Spacetime causality in the study of the Hankel transform

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Abstract

We study Hilbert space aspects of the Klein-Gordon equation in two-dimensional spacetime. We associate to its restriction to a spacelike wedge a scattering from the past light cone to the future light cone, which is then shown to be (essentially) the Hankel transform of order zero. We apply this to give a novel proof, solely based on the causality of this spatio-temporal wave propagation, of the theorem of de Branges and V. Rovnyak concerning Hankel pairs with a support property. We recover their isometric expansion as an application of Riemann's general method for solving Cauchy-Goursat problems of hyperbolic type.

keywords: Klein-Gordon equation; Hankel and Fourier transforms; Scattering.

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1 Introduction

We work in two-dimensional spacetime with metric $c^2dt^2 - dx^2$. We shall use units such that $c = 1$. Points are denoted $P = (t, x)$. And the d'Alembertian operator $\Box$ is $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$. We consider the Klein-Gordon equation (with $m = 1$, $\hbar = 1$; actually we shall only study the classical wave field, no quantization is involved in this paper):

$$\Box\phi + \phi = 0 \tag{1}$$

We have an energy density:[1]

$$\mathcal{E} = |\phi|^2 + \left|\frac{\partial \phi}{\partial x}\right|^2 + \left|\frac{\partial \phi}{\partial t}\right|^2 \tag{2}$$

which gives a conserved quantity:

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{E}(\phi)(t,x)\;dx\;, \tag{3}$$

in the sense that if the Cauchy data at time $t = 0$ has $E < \infty$ then E is finite (and constant. . .) at all times (past and future). We shall mainly work with such finite energy solutions. Although we failed in locating a reference for the following basic observation, we can not imagine it to be novel:

Theorem 1. *If ϕ is a finite energy solution to the Klein-Gordon equation then:*

$$\lim_{t\to+\infty}\int_{|x|>t} \mathcal{E}(\phi)(t,x)\;dx = 0\;.$$

Obviously this would be completely wrong for the zero mass equation. We shall give a (simple) self-contained proof, because it is the starting point of all that we do here. Let us nevertheless state that the result follows immediately from Hörmander's fine pointwise estimates ([8, 9]; see also the paper of S. Klainerman [11] and the older papers of S. Nelson [14, 15].) I shall not reproduce the strong pointwise results of Hörmander, as they require notations and preliminaries. Let me simply mention that Hörmander's Theorem 2.1 from [8] can be applied to the positive and negative frequency parts of a solution with Cauchy data which is gaussian times polynomial. So theorem 1 holds for them, and it holds then in general, by an approximation argument.

The energy conservation follows from:

$$\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x}\mathcal{P} = 0 \qquad \text{with} \qquad \mathcal{P} = -\frac{\partial \phi}{\partial x}\overline{\frac{\partial \phi}{\partial t}} - \overline{\frac{\partial \phi}{\partial x}}\frac{\partial \phi}{\partial t} \tag{4}$$

If we apply Gauss' theorem to the triangle with vertices $O = (0,0)$, $A = (t,t)$, $B = (t,-t)$, we obtain ($t > 0$):

$$\int_{|x|\le t} \mathcal{E}(\phi)(t,x)dx = \int_{-t}^{0} (|\phi(|x|,x)|^2 + \left|\frac{d}{dx}\phi(|x|,x)\right|^2)\,dx + \int_{0}^{t} (|\phi(x,x)|^2 + \left|\frac{d}{dx}\phi(x,x)\right|^2)\,dx$$

[1] as this paper is principally of a mathematical nature, we do not worry about an overall $\frac{1}{2}$ factor.

This proves that $\int_{|x|>t}\mathcal{E}(\phi)(t,x)\,dx$ decreases as $t\to+\infty$. It shows also that theorem 1 is equivalent to:

$$E=\frac{1}{2\pi}\int_{-\infty}^{0}(|\phi(|x|,x)|^2+\left|\frac{d}{dx}\phi(|x|,x)\right|^2)\,dx+\frac{1}{2\pi}\int_{0}^{\infty}(|\phi(x,x)|^2+\left|\frac{d}{dx}\phi(x,x)\right|^2)\,dx \quad (5)$$

Otherwise stated, there is a unitary representation of ϕ on the future light cone. Here is now the basic idea: as solutions to hyperbolic equations propagate causally, equation (5) gives a unitary representation from the Hilbert space of Cauchy data at time $t=0$ with support in $x\geq 0$ to the Hilbert space of functions $p(v)=\phi(v,v)$ on $[0,+\infty[$ with squared norm $\frac{1}{2\pi}\int_0^\infty(|p(v)|^2+|p'(v)|^2)\,dv$. Instead of Cauchy data vanishing for $x<0$, it will be useful to use Cauchy data invariant under $(t,x)\to(-t,-x)$. Then p will be considered as an even, and p' as an odd, function, and $\frac{1}{2\pi}\int_0^\infty(|p(v)|^2+|p'(v)|^2)\,dv$ will be $\frac{1}{2}E(\phi)$, for $\phi(t,x)=\phi(-t,-x)$. We can also consider the past values $g(u)=\phi(-u,u)$, $t=-u$, $x=u$, $0\leq u<\infty$. So there is a unitary map from such g's to the p's:

Theorem 2. *Let $g(u)$, $u>0$, and $p(v)$, $v>0$ be such that $\int_0^\infty|g(u)|^2+|g'(u)|^2du<\infty$, $\int_0^\infty|p(v)|^2+|p'(v)|^2dv<\infty$. The necessary and sufficient condition for $A(r)=\sqrt{r}\,g(\frac{r^2}{2})$ and $B(s)=-\sqrt{s}\,p'(\frac{s^2}{2})$ to be Hankel transforms of order zero of one another ($A(r)=\int_0^\infty\sqrt{rs}J_0(rs)B(s)\,ds$) is for g and p to be the values on the past and future boundaries of the Rindler wedge $0<|t|<x$ of a finite energy solution $\phi(t,x)$ of the Klein-Gordon equation ($g(u)=\phi(-u,u)$, $p(v)=\phi(v,v)$.) For any $a>0$ the vanishing on $0<x<2a$ of the Cauchy data for $\phi(t,x)$ at $t=0$ is the necessary and sufficient condition for the simultaneous vanishing of $g(u)$ for $0<u<a$ and of $p(v)$ for $0<v<a$.*

The statements relative to the support properties are corollaries to the relativistic causality of the propagation of solutions to the Klein-Gordon equation. Regarding the function B, if $k(v)=-p'(v)$ vanishes identically on $(0,a)$, then $p(v)$ is constant there, and this constant has to be 0 if $g(u)$ is also identically zero on $(0,a)$: indeed the finite energy solution ϕ is continuous on spacetime (this follows from the well-known explicit formulas (32)). We employed temporarily $A(r)=\sqrt{r}\,g(\frac{r^2}{2})$ and $B(s)=-\sqrt{s}\,p'(\frac{s^2}{2})$ in the statement of Theorem 2 in order to express the matter with the zero order Hankel transform. It proves more natural to stay with $g(u)$ and $k(v)=-p'(v)$. They are connected by the integral formula: $g(u)=\int_0^\infty J_0(2\sqrt{uv})k(v)\,dv$, so this motivates the definition of the $\mathcal{H}$ transform:

$$\mathcal{H}(f)(x)=\int_0^\infty J_0(2\sqrt{xy})f(y)\,dy \quad (6)$$

The $\mathcal{H}$ transform is a unitary operator on $L^2(0,+\infty;dx)$ which is self-reciprocal. As is well-known $\sqrt{x}e^{-\frac{1}{2}x^2}$ is an invariant function for the Hankel transform of order zero, so, for the $\mathcal{H}$ transform we have e^{-x} as invariant function in $L^2(0,\infty;dx)$. The $\mathcal{H}$ operator is "scale-reversing": by this we mean that $\mathcal{H}(f(\lambda y))(x) = \lambda^{-1}\mathcal{H}(f)(\lambda^{-1}x)$, or, equivalently, that the operator $\mathcal{H}\cdot I$ is scale invariant, where I is the unitary operator $f(x)\mapsto \frac{1}{x}f(\frac{1}{x})$. As we explain later, $\mathcal{H}$ is the unique scale-reversing operator on $L^2(0,\infty;dx)$ having among its self-reciprocal functions the function e^{-x}. Let us restate Theorem 2 as it applies to $\mathcal{H}$:

Theorem 3. *Let $\phi(t,x)$ be a finite energy solution of the Klein-Gordon equation. Let $g(u)=\phi(-u,u)$ for $u>0$ and $p(v)=\phi(v,v)$ for $v>0$ be the values taken by ϕ on the past, respectively future, boundaries of the Rindler wedge $0<|t|<x$. Then $k(v)=-p'(v)$ is the $\mathcal{H}$ transform of $g(u)$: $k(v)=\int_0^\infty J_0(2\sqrt{uv})g(u)\,du$. For any $a>0$ the vanishing for $0<x<2a$, $t=0$, of the Cauchy data for $\phi(t,x)$ is the necessary and sufficient condition for the simultaneous vanishing of $g(u)$ for $0<u<a$ and $p(v)$ for $0<v<a$.*

In this manner a link has been established between the relativistic causality and a mathematical theorem of de Branges [3], and V. Rovnyak [16] (see further [17]). They proved an explicit isometric representation of $L^2(0,+\infty;dx)$ onto $L^2(0,+\infty;dy)\oplus L^2(0,+\infty;dy)$, $h\mapsto(f,g)$, such that the zero order Hankel transform on $L^2(0,+\infty;dx)$ is conjugated to the simple map $(f,g)\to(g,f)$, and such that the pair $(f(y),g(y))$ vanishes identically on $(0,a)$ if and only $h(x)$ and its Hankel transform of order zero both identically vanish on $(0,a)$. Their formulas ((5) and (7) of [3] should be corrected to read as (3) and (2) of [16]) are:

$$f(y) = \int_y^\infty h(x)J_0(y\sqrt{x^2-y^2})\sqrt{xy}\,dx \tag{7a}$$

$$g(y) = h(y) - \int_y^\infty h(x)y\,\frac{J_1(y\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}}\sqrt{xy}\,dx \tag{7b}$$

$$h(x) = g(x) + \int_0^x f(y)J_0(y\sqrt{x^2-y^2})\sqrt{xy}\,dy - \int_0^x g(y)\frac{J_1(y\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}}\sqrt{xy}\,y\,dy \tag{7c}$$

$$\int_0^\infty |h(x)|^2\,dx = \int_0^\infty (|f(y)|^2+|g(y)|^2)\,dy \tag{7d}$$

We shall give an independent, self-contained proof, that these formulas are mutually compatible and have the stated relation to the Hankel tranform of order zero. The main underlying idea has been to realize the Hankel transform of order zero as a scattering related to a causal propagation of waves. The support condition initially considered by de Branges and Rovnyak has turned out to be related to relativistic causality, and the looked-after scattering has been realized as the transition from the past to the future boundary of the Rindler wedge $0 < |t| < x$. Also, in the technique of proof we apply, in a perhaps unusual manner, the classical Riemann method ([10, IV§1], [6, VI§5]) from the theory of hyperbolic equations. Let us reformulate here the isometric expansion of de Branges-Rovnyak into a version which applies to the $\mathcal{H}$ transform. For this we write, for $x > 0$,

$$h(x) = \sqrt{x}\, k(\frac{x^2}{2}), \qquad f(x) = \sqrt{x} F(x^2), \qquad g(x) = \sqrt{x} G(x^2)$$

Then the equations above become:

$$F(x) = \int_{x/2}^{\infty} J_0(\sqrt{x(2v-x)}) k(v)\, dv \tag{8a}$$

$$G(x) = k(\frac{x}{2}) - \int_{x/2}^{\infty} x\, \frac{J_1(\sqrt{x(2v-x)})}{\sqrt{x(2v-x)}}\, k(v)\, dv \tag{8b}$$

$$k(v) = G(2v) + \frac{1}{2}\int_0^{2v} J_0(\sqrt{x(2v-x)}) F(x)\, dx - \frac{1}{2}\int_0^{2v} x\, \frac{J_1(\sqrt{x(2v-x)})}{\sqrt{x(2v-x)}} G(x)\, dx \tag{8c}$$

$$\int_0^{\infty} 2|k(v)|^2\, dv = \int_0^{\infty} (|F(x)|^2 + |G(x)|^2)\, dx \tag{8d}$$

The de Branges Rovnyak theorem is thus the equivalence between equations (8a), (8b) and (8c), the validity of (8d), the fact that the pair (F, G) is identically zero on $(0, 2a)$ if and only if both k and $\mathcal{H}(k)$ vanish identically on $(0, a)$, and finally the fact that permuting F and G is equivalent to $k \leftrightarrow \mathcal{H}(k)$.

It proves convenient to work with the first order “Dirac” system:

$$\frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial x} = +\phi \tag{9a}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} = -\psi \tag{9b}$$

Let us write $\begin{bmatrix} \psi(0,x) \\ \phi(0,x) \end{bmatrix} = \begin{bmatrix} G(x) \\ F(x) \end{bmatrix}$. We shall use $K(\psi, \phi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (|F(x)|^2 + |G(x)|^2)\, dx$ as the Hilbert space (squared) norm. We shall require $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \psi}{\partial x}$ to be in L^2 at $t = 0$ (then ϕ and ψ are continuous on space-time). Our previous $E(\phi)$ is not invariant under Lorentz boosts: it is only the first component of a Lorentz vector $(E(\phi), P(\phi))$ (see equation (17) for the expression of P). And it turns out that in fact $K(\psi, \phi) = E(\phi) - P(\phi) = E(\psi) + P(\psi)$. The point is that in order to define an action of the Lorentz group on the solutions of the Dirac system it is necessary to rescale in opposite ways ψ and ϕ. When done symmetrically, K then becomes an invariant under the Lorentz boosts. This relativistic covariance of the spinorial quantity $\begin{bmatrix} \psi \\ \phi \end{bmatrix}$ is important for the proof of the next theorem:

Theorem 4. *Let F and G be two functions with $\int_0^\infty |F|^2+|F'|^2+|G|^2+|G'|^2dx<\infty$. Let $\left[\begin{smallmatrix}\psi\\ \phi\end{smallmatrix}\right]$ be the unique solution in the Rindler wedge $x>|t|>0$ of the first order system:*

$$\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial x}=+\phi \tag{10a}$$

$$\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}=-\psi \tag{10b}$$

with Cauchy data $\phi(0,x)=F(x)$, $\psi(0,x)=G(x)$. The boundary values:

$$g(u)=\phi(-u,u)\quad(u>0),\qquad \text{and}\qquad k(v)=\psi(v,v)\quad(v>0),$$

verify $\int_0^\infty |g(u)|^2+|g'(u)|^2du<\infty$, $\int_0^\infty |k(v)|^2+|k'(v)|^2dv<\infty$ and are a $\mathcal{H}$ transform pair. For any $a>0$ the identical vanishing of $F(x)$ and $G(x)$ for $0<x<2a$ is equivalent to the identical vanishing of $g(u)$ for $0<u<a$ and of $k(v)$ for $0<v<a$. All $\mathcal{H}$ pairs with $\int_0^\infty |g|^2+|g'|^2du<\infty$, $\int_0^\infty |k|^2+|k'|^2dv<\infty$ are obtained in this way. The functions $F(x)$, $G(x)$, $g(u)$ and $k(v)$ are related by the following formulas:

$$F(x)=\int_{x/2}^\infty J_0(\sqrt{x(2v-x)})k(v)\,dv=g(\frac{x}{2})-\int_{x/2}^\infty x\,\frac{J_1(\sqrt{x(2u-x)})}{\sqrt{x(2u-x)}}\,g(u)\,du \tag{10c}$$

$$G(x)=k(\frac{x}{2})-\int_{x/2}^\infty x\,\frac{J_1(\sqrt{x(2v-x)})}{\sqrt{x(2v-x)}}\,k(v)\,dv=\int_{x/2}^\infty J_0(\sqrt{x(2u-x)})g(u)\,du \tag{10d}$$

$$g(u)=F(2u)+\frac{1}{2}\int_0^{2u} J_0(\sqrt{x(2u-x)})G(x)\,dx-\frac{1}{2}\int_0^{2u} x\,\frac{J_1(\sqrt{x(2u-x)})}{\sqrt{x(2u-x)}}F(x)\,dx \tag{10e}$$

$$k(v)=G(2v)+\frac{1}{2}\int_0^{2v} J_0(\sqrt{x(2v-x)})F(x)\,dx-\frac{1}{2}\int_0^{2v} x\,\frac{J_1(\sqrt{x(2v-x)})}{\sqrt{x(2v-x)}}G(x)\,dx \tag{10f}$$

$$\int_0^\infty 2|k(v)|^2\,dv=\int_0^\infty (|F(x)|^2+|G(x)|^2)\,dx=\int_0^\infty 2|g(u)|^2\,du \tag{10g}$$

$$k(v)=\int_0^\infty J_0(2\sqrt{uv})g(u)\,du\qquad\qquad g(u)=\int_0^\infty J_0(2\sqrt{uv})k(v)\,dv \tag{10h}$$

The integrals converge as improper Riemann integrals.

The Lorentz boost parameter can serve as "time" as K is conserved under it. In this manner going-over from ϕ on the past light cone to ψ on the future light cone becomes a scattering. We shall explain its formulation in the Lax-Phillips [12] terminology.

In conclusion we can say that this paper identifies the unique scale reversing operator $\mathcal{H}$ on $L^2(0,+\infty;\,dx)$ such that e^{-x} is self-reciprocal as the scattering from the past (positive x)-light-cone to the future (positive x)-light-cone for finite energy solutions of the Dirac-Klein-Gordon equation in two-dimensional space-time. Some further observations and remarks will be found in the concluding section of the paper. The operator $\mathcal{H}$, which is involved in some functional equations of number theory, is studied further by the author in [5].

2 Plane waves

Throughout this paper we shall use the following light cone coordinates, which are positive on the right wedge:

$$v = \frac{x+t}{2} \qquad u = \frac{x-t}{2} \tag{11a}$$

$$x = u + v \qquad t = -u + v \qquad t^2 - x^2 = 4(-u)v \qquad \Box = -\frac{\partial^2}{\partial u \partial v} \tag{11b}$$

We write sometimes $\phi(t,x) = \phi[u,v]$.

Let us begin the proof of Theorem 1. We can build a solution to the Klein-Gordon equation by superposition of plane waves:

$$\phi(t,x) = \int_{-\infty}^{+\infty} e^{+i(\lambda u - \frac{1}{\lambda} v)} \alpha(\lambda)\, d\lambda = \int_{-\infty}^{+\infty} e^{-i(\omega t - \mu x)} \alpha(\lambda)\, d\lambda \tag{12a}$$

$$\text{with} \qquad \omega = \frac{1}{2}(\lambda + \frac{1}{\lambda}), \qquad \mu = \frac{1}{2}(\lambda - \frac{1}{\lambda}) \tag{12b}$$

The full range $-\infty < \lambda < +\infty$ allows to keep track simultaneously of the "positive frequency" ($\lambda > 0$, $\omega \geq 1$), and "negative frequency" ($\lambda < 0$, $\omega \leq -1$) parts.

At first we only take α to be a smooth, compactly supported function of λ, vanishing identically in a neighborhood of $\lambda = 0$. Then the corresponding ϕ is a smooth, finite energy solution of the Klein Gordon equation. Let us compute this energy. At $t = 0$ we have

$$\phi(0,x) = \int_{-\infty}^{+\infty} e^{+i\mu x} \alpha(\lambda)\, d\lambda \qquad \frac{\partial \phi}{\partial t}(0,x) = -i \int_{-\infty}^{+\infty} e^{+i\mu x} \frac{1}{2}(\lambda + \frac{1}{\lambda}) \alpha(\lambda)\, d\lambda$$

So we will apply Plancherel's theorem, after the change of variable $\lambda \to \mu$. We must be careful that if λ is sent to μ, then $\lambda' = -\frac{1}{\lambda}$, is too. Let $\lambda_1 > 0$ and $\lambda_2 < 0$ be the ones being sent to μ. Let us also define:

$$a(\mu) = \frac{\alpha(\lambda_1)}{\frac{1}{2}(1 + \frac{1}{\lambda_1^2})}, \qquad b(\mu) = \frac{\alpha(\lambda_2)}{\frac{1}{2}(1 + \frac{1}{\lambda_2^2})}$$

Then:

$$\phi(0,x) = \int_{-\infty}^{+\infty} e^{+i\mu x} (a(\mu) + b(\mu)) d\mu \qquad \frac{\partial \phi}{\partial t}(0,x) = -i \int_{-\infty}^{+\infty} e^{+i\mu x} \frac{1}{2}(\lambda_1 + \frac{1}{\lambda_1})(a(\mu) - b(\mu)) d\mu$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(|\phi|^2+|\frac{\partial}{\partial x}\phi|^2)dx = \int_{-\infty}^{+\infty}|a(\mu)+b(\mu)|^2(1+\mu^2)\,d\mu$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\frac{\partial}{\partial t}\phi|^2dx = \int_{-\infty}^{+\infty}|a(\mu)-b(\mu)|^2\left(\frac{1}{2}(\lambda_1+\frac{1}{\lambda_1})\right)^2\,d\mu$$

Observing that $1+\mu^2 = \left(\frac{1}{2}(\lambda_1+\frac{1}{\lambda_1})\right)^2 = \left(\frac{1}{2}(\lambda_2+\frac{1}{\lambda_2})\right)^2$, this gives

$$E(\phi) = 2\int_{-\infty}^{+\infty}(|a(\mu)|^2+|b(\mu)|^2)\lambda_1^2\left(\frac{1}{2}(1+\frac{1}{\lambda_1^2})\right)^2\,d\mu$$
$$= 2\int_0^{\infty}|a(\mu)|^2\lambda_1^2\left(\frac{1}{2}(1+\frac{1}{\lambda_1^2})\right)^3\,d\lambda_1 + 2\int_{-\infty}^{0}|b(\mu)|^2\lambda_2^2\left(\frac{1}{2}(1+\frac{1}{\lambda_2^2})\right)^3\,d\lambda_2$$
$$= 2\int_0^{\infty}|\alpha(\lambda)|^2\frac{1}{2}(1+\lambda^2)\,d\lambda + 2\int_{-\infty}^{0}|\alpha(\lambda)|^2\frac{1}{2}(1+\lambda^2)\,d\lambda$$

$$E(\phi) = \int_{-\infty}^{+\infty}|\alpha(\lambda)|^2(1+\lambda^2)\,d\lambda \tag{13}$$

Let us now compute the energy on the future light cone. We write $g(u)=\phi(-u,u)$, $p(v)=\phi(+v,v)$. We have:

$$g(u) = \int_{-\infty}^{+\infty}e^{+i\lambda u}\alpha(\lambda)\,d\lambda \tag{14}$$

Let $\alpha=\alpha_+ + \alpha_-$ be the decomposition of α as the sum of α_+, belonging to the Hardy space of the upper half-plane $\Im(\lambda)>0$ and of α_-, belonging to the Hardy space of the lower half-plane. We have:

$$\frac{1}{2\pi}\int_{-\infty}^{0}|g(u)|^2\,du = \int_{-\infty}^{+\infty}|\alpha_+(\lambda)|^2\,d\lambda \tag{15a}$$

We must be careful regarding $\frac{1}{2\pi}\int_{-\infty}^{0}|g'(u)|^2\,du$. We have $g'(u)\,\mathbf{1}_{u<0}(u)-g(0)\delta(u) = \int_{-\infty}^{+\infty}e^{+i\lambda u}i\lambda\alpha_+(\lambda)\,d\lambda$ as a distribution identity so

$$\frac{1}{2\pi}\int_{-\infty}^{0}|g'(u)|^2\,du = \int_{-\infty}^{+\infty}\left|\lambda\alpha_+(\lambda)+\frac{g(0)}{2\pi i}\right|^2\,d\lambda$$

On the other hand $(t\alpha(t))_+(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{t\alpha(t)\,dt}{t-\lambda} = \lambda\alpha_+(\lambda)+\frac{g(0)}{2\pi i}$ so the formula is:

$$\frac{1}{2\pi}\int_{-\infty}^{0}|g'(u)|^2\,du = \int_{-\infty}^{+\infty}|(t\alpha)_+(\lambda)|^2\,d\lambda \tag{15b}$$

Similarly, as $p(v)=\int_{-\infty}^{+\infty}e^{+i\lambda v}\alpha(-\frac{1}{\lambda})\frac{1}{\lambda^2}\,d\lambda$, and defining $\beta(\lambda)=\alpha(-\frac{1}{\lambda})\frac{1}{\lambda^2}$ we obtain:

$$\frac{1}{2\pi}\int_0^{\infty}(|p(v)|^2+|p'(v)|^2)dv = \int_{-\infty}^{+\infty}(|\beta_-(\lambda)|^2+|(t\beta)_-(\lambda)|^2)\,d\lambda$$

Now, from $t\beta(t) = \frac{1}{t}\alpha(\frac{-1}{t})$ it is seen that $(t\beta)_-(\lambda) = \frac{1}{\lambda}\alpha_-(\frac{-1}{\lambda})$, so $\int_{-\infty}^{+\infty} |(t\beta)_-(\lambda)|^2\, d\lambda = \int_{-\infty}^{+\infty} |\alpha_-(\lambda)|^2\, d\lambda$. And, as $t\alpha(t) = \frac{1}{t}\beta(\frac{-1}{t})$ one has in a similar manner $\int_{-\infty}^{+\infty} |(t\alpha)_+(\lambda)|^2\, d\lambda = \int_{-\infty}^{+\infty} |\beta_+(\lambda)|^2\, d\lambda$. Combining, we get

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{0} (|g(u)|^2 + |g'(u)|^2)\, du + \frac{1}{2\pi}\int_{0}^{\infty} (|p(v)|^2 + |p'(v)|^2) dv \\ &= \int_{-\infty}^{+\infty} (|\alpha_+(\lambda)|^2 + |\beta_+(\lambda)|^2 + |\beta_-(\lambda)|^2 + |\alpha_-(\lambda)|^2)\, d\lambda\end{aligned}$$

and as the two Hardy spaces are mutually perpendicular in $L^2(-\infty, +\infty; d\lambda)$ we finally obtain:

$$\int_{-\infty}^{+\infty} |\alpha(\lambda)|^2 (1+\lambda^2)\, d\lambda$$

as the energy on the future light cone.

So, with this, the theorem that $E(\phi)$ is entirely on the future light cone is proven for the ϕ's corresponding to α's which are smooth and compactly supported away from $\lambda = 0$. Obviously the Cauchy data for such ϕ's is a dense subspace of the full initial data Hilbert space. As energy is conserved as $t \to \infty$, the fact that $\lim_{t\to\infty} \int_{|x|>t} \mathcal{E}(\phi) dx = 0$ holds for all finite energy ϕ's then follows by approximation. Furthermore we see that a finite energy solution is uniquely written as a wave packet:

$$\phi(t,x) = \int_{-\infty}^{+\infty} e^{+i(\lambda u - \frac{1}{\lambda} v)} \alpha(\lambda)\, d\lambda \qquad E(\phi) = \int_{-\infty}^{+\infty} (1+\lambda^2)|\alpha(\lambda)|^2\, d\lambda < \infty \tag{16}$$

At this stage Theorem 1 is established.

When studying the Klein-Gordon equation in the right wedge $x > 0$, $|t| < x$, we can arbitrarily extend the Cauchy data to $x < 0$. Setting it to 0 on $x < 0$, however, will be compatible with the finite energy condition only if $\phi(0, 0^+) = 0$. If this is the case then this choice makes $g(u)$ vanish for $u < 0$ and $p(v)$ vanish for $v < 0$, which is a Hardy space constraint on α, in fact it means that α belongs to the Hardy space of the lower half-plane. Another manner to extend the Cauchy data to $x < 0$ is to make it invariant under the PT operation $(t,x) \to (-t,-x)$. This has the advantage, if $\int_0^\infty |\phi(0,x)|^2 + |\frac{\partial}{\partial x}\phi(0,x)|^2\, dx < \infty$, to produce Cauchy data of finite energy on the full line $-\infty < x < \infty$. The condition on α is to be even. We shall often use this convention when studying the Klein-Gordon equation in the right wedge.

3 Energy and momentum

The momentum density $\mathcal{P} = -\frac{\partial\phi}{\partial x}\overline{\frac{\partial\phi}{\partial t}} - \overline{\frac{\partial\phi}{\partial x}}\frac{\partial\phi}{\partial t}$ also satisfies a conservation law:

$$\frac{\partial}{\partial t}\mathcal{P} + \frac{\partial}{\partial x}\left(-|\phi|^2 + \left|\frac{\partial\phi}{\partial x}\right|^2 + \left|\frac{\partial\phi}{\partial t}\right|^2\right) = 0$$

So

$$P = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\partial\phi}{\partial x}\overline{\frac{\partial\phi}{\partial t}} + \overline{\frac{\partial\phi}{\partial x}}\frac{\partial\phi}{\partial t}\right)dx \tag{17}$$

is also a conserved quantity. We have:

$$E - P = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(|\phi|^2 + \left|\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right|^2\right)dx \tag{18a}$$

$$E + P = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(|\phi|^2 + \left|\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial t}\right|^2\right)dx \tag{18b}$$

Applying Gauss' theorem to $\mathcal{P}$ we obtain for $t > 0$:

$$\int_{|x|\leq t}\mathcal{P}(\phi)(t,x)dx = \int_{-t}^{0}(-|\phi(-x,x)|^2 + \left|\frac{d}{dx}\phi(-x,x)\right|^2)\,dx + \int_0^t(|\phi(x,x)|^2 - \left|\frac{d}{dx}\phi(x,x)\right|^2)\,dx$$

The integral of $|\mathcal{P}|$ for $|x| > t$ tends to zero for $t \to +\infty$ as it is bounded above by the one for $\mathcal{E}$. So:

$$P = \frac{1}{2\pi}\int_{-\infty}^{0}(-|g(u)|^2 + |g'(u)|^2)\,du + \frac{1}{2\pi}\int_0^{\infty}(|p(v)|^2 - |p'(v)|^2)\,dv \tag{19}$$

with, again, $g(u) = \phi(-u,u)$, $p(v) = \phi(+v,v)$. Hence:

$$E - P = \frac{1}{\pi}\int_{-\infty}^{0}|g(u)|^2\,du + \frac{1}{\pi}\int_0^{\infty}|p'(v)|^2\,dv \tag{20a}$$

$$E + P = \frac{1}{\pi}\int_{-\infty}^{0}|g'(u)|^2\,du + \frac{1}{\pi}\int_0^{\infty}|p(v)|^2\,dv \tag{20b}$$

From (15a) and the similar formulas relative to p we can express all four integrals in terms of $\alpha(\lambda)$. Doing so we find after elementary steps:

$$E - P = 2\int_{-\infty}^{+\infty}|\alpha(\lambda)|^2\,d\lambda \qquad E + P = 2\int_{-\infty}^{+\infty}\lambda^2\,|\alpha(\lambda)|^2\,d\lambda \tag{21}$$

So:

$$P = \int_{-\infty}^{+\infty}(\lambda^2 - 1)|\alpha(\lambda)|^2\,d\lambda \tag{22}$$

This confirms that a λ with $|\lambda| \geq 1$ gives a "right-moving" component of the wave packet (its phase is constant for $\omega t - \mu x = C$, $\omega = \frac{1}{2}(\lambda + \frac{1}{\lambda})$, $\mu = \frac{1}{2}(\lambda - \frac{1}{\lambda})$.) The values of λ with $|\lambda| \leq 1$ give "left-moving" wave components. As a check, we can observe that it is impossible to have a purely right-moving packet with vanishing Cauchy data for $t = 0$, $x < 0$, because as we saw above, for such Cauchy data α has to belong to the Hardy space of the lower half-plane and can thus (by a theorem of Wiener) not vanish identically on $(-1, 1)$. A purely right-moving packet starting entirely on $x > 0$ would have a hard time hitting the light cone, and this would emperil Theorem 1. Such wave-packets exist for the zero-mass equation, one way of reading Theorem 1 is to say that they don't exist for non-vanishing real mass.

Let us consider the effect of a Lorentz boost on E and P. We take $\Lambda = e^{\xi}$ ($\xi \in \mathbb{R}$) and replace ϕ by:

$$\phi_\Lambda(t, x) = \phi(\cosh(\xi)t + \sinh(\xi)x, \sinh(\xi)t + \cosh(\xi)x) \tag{23a}$$

$$\phi_\Lambda[u, v] = \phi[\frac{1}{\Lambda}u, \Lambda v] \tag{23b}$$

$$g_\Lambda(u) = \phi_\Lambda[u, 0] = g(\frac{1}{\Lambda}u) \qquad p_\Lambda(v) = p(\Lambda v) \tag{23c}$$

$$\alpha(\lambda) \mapsto \alpha_\Lambda(\lambda) = \Lambda\alpha(\Lambda\lambda) \tag{23d}$$

$$E_\Lambda - P_\Lambda = \Lambda \cdot (E - P) \qquad\qquad E_\Lambda + P_\Lambda = \frac{1}{\Lambda}(E + P) \tag{23e}$$

$$E_\Lambda = \cosh(\xi)E - \sinh(\xi)P \tag{23f}$$

$$P_\Lambda = -\sinh(\xi)E + \cosh(\xi)P \tag{23g}$$

So the conserved quantities E and P are not Lorentz invariant but the Einstein rest mass squared $E^2 - P^2$ is.

4 Scale reversing operators

We begin the proof of Theorem 2. Let us consider the manner in which the function $g(u)$ for $u > 0$ is related to the function $p(v) > 0$. We know that they are in unitary correspondence for the norms $\int_{u>0} |g|^2 + |g'|^2\, du$ and $\int_{v>0} |p|^2 + |p'|^2\, dv$, and the formulas (20a) for $E - P$ and $E + P$ suggest that one should pair g with p' and g' with p. In fact if we take into consideration the wave which has values $\phi(t, x) = e^{-|x|}$ for space-like points, we are rather led to pair g with $-p'$ and g' with $-p$ (the values of ϕ at time-like points are more involved and we don't need to know about them here; suffice it to say that certainly e^{-x} solves Klein-Gordon, so it gives the unique solution in the right wedge with $\phi(0,x) = e^{-x}$, $\frac{\partial \phi}{\partial t}(0,x) = 0$.)

Let us denote by $\mathcal{H}$ the operator which acts as $g \mapsto -p'$, on even g's. Under a Lorentz boost: $g \mapsto g_\Lambda(u) = g(\frac{1}{\Lambda}u)$, $-p' \mapsto -\Lambda p'(\Lambda v)$ and also the assignment $g \mapsto -p'$ is unitary for the L^2 norm:

$$g(u) = \int_{-\infty}^{+\infty} e^{iu\lambda}\alpha(\lambda)\,d\lambda \qquad p(v) = \int_{-\infty}^{+\infty} e^{i\lambda v}\alpha(-\frac{1}{\lambda})\frac{1}{\lambda^2}\,d\lambda$$

$$-p'(v) = -i\int_{-\infty}^{+\infty} e^{i\lambda v}\alpha(-\frac{1}{\lambda})\frac{1}{\lambda}\,d\lambda$$

Going from g to α is unitary, from α to $-i\alpha(-\frac{1}{\lambda})\frac{1}{\lambda}$ also, and back to $-p'$ also, in the various L^2 norms. So the assignment from g to $-p'$ is unitary.

Identifiying the L^2 space on $u > 0$ with the L^2 space on $v > 0$, through $v = u$, $\mathcal{H}$ is a unitary operator on $L^2(0,+\infty; du)$. Furthermore it is "scale reversing": we say that an operator $\mathcal{K}$ (bounded, more generally, closed) is scale reversing if its composition $\mathcal{K}I$ with $I : g(u) \mapsto \frac{g(1/u)}{u}$ commutes with the unitary group of scale changes $g \mapsto \sqrt{\Lambda}g(\Lambda u)$. The Mellin transform $g \mapsto \widehat{g}(s) = \int_0^\infty g(u)u^{-s}\,du$, for $s = \frac{1}{2} + i\tau$, $\tau \in \mathbb{R}$, is the additive Fourier transform of $e^{t/2}g(e^t) \in L^2(-\infty,+\infty; dt)$. The operator $\mathcal{K}I$ commutes with multiplicative translations hence is diagonalized by the Mellin transform: we have a certain (bounded for $\mathcal{K}$ bounded) measurable function χ on the critical line $\Re(s) = \frac{1}{2}$ such that for any $g(u) \in L^2(0,\infty; du)$, and almost everywhere on the critical line:

$$(\mathcal{K}g)^\wedge(s) = (\mathcal{K}I(Ig))^\wedge(s) = \chi(s)(Ig)^\wedge(s) = \chi(s)\widehat{g}(1-s)$$

Let us imagine for a minute that we know a g which is invariant under $\mathcal{K}$ and which, furthermore has $\widehat{g}(s)$ almost everywhere non vanishing (by a theorem of Wiener, this means exactly that the linear span of its orbit under the unitary group of scale changes is dense in L^2). Then we know $\chi(s)$ hence, we know $\mathcal{K}$. So $\mathcal{K}$ is uniquely determined by the knowledge of one such invariant function.

In the case of our operator $\mathcal{H}$ which goes from the data of $g(u)$, $u > 0$, to the data of $k(v) = -p'(v)$, $v > 0$, where g and p are the boundary values of a finite energy solution of the Klein-Gordon equation in the right wedge, we know that it is indeed unitary, scale reversing, and has e^{-u} as a self-reciprocal function (so, here, $\chi(s) = \frac{\Gamma(1-s)}{\Gamma(s)}$).

On the other hand the Hankel transform of order zero is unitary, scale reversing, and has $\sqrt{u}e^{-u^2/2}$ as self-reciprocal invariant function. So we find that the assignment of $-\sqrt{v}\,k'(\frac{v^2}{2})$ to $\sqrt{u}\,g(\frac{u^2}{2})$ is exactly the Hankel transform of order zero. This may also be proven directly by the method we will employ in section 7.

5 Causality and support conditions

The Theorem 2 is almost entirely proven: if the Cauchy data vanishes identically for $0 < x < 2a$, then by unicity and causal propagation, $g(u) = \phi(-u, u)$ vanishes identically for $0 < u < a$ and $p(v) = \phi(+v, v)$ vanishes identically for $0 < v < a$. Conversely, if A and B from Theorem 2 vanish identically for $0 < r, s < \sqrt{2a}$, then $g(u)$ and $-p'(v)$ vanish identically for $0 < u < a$ and $0 < v < a$. We explained in the introduction that p itself also vanishes identically for $0 < v < a$. Then $\phi[u, v] = \iint_{\substack{0\leq r\leq u\\ 0\leq s\leq v}} \phi[r, s]\,drds$ for $0 \leq u \leq a$, $0 \leq v \leq a$, hence ϕ vanishes identically in this range, and the Cauchy data for ϕ at $t = 0$, $0 < x < 2a$, vanishes identically. The proof of Theorem 2 (hence also in its equivalent form 3) is complete.

We would like also to relax the finite energy condition on ϕ. Let us imagine that our g, say even, is only supposed L^2. It has an L^2 Fourier transform α such that $g(u) = \int_{-\infty}^{+\infty} e^{+iu\lambda}\alpha(\lambda)\,d\lambda$. Let us approximate α by an L^2 converging sequence of α_n's, corresponding to finite energy Klein-Gordon solutions ϕ_n. We have by (18a) and (21):

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(|\phi_n - \phi_m|^2 + \left|\frac{\partial(\phi_n - \phi_m)}{\partial x} + \frac{\partial(\phi_n - \phi_m)}{\partial t}\right|^2\right)dx = 2\int_{-\infty}^{+\infty}|\alpha_n - \alpha_m|^2\,d\lambda$$

So the ϕ_n converge for $t = 0$ in the L^2 sense, and also the $\frac{\partial\phi_n}{\partial x} + \frac{\partial\phi_n}{\partial t}$. We can then consider, as is known to exist, the distribution solution ϕ with this Cauchy data.

Let us suppose that we start from an even g which, together with its $\mathcal{H}$ transform, vanish in $(0, a)$. First we show that we can find, with $0 < b_n < 1$, $b_n \to 1$, a sequence of g_n's, such that g_n' is in L^2, and $g_n \to g$ in L^2, with the g_n's satisfying the support condition for $(0, b_n a)$. We obtain such g_n by multiplicative convolution of g with a test function supported in $(b_n, \frac{1}{b_n})$. At the level of Mellin transforms, this multiplies by a Schwartz function. As $u\frac{d}{du}$ corresponds to multiplication by $-s$ certainly the $u\frac{d}{du}$ of our g_n's are in L^2. But then $\frac{d}{du}g_n$ itself is in L^2 as we know that it vanishes in $(0, b_n a)$. And its $\mathcal{H}$ transform also vanishes there.

So the corresponding ϕ_n's will have for $t = 0$ vanishing Cauchy data in intervals only arbitrarily slightly smaller than $(0, 2a)$. The L^2 functions $\phi(0, x)$ and $(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t})(0, x)$ will thus vanish identically, in $(0, 2a)$. Conversely if we have two L^2 functions L and M vanishing in $(0, 2a)$ we can approximate then by Schwartz functions L_n and M_n vanishing in $(0, b_n 2a)$ $(0 < b_n < 1,\ b_n \to 1)$, solve the Cauchy problem with data $\phi = L_n$ and $\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t} = M_n$ at $t = 0$, consider the corresponding g_n's which vanish identically for $0 < u < b_n a$ and get an L^2 limit g vanishing identically in $(0, a)$. The $\mathcal{H}$ transform of g will be the limit in L^2 of the $\mathcal{H}$ transforms of the g_n, so it will also vanish in $(0, a)$.

In conclusion the space-time representation of Hankel pairs with support condition as given in Theorem 2 extends to the general case of L^2 Hankel pairs if one allows Klein-Gordon solutions of possibly infinite energy but such that $\phi(0, x)$ and $\frac{\partial\phi}{\partial x}(0, x) + \frac{\partial\phi}{\partial t}(0, x)$ are in L^2.

6 The Dirac system and its associated scattering

We return to finite energy solutions which are associated to functions α verifying the condition $\int_{-\infty}^{+\infty}(1+\lambda^2)|\alpha(\lambda)|^2\, d\lambda < \infty$. Let us consider in fact a pair $\begin{bmatrix}\psi\\ \phi\end{bmatrix}$ of such finite energy solutions satisfying the first order system:

$$\frac{\partial\psi}{\partial t} - \frac{\partial\psi}{\partial x} = +\phi \qquad\qquad \frac{\partial\psi}{\partial u} = -\phi \tag{24a}$$

$$\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} = -\psi \qquad\qquad \frac{\partial\phi}{\partial v} = -\psi \tag{24b}$$

If α corresponds to ϕ and β corresponds to ψ, then there is the relation: $\alpha(\lambda) = -i\lambda\beta(\lambda)$ so we must have $\int_{-\infty}^{+\infty} \frac{1}{\lambda^2}|\alpha(\lambda)|^2\, d\lambda < \infty$. To enact a Lorentz boost we could imagine replacing ϕ and ψ by

$$\phi(\cosh(\xi)t + \sinh(\xi)x, \sinh(\xi)t + \cosh(\xi)x) = \phi[e^{-\xi}u, e^{\xi}v]$$

$$\psi(\cosh(\xi)t + \sinh(\xi)x, \sinh(\xi)t + \cosh(\xi)x) = \psi[e^{-\xi}u, e^{\xi}v]$$

but this does not give a solution of the Dirac type system (24). To obtain a solution we must rescale ϕ, or ψ, or both. We choose:[2]

$$\phi_\xi[u,v] = e^{-\xi/2}\,\phi[e^{-\xi}u, e^{\xi}v] \qquad \psi_\xi[u,v] = e^{\xi/2}\,\psi[e^{-\xi}u, e^{\xi}v] \tag{25}$$

In other words, if we want to consider our ϕ as a component of such a system we must cease treating it as a scalar. It is a (spinorial) quantity which transforms as indicated under a Lorentz boost. We note further that with this modification both $E(\phi)-P(\phi)$ and $E(\psi)+P(\psi)$ are Lorentz invariant. In fact they are identical: $E(\phi)-P(\phi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\phi|^2 + \left|\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial t}\right|^2 dx$, $E(\psi)+P(\psi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\psi|^2 + \left|\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial t}\right|^2 dx$, hence:

$$E(\phi)-P(\phi) = E(\psi)+P(\psi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(|\phi(0,x)|^2+|\psi(0,x)|^2\right)\,dx \tag{26}$$

We again focus on what happens in the right wedge. Thus, we can as well take ϕ to be PT invariant. But then as $\psi = -\frac{\partial\phi}{\partial v}$, ψ must acquire a sign under the PT transformation: $\psi(-x,-t) = -\psi(x,t)$. So the function $g(u) = \phi(-u,u) = \phi[u,0]$ is even but the function $k(v) = \psi(v,v) = \psi[0,v]$ is odd. In fact $k(v) = -p'(v)$ with our former notation. So we know that the PT invariant ϕ is uniquely determined by $g(u)$ for $u>0$ which gives under the $\mathcal{H}$ transform the function $k(v)$ for $v>0$ which must be considered odd and correspond to the PT anti-invariant ψ. We note that if $k(0^+)\neq 0$ then this ψ is not of finite energy. Using only that ϕ is finite energy, we have from equation (20a):

$$E(\phi)-P(\phi) = \frac{1}{\pi}\int_{-\infty}^{0}|g(u)|^2\,du + \frac{1}{\pi}\int_0^\infty |k(v)|^2\,dv$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(|\phi(0,x)|^2+|\psi(0,x)|^2\right)\,dx = E(\phi)-P(\phi) = \frac{1}{\pi}\int_0^\infty |g(u)|^2\,du + \frac{1}{\pi}\int_0^\infty |k(v)|^2\,dv$$

$$\int_0^\infty\left(|\phi(0,x)|^2+|\psi(0,x)|^2\right)\,dx = 2\int_0^\infty |g(u)|^2\,du \tag{27a}$$

$$\int_0^\infty\left(|\phi(0,x)|^2+|\psi(0,x)|^2\right)\,dx = 2\int_0^\infty |k(v)|^2\,dv \tag{27b}$$

We now begin the proof of Theorem 4. To prove that $\int_0^\infty |F(x)|^2+|G(x)|^2\,dx = 2\int_0^\infty |g(u)|^2\,du = 2\int_0^\infty |k(v)|^2\,dv$, we extend F to be even and G to be odd. Then ϕ is PT even of finite energy, and ψ is PT odd and equations (27a) and (27b) apply. Note that if $G(0^+)\neq 0$ then ψ is not of finite energy but only the fact that ϕ is of finite energy was used for (27a) and (27b). That $k=\mathcal{H}(g)$ and $\int_0^\infty |g(u)|^2+|g'(u)|^2du<\infty$ hold

[2] this conflicts with our previous notation $\phi_\Lambda[u,v] = \phi[\frac{1}{\Lambda}u, \Lambda v]$; no confusion should arise.

are among our previous results. If we choose G to be even and F to be odd, then it is ψ which is of finite energy and so $\int_0^\infty |k(v)|^2 + |k'(v)|^2 dv < \infty$ holds true. We can also prove $\int_0^\infty |g|^2 + |g'|^2 du < \infty$, $\int_0^\infty |k|^2 + |k'|^2 dv < \infty$ after extending F and G such that $\int_{-\infty}^\infty |F|^2 + |F'|^2 + |G|^2 + |G'|^2 dx < \infty$ so that both ϕ and ψ are then of finite energy. The boundary values $g(u)$, $u > 0$, and $k(v)$, $v > 0$ do not depend on choices. Furthermore the vanishing of F and G on $(0, 2a)$ at $t = 0$ is equivalent by our previous arguments to the vanishing of g and k on $(0, a)$. To show that all $\mathcal{H}$ pairs with $\int_0^\infty |g|^2 + |g'|^2 du < \infty$, $\int_0^\infty |k|^2 + |k'|^2 dv < \infty$ are obtained, let k_1 be the odd function with $k_1(v) = k(v) - k(0^+)e^{-v}$ for $v > 0$ and let g_1 be the even function with $g_1(u) = g(u) - k(0^+)e^{-u}$ for $u \geq 0$. Then $k_1 = \mathcal{H}(g_1)$ and $\int_{-\infty}^\infty |g_1|^2 + |g_1'|^2 du < \infty$ and $\int_{-\infty}^\infty |k_1|^2 + |k_1'|^2 dv < \infty$. They thus correspond to ϕ_1 and ψ_1 both of finite energy. We define for $x > 0$: $F(x) = \phi_1(0, x) + k(0^+)e^{-x}$ and $G(x) = \psi_1(0, x) + k(0^+)e^{-x}$, it then holds that $\int_0^\infty |F|^2 + |F'|^2 + |G|^2 + |G'|^2 dx < \infty$ and $\left[\begin{smallmatrix}\psi\\ \phi\end{smallmatrix}\right] = \left[\begin{smallmatrix}\psi_1 + k(0^+)e^{-x}\\ \phi_1 + k(0^+)e^{-x}\end{smallmatrix}\right]$ is the unique solution in the Rindler wedge of the Dirac system with Cauchy data $\left[\begin{smallmatrix}G\\ F\end{smallmatrix}\right]$ on $x > 0$, $t = 0$, and it has $g(u)$ and $k(v)$ as boundary values. To complete the proof of Theorem 4 there only remains to show the formulas relating F, G, g, and k and this will be done in the next section.

On the Hilbert space $L^2(0, \infty; dx) \oplus L^2(0, \infty; dx)$ of the pairs (F, G), we can define a unitary group $U(\xi)$, $-\infty < \xi < \infty$, as follows: we define its action at first for (F, G) with $F', G' \in L^2$. Let $\left[\begin{smallmatrix}\psi\\ \phi\end{smallmatrix}\right]$ be the solution of first order system (24) such that $\phi(0, x) = F(x)$, $\psi(0, x) = G(x)$. Then we take:

$$U(\xi)(F, G) = (\phi_{-\xi}|_{t=0}\,, \psi_{-\xi}|_{t=0}) \tag{28}$$

where (25) has been used. As ξ increases from $-\infty$ to $+\infty$ this has the effect of transporting ϕ and ψ forward along the Lorentz boosts trajectories. We can also implement $U(\xi)$ as a unitary group acting on the L^2 space of the $g(u) = \phi(-u, u)$ functions, or on the space of the $k(v) = \psi(v, v)$ functions. We then have, taking into account (25) (and $-\xi$):

$$g_\xi(u) = e^{\frac{\xi}{2}} g(e^\xi u) \qquad\qquad k_\xi(v) = e^{-\frac{\xi}{2}} k(e^{-\xi} v) \tag{29}$$

Following the terminology of Lax-Phillips [12] (the change of variable $u \to \log(u)$ would reduce to the additive language of [12]) we shall say that $(F, G) \mapsto I(g)$ provides an incoming (multiplicative) translation representation ($U(\xi)$ moves the graph of $e^{y/2} I(g)(e^y) = e^{-y/2} g(e^{-y})$ to the right by an amount of additive time ξ) and $(F, G) \mapsto k$ is an outgoing translation representation. We use $(Ig)(u) = \frac{1}{u} g(\frac{1}{u})$ as it is translated by

$U(\xi)$ in the same direction as k. The assignment $Ig \to k$ will be called the "scattering matrix" $\mathcal{S}$ (it is canonical only up to a translation in "time", which means here only up to a scale change in u). With our previous notation it is $S = \mathcal{H}I$. Let us give a "spectral" representation of S. For this we represent g as a superposition of (multiplicative) harmonics, $g(u) = \frac{1}{2\pi}\int_{\Re(s)=\frac{1}{2}} \widehat{g}(s)u^{s-1}\,|ds|$, with $\widehat{g}(s) = \int_0^\infty g(u)\,u^{-s}\,du$, $s = \frac{1}{2} + i\tau$. Then the unitary operator S will be represented as multiplication by a unit modulus function $\chi(s)$. Multiplication by $\chi(s)$ must send the Mellin transform $\Gamma(s)$ of $I(e^{-u})$ to the Mellin transform $\Gamma(1-s)$ of e^{-u}, in other words:

$$\chi(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \tag{30}$$

We thus see that the first order system in the wedge of two dimensional space-time provides an interpretation of this function (for $\Re(s) = \frac{1}{2}$) as a scattering matrix. To obtain the Hankel transform of order zero, and not its succédané $\mathcal{H}$, one writes $s = \frac{1}{4} + \frac{w}{2}$, where again $\Re(w) = \frac{1}{2}$. In fact, with our normalizations, the scattering matrix corresponding to the tranform $g(t) \mapsto f(u) = \int_0^\infty \sqrt{ut}J_0(ut)g(t)\,dt$ is the function $2^{\frac{1}{2}-w}\frac{\Gamma(\frac{3}{4}-\frac{w}{2})}{\Gamma(\frac{1}{4}+\frac{w}{2})}$ on the critical line $\Re(w) = \frac{1}{2}$.

7 Application of Riemann's method

The completion of the proof of Theorem 4 will now be provided. I need to briefly review Riemann's method ([10, IV§1], [6, VI§5]), although it is such a classical thing, as I will use it in a special manner later. In the case of the (self-adjoint) Klein-Gordon equation $\frac{\partial^2\phi}{\partial u\partial v} = +\phi$, $t^2 - x^2 = 4(-u)v$, Riemann's method combines:

1. whenever ϕ and ψ are two solutions, the differential form $\omega = \phi\frac{\partial\psi}{\partial u}\,du + \psi\frac{\partial\phi}{\partial v}\,dv$ is closed,
2. it is advantageous to use either for ϕ or for ψ the special solution (Riemann's function) $R(P,Q)$ which reduces to the constant value 1 on each of characteristics issued from a given point P. Here $R(P,Q) = R(P-Q,0) = R(Q-P,0)$, $R((t,x),0) = J_0(\sqrt{t^2-x^2}) = J_0(2\sqrt{-uv})$.

Usually one uses Riemann's method to solve for ϕ when its Cauchy data is given on a curve transversal to the characteristics. But one can also use it when the data is on the characteristics (Goursat problem). Also, one usually symmetrizes the formulas obtained in combining the information from using $\phi\frac{\partial R}{\partial u}\,du + R\frac{\partial \phi}{\partial v}\,dv$ with the information from using $R\frac{\partial \phi}{\partial u}\,du + \phi\frac{\partial R}{\partial v}\,dv$. For our goal it will be better not to symmetrize in this manner. Let us recall as a warming-up how one can use Riemann's method to find $\phi(t,x)$ for $t > 0$ when ϕ and $\frac{\partial \phi}{\partial t}$ are known for $t = 0$. Let $P = (t,x)$, $A = (0, x-t)$, $B = (0, x+t)$, and $R(Q) = R(P-Q)$.

$$\phi(P) - \phi(A) = \int_{A\to P} \frac{\partial \phi}{\partial v} dv = \int_{A\to P} R\frac{\partial \phi}{\partial v} dv + \phi\frac{\partial R}{\partial u} du = \int_{A\to B} + \int_{B\to P} = \int_{A\to B}$$

Hence:

$$\phi(P) = \phi(A) + \int_{A\to B} (R\frac{\partial \phi}{\partial v} + \phi\frac{\partial R}{\partial u})\frac{dx}{2}$$

Using $R\frac{\partial \phi}{\partial u}\,du + \phi\frac{\partial R}{\partial v}\,dv$ we get in the same manner:

$$\phi(P) = \phi(B) - \int_{A\to B} (R\frac{\partial \phi}{\partial u} + \phi\frac{\partial R}{\partial v})\frac{dx}{2} \tag{31}$$

After averaging:

$$\phi(P) = \frac{\phi(A)+\phi(B)}{2} + \frac{1}{2}\int_{A\to B} (R\frac{\partial \phi}{\partial t} - \phi\frac{\partial R}{\partial t})dx$$

This gives the classical formula ($t > 0$):

$$\begin{aligned}\phi(t,x) = \frac{\phi(0,x-t)+\phi(0,x+t)}{2} &- \frac{1}{2}\int_{x-t}^{x+t} t\,\frac{J_1(\sqrt{t^2-(x-x')^2})}{\sqrt{t^2-(x-x')^2}}\phi(0,x')dx' \\ &+\frac{1}{2}\int_{x-t}^{x+t} J_0(\sqrt{t^2-(x-x')^2})\frac{\partial \phi}{\partial t}(0,x')dx'\end{aligned} \tag{32}$$

I have not tried to use it to establish theorem 1. Anyway, when ϕ, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial t}$ all belong to L^2 at $t = 0$, this formula shows that $\phi(P)$ is continuous in P for $t > 0$. Replacing $t = 0$ with $t = -T$, we find that ϕ is continuous on spacetime.

Let us now consider the problem, with the notations of Theorem 4, of determining $k(v) = \psi(v,v)$ for $v > 0$ when $F(x) = \phi(0,x) = -\frac{\partial \psi}{\partial u}(0,x)$ and $G(x) = \psi(0,x) = -\frac{\partial \phi}{\partial v}(0,x)$ are known for $x > 0$. We use $P = (v_0, v_0)$, $A = (0,0)$, $B = (0, 2v_0)$. We then have:

$$R(t,x) = J_0(\sqrt{(v_0-t)^2-(v_0-x)^2}) = J_0(2\sqrt{u(v_0-v)}) \qquad R(0,x) = J_0(\sqrt{x(2v_0-x)})$$

$$\frac{\partial R}{\partial v} = \frac{J_1(2\sqrt{u(v_0-v)})}{2\sqrt{u(v_0-v)}}2u \qquad \frac{\partial R}{\partial v}(0,x) = \frac{J_1(\sqrt{x(2v_0-x)})}{\sqrt{x(2v_0-x)}}x$$

Hence, using (31) (for ψ):

$$\psi(v,v) = G(2v) + \frac{1}{2}\int_0^{2v} (J_0(\sqrt{x(2v_0-x)})F(x) - x\,\frac{J_1(\sqrt{x(2v_0-x)})}{\sqrt{x(2v_0-x)}}G(x))\,dx \tag{33}$$

We then consider the converse problem of expressing $G(x) = \psi(0,x)$ in terms of $k(v) = \psi(v,v)$. We choose $x_0 > 0$, and consider the rectangle with vertices $P = (\frac{1}{2}x_0, \frac{1}{2}x_0)$, $Q = (0, x_0)$, $Q' = (X, X + x_0)$, $P' = (X + \frac{1}{2}x_0, X + \frac{1}{2}x_0)$ for $X \gg 0$. We take Riemann's function S to be 1 on the edges $P \to Q$ and $Q \to Q'$. We then write:

$$\psi(Q) - \psi(P) = \int_{P\to Q} \frac{\partial \psi}{\partial u}\, du = \int_{P\to Q} (S\frac{\partial \psi}{\partial u}\, du + \psi\frac{\partial S}{\partial v} dv)$$

$$= \int_{P\to P'} + \int_{P'\to Q'} + \int_{Q'\to Q} = \int_{P\to P'} \psi\frac{\partial S}{\partial v}\, dv + \int_{P'\to Q'} S\frac{\partial \psi}{\partial u}\, du$$

$$G(x_0) = \psi(\frac{x_0}{2}, \frac{x_0}{2}) + \int_{P\to P'} \psi\frac{\partial S}{\partial v}\, dv - \int_{P'\to Q'} S\phi\, du \tag{34}$$

Now, $|S| \leq 1$ on the segment leading from P' to Q', so we can bound the last integral, using Cauchy-Schwarz, then the energy integral, and finally the theorem 1. So this term goes to 0. On the light cone half line from P to ∞ we have:

$$S(v,v) = J_0(\sqrt{x_0(2v - x_0)}) \qquad \frac{\partial S}{\partial v} = -\frac{J_1(\sqrt{x_0(2v - x_0)})}{\sqrt{x_0(2v - x_0)}}\, x_0$$

$$G(x_0) = \psi(\frac{x_0}{2}, \frac{x_0}{2}) - \int_{x_0/2}^{\infty} \frac{J_1(\sqrt{x_0(2v - x_0)})}{\sqrt{x_0(2v - x_0)}}\, x_0 \psi(v,v)\, dv \tag{35}$$

Our last task is to obtain the formula for $F(x_0)$. We use the same rectangle and same function S.

$$\phi(Q) - \phi(Q') = \int_{Q'\to Q} \frac{\partial \phi}{\partial v} dv = \int_{Q'\to Q} S\frac{\partial \phi}{\partial v} dv + \phi\frac{\partial S}{\partial u} du = \int_{Q'\to P'} \phi\frac{\partial S}{\partial u} du + \int_{P'\to P} S\frac{\partial \phi}{\partial v} dv + 0$$

On the segment $Q' \to P'$ we integrate by parts to get:

$$\int_{Q'\to P'} \phi\frac{\partial S}{\partial u} du = \phi(P')S(P') - \phi(Q') - \int_{Q'\to P'} \frac{\partial \phi}{\partial u} S\, du$$

Again we can bound S by 1 and apply Cauchy-Schwarz to $\int_{Q'\to P'} \frac{\partial \phi}{\partial u} S\, du$. Then we observe that $\int_{Q'\to P'} |\frac{\partial \phi}{\partial u}|^2 |du|$ is bounded above by the energy integral, which itself is bounded above by the energy integral on the horizontal line having P' as its left end. By Theorem 1 this goes to 0. And regarding $\phi(P')$ one has $\lim_{v\to+\infty} \phi(v,v) = 0$ as $\phi(v,v)$ and its derivative belong to $L^2(0,+\infty; dv)$. We cancel the $\phi(Q')$'s on both sides of our equations and obtain:

$$\phi(Q) = -\int_{P\to(\infty,\infty)} S\frac{\partial \phi}{\partial v} dv = +\int_{P\to(\infty,\infty)} S\psi dv$$

Hence

$$F(x_0) = \int_{x_0/2}^{\infty} J_0(\sqrt{x_0(2v-x_0)})\psi(v,v)\,dv \tag{36}$$

In conclusion: the functions $F(x) = \phi(0,x)$, $G(x) = \psi(0,x)$, and $k(v) = \psi(v,v)$ of Theorem 4 are related by the following formulas:

$$F(x) = \int_{x/2}^{\infty} J_0(\sqrt{x(2v-x)})k(v)\,dv \tag{37a}$$

$$G(x) = k(\frac{x}{2}) - \int_{x/2}^{\infty} x\,\frac{J_1(\sqrt{x(2v-x)})}{\sqrt{x(2v-x)}}\,k(v)\,dv \tag{37b}$$

$$k(v) = G(2v) + \frac{1}{2}\int_0^{2v} J_0(\sqrt{x(2v-x)})F(x)\,dx - \frac{1}{2}\int_0^{2v} x\,\frac{J_1(\sqrt{x(2v-x)})}{\sqrt{x(2v-x)}}G(x)\,dx \tag{37c}$$

Exchanging F and G is like applying a time reversal so it corresponds exactly to exchanging $k(v) = \psi(v,v)$ with $g(u) = \phi(-u,u)$. So the proof of Theorem 4 is complete.

8 Conformal coordinates and concluding remarks

The Rindler coordinates (ξ,η) in the right wedge are defined by the equations $x = \eta\cosh\xi$, $t = \eta\sinh\xi$. Let us use the conformal coordinate system:

$$\xi = \frac{1}{2}\log\frac{x+t}{x-t} \qquad\qquad \zeta = \frac{1}{2}\log(x^2-t^2) - \log 2 = \log\frac{\eta}{2}$$

where $-\infty < \xi < +\infty$, $-\infty < \zeta < +\infty$. The variable ξ plays the rôle of time for our scattering. The reason for $-\log 2$ in ζ is the following: at $t = 0$ this gives $e^\zeta = \frac{1}{2}x = u = v$. The differential equations we shall write are related to the understanding of the vanishing condition for an $\mathcal{H}$ pair on an interval $(0,a)$. And $a = \frac{1}{2}(2a)$ hence the $-\log 2$ (to have equations identical with those in [5].) The Klein-Gordon equation becomes:

$$\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\zeta^2} + 4e^{2\zeta}\phi = 0 \tag{38}$$

If we now look for "eigenfunctions", oscillating harmonically in time, $\phi = e^{-i\gamma\xi}\,\Phi(\zeta)$, $\gamma\in\mathbb{R}$, we obtain a Schrödinger eigenvalue equation:

$$-\Phi''(\zeta) + 4e^{2\zeta}\Phi(\zeta) = \gamma^2\Phi(\zeta) \tag{39}$$

This Schrödinger operator has a potential function which can be conceived of as acting as a repulsive exponential barrier for the de Broglie wave function of a quantum mechanical particle coming from $-\infty$ and being ultimately bounced back to $-\infty$. The solutions of (39) are the modified Bessel functions ([18]) of imaginary argument $i\gamma$ in the variable $2e^\zeta$. For each $\gamma \in \mathbb{C}$ the unique (up to a constant factor) solution of (39) which is square integrable at $+\infty$ is $K_{i\gamma}(2e^\zeta)$.

From Theorem 4 it is more convenient to express the $\mathcal{H}$ transform as a scattering for the two-component, "Dirac", differential system. The spinorial nature of $\left[\begin{smallmatrix}\psi\\ \phi\end{smallmatrix}\right]$ leads under the change of coordinates $(t,x)\mapsto(\xi,\zeta)$ to $e^{\frac{\zeta}{2}}e^{-\frac{\xi}{2}}\phi$ rather than ϕ, and to $e^{\frac{\zeta}{2}}e^{+\frac{\xi}{2}}\psi$ rather than ψ. In order to get quantities which, in the past at $\xi\to-\infty$, look like ϕ and, in the future at $\xi\to+\infty$, look like ψ we consider the linear combinations:

$$\mathcal{A} = \frac{1}{2}\,e^{\frac{\zeta}{2}}\,(+e^{-\frac{\xi}{2}}\phi + e^{\frac{\xi}{2}}\psi) \tag{40a}$$

$$\mathcal{B} = \frac{i}{2}\,e^{\frac{\zeta}{2}}\,(-e^{-\frac{\xi}{2}}\phi + e^{\frac{\xi}{2}}\psi) \tag{40b}$$

Their differential system is:

$$+i\frac{\partial\mathcal{A}}{\partial\xi} = +\left(\frac{\partial}{\partial\zeta} - 2e^\zeta\right)\mathcal{B} \tag{41a}$$

$$+i\frac{\partial\mathcal{B}}{\partial\xi} = -\left(\frac{\partial}{\partial\zeta} + 2e^\zeta\right)\mathcal{A} \tag{41b}$$

Or, if we look for solutions oscillating in time as $e^{-i\gamma\xi}$:

$$\left(\frac{\partial}{\partial\zeta} - 2e^\zeta\right)\mathcal{B} = \gamma\mathcal{A} \tag{42a}$$

$$\left(-\frac{\partial}{\partial\zeta} - 2e^\zeta\right)\mathcal{A} = \gamma\mathcal{B} \tag{42b}$$

and this gives Schrödinger equations:

$$-\frac{\partial^2\mathcal{A}}{\partial\zeta^2} + (4e^{2\zeta} - 2e^\zeta)\mathcal{A} = \gamma^2\mathcal{A} \tag{43a}$$

$$-\frac{\partial^2\mathcal{B}}{\partial\zeta^2} + (4e^{2\zeta} + 2e^\zeta)\mathcal{B} = \gamma^2\mathcal{B} \tag{43b}$$

So we have two exponential barriers, and two associated "scattering functions" giving the induced phase shifts. From our previous discussion of the scattering in the Lax-Phillips formalism we can expect from equation (30) that a formalism of Jost functions will confirm these functions to be

$$\mathcal{S}(\gamma) = \frac{\Gamma(\frac{1}{2} - i\gamma)}{\Gamma(\frac{1}{2} + i\gamma)} \qquad (\gamma\in\mathbb{R})\ , \tag{44}$$

for the equation associated with $\mathcal{A}$ and $-\mathcal{S}(\gamma)$ for the equation associated with $\mathcal{B}$. And indeed the solution $\begin{bmatrix}\mathcal{A}_\gamma\\ \mathcal{B}_\gamma\end{bmatrix}$ of the system (42) which is square-integrable at $+\infty$ is given by the formula

$$\begin{bmatrix}\mathcal{A}_\gamma(\zeta)\\ \mathcal{B}_\gamma(\zeta)\end{bmatrix} = \begin{bmatrix} e^{\frac{\zeta}{2}}\left(K_s(2e^\zeta) + K_{1-s}(2e^\zeta)\right)\\ i\, e^{\frac{\zeta}{2}}\left(K_s(2e^\zeta) - K_{1-s}(2e^\zeta)\right)\end{bmatrix} \qquad (s = \frac{1}{2} + i\gamma) \tag{45}$$

Let $j_\gamma(\zeta)$ be the solution of (43a) which satisfies the Jost condition $j_\gamma(\zeta) \sim e^{-i\gamma\zeta}$ as $\zeta \to -\infty$. Then the exact relation holds (a detailed treatment is given in [5]):

$$\mathcal{A}_\gamma(\zeta) = \frac{1}{2}\left(\Gamma(s) j_\gamma(\zeta) + \Gamma(1-s) j_{-\gamma}(\zeta)\right) \qquad (s = \frac{1}{2} + i\gamma) \tag{46}$$

We interpret this as saying that the $\mathcal{A}$-wave comes from $-\infty$ and is bounced back with a phase-shift which at frequency γ equals $\arg \frac{\Gamma(\frac{1}{2}-i\gamma)}{\Gamma(\frac{1}{2}+i\gamma)} = \arg \mathcal{S}(\gamma)$. For the $\mathcal{B}$ equation one obtains $-\mathcal{S}(\gamma)$ as the phase shift function.

We have associated in [4] Schrödinger equations to the cosine and sine kernels whose potential functions also have exponential vanishing at $-\infty$ and exponential increase at $+\infty$, and whose associated scattering functions are the functions arising in the functional equations of the Riemann and Dirichlet L-functions. The equations (13a), (13b) of [4] are analogous to (40a), (40b) above, and (14a), (14b) of [4] are analogous to (42a) and (42b) above. The analogy is no accident. The reasoning of [4] leading to the consideration of Fredholm determinants when trying to understand self- and skew-reciprocal functions under a scale reversing operator on $L^2(0, +\infty; dx)$ is quite general. The (very simple) potential functions in the equations (43a) and (43b) can be written in terms of Fredholm determinants associated with the $\mathcal{H}$ transform. The detailed treatment is given in [5].

The function $\mathcal{S}(\gamma)$ arises in number theoretical functional equations (for the Dedekind zeta functions of imaginary quadratic fields). We don't know if its interpretation obtained here in terms of the Klein-Gordon equation may lead us to legitimately hope for number theoretical applications. An interesting physical context where $S(\gamma)$ has appeared is the method of angular quantization in integrable quantum field theory [13, App. B]. And, of course the group of Lorentz boosts and the Rindler wedge are connected by the Bisognano-Wichman theorem [1, 2, 7].

The potentials associated in [4] to the cosine and sine kernels are, contrarily to the simple-minded potentials obtained here, mainly known through their expressions as Fredholm determinants, and these are intimately related to the Fredholm determinant of the

Dirichlet kernel, which has been found to be so important in random matrix theory. It is thus legitimately considered an important problem to try to acquire for the cosine and sine kernels the kind of understanding which has been achieved here for the $\mathcal{H}$ transform. Will it prove possible to achieve this on (a subset, with suitable conformal coordinates) of (possibly higher dimensional) Minkowski space?

We feel that some kind of non-linearity should be at work. A tantalizing thought presents itself: perhaps the kind of understanding of the Fourier transform which is hoped for will arise from the study of the causal propagation and scattering of (quantum mechanical?) waves on a certain curved Einsteinian spacetime.

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