On the summability of formal solutions of non-linear partial differential equations with shrinkings

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Abstract

We investigate the existence and the Borel summability of formal power series in one variable with holomorphic coefficients, solutions to non-linear partial differential equations with shrinkings having non-Kowalevski type in the complex domain. Sufficient conditions are given in terms of the shape of the functional partial differential equations and initial conditions. Existence and asymptotic behaviour of local sectorial holomorphic solutions to these non-linear functional partial differential equations are obtained as byproduct. We apply the results to the study of asymptotic behaviour of solutions to some advanced-argument partial differential equations in a neighborhood of infinity.

Key words: asymptotic expansions, Cauchy problems, partial differential equations, Gevrey estimates, power series solutions, Borel summability, shrinkings, deviating variables. 2000 MSC: 35C10, 35C20.

1 Introduction

In this paper, we study non-linear partial differential equations with deviating variables

$$(E) \qquad \partial_z^S \hat{X}(t,z) = A(t,z,((\partial_t^{k_0}\partial_z^{k_1}\hat{X})(\phi_{k_0,k_1}(t),z))_{(k_0,k_1)\in\mathcal{S}}),$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ and A is an analytic function in the variables t,z in some neighborhood of the origin in $\mathbb{C}^2$ and polynomial in its other arguments. The "deviations" $\phi_{k_0,k_1}(t)$ are holomorphic functions in some neighborhood of the origin in $\mathbb{C}$.

Our approach is based on resummation methods of formal series solutions of (E), known as Borel summability, starting with the fundamental contributions of Écalle, see [10]. This point of view has shown a great success in the framework of differential equations, difference and difference-differential equations, see [4],[6],[12],[14],[18],[19]. But in the context of functional partial differential equations, very little is known.

The goal of this paper is to provide a general approach to construct Borel summable formal series solutions of differential equations of the type (E). In a first part of the paper, we give sufficient conditions for the existence and unicity of formal series in the variable t,

$$\hat{X}(t,z) = \sum_{j=0}^{+\infty} \frac{t^j}{j!} X_j(z),$$

which are solutions of (E), where $X_j(z)$, $j \geq 0$, are analytic functions in some neighborhood of the origin in $\mathbb{C}$ (Proposition 2).

In a second part, we investigate $1-$summability properties of these formal series. We give new sufficient conditions, both on the shape of the equation (E) and on initial conditions, under which these constructed formal series solutions of (E) are $1-$summable (Theorem 1). In the proof, we use a key proposition (Proposition 1), already introduced in a previous paper, see [5]. We will also make use of the classical Cauchy-Kowalevskaya theorem (see [13]), stated below.

Theorem CK *Consider a non-linear partial differential equation of the form*

$$(1') \qquad \partial_x^\beta u(t,x) = F(t,x,(\partial_t^{\alpha_0}\partial_x^{\alpha_1}u(t,x))_{(\alpha_0,\alpha_1)\in\mathcal{I}})$$

where $\mathcal{I} = \{(\alpha_0,\alpha_1)\in\mathbb{N}^2 : \alpha_0+\alpha_1\leq\beta, (\alpha_0,\alpha_1)\neq(0,\beta)\}$, and F is analytic in the variables t,x in a neighborhood of the origin and polynomial in its other arguments, with initial conditions

$$(2') \qquad \partial_x^k u(t,0) = \varphi_k(t),$$

for $0\leq k\leq\beta-1$, where φ_k are analytic functions in a neighborhood of the origin. Then, the problem $(1')$, $(2')$ has a unique solution u that is analytic in a neighborhood of the origin.

The leading idea will be the same as in the paper [17], where we have dealt with linear partial differential equation, with an additional difficulty because of the presence of "deviations", ϕ_{k_0,k_1}. To this aim, we will use a higher order chain rule which has been introduced by T. Yamanaka, see [20], and which is much more suitable than the classical Faà di Bruno formula to deal with Gevrey estimates.

One consequence of Theorem 1 is the existence and asymptotic behaviour of holomorphic solutions $X(t,z)$ of (E) (where X is the 1-sum of $\hat{X}$), for t in a small sector and z in a neighborhood of the origin. In the last part of the paper, we apply our results to the study of existence and asymptotic properties of solutions to some non-linear advanced-argument partial differential equations. There exists a huge literature on the subject of asymptotic analysis of solutions to functional ordinary or partial differential equations. For recent results in this field, we refer to [2],[7],[8],[9],[11],[16].

2 Summability criterions

In this section, we recall a proposition already obtained in a previous paper [5], which gives necessary and sufficient conditions for the summability of formal series with variable coefficients.

We consider a sequence $(X_j)_{j\geq0}$ of holomorphic functions $X_j : \mathcal{D}_\rho \longrightarrow \mathbb{C}$ where $\mathcal{D}_\rho = \{z\in\mathbb{C} : |z| < \rho\}$ with $\rho\in\mathbb{R}_+$.

Definition 1 *The formal series*

$$(\hat{X}) \qquad \hat{X}(t,z) = \sum_{j=0}^{\infty}\frac{t^j}{j!}X_j(z)$$

is said to be $k-$summable in the direction $d\in\mathbb{R}$ if

i) *There exist $\rho\in\mathbb{R}_+$ and $R(\rho)>0$, such that the following formal series, called formal Borel transform with respect to the variable t of order k,*

$$W(t,z) = \sum_{j=0}^{\infty}\frac{X_j(z)t^j}{j!\Gamma(1+j/k)},$$

is absolutely convergent for $z \in \mathcal{D}_\rho$ *and* $|t| < R(\rho)$,

ii) *There exists* $\delta > 0$ *such that for all* $z \in \mathcal{D}_\rho$, *the series* $W(t,z)$ *can be analytically continued with respect to* t *in a sector* $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$. *Moreover, there exist* $\delta_1 < \delta$, $C > 0$, *and* $K > 0$ *such that*

$$\sup_{z \in \bar{\mathcal{D}}_\rho} |W(t,z)| \leq Ce^{K|t|^k}$$

for all $t \in S_{d,\delta_1}$.

If this is so, the Laplace transform of order k of $W(t,z)$,

$$(X) \qquad X(t,z) = t^{-k} \int_0^{\infty(\gamma)} W(\tau,z) e^{-(\tau/t)^k} d\tau^k,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the $k-$ sum of the formal series $\hat{X}(t,z)$.

Proposition 1 *The following statements are equivalent.*

i) *The formal series* $\hat{X}$ *is* $k-$*summable with respect to* t *in direction* d.

ii) *For all* $n \in \mathbb{N}$, *the series*

$$W_n(t) = \partial_z^n W(t,z)_{|z=0}$$

all converge for $|t| \leq \tau$, *for some* $\tau > 0$ *that is independent of* n. *Moreover, there exists* $\delta_1 > 0$ *so that all the functions* $W_n(t)$ *can be holomorphically continued with respect to* t *into a sector* $S_{d,\delta_1} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta_1\}$. *Finally, for every* $\delta_2 < \delta_1$, *there exist constants* $C, K > 0$, *independent of* n, *so that*

$$|W_n(t)| \leq C^{|n|} n! e^{K|t|^k},$$

for all $t \in S_{d,\delta_2}$ *and* $n \in \mathbb{N}$.

iii) *For all* $n \in \mathbb{N}$, *the formal series*

$$\hat{X}_n(t) = \partial_z^n \hat{X}(t,z)_{|z=0}$$

are k*-summable in the direction* d. *Moreover, the sums* $X_n(t)$ *are holomorphic in a region* G, *independent of* n, *where* G *has bisecting direction* d *and opening greater than* π/k, *and for every closed subsector* $\bar{S} \subset G$ *with bisecting direction* d, *there exist constants* $C, K > 0$, *independent of* n, *such that*

$$\sup_{t \in \bar{S}} |\partial_t^l X_n(t)| \leq CK^{n+l} n! l! \Gamma(1 + l/k)$$

for all $n, l \in \mathbb{N}$.

3 Formal series solutions to non-linear partial differential equations with deviating variables

In this section we are looking for formal series $\hat{X}(t,z)$ of the form $(\hat{X})$ which are solutions of the non-linear partial differential equation with deviating variables,

$$(SPDE) \qquad \partial_z^S \hat{X}(t,z) = A(t, z, ((\partial_t^{k_0} \partial_z^{k_1} \hat{X})(\phi_{k_0,k_1}(t), z))_{(k_0,k_1) \in \mathcal{S}}),$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$. The function $A(t, z, (z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}})$ is assumed to be holomorphic in the variables $(t, z) \in D_{2R(\rho)} \times D_\rho$ and polynomial in the variables $(z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}$. The function A has an expansion of the form

$$A(t, z, (z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) = \sum_{\mathcal{M}_1} b_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}}(t) z^{T_1} \Pi_{(k_0,k_1)\in\mathcal{S}} (z_{k_0,k_1})^{T_{k_0,k_1}},$$

where

$$\mathcal{M}_1 = \{(T_1, (T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N} \times \mathcal{P}\},$$

where $\mathcal{P}$ is a finite subset of $\mathbb{N}^{\mathrm{Card}(\mathcal{S})}$. The functions $\phi_{k_0,k_1}(t)$ are holomorphic on a common neighborhood of the origin in $\mathbb{C}$ such that $\phi_{k_0,k_1}(0) = 0$.

From the Cauchy formula, we know that for all $\nu_1, T_1, (T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}} \geq 0$, we get

$$(CI) \qquad \sup_{t\in\bar{D}_{R(\rho)}} \frac{|\partial_t^{\nu_1} b_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}}(t)|}{(\nu_1!)} = |b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}},\nu_1},$$

with the property that the functions

$$A_K(t, z, (z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) = \sum_{\mathcal{M}_1} |b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}}(t) z^{T_1} \Pi_{(k_0,k_1)\in\mathcal{S}} (z_{k_0,k_1})^{T_{k_0,k_1}},$$

where

$$|b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}}(t) = \sum_{\nu_1\geq 0} |b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}},\nu_1} t^{\nu_1},$$

are holomorphic in the variables t, z in a neighborhood of the origin and polynomial in the variables $(z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}$. We also define the functions,

$$R_{k_0,k_1}(t) = \sum_{l\geq 0} r_{k_0,k_1}^l t^l,$$

where r_{k_0,k_1} are positive real numbers for all $k_0, k_1 \geq 0$.

In the next lemma we describe formal series solutions of $(SPDE)$ in term of differential relations.

Lemma 1 *Rewrite the formal series $(\hat{X})$ in a formal series with respect to the variable z,*

$$\hat{X}(t, z) = \sum_{n_1\geq 0} \frac{\hat{X}_{n_1}(t)}{n_1!} z^{n_1}.$$

Then, $\hat{X}(t, z)$ is a formal series solution of $(SPDE)$ if and only if the following differential relation (SDR) is satisfied

$$\frac{\hat{X}_{m_1+S}(t)}{m_1!} =$$

$$\sum_{\mathcal{M}_1'} \sum_{\mathcal{M}_2} b_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}}(t) \Pi_{(k_0,k_1)\in\mathcal{S}} (\sum_{\mathcal{M}_3} \frac{(\partial_t^{k_0} \hat{X}_{n_1+k_1})(\phi_{k_0,k_1}(t))}{n_1!} \cdots \frac{(\partial_t^{k_0} \hat{X}_{(n_{T_{k_0,k_1}}+k_1)})(\phi_{k_0,k_1}(t))}{(n_{T_{k_0,k_1}})!})$$

for all $m_1 \geq 0$*, where*

$$\mathcal{M}'_1 = \{(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}} \in \mathcal{P}\},$$

$$\mathcal{M}_2 = \{(T_1,(m_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}^{1+\mathrm{Card}(\mathcal{S})} : \sum_{(k_0,k_1)\in\mathcal{S}} m_{k_0,k_1} + T_1 = m_1\}$$

and

$$\mathcal{M}_3 = \{(n_1,\ldots,n_{(T_{k_0,k_1})}) \in \mathbb{N}^{T_{k_0k_1}} : n_1+\cdots+n_{(T_{k_0,k_1})} = m_{k_0k_1}\}.$$

In the next proposition, which is an easy consequence of Lemma 1, we give sufficient conditions for the existence and uniqueness of formal series $(\hat{X})$ that are solutions of $(SPDE)$.

Proposition 2 *We assume that the finite set* $\mathcal{S}$ *satisfies*

$$\mathcal{S} \subset \{(k_0,k_1) \in \mathbb{N}^2 : k_1 \leq S-1\}.$$

Moreover, assume that formal series $\hat{X}_\mu(t)$*, for* $0 \leq \mu \leq S-1$*, are given. Then, there exists a unique sequence* $(\hat{X}_\mu(t))_{\mu\geq 0}$ *of formal series which satisfies the differential recurrence* (SDR)*. As a result, for given formal series* $(\hat{X}_\mu(t))$*, for* $0 \leq \mu \leq S-1$*, there exists a unique formal series* $(\hat{X})$*, solution of* $(SPDE)$*.*

4 Summability of formal solutions of non-linear partial differential equations with deviating variables

In this section, we describe sufficient conditions under which the formal series solutions constructed in the previous section are 1−summable.

Firstly, we recall estimates for high order derivatives of compositions of functions obtained as a consequence of the higher order chain rule introduced by T. Yamanaka in [20], Theorem 2.1.

Lemma 2 *Let* D *be an open disc centered at the origin in* $\mathbb{C}$ *and* G *be an open set in* $\mathbb{C}$*. Let* $v : D \to G$ *and* $w : G \to \mathbb{C}$ *be holomorphic functions, such that* $|\partial_x v(x)| \leq 1$ *for all* $x \in D$*. Then, there exist an open disc* $D_1 \subset D$ *centered at the origin and* $r > 0$*, such that the estimates hold,*

$$|\partial_x^n (w \circ v)(x)| \leq \sum_{a_1+a_2=n} \frac{n!}{a_1!} |(\partial_x^{a_1} w)(v(x))| r^{a_2},$$

for all $x \in D_1$ *and* $n \in \mathbb{N}$*.*

we note that similar computations have been made in the paper [15], pp. 667–668.

The next result gives sufficient conditions for the existence of 1−summable formal series solutions $(\hat{X})$ of $(SPDE)$. In particular, we have to give additional "shrinking" conditions on the functions ϕ_{k_0,k_1} to get summability properties for formal series solutions of $(SPDE)$.

Theorem 1 *Consider the semi-linear partial differential equation with shrinkings,*

$$(SPDE) \qquad \partial_z^S \hat{X}(t,z) = A(t,z,((\partial_t^{k_0}\partial_z^{k_1}\hat{X})(\phi_{k_0,k_1}(t),z))_{(k_0,k_1)\in\mathcal{S}}),$$

where

$$\mathcal{S} = \{(k_0,k_1) \in \mathbb{N}^2 : 2k_0 + k_1 \leq S, (k_0,k_1) \neq (0,S)\}.$$

a) Assume that the formal series $\hat{X}_\mu(t)$, for $0 \le \mu \le S-1$, are $1-$summable with respect to t in a direction $d \in \mathbb{R}$.

b) Assume that their sums $X_\mu(t)$ are holomorphic in a sectorial region $G \subset D_{R(\rho)}$, independent of μ, which has bisecting direction d and opening greater than π and satisfy the following estimates.

For all closed sector $\bar{S} \subset G$ with bisecting direction d, there exist $C, \tilde{K} > 0$ such that

$$\sup_{t \in \bar{S}} |\partial_t^{n_0} X_{m_1}(t)| \le C \tilde{K}^{n_0+m_1} (2n_0)! m_1!,$$

for all $0 \le m_1 \le S-1$ and all $n_0 \in \mathbb{N}$.

c) Assume that the functions $\phi_{k_0,k_1}(t)$ are holomorphic in a common neighborhood O of the origin in $\mathbb{C}$, with $\phi_{k_0,k_1}(0) = 0$. Moreover, assume that $|\partial_t \phi_{k_0,k_1}(t)| \le 1$ on O and that $\phi_{k_0,k_1}(\bar{S}) \subset \bar{S}$, for all closed sector $\bar{S} \subset G$ with bisecting direction d.

Then, the formal series $\hat{X}(t,z)$, solution of $(SPDE)$, is $1-$summable with respect to t in the direction $d \in \mathbb{R}$.

Proof The goal of the proof is to show that the conditions **iii)** in Proposition 1 are fulfilled. First of all, from the paper [1], Theorem 6, we deduce that the composition of a $1-$summable formal series $\hat{X}(t)$ in direction d, with an holomorphic function ϕ which satisfies the assumption **c**), $\hat{X}(\phi(t))$ is again $1-$summable in direction d. From general classical results, see [4], we also know that derivation, addition and product also conserve $1-$summability of formal series. From these results, the assumption **a**) on initial conditions, the relation (SDR) and Proposition 2, we deduce that the formal series $\hat{X}_\mu(t)$ are $1-$summable in the direction $d \in \mathbb{R}$, for all $\mu \ge 0$. Moreover, their sums $X_\mu(t)$ are holomorphic in the sectorial region G.

The rest of the proof will consist in proving that the estimates in **iii)** of Proposition 1 are satisfied for the formal series $\hat{X}_\mu(t)$. We consider now the sequence of positive real numbers

$$w_{n_0,m_1} = \sup_{t \in \bar{S}} |\partial_t^{n_0} X_{m_1}(t)|,$$

for $n_0, m_1 \ge 0$, where $\bar{S}$ is a closed sector in G with bisecting direction d. From the Leibniz formula, we have that

$$A_{k_0,k_1}(t) = \partial_t^{\nu_{k_0,k_1}} \Big(\sum_{\mathcal{M}_3} \frac{(\partial_t^{k_0} X_{n_1+k_1})(\phi_{k_0,k_1}(t))}{n_1!} \cdots \frac{(\partial_t^{k_0} X_{(n_{T_{k_0,k_1}}+k_1)})(\phi_{k_0,k_1}(t))}{(n_{T_{k_0,k_1}})!}\Big)$$

$$= \sum_{\mathcal{M}_3} \sum_{\mathcal{M}_4} \frac{\nu_{k_0k_1}!}{\mu_1! \cdots \mu_{T_{k_0k_1}}!} \frac{\partial_t^{\mu_1}[(\partial_t^{k_0} X_{n_1+k_1})(\phi_{k_0,k_1}(t))]}{n_1!} \cdots \frac{\partial_t^{\mu_{T_{k_0k_1}}}[(\partial_t^{k_0} X_{(n_{T_{k_0,k_1}}+k_1)})(\phi_{k_0,k_1}(t))]}{(n_{T_{k_0,k_1}})!}$$

where

$$\mathcal{M}_3 = \{(n_1, \ldots, n_{T_{k_0k_1}}) \in \mathbb{N}^{T_{k_0k_1}} : n_1 + \cdots + n_{T_{k_0k_1}} = m_{k_0k_1}\},$$

$$\mathcal{M}_4 = \{(\mu_1, \ldots, \mu_{T_{k_0k_1}}) \in \mathbb{N}^{T_{k_0k_1}} : \mu_1 + \cdots + \mu_{T_{k_0k_1}} = \nu_{k_0k_1}\}.$$

By the lemma 2 and the assumption **c**), we deduce that for all k_0, k_1, there exist $r_{k_0,k_1} > 0$ such that,

$$|\partial_t^{\mu_m}[(\partial_t^{k_0} X_{n_m+k_1})(\phi_{k_0,k_1}(t))]| \le \sum_{a_1+a_2=\mu_m} \frac{\mu_m!}{a_1!} |(\partial_t^{a_1+k_0} X_{n_m+k_1}(\phi_{k_0,k_1}(t))| r_{k_0,k_1}^{a_2}.$$

From the latter inequalities, we deduce

$$|A_{k_0,k_1}(t)| \le \sum_{\mathcal{M}_3}\sum_{\mathcal{M}_4} \nu_{k_0k_1}! \prod_{m=1}^{T_{k_0k_1}} \Big(\sum_{a_1+a_2=\mu_m} \frac{|(\partial_t^{a_1+k_0} X_{n_m+k_1}(\phi_{k_0,k_1}(t))|}{a_1! n_m!} r_{k_0,k_1}^{a_2} \Big).$$

If we put

$$B_{k_0,k_1}(t) = \sum_{\mathcal{M}_3} \frac{(\partial_t^{k_0} X_{n_1+k_1})(\phi_{k_0,k_1}(t))}{n_1!} \cdots \frac{(\partial_t^{k_0} X_{(n_{T_{k_0,k_1}}+k_1)})(\phi_{k_0,k_1}(t))}{(n_{T_{k_0,k_1}})!},$$

we get, again from the Leibniz formula,

$$\partial_t^{n_0}(b_{T_1,(T_{k_0k_1})_{(k_0,k_1)\in\mathcal{S}}}(t)\Pi_{(k_0,k_1)\in\mathcal{S}} B_{k_0,k_1}(t)) \\ = \sum_{\mathcal{M}_5} \frac{n_0!}{\nu_1! \Pi_{(k_0,k_1)\in\mathcal{S}}(\nu_{k_0k_1}!)} \partial_t^{\nu_1} b_{T_1,(T_{k_0k_1})_{(k_0,k_1)\in\mathcal{S}}}(t) \Pi_{(k_0,k_1)\in\mathcal{S}} A_{k_0,k_1}(t)$$

where

$$\mathcal{M}_5 = \{(\nu_1,(\nu_{k_0k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}^{1+\mathrm{Card}(\mathcal{S})} : \nu_1 + \sum_{(k_0,k_1)\in\mathcal{S}} \nu_{k_0k_1} = n_0\}.$$

On the other hand, by classical estimates, we have that

$$\frac{n_0!}{a_1!\cdots a_{T_{k_0k_1}}!} \le \frac{(2n_0)!}{(2a_1)!\cdots(2a_{T_{k_0k_1}})!},$$

for all $n_0 \ge 0$ and $a_1 + \cdots + a_{T_{k_0k_1}} \le n_0$. Due to the latter inequalities and (CI), we finally get the inequalities (MRI) for the numbers w_{n_0,m_1},

$$\frac{w_{n_0,m_1}}{(2n_0)! m_1} \\ \le \sum_{\mathcal{M}_1'}\sum_{\mathcal{M}_2}\sum_{\mathcal{M}_5} |b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}},\nu_1} \Pi_{(k_0,k_1)\in\mathcal{S}} \left(\sum_{\mathcal{M}_3}\sum_{\mathcal{M}_4} \Pi_{m=1}^{T_{k_0k_1}} \Big(\sum_{a_1+a_2=\mu_m} \frac{w_{a_1+k_0,n_m+k_1}}{(2a_1)! n_m!} r_{k_0,k_1}^{a_2} \Big) \right)$$

for all $n_0, m_1 \ge 0$, where

$$\mathcal{M}_1' = \{(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}} \in \mathcal{P}\},$$

$$\mathcal{M}_2 = \{(T_1,(m_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}^{1+\mathrm{Card}(\mathcal{S})} : \sum_{(k_0,k_1)\in\mathcal{S}} m_{k_0,k_1} + T_1 = m_1\},$$

$$\mathcal{M}_5 = \{(\nu_1,(\nu_{k_0k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}^{1+\mathrm{Card}(\mathcal{S})} : \nu_1 + \sum_{(k_0,k_1)\in\mathcal{S}} \nu_{k_0k_1} = n_0\},$$

$$\mathcal{M}_3 = \{(n_1,\ldots,n_{(T_{k_0,k_1})}) \in \mathbb{N}^{T_{k_0k_1}} : n_1 + \cdots + n_{(T_{k_0,k_1})} = m_{k_0k_1}\},$$

$$\mathcal{M}_4 = \{(\mu_1,\ldots,\mu_{T_{k_0k_1}}) \in \mathbb{N}^{T_{k_0k_1}} : \mu_1 + \cdots + \mu_{T_{k_0k_1}} = \nu_{k_0k_1}\}.$$

Consider now the auxiliary partial differential equation (CK),

$$\partial_z^S E(t,z) = \sum_{\mathcal{M}_1} |b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}}(t^2) z^{T_1} \Pi_{(k_0,k_1)\in\mathcal{S}} (R_{k_0,k_1}(t^2)\partial_t^{2k_0}\partial_z^{k_1} E(t,z))^{T_{k_0,k_1}},$$

where

$$\mathcal{M}_1 = \{(T_1, (T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}\times\mathcal{P}\},$$

which satisfies the conditions of Theorem CK from the introduction. From Theorem CK, we deduce that there exists a unique function $E(t,z)$, solution of (CK), with E holomorphic with respect to t, z in a neighborhood of the origin, if $\partial_z^k E(t,0)$ are prescribed holomorphic functions in a neighborhood of the origin, for all $0 \le k \le S-1$. In the rest of the proof, we prescribe the functions $\partial_z^k E(t,0)$, $0 \le k \le S-1$, to be of the form $\partial_z^k E(t,0) = V_k(t^2)$, where V_k are holomorphic in a neighborhood of the origin. From the latter assumptions and due to the fact that the coefficients of the equation (CK) are also of the form $\alpha(t^2,z)$, we deduce that the solution E of (CK) can be written in the form $E(t,z) = V(t^2,z)$, where V is holomorphic in a neighborhood of the origin.
We expand $V(t^2,z)$ into a Taylor series in the variables t,z,

$$V(t^2,z) = \sum_{n_0,n_1\ge 0} v_{n_0,n_1}\frac{t^{2n_0}}{(2n_0)!}\frac{z^{n_1}}{n_1!},$$

we get that

$$R_{k_0,k_1}(t^2)\partial_t^{2k_0}\partial_z^{k_1}V(t^2,z) = \sum_{n_0,n_1\ge 0}\Big(\sum_{a_1+a_2=n_0}\frac{v_{a_1+k_0,n_1+k_1}}{(2a_1)!n_1!}r_{k_0,k_1}^{a_2}\Big)t^{2n_0}z^{n_1}.$$

From this, we deduce a multivariate non-linear recurrence (MR) satisfied by the coefficients v_{n_0,n_1},

$$\frac{v_{n_0,m_1}}{(2n_0)!m_1}$$
$$= \sum_{\mathcal{M}_1'}\sum_{\mathcal{M}_2}\sum_{\mathcal{M}_5}|b|_{T_1,(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}},\nu_1}\Pi_{(k_0,k_1)\in\mathcal{S}}\left(\sum_{\mathcal{M}_3}\sum_{\mathcal{M}_4}\Pi_{m=1}^{T_{k_0k_1}}\Big(\sum_{a_1+a_2=\mu_m}\frac{v_{a_1+k_0,n_m+k_1}}{(2a_1)!n_m!}r_{k_0,k_1}^{a_2}\Big)\right)$$

for all $n_0, m_1 \ge 0$, where

$$\mathcal{M}_1' = \{(T_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}} \in \mathcal{P}\},$$
$$\mathcal{M}_2 = \{(T_1,(m_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}^{1+\mathrm{Card}(\mathcal{S})} : \sum_{(k_0,k_1)\in\mathcal{S}} m_{k_0,k_1} + T_1 = m_1\},$$
$$\mathcal{M}_5 = \{(\nu_1,(\nu_{k_0k_1})_{(k_0,k_1)\in\mathcal{S}}) \in \mathbb{N}^{1+\mathrm{Card}(\mathcal{S})} : \nu_1 + \sum_{(k_0,k_1)\in\mathcal{S}}\nu_{k_0k_1} = n_0\},$$
$$\mathcal{M}_3 = \{(n_1,\dots,n_{(T_{k_0,k_1})}) \in \mathbb{N}^{T_{k_0k_1}} : n_1+\dots+n_{(T_{k_0,k_1})} = m_{k_0k_1}\},$$
$$\mathcal{M}_4 = \{(\mu_1,\dots,\mu_{T_{k_0k_1}}) \in \mathbb{N}^{T_{k_0k_1}} : \mu_1+\dots+\mu_{T_{k_0k_1}} = \nu_{k_0k_1}\}.$$

From the holomorphy of V, we deduce the following estimates for the coefficients v_{n_0,m_1}. There exist $C,\tilde{K} > 0$, such that

$$(Es) \qquad |v_{n_0,m_1}| \le C\tilde{K}^{n_0+m_1}(2n_0)!m_1!,$$

for all $n_0, m_1 \ge 0$.

From Theorem CK, we see that the sequence $(v_{n_0,m_1})_{n_0,m_1}$ is uniquely determined by the sequences $(v_{n_0,m_1})_{n_0,m_1}$, $0 \le m_1 \le S-1$. We choose now the latter sequences as follows.

$$v_{n_0,m_1} := w_{n_0,m_1},$$

for all $0 \leq m_1 \leq S-1$, $n_0 \geq 0$.

From the assumption **b**) of Theorem 1, we deduce that the series

$$E(t,0), \dots, \partial_z^{S-1} E(t,0)$$

are convergent in a neighborhood of the origin. Moreover, due to the relations (MRI) and (MR), we deduce that

$$w_{n_0,m_1} \leq v_{n_0,m_1},$$

for all $n_0, m_1 \geq 0$.

From the estimates (Es), we finally deduce that there exist $C, \tilde{K} > 0$, such that the sequence w_{n_0,m_1} satisfies

$$w_{n_0,m_1} \leq C\tilde{K}^{n_0+m_1}(2n_0)!m_1!,$$

for all $n_0, m_1 \geq 0$.

Finally, we conclude that the condition **iii)** *in Proposition 1 is satisfied for the formal series* $\hat{X}_{n_1}(t)$, so that the formal series $\hat{X}(t,z)$, solution of $(SPDE)$, is $1-$summable with respect to t in the direction $d \in \mathbb{R}$. □

5 Application to non-linear advanced-argument partial differential equations

Consider the following Moebius transformation,

$$\phi(t) = \frac{Qt}{1+Ht},$$

defined on $\mathbb{C} \setminus \{-1/H\}$, with $0 < Q < 1$ and $H \in \mathbb{R}_+$. It is easy to see that the image $\phi(D)$ of a line D which goes through the point $t = 0$ and is not equal to $\mathbb{R}$ in $\mathbb{C}$, is a circle which is tangent to D and goes through the points $t = 0$ and $t = Q/H$. Moreover, there exists a neighborhood O of the origin in $\mathbb{C}$ such that

$$|\phi(t)| \leq |t| \quad , \quad |\partial_t \phi(t)| \leq 1,$$

for all $t \in O$. Let G be a sectorial region $G \subset O$ which has bisecting direction $d = 0$ and opening greater than π. We deduce that $\phi(\bar{S}) \subset \bar{S}$ for all closed sector $\bar{S} \subset G$ with bisecting direction $d = 0$. So that the function ϕ satisfies the assumption **c**) of Theorem 1.

The theorem 1 gives sufficient conditions for $1-$summability in the direction $d = 0$ of formal solutions $\hat{X}(t,z)$ of the form $(\hat{X})$ to partial differential equations with deviating variables of the form,

$$(SPDE) \qquad \partial_z^S \hat{X}(t,z) = A(t,z,((\partial_t^{k_0}\partial_z^{k_1}\hat{X})(\frac{Q_{k_0,k_1}t}{1+H_{k_0,k_1}t}, z))_{(k_0,k_1)\in\mathcal{S}}),$$

where the function $A(t,z,(z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}})$ is assumed to be holomorphic in the variables (t,z) in a neighborhood of the origin in $\mathbb{C}^2$ and polynomial in the variables $(z_{k_0,k_1})_{(k_0,k_1)\in\mathcal{S}}$, and $0 < Q_{k_0,k_1} < 1$ and $H_{k_0,k_1} \in \mathbb{R}_+$ for all k_0, k_1.

From this, we get $1-$summable formal solutions $\hat{u}(x,z_1,z_2) = \hat{X}(1/x,z_1,z_2)$ in direction $d = 0$ of so-called *non-linear advanced-argument partial differential equations* of the form

$$(DD) \qquad \partial_z^S \hat{u}(x,z) = B(x,z,((\partial_x^{k_0}\partial_z^{k_1}\hat{u})(\frac{x}{Q_{k_0,k_1}} + \frac{H_{k_0,k_1}}{Q_{k_0,k_1}}, z))_{(k_0,k_1)\in\mathcal{S}}).$$

From the classical theory, see [4], one deduces asymptotic properties of actual solutions of (DD), $u(x,z) = X(1/x,z)$,where X is the $1-$sum of $\hat{X}$. In particular, there exist constants $C, M > 0$ such that

$$|u(x,z) - \sum_{p=0}^{n-1} \frac{X_p(z)}{p!}(1/x)^p| \leq CM^n n! |1/x|^n$$

for all $n \geq 1$ and for large x in a sector having bisecting direction $d = 0$ and z in a neighborhood of the origin in $\mathbb{C}$.

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