the nonlinear terms. Accordingly, the purpose of the present paper is to establish the following result:

Theorem 2. *Let $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a hyperbolic linear operator, Λ be a set of C^1 mappings $\lambda : \mathbb{R}^n \mapsto \mathbb{R}^n$ with the first derivative vanishing at 0 and with bounded support, and $Q_L \subset \Lambda$ be a set of weakly resonant nonlinearities. Let $\mathcal{H} : \Lambda \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$ be a linearization. Then,*

- *If for some $\lambda \in \Lambda$ and some $\mu \in \Lambda$ the sequence in formula 4 of Remark 7 is uniformly convergent, then it represents the derivative $(\mathrm{D}\,\mathcal{H})_\lambda\,(\mu)$.*
- *$\mathcal{H} \in C^1(Q_L)$ and the derivative of $\mathcal{H} \in Q_L$ is given by formula 4.*

The proof of this fact is organized as follows. In Section 2 we define the derivative of linearization at a given point in a given direction (Definition 3); and for a special type of perturbations Q_L (weakly resonant quadratic maps with bounded support) we prove the differentiability of the linearization. We also provide a formula, which expresses the derivative in terms of nonlinear perturbations.

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In order to do this, first, in Section 2, we invoke a result of Guysinsky, Hasselblatt and Rayskin ([GHR]), which allows us to write the conjugation via a natural correspondence between orbits under the linear and nonlinear dynamics. Using this formula, we are able to write in Section 3 a formal expression for the first derivative of the conjugation with respect to the perturbations. The next part of the paper (Section 4) is devoted to the proof of convergence of the formal series to the value of the derivative. In this part we use results from [GHR] to estimate the growth of coefficients of the orbits. This is the place where we use the assumption that nonlinear terms are weakly resonant. However, we believe that the convergence of the derivative's formal power series can be proved for any nonlinear perturbations with bounded support.

2. Definitions

Let $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a hyperbolic linear operator. Define the set of perturbations

$$\Lambda = \{\lambda \in C^1(\mathbb{R}^n, \mathbb{R}^n) : \ \text{supp}(\lambda) \text{ is bounded, } \lambda = O(x^2)\}. \tag{1}$$

The proof of the Hartman-Grobman theorem [G] shows that for each L and for each $\lambda \in \Lambda$, there exists a unique conjugating map $H_\lambda : \mathbb{R}^n \mapsto \mathbb{R}^n$ of the form $H_\lambda = \operatorname{Id} + h_\lambda$, such that $LH_\lambda = H_\lambda \circ (L + \lambda)$.
We can define the derivative of conjugating diffeomorphism on the set of perturbations Λ as follows:

Definition 3. Consider the following continuous mapping:

$$\mathcal{H} : \begin{array}{ccc} \Lambda & \mapsto & C^0(\mathbb{R}^n, \mathbb{R}^n) \\ \lambda & \mapsto & h_\lambda. \end{array}$$

Pick $\lambda,\ \mu \in \Lambda$. If the following limit

$$(\mathrm{D}\,\mathcal{H})_\lambda(\mu) = \lim_{t \to 0} \frac{h_{\lambda + t\mu} - h_\lambda}{t}, \quad t \in \mathbb{R}$$

exists, we call it the derivative of $\mathcal{H}$ at λ in the direction μ.

In order to construct the derivative $\mathrm{D}\,\mathcal{H}$ for the given diffeomorphism $L + \lambda$, we shell use an expression for the linearization $\operatorname{Id} + h_\lambda$, which was derived in [GHR]. We proceed with a construction of this formula.
Let

$$F^k := L^{-k}(L + \lambda + t\mu)^k = \left(\operatorname{Id} + \hat{\lambda} + t\hat{\mu}\right)^k$$

and

$$G^k := L^{-k}(L + \lambda)^k = \left(\operatorname{Id} + \hat{\lambda}\right)^k$$

be the k-th iterations (compositions) of these maps (i.e., $(\operatorname{Id} + \lambda + t\mu)^k = (\operatorname{Id} + \lambda + t\mu) \circ (\operatorname{Id} + \lambda + t\mu)^{k-1}$). In order to simplify our notations we shell use λ for $\hat{\lambda}$ and μ for $\hat{\mu}$ in the future.
It was shown in part 5 of [GHR], there exist $n_F \in \mathbb{N}$ and $n_G \in \mathbb{N}$ such that the linearizations of the maps

$$L + \lambda + t\mu$$

and

$$L + \lambda$$

can be written as F^{n_F} and G^{n_G} correspondingly. (Moreover, one can substitute $n = \max\{n_G, n_F\}$ for n_G or n_F.) For a better clarity we repeat here, how the conjugations F^n and G^n were constructed in [GHR].
Consider $\mathfrak{f}(x) = Lx + \lambda(x)$. Assume that support of λ is such that, if $\|x\| \geq \delta$ then $\mathfrak{f}(x) = Lx$.

Take $\delta' < \delta$ sufficiently small, and for $\|x\| < \delta'$ define

$$n_+(x) := 1 + \max\{n \in \mathbb{N} \mid \|\mathfrak{f}^i(x)\| < \delta \text{ for } 0 \leq i \leq n\}.$$

Then $\lim_{\|x\|\to 0} n_+(x) = \infty$ and $n_+(\mathfrak{f}(x)) = n_+(x) - 1$. Denote by $(.)_\pm$ the coordinate projections to the expanding and contracting directions, respectively. Since the linear part of $\mathfrak{f}$ is hyperbolic, n_+ is finite off the contracting direction. For $\|x\| < \delta'$ and with $h_n := L^{-n}\mathfrak{f}^n$ we set $h_+ = 0$ on the contracting direction and

$$h_+(x) := (h_{n_+(H^+(x))}(H^+(x)))_+$$

on the expanding direction. One can prove continuity of h_+ on the contracting direction. Discontinuities of n_+ do not produce discontinuities of h_+ because $h_{n_+(H^+(x))}(H^+(x)) = h_{n_+(H^+(x))+1}(H^+(x))$.

We can also define n_- and h_- analogously to n_+ and h_+ and set $h := (h_+, h_-)$. It was shown in [GHR], that h is differentiable at 0 and is a conjugacy between $\mathfrak{f}$ and its linear part L.

3. Formula for the derivative

In this section we derive a formal power series, which (if convergent) represents the derivative $\mathrm{D}\,\mathcal{H}$. The convergence is proved in Section 4, although only for a special type of maps with some quadratic weakly resonant nonlinearities. These quadratic nonlinearities are described in the next definition.

Definition 4. Consider the root space decomposition of L. We shell say that λ contains only weakly resonant terms, if whenever (in coordinates adapted to the root space decomposition) the ith component of λ contains a term $a \cdot xy$ with $a \neq 0$ then the eigenvalues L_1 and L_2 associated with the root spaces corresponding to x and y, respectively, are related to the eigenvalue L_i associated with the root space containing the ith unit vector by $|L_i| = |L_1||L_2|$.

We denote the set of weakly resonant quadratic perturbations with finite support by Q_L, i.e.

$$Q_L = \{\lambda \in \Lambda,\ \lambda \text{ contains only weakly resonant quadratic terms}\}. \tag{2}$$

The following theorem provides a sufficient condition for smooth dependence of the linearization H on the perturbations λ.

Theorem 5. *Let $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a hyperbolic linear operator, and $\mathcal{H}$ be the mapping as in Definition 3, i.e.,*

$$\mathcal{H} : \Lambda \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$$

Then,

- *If for some $\lambda \in \Lambda$ and some $\mu \in \Lambda$ the sequence in formula 4 of Remark 7 is uniformly convergent, then it represents the derivative $(\mathrm{D}\,\mathcal{H})_\lambda(\mu)$.*
- *$\mathcal{H} \in C^1(Q_L)$ and the derivative of $\mathcal{H}$ is given by formula 4.*

The proof of this theorem rests on the following two formulas, shown in Lemma 6 and Remark 7. The latter is the formula for $(\mathrm{D}\,\mathcal{H})_\lambda(\mu)$.

Lemma 6.
(3)
$$
\begin{aligned}
F^n - G^n = t\sum_{k=0}^{n-1}\mu(F^k)\\
&+ t[1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^2)]\lambda'(G^1)\cdot\mu(F^0)\\
&+ t[1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^3)]\lambda'(G^2)\cdot\big(\mu(F^1)+\mu(F^0)\big)\\
&+ t[1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^4)]\lambda'(G^3)\cdot\big(\mu(F^2)+\mu(F^1)+\mu(F^0)\big)\\
&\vdots\\
&+ t\lambda'(G^{n-1})\cdot\big(\mu(F^{n-2})+\mu(F^{n-3})+...+\mu(F^0)\big).
\end{aligned}
$$

In the expression above $\lambda'(G^k)$ *denotes* $\int_0^1(D\lambda)_{\sigma(s)}ds$*, where* $\sigma:[0,1]\mapsto\mathbb{R}^n$ *is a path connecting the points* $G^k=\sigma(0)$ *and* $F^k=\sigma(1)$*.*

Remark 7. It immediately follows from Lemma 6, that if the following limit exists, than it equals to the expression on the right hand side:
(4)
$$
\begin{aligned}
\lim_{t\to 0}\frac{F^n-G^n}{t} = \sum_{k=0}^{n-1}\mu(G^k)\\
&+ [1+(D\lambda)_{G^{n-1}}]\cdot[1+(D\lambda)_{G^{n-2}}]...[1+(D\lambda)_{G^2}](D\lambda)_{G^1}\cdot\mu(G^0)\\
&+ [1+(D\lambda)_{G^{n-1}}][1+(D\lambda)_{G^{n-2}}]...[1+(D\lambda)_{G^3}](D\lambda)_{G^2}\cdot\big(\mu(G^1)+\mu(G^0)\big)\\
&+ [1+(D\lambda)_{G^{n-1}}][1+(D\lambda)_{G^{n-2}}]...[1+(D\lambda)_{G^4}](D\lambda)_{G^3}\cdot\big(\mu(G^2)+\mu(G^1)+\mu(G^0)\big)\\
&\vdots\\
&+ (D\lambda)_{G^{n-1}}\cdot\big(\mu(G^{n-2})+\mu(G^{n-3})+...+\mu(G^0)\big).
\end{aligned}
$$

Proof. In order to prove Lemma 6, we shell use the induction on n. For the base of the induction we take $n=1$, and obtain

$$F-G=t\mu,$$

which is obviously correct.
Suppose our formula 3 holds for $i=1,...,n$. Consider $i=n+1$:

$$
\begin{aligned}
F^{n+1}-G^{n+1} &= F(F^n)-G(G^n)\\
&= t\mu(F^n)+F^n-G^n+\lambda'(G^n)\cdot[F^n-G^n]\\
&= t\mu(F^n)+[1+\lambda'(G^n)]\cdot[F^n-G^n]
\end{aligned}
\tag{5}
$$

Now, we can substitute the inductive assumption for the expression $F^n - G^n$.

$$\begin{aligned}F^{n+1} - G^{n+1} = {} & t\mu(F^n) \\ & + t[1+\lambda'(G^n)]\sum_{k=0}^{n-1}\mu(F^k) \\ & + t[1+\lambda'(G^n)][1+\lambda'(G^{n-1})]...[1+\lambda'(G^2)]\lambda'(G^1)\cdot\mu(F^0)+ \\ & + t[1+\lambda'(G^n)][1+\lambda'(G^{n-1})]...[1+\lambda'(G^3)]\lambda'(G^2)\cdot\big(\mu(F^1)+\mu(F^0)\big)+ \\ & + t[1+\lambda'(G^n)][1+\lambda'(G^{n-1})]...[1+\lambda'(G^4)]\lambda'(G^3)\cdot\big(\mu(F^2)+\mu(F^1)+\mu(F^0)\big)+ \\ & \vdots \\ & + t[1+\lambda'(G^n)]\lambda'(G^{n-1})\cdot\big(\mu(F^{n-2})+\mu(F^{n-1})+...+\mu(F^0)\big)\end{aligned}$$

Simplifying this expression, we obtain

$$\begin{aligned}F^{n+1} - G^{n+1} = {} & t\sum_{k=0}^{n}\mu(F^k) \\ & + t[1+\lambda'(G^n)][1+\lambda'(G^{n-1})]...[1+\lambda'(G^2)]\lambda'(G^1)\cdot\mu(F^0)+ \\ & + t[1+\lambda'(G^n)][1+\lambda'(G^{n-1})]...[1+\lambda'(G^3)]\lambda'(G^2)\cdot\big(\mu(F^1)+\mu(F^0)\big)+ \\ & + t[1+\lambda'(G^n)][1+\lambda'(G^{n-1})]...[1+\lambda'(G^4)]\lambda'(G^3)\cdot\big(\mu(F^2)+\mu(F^1)+\mu(F^0)\big)+ \\ & \vdots \\ & + t[1+\lambda'(G^n)]\lambda'(G^{n-1})\cdot\big(\mu(F^{n-2})+\mu(F^{n-3})+...+\mu(F^0)\big) \\ & + t\lambda'(G^n)\cdot\sum_{k=0}^{n-1}\mu(F^k)\end{aligned}$$

Thus, our assumption was correct and proves the theorem. □

4. Convergence

It is clear that in the formula 3 of Lemma 6 (as well as in the formula 4 of the Remark 7) $n = n(x) \to \infty$ as $x \to 0$. But it was proved in [GHR], that $n(x)$ has at most logarithmic rate of growth (see inequality 7 below). This idea allows us to show, that the equation 4 converges to the derivative as $t \to 0$ uniformly in x. In order to prove that, we shell use the estimates from the lemmas 16 and 13 of [GHR]. These lemmas establish the bounds for the expressions $G^n - \mathrm{Id}$ and $F^n - \mathrm{Id}$, as well as for $n(x)$. The inequalities from the lemmas were proved for perturbations from the class Q_L, and this is the place where we use the assumption that $\lambda, \mu \in Q_L$. One of the inequalities from lemma 16 shows, that

$$||(G^{n_+}(x)-x)_+|| \le \sum_{m\ge 2} q^m ||x||, \quad ||(F^{n_+}(x)-x)_+|| \le \sum_{m\ge 2} q^m ||x||, \tag{6}$$

with $q = e^{-\gamma(n_+ + C)/2}$ and some positive C and γ.

Another pair of inequalities (lemma 13 from [GHR]) for the expanding coordinates x_i provides a logarithmic bound for n:

$$|x_i| < e^{H^2 \log ||F^j|| - \gamma n}, \quad |x_i| < e^{H^2 \log ||G^j|| - \gamma n} \tag{7}$$

These inequalities hold for each sufficiently small expanding coordinate x_i, and some positive constants H, γ.

There are similar estimates for the contracting coordinates. In this section we present a proof for the expanding direction only. In order to simplify our notations, we shell drop the subscripts "+".

Lemma 8. *Suppose $\lambda, \mu \in Q_L$. Then, the sequence*

$$\frac{F^{n(x)}(x) - G^{n(x)}(x)}{t} = \frac{F_t^{n(x)}(x) - G^{n(x)}(x)}{t}$$

converges as $t \to 0$ uniformly in x.

Proof. We shell show that the sequence $\frac{F_t^{n(x)}(x)-G^{n(x)}(x)}{t}$ tends to $(\mathrm{D}\,\mathcal{H})_\lambda(\mu)$ uniformly for $x \in (\operatorname{supp}\lambda \cap \operatorname{supp}\mu)$. To simplify notations we shell write n for $n(x)$.

$$\begin{aligned}
&\frac{F_t^n(x) - G^n(x)}{t} - (\mathrm{D}\,\mathcal{H})_\lambda(\mu) = \sum_{k=0}^{n-1}[\mu(F_t^k) - \mu(G^k)] \\
&+ [1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^2)][\lambda'(G^1)\mu(F_t^0) - (D\lambda)_{G^1}\mu(G^0)] \\
&+ \big([1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^2)] \\
&-\ [1+(D\lambda)_{G^{n-1}}][1+(D\lambda)_{G^{n-2}}]...[1+(D\lambda)_{G^2}]\big)\,(D\lambda)_{G^1}\mu(G^0) \\
&+ [1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^3)] \\
&\cdot [\lambda'(G^2)\left(\mu(F_t^1) + \mu(F_t^0)\right) - (D\lambda)_{G^2}\left(\mu(G^0)+\mu(G^1)\right)] \\
&+ \big([1+\lambda'(G^{n-1})][1+\lambda'(G^{n-2})]...[1+\lambda'(G^3)] \\
&-\ [1+(D\lambda)_{G^{n-1}}][1+(D\lambda)_{G^{n-2}}]...[1+(D\lambda)_{G^3}]\big)\cdot(D\lambda)_{G^2}\left(\mu(G^0)+\mu(G^1)\right) \\
&\ \vdots \\
&+ \lambda'(G^{n-1})\left(\mu(F_t^0)+\mu(F_t^1)+...+\mu(F_t^{n-2})\right) \\
&- (D\lambda)_{G^{n-1}}\left(\mu(G^0)+\mu(G^1)+...+\mu(G^{n-2})\right).
\end{aligned} \tag{8}$$

Let

$$S_m := \sum_{k=0}^{m-1}\left(\mu(F_t^k) - \mu(G^k)\right)$$

Let $\mu'(G^k)$ be defined analogously to $\lambda'(G^k)$, i.e., $\mu'(G^k) = \int_0^1 (D\mu)_{\sigma(s)}ds$, where $\sigma : [0,1] \mapsto \mathbb{R}^n$ is a path connecting the points $G^k = \sigma(0)$ and $F^k = \sigma(1)$. Then,

$$\begin{aligned}
S_m &\le \sum_{k=0}^{m-1} ||\mu'(G^k)(F_t^k - G^k)|| \\
&\le \sum_{k=0}^{m-1} ||\mu'(G^k)||\left(\left\|F_t^k - \mathrm{Id}\right\| + \left\|G^k - \mathrm{Id}\right\|\right) \\
&\le \sum_{k=0}^{m-1} \max_{\operatorname{supp}(\mu)} ||D\mu|| 2\frac{q^2}{1-q}||x|| = \max_{\operatorname{supp}(\mu)} ||D\mu|| 2\frac{q^2}{1-q} m(x)||x||
\end{aligned}$$

where $q = e^{-\gamma(n_+ + C)/2}$ and $m(x)$ has a logarithmic growth, as $x \to 0$. This implies that

$$\lim_{x\to 0} S_m = 0. \tag{9}$$

On the other hand, for $||x||$ bounded away from 0, there exists a positive constant N, such that $|m(x)| < N$, and we obtain the following estimates:

$$S_m \leq N \max_{0\leq k\leq N} \left(\max_{\mathrm{supp}(\mu)} ||D\mu|| ||(F_t^k - G^k)|| \right) \to 0, \tag{10}$$

uniformly in x as $t \to 0$.
Thus, combining estimates 9 and 10, we obtain that S_m tends to 0 uniformly as $t \to 0$.
We also note that from the above estimates it is easy to see that

$$\sum_{k=0}^{m(x)-1} ||F_t^k(x) - G^k(x)|| \to 0 \tag{11}$$

uniformly in x as $t \to 0$.
Now, we shell define and estimate R_n, which we can use later for a T_n bound (see below).

$$R_n := \sum_{m=1}^{n} ||S_m|| \leq \sum_{k=0}^{n-1} (n-k)||\mu'(G^k)(F_t^k - G^k)|| \tag{12}$$

converges to 0 uniformly as $t \to 0$, if $n(x)$ has logarithmic growth.

Now, we shell consider

$$T_m := \lambda'(D^m) \sum_{k=0}^{m-1} \mu(F_t^k) - (D\lambda)_{G^m} \sum_{k=0}^{m-1} \mu(G^k)$$

Then,

$$||T_m|| \leq ||S_m|| \cdot ||\lambda'(G^m)|| + ||\mu(G^0) + ... + \mu(G^{m-1})|| \cdot ||\lambda'(G^m) - (D\lambda)_{G^m}||$$

$$\sum_{m=1}^{n-1} ||T_m|| \leq R_{n-1} \max ||\lambda'|| + n(x)||\mu(G^0) + ... + \mu(G^{m-1})|| \cdot ||\lambda'(G^m) - (D\lambda)_{G^m}||$$

Since $||G^k(x) - x|| \leq \frac{q^2}{1-q}||x||$, $\mu = O(x^2)$ (for $x \to 0$) and $n(x)$ has logarithmic growth,

$$\lim_{t\to 0} n(x)||\mu(G^0) + ... + \mu(G^{m-1})|| = 0 \tag{13}$$

On the other hand, for $||x||$ bounded away from 0, $n(x)$ is bounded and

$$n(x)||\mu(G^0) + ... + \mu(G^{m-1})|| \cdot ||\lambda'(G^m) - (D\lambda)_{G^m}|| \tag{14}$$

tends to 0 as $t \to 0$, because the last term, $||\lambda'(G^m) - (D\lambda)_{G^m}||$, is small.
Combining the last two estimates 13 and 14, we conclude that $\sum_{m=1}^{n-1} ||T_m||$ tends to 0 uniformly, as $t \to 0$.

Consider

$$P_m := \prod_{k=m-1}^{2} (1 + \lambda'(G^k))$$

As $x \to 0$, $\lambda'(G^k(x)) = O(x)$ (like in the above calculations). Then, if $m(x)$ has logarithmic growth, the norm of the above product is uniformly bounded. (We shell call this bounding constant P).

Finally, let

$$U_n := \prod_{m=n-1}^{2} [1 + \lambda'(G^m)] - \prod_{m=n-1}^{2} [1 + (D\lambda)_{G^m}]$$

It is easy to see that

$$\begin{aligned}
||U_n|| \leq \sum_{m=n-1}^{2} \Big\| & [1 + \lambda'(G^{n-1})]...[1 + \lambda'(G^{m+1})] \\
& [\lambda'(G^m) - (D\lambda)_{G^m}] \\
& [1 + (D\lambda)_{G^{m-1}}]...[1 + (D\lambda)_{G^2}] \Big\| \\
& \leq P \sum_{m=n-1}^{2} ||[\lambda'(G^m) - (D\lambda)_{G^m}]|| \\
& \leq P \max_{1<m<n} \left(\max_{x,y \in \mathrm{supp}} ||(D\lambda)_y - (D\lambda)_{G^m(x)}|| \right) \sum_{m=n-1}^{2} ||F_t^m - G^m||
\end{aligned}$$

It follows from the estimate 11 that U_n tends to 0 uniformly in x as $t \to 0$. Moreover, using the inequalities, similar to the inequalities 12, one can see, that

$$\sum_{m=1}^{n(x)} ||U_m(x)|| \to 0$$

uniformly in x, as $t \to 0$.
Substituting the estimates for S_n, T_m, P_m and U_m in the equation 8 we can see that

$$\begin{aligned}
\left\| \frac{F_t^{n(x)}(x) - G^{n(x)}(x)}{t} - (\mathrm{D}\,\mathcal{H})_\lambda (\mu) \right\| & \leq ||S_n|| + \sum_{m=1}^{n-1} ||T_m|| \cdot ||P_m|| + O(\sum_{m=1}^{n-1} ||U_m||) \\
& \leq ||S_n|| + P \sum_{m=1}^{n-1} ||T_m|| + O(\sum_{m=1}^{n-1} ||U_m||)
\end{aligned}$$

tends to 0 uniformly in x, as $t \to 0$.

□

Conjecture 9. *Let $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a hyperbolic linear operator and Λ be the set defined in 1. Then, $\mathcal{H} \in C^1(\Lambda)$ in the sense of Definition 3 and the derivative $(\mathrm{D}\,\mathcal{H})_\lambda (\mu)$ is represented by the formula 4 of Remark 7.*

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