Operator Theory on a Commutation Relation

Asao Arai*
Department of Mathematics, Hokkaido University
Sapporo 060-0810
Japan
E-mail: arai@math.sci.hokudai.ac.jp

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Abstract

An operator theory is developed for a pair (A, H) with A being a symmetric operator on a Hilbert space $\mathcal{H}$ and H an injective self-adjoint operator on $\mathcal{H}$ such that $D(H^2) \subset D(A)$ (for a linear operator T, $D(T)$ denotes its domain) and $\langle H^2\psi, A\phi\rangle = \langle A\psi, H^2\phi\rangle$, $\psi, \phi \in D(H^2)$, where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$. One of the main results of the present paper is concerned with a decomposition theorem of $\bar{A}$ (the closure of A): Under suitable conditions, there exists a unique pair (A_+, A_-) of self-adjoint operators such that the following hold: (i) $\bar{A} = A_+ + A_-$; (ii) A_+ strongly commutes with H; (iii) A_- strongly anticommutes with H. Moreover, under some additional condition, A_+ strongly anticommutes with A_-. It is shown that, under suitable conditions, the spectrum of H is symmetric with respect to the origin in $\mathbb{R}$. As an application, we define a class of linear operators X on $\mathcal{H}$ for which $e^{itH}Xe^{-itH}$ $(t \in \mathbb{R})$ has an explicit representation. A class of pairs (A, H) to which the general theory can be applied is given with H being an abstract form of Dirac type operators.

Keywords: self-adjoint operator, strong commutativity, strong anticommutativity, decomposition theorem, spectrum, spectral reflection symmetry, Heisenberg operator, canonical commutation relation, Dirac operator

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1 Introduction

In this paper we develop an operator theory for a pair (A, H) with A being a symmetric operator on a Hilbert space $\mathcal{H}$ and H an injective self-adjoint operator on $\mathcal{H}$ such that

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$D(H^2) \subset D(A)$ (for a linear operator T, $D(T)$ denotes its domain) and

$$\left\langle H^2\psi, A\phi\right\rangle = \left\langle A\psi, H^2\phi\right\rangle, \quad \psi, \phi \in D(H^2), \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$. If A is bounded, then (1.1) is equivalent to $AH^2 \subset H^2 A$, which means the commutativity of A and H^2 (in the present paper, we mean by "a bounded operator on $\mathcal{H}$" a bounded linear operator *everywhere defined* on $\mathcal{H}$, unless otherwise stated). Hence (1.1) can be regarded as a generalized version of this commutativity to the case where A is not necessarily bounded.

We introduce a subset of linear operators on $\mathcal{H}$:

Definition 1.1 We say that A is in the set $\{H^2\}'_{\mathrm{w}}$ if A is symmetric with $D(H^2) \subset D(A)$ and (1.1) holds (in this definition, H is not necessarily injective).

Remark 1.1 It is easy to see that, for each bounded, real-valued Borel measurable function f on $\mathbb{R}$ and $\alpha \in [0, 2]$, the operator $f(H)|H|^\alpha$ is an element of $\{H^2\}'_{\mathrm{w}}$. But the case $A = f(H)|H|^\alpha$ is trivial. Hence, in what follows, we have in mind the case where A is not such a form. See Proposition 2.5 below.

To outline the contents of the present paper, we first recall two key notions:

Definition 1.2 Let S and T be self-adjoint operators on a Hilbert sapce. We denote the spectral measure of S by $E_S(\cdot)$

(i) We say that S and T strongly commute if their spectral measures commute, i.e., $E_S(J)E_T(K) = E_T(K)E_S(J)$ for all Borel sets J, $K \subset \mathbb{R}$ (e.g., [11, p.271], [12]).

(ii) We say that S strongly anticommutes with T if $e^{itT}S \subset Se^{-itT}$ for all $t \in \mathbb{R}$ ([10, 14]).

Remark 1.2 One can prove that, if S strongly anticommutes with T, then $e^{itS}T \subset Te^{-itS}$ for all $t \in \mathbb{R}$ [10], i.e., T strongly anticommutes with S. Hence Definition 1.2(ii) is symmetric in S and T.

One of the main results of this paper is stated as follows. Let H be an injective self-adjoint operator and $A \in \{H^2\}'_{\mathrm{w}}$. Then there exists a unique pair (A_+, A_-) of self-adjoint operators such that the following hold: (i) $\bar{A} = A_+ + A_-$, where $\bar{A}$ is the closure of A

; (ii) A_+ strongly commutes with H ; (iii) A_- strongly anticommutes with H. If some additional condition is satisfied, then A_+ strongly anticommutes with A_-.

This is a kind of decomposition theorem of operators relative to the self-adjoint operator H. It is well known that the strong commutativity and the strong anticommutativity of self-adjoint operators make it possible to perform more detailed analyses in various contexts of operator theory (e.g., [1]-[6], [10, 12, 14]). Thus such decomposition theorems as stated above may be very useful.

We also investigate the spectral properties of H. It is shown that, under suitable conditions, the spectrum of H is symmetric with respect to the origin in $\mathbb{R}$.

As an application, we define a class of linear operators X on $\mathcal{H}$ for which the Heisenberg operator $e^{itH}Xe^{-itH}$ of X ($t \in \mathbb{R}$) with respect to H has an explicit representation. Moreover, we present a class of pairs (A, H) to which the general theory can be applied with H being an abstract form of Dirac type operators.

One of the motivations for this work comes from trying to understand general operator-theoretical structures underlying the so-called Zitterbewegung in relativistic quantum mechanics ([8], [13, §1.6]). But the theory presented in this paper is very general. It may have many applications both in operator theory and in quantum physics[1].

The present paper is orginaized as follows. In Section 2, we prove some elementary facts on the set $\{H^2\}'_{\mathrm{w}}$. At the end of this section, a purely operator-theoretical motivation for this work is given. In Section 3 we consider a similarity transformation for a linear operator A given by the correspondence :$A \mapsto A_H := HAH^{-1}$ under the condition that H is injective. Some properties of the operator A_H are discussed. Section 4 is devoted to proof of the decomposition theorem stated above. In Section 5 we prove that, under additional conditions, the spectrum and the point spectrum of H are symmetric with respect to the origin in $\mathbb{R}$. In Section 6 we introduce a class of linear operators X for which the Heisenberg operator $X(t) = e^{itH}Xe^{-itH}$ ($t \in \mathbb{R}$) has an explicit representation. The asymptotic behavor of $X(t)$ as $|t| \to \infty$ is clarified in the sense of sesquilinear form. In Section 7 we discuss a connection with the generalized weak Weyl relation introduced in [7]. We show that, under suitable conditions,

$$\lim_{|t|\to\infty} \left\langle \psi, e^{-itH} A_+ \phi \right\rangle = 0$$

[1]The present author plans to discuss, in a separate paper, an application to quantum electrodynamics in connection with Koba's work [8].

for all $\psi, \phi \in \mathcal{H}$ and H has no eigenvalues. In the last section we present a class of pairs (A, H) to which the general theory developed in the previous sections can be applied.

2 Preliminaries

We first recall a useful fact:

Lemma 2.1 (Nussbaum) *Let T be a self-adjoint operator on $\mathcal{H}$ and S be a symmetric operator on $\mathcal{H}$ such that $D(T) \subset D(S)$ and $\langle T\psi, S\phi\rangle = \langle S\psi, T\phi\rangle$, $\psi, \phi \in D(T)$. Then S is essentially self-adjoint on $D(T)$ and $\bar{S}$ (the closure of S) strongly commutes with T.*

Proof. See [9, Corollary 1] or [10, Proposition 4.1]. ■

Let H be a self-adjoint opeartor on $\mathcal{H}$.

Proposition 2.2 *For all $A \in \{H^2\}'_{\mathrm{w}}$, A is essentially self-adjoint on $D(H^2)$ and $\bar{A}$ strongly commutes with H^2. Moreover $\bar{A} \in \{H^2\}'_{\mathrm{w}}$.*

Proof. The first half of this proposition follows from a simple application of Lemma 2.1 to the case $T = H^2$ and $S = A$.

Since $D(A) \subset D(\bar{A})$, we have $D(H^2) \subset D(\bar{A})$. By the strong commutativity of $\bar{A}$ and H^2, we have $e^{itH^2}\bar{A} \subset \bar{A}e^{itH^2}$ for all $t \in \mathbb{R}$. Hence, for all $\psi, \phi \in D(H^2)$ and $t \in \mathbb{R}$, we have $\left\langle e^{itH^2}\psi, \bar{A}\phi\right\rangle = \left\langle \bar{A}\psi, e^{-itH^2}\phi\right\rangle$. Differentiating the both sides at $t = 0$, we obtain $\left\langle H^2\psi, \bar{A}\phi\right\rangle = \left\langle \bar{A}\psi, H^2\phi\right\rangle$. Thus $\bar{A} \in \{H^2\}'_{\mathrm{w}}$. ■

The set $\{H^2\}'_{\mathrm{w}}$ has a real linear structure:

Proposition 2.3 *Let $A, B \in \{H^2\}'_{\mathrm{w}}$. Then $A + B \in \{H^2\}'_{\mathrm{w}}$ and $\lambda A \in \{H^2\}'_{\mathrm{w}}$ for all $\lambda \in \mathbb{R}$.*

Proof. By definition, $D(A + B) = D(A) \cap D(B)$ and $D(\lambda A) = D(A), \forall\lambda \in \mathbb{R}$. Hence $D(H^2) \subset D(A + B)$ and $D(H^2) \subset D(\lambda A), \forall\lambda \in \mathbb{R}$. Then it is easy to see that $A + B$ (resp. λA) is symmetric and (1.1) with A replaced by $A + B$ (resp. λA) holds. ■

Propositions 2.2 and 2.3 immediately yield the following result:

Corollary 2.4 *Let $A_1, \cdots, A_n \in \{H^2\}'_{\mathrm{w}}$ $(n = 1, 2, \cdots)$. Then, for all $a_1, \cdots, a_n \in \mathbb{R}$, the operator $\sum_{j=1}^n a_jA_j$ is essentially self-adjoint on $D(H^2)$ and $\overline{\sum_{j=1}^n a_jA_j}$ strongly commutes with H^2.*

The following proposition shows the structure of a subset of $\{H^2\}'_{\mathrm{w}}$.

Proposition 2.5 *Let B (resp. C) be a self-adjoint operator strongly commuting (resp. anticommuting) with H and $D(H^2) \subset D(B) \cap D(C)$. Let $A := B + C$. Then, $A \in \{H^2\}'_{\mathrm{w}}$. In particular, A is essentially self-adjoint on $D(H^2)$ and $\bar{A}$ strongly commutes with H^2.*

Proof. The strong commutativity of B with H implies that of B with H^2. Hence, (1.1) with A replaced by B follows from the two variable functional calculus on the pair (H^2, B). Thus $B \in \{H^2\}'_{\mathrm{w}}$. As for the operator C, we have $e^{itC}H \subset He^{-itC}$ for all $t \in \mathbb{R}$. This implies that $e^{itC}H^2 \subset H^2 e^{itC}$ for all $t \in \mathbb{R}$. Hence C strongly commutes with H^2. Hence, as in the case of B, C is in $\{H^2\}'_{\mathrm{w}}$. These results and Proposition 2.3 imply that $A \in \{H^2\}'_{\mathrm{w}}$. Hence Proposition 2.2 imply the last statement of the present proposition. ■

From a purely operator-theoretical point of view, it would be interesting to investigate an inverse problem concerning Proposition 2.5: Under what conditions does a self-adjoint operator A have a decomposition as described in Proposition 2.5 ? This is one of the motivations for the present work.

3 A similarity transformation

In what follows, we need the following condition:

Hypothesis (H): H is injective.

We assume (H). Then, for a linear operator A on $\mathcal{H}$, we define a linear operator A_H by

$$A_H := HAH^{-1}, \tag{3.1}$$

which is a similarity transform of A by H.

Lemma 3.1 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Then:*

(i) For all $\psi, \phi \in D(A_H) \cap D(H)$, $\langle \psi, A_H\phi \rangle = \langle A_H\psi, \phi \rangle$.

(ii) For all $\psi, \phi \in D(H^2) \cap D(A_H)$,

$$\left\langle H^2\psi, A_H\phi \right\rangle = \left\langle A_H\psi, H^2\phi \right\rangle. \tag{3.2}$$

Proof. (i) Let $\psi, \phi \in D(A_H) \cap D(H)$. Then $H^{-1}\psi, H^{-1}\phi \in D(H^2)$ and

$$\begin{aligned} \langle \psi, A_H \phi \rangle &= \left\langle H^2(H^{-1}\psi), A(H^{-1}\phi) \right\rangle = \left\langle A(H^{-1}\psi), H^2(H^{-1}\phi) \right\rangle \\ &= \langle A_H \psi, \phi \rangle , \end{aligned}$$

where (1.1) is used to obtain the second equality.

(ii) Let $\phi \in D(H^2) \cap D(A_H)$. Then $H^{-1}\phi \in D(A) \subset D(\bar{A})$. Hence, by the strong commutativity of H^2 and $\bar{A}$ (Proposition 2.2), $e^{-itH^2}H^{-1}\phi \in D(\bar{A})$ and $\bar{A}e^{-itH^2}H^{-1}\phi = e^{-itH^2}AH^{-1}\phi$. Note that $e^{-itH^2}H^{-1}\phi \in D(H^2)$ and $e^{-itH^2}\phi \in D(H^{-1})$. Hence

$$\bar{A}e^{-itH^2}H^{-1}\phi = AH^{-1}e^{-itH^2}\phi.$$

Hence $AH^{-1}e^{-itH^2}\phi = e^{-itH^2}AH^{-1}\phi$. Since $AH^{-1}\phi \in D(H)$, it follows that $e^{-itH^2}AH^{-1}\phi$ is in $D(H)$ and $He^{-itH^2}AH^{-1}\phi = e^{-itH^2}A_H\phi$. Hence $AH^{-1}e^{-itH^2}\phi$ is in $D(H)$ and $A_He^{-itH^2}\phi = e^{-itH^2}A_H\phi$. Therefore, for all $\psi \in D(H^2) \cap D(A_H)$, we have

$$\left\langle e^{itH^2}\psi, A_H\phi \right\rangle = \left\langle A_H\psi, e^{-itH^2}\phi \right\rangle .$$

Differentiating the both sides at $t = 0$, we obtain (3.2). ∎

Lemma 3.1 and Lemma 2.1 imply the following theorem:

Theorem 3.2 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Suppose that $D(H^2) \subset D(A_H)$. Then $A_H \in \{H^2\}'_{\mathrm{w}}$. In particular, A_H is essentially self-adjoint on $D(H^2)$ and $\bar{A}_H$ strongly commutes with H^2.*

Corollary 3.3 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Then $A_{H^{-1}}\psi = A_H\psi$ for all $\psi \in D(A_H) \cap D(A_{H^{-1}})$. In particular, if $D(H^2) \subset D(A_H) \cap D(A_{H^{-1}})$, then $A_{H^{-1}} \in \{H^2\}'_{\mathrm{w}}$.*

Proof. By Proposition 2.2, A is essentially self-adjoint on $D(H^2)$ and $\bar{A}$ strongly commutates with H^2. Hence $\bar{A}$ strongly commutes with $|H|$ too. Let $H = U|H|$ be the polar decomposition of H. Then U is unitary and self-adjoint, since H is injective. By the two variable functional calculus, we have $|H|A|H|^{-1}\psi = |H|^{-1}A|H|\psi$ for all $\psi \in D(|H|A|H|^{-1}) \cap D(|H|^{-1}A|H|)$. Hence, for all $\psi \in D(A_H) \cap D(A_{H^{-1}})$, $A_H\psi = U|H|A|H|^{-1}U\psi = U|H|^{-1}A|H|U\psi = A_{H^{-1}}\psi$. ∎

4 Decomposition theorems

Assume (H). Then, for a linear operator A on $\mathcal{H}$, we can define

$$A_{\pm,0} := \frac{1}{2}(A \pm A_H), \tag{4.1}$$

where A_H is defined by (3.1). It is easy to see that

$$A \supset A_{+,0} + A_{-,0}. \tag{4.2}$$

For a linear operator L and a subspace $D \subset D(L)$, we denote by $L|D$ the restriction of L to D.

Theorem 4.1 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Suppose that $D(H^2) \subset D(A_H)$. Then:*

(i) $A_{\pm,0}$ are essentially self-adjoint on $D(H^2)$ and

$$A_\pm := \bar{A}_{\pm,0} \tag{4.3}$$

strongly commute with H^2.

(ii) The operator $A_+ + A_-$ is essentially self-adjoint on $D(H^2)$ and the operator equality

$$\bar{A} = \overline{A_+ + A_-} \tag{4.4}$$

holds.

Proof. (i) By Theorem 3.2, $A_H \in \{H^2\}'_{\mathrm{w}}$. Hence, by Corollary 2.4, the assertions follow.

(ii) By (4.2) and (i), we have $A|D(H^2) \subset A_+ + A_-$. By this relation and Proposition 2.2, $A_+ + A_-$ is essentially self-adjoint on $D(H^2)$ and (4.4) holds. ∎

If $A_+ + A_-$ is closed, then (4.4) takes the form $\bar{A} = A_+ + A_-$, which gives a decomposition of $\bar{A}$. Hence a next task to be done is to investigate conditions which ensure the closedness of $A_+ + A_-$.

Lemma 4.2 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Suppose that $D(H) \subset D(A)$. Then, for all $\psi \in D(A_H) \cap D(H)$ and $\phi \in D(H)$,*

$$\langle H\psi, A\phi\rangle = \langle A_H\psi, H\phi\rangle. \tag{4.5}$$

Proof. Let $\phi \in D(H^2)$ first. Then $\langle H\psi, A\phi\rangle = \langle H^2(H^{-1}\psi), A\phi\rangle = \langle AH^{-1}\psi, H^2\phi\rangle = \langle A_H\psi, H\phi\rangle$. Hence (4.5) holds.

We next consider the case $\phi \in D(H)$. By the condition $D(H) \subset D(A)$ and the closed graph theorem, there exists a constant $c \geq 0$ such that

$$\|A\psi\| \leq c(\|H\psi\| + \|\psi\|), \quad \psi \in D(H). \tag{4.6}$$

Since $D(H^2)$ is a core of H, for each $\phi \in D(H)$, there exists a sequence $\{\phi_n\}_{n=1}^{\infty}$ with $\phi_n \in D(H^2)$ such that $\lim_{n\to\infty} \phi_n = \phi$ and $\lim_{n\to\infty} H\phi_n = H\phi$. By the preceding result, we have $\langle H\psi, A\phi_n\rangle = \langle A_H\psi, H\phi_n\rangle$. By (4.6), $A\phi_n \to A\phi$ as $n \to \infty$. Thus (4.5) follows. ■

Lemma 4.3 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Suppose that $D(H) \subset D(A)$. Then:*

(i) For all $\psi, \phi \in D(A_H) \cap D(H)$,

$$\langle H\psi, A_{+,0}\phi\rangle = \langle A_{+,0}\psi, H\phi\rangle\,. \tag{4.7}$$

(ii) For all $\psi, \phi \in D(A_H) \cap D(H)$,

$$\langle H\psi, A_{-,0}\phi\rangle + \langle A_{-,0}\psi, H\phi\rangle = 0. \tag{4.8}$$

Proof. Direct computations using (4.5). ■

Theorem 4.4 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Suppose that $D(H) \subset D(A_H) \cap D(A)$. Then $A_{+,0}$ is essentially self-adjoint on $D(H)$ and A_+ strongly commutes with H.*

Proof. This follows from Lemma 4.3(i) and Lemma 2.1. ■

We need the following fact:

Lemma 4.5 (S. Pedersen [10, Theorem 4.3]) *Let A and B be symmetric operators on $\mathcal{H}$ and $\mathcal{D}$ be a dense subspace such that $\mathcal{D} \subset D(A) \cap D(B)$. Suppose that $(A \pm B)|\mathcal{D}$ are essentially self-adjoint and $\langle A\psi, B\phi\rangle + \langle B\psi, A\phi\rangle = 0$ for all $\psi, \phi \in \mathcal{D}$. Then A and B are essentially self-adjoint. Moreover, $\bar{A}$ and $\bar{B}$ strongly anticommute.*

Theorem 4.6 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$. Suppose that $D(H) \subset D(A) \cap D(A_H)$. Then $A_{-,0}$ is essentially self-adjoint and A_- strongly anticommutes with H.*

Proof. Let $T_\pm := H \pm A_{-,0}$. Then $D(T_\pm) = D(H)$ and $T_\pm$ are symmetric. For all $\psi, \phi \in D(H^2)$, we have $\langle T_\pm\psi, H^2\phi\rangle = \langle H^2\psi, T_\pm\phi\rangle$. Hence, by Lemma 2.1, $T_\pm$ are essentially self-adjoint on $D(H^2)$. By Lemma 4.3(ii), we have $\langle H\psi, A_{-,0}\phi\rangle + \langle A_{-,0}\psi, H\phi\rangle = 0$ for all $\psi, \phi \in D(H^2)$. Hence, by Lemma 4.5, $A_{-,0}$ is essentially self-adjoint and $A_- = \bar{A}_{-,0}$ strongly anticommutes with H. ■

Theorem 4.7 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$ with $D(H) \subset D(A) \cap D(A_H)$. Suppose that there exists a constant $C > 0$ such that*

$$\|A_H\psi\|^2 \le \|A\psi\|^2 + 2C^2\|\psi\|^2, \quad \psi \in D(H^2). \tag{4.9}$$

Then $A_{\pm,0}$ are essentially self-adjoint with $D(\bar{A}) = D(A_+) \cap D(A_-)$ and the operator equality

$$\bar{A} = A_+ + A_- \tag{4.10}$$

holds. Moreover, for all $\psi \in D(A_+) \cap D(A_-)$

$$\|A_+\psi\|^2 + \|A_-\psi\|^2 \le \|\bar{A}\psi\|^2 + C^2\|\psi\|^2. \tag{4.11}$$

Proof. Let $\psi \in D(H^2)$. Then

$$\|(A_{+,0} + A_{-,0})\psi\|^2 = \|A_{+,0}\psi\|^2 + \|A_{-,0}\psi\|^2 + \langle A_{+,0}\psi, A_{-,0}\psi\rangle + \langle A_{-,0}\psi, A_{+,0}\psi\rangle .$$

It is easy to see that

$$\langle A_{+,0}\psi, A_{-,0}\psi\rangle + \langle A_{-,0}\psi, A_{+,0}\psi\rangle = \frac{1}{2}(\|A\psi\|^2 - \|A_H\psi\|^2), \tag{4.12}$$

which, together with (4.9), implies (4.11) with $\psi \in D(H^2)$. Then, by a limiting argument using the fact that $D(H^2)$ is a core of $\bar{A}$, we see that $D(\bar{A}) \subset D(A_+) \cap D(A_-)$ and (4.11) holds for all $\psi \in D(\bar{A})$. Since $\overline{A_+ + A_-} = \overline{A_{+,0} + A_{-,0}} = \bar{A}$, it follows that $D(A_+) \cap D(A_-) \subset D(\bar{A})$. Thus the desired results follow. ■

The decomposition (4.10) is unique in the following sense:

Theorem 4.8 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$ with $D(H^2) \subset D(A_H)$. Suppose that there exists a pair (B_+, B_-) of self-adjoint opeartors having the following properties:*

(B.1) $\bar{A} = B_+ + B_-$.

(B.2) $D(H^2)$ is a common core of $B_\pm$.

(B.3) B_+ strongly commutes with H and B_- strongly anticommutes with H.

Then $A_+ = B_+$ and $A_- = B_-$.

Proof. Let $S_1 = A_+ - B_+$ and $S_2 = B_- - A_-$. Then, by (B.1) and (4.2), we have $S_1\psi = S_2\psi$ for all $\psi \in D(H^2)$. It is easy to see that, for all $t \in \mathbb{R}$, $e^{itH}S_1 \subset S_1 e^{itH}$ and $e^{itH}S_2 \subset S_2 e^{-itH}$. Hence, for all $t \in \mathbb{R}$,

$$e^{itH}S_1\psi = S_1 e^{itH}\psi = S_2 e^{itH}\psi = e^{-itH}S_2\psi = e^{-itH}S_1\psi, \tag{4.13}$$

i.e., $e^{2itH}S_1\psi = S_1\psi$. This implies that $S_1\psi \in D(H)$ and $HS_1\psi = 0$. Hence $S_1\psi = 0$. In particular, we have $A_+\psi = B_+\psi$ for all $\psi \in D(H^2)$. Since $D(H^2)$ is a common core of A_+ and B_+, it follows that $A_+ = B_+$ (operator equality). By this result and (4.13), we have $S_2\psi = 0$ for all $\psi \in D(H^2)$. Then, in the same way as in the case of (A_+, B_+), we see that the operator equality $A_- = B_-$ holds. ■

Proposition 4.9 *Let the same assumption as in Theorem 4.1 be satisfied. Suppose that*

$$\|A_H\psi\|^2 = \|A\psi\|^2, \quad \psi \in D(H^2). \tag{4.14}$$

Then A_+ and A_- strongly anticommute.

Proof. By Theorem 4.1, $(A_+ + A_-)|D(H^2)$ is essentially self-adjoint. Theorem 3.2 implies that $(A_+ - A_-)|D(H^2)$ $(= A_H|D(H^2))$ is essentially self-adjoint. Moreover, by (4.12) and (4.14), we have $\langle A_+\psi, A_-\phi\rangle + \langle A_-\psi, A_+\phi\rangle = 0$, $\psi, \phi \in D(H^2)$. Hence, we can apply Lemma 4.5 to conclude that A_+ and A_- strongly anticommute. ■

Corollary 4.10 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$ with $D(H) \subset D(A) \cap D(A_H)$. Suppose that (4.14) holds. Then $A_{\pm,0}$ are essentially self-adjoint with $D(\bar{A}) = D(A_+) \cap D(A_-)$, and A_+ and A_- strongly anticommute. Moreover, the operator equality*

$$\bar{A} = A_+ + A_- \tag{4.15}$$

holds and

$$\|\bar{A}\psi\|^2 = \|A_+\psi\|^2 + \|A_-\psi\|^2, \quad \psi \in D(\bar{A}). \tag{4.16}$$

Proof. It is obvious that (4.14) implies (4.9). Hence the conclusion of Theorem 4.7 holds. The strong anticommutativity of A_+ and A_- follows from Proposition 4.9. Equation (4.16) is just an application of a general theorem on two strongly anticommuting self-adjoint operators (Lemma 4.11 below). ■

Lemma 4.11 (Vasilescu[14, Theorem 2.1]) *If S ad T are strongly anticommuting self-adjoint operators on a Hilbert space, then $S+T$ is self-adjoint and*

$$(S+T)^2 = S^2 + T^2, \tag{4.17}$$

$$\|(S+T)\psi\|^2 = \|S\psi\|^2 + \|T\psi\|^2, \quad \forall \psi \in D(S) \cap D(T). \tag{4.18}$$

5 Spectral properties of H

In this section we describe some implications of Thereom 4.6. But, before doing that, we first prove some general facts in the theory of strongly anticommuting self-adjoint operators.

For a linear operator L on a Hilbert space, we denote by $\sigma(L)$ (resp. $\sigma_{\rm p}(L)$) the spectrum (resp. the point spectrum) of L.

We say that a subset Σ of $\mathbb{R}$ is symmetric with respect to the origin if, for all $\lambda \in \Sigma$, $-\lambda \in \Sigma$.

Let S and T be strongly anticommuting self-adjoint operators on $\mathcal{H}$. We denote by $\mathrm{Ran}(S)$ the range of S.

Lemma 5.1 *(i) T is reduced by $\overline{\mathrm{Ran}(S)}$.*

(ii) We denote by T_S the reduced part of T to $\overline{\mathrm{Ran}(S)}$. Suppose that $S \neq 0$. Then T_S is unitarily equivalent to $-T_S$. In particular, $\sigma(T_S)$ (resp. $\sigma_{\rm p}(T_S)$) is symmetric with respect to the origin.

Proof. (i) Let $S = U_S|S|$ be the polar decomposition of S with $\ker U_S = \ker S$. Then, by [10, Theorem 3.1],

$$U_S T \subset -TU_S \tag{5.1}$$

and $|S|$ strongly commutes with T. The latter implies that T is reduced by $\overline{\mathrm{Ran}(|S|)}$ ([7, Lemma 6.5]). Since S and $|S|$ are self-adjoint, we have the following orthogonal decompositions of $\mathcal{H}$: $\mathcal{H} = \ker S \oplus \overline{\mathrm{Ran}(S)} = \ker|S| \oplus \overline{\mathrm{Ran}(|S|)}$. We have $\ker S = \ker|S|$. Hence $\overline{\mathrm{Ran}(|S|)} = \overline{\mathrm{Ran}(S)}$. Therefore T is reduced by $\overline{\mathrm{Ran}(S)}$.

(ii) Since $S \neq 0$, the operator $V := U_S|\overline{\mathrm{Ran}(S)}$ is a unitary operator on $\overline{\mathrm{Ran}(S)}$. By (5.1), we have $VT_SV^{-1} = -T_S$. Hence T_S is unitarily equivalent to $-T_S$. The statement of the spectral properties of T_S follows from the unitary invariance of the spectrum and the point spectrum of a linear operator. ■

Let

$$P_+(T) := E_T((0,\infty)), \quad P_-(T) := E_T((-\infty,0)), \quad P_0(T) := E_T(\{0\}) \tag{5.2}$$

and

$$\mathcal{H}_\pm(T) := \mathrm{Ran}(P_\pm(T)). \tag{5.3}$$

Then we have

$$\mathcal{H} = \mathcal{H}_+(T) \oplus \mathcal{H}_-(T) \oplus \ker T. \tag{5.4}$$

Lemma 5.2 *Let S and T be as above. Then*

(i) $P_\pm(T)S \subset SP_\mp(T)$ and $P_0(T)S \subset SP_0(T)$.

(ii) S maps $D(S) \cap \mathcal{H}_\pm(T)$ (resp. $\ker T$) to $\mathcal{H}_\mp(T)$ (resp. $\ker T$).

Proof. (i) Since S and T strongly anticommute, it follows from [10, Proposition 1.4] that $E_T(J)S \subset SE_T(-J)$ for all Borel sets $J \subset \mathbb{R}$, where $-J := \{-\lambda|\lambda \in J\}$. Applying this fact with $J = (0,\infty), (-\infty,0), \{0\}$, we obtain the desired results.

(ii) This follows from part (ii). ■

We now return to the operator theory developed in the previous sections.

Theorem 5.3 *Assume (H). Suppose that there exists an operator $A \in \{H^2\}'_{\mathrm{w}}$ having the following properties:*

(a) $D(H) \subset D(A) \cap D(A_H)$.

(b) $A_- \neq 0$.

Then:

(i) H is reduced by $\overline{\mathrm{Ran}(A_-)}$

We denote by $\widetilde{H}$ the reduced part of H to $\overline{\mathrm{Ran}(A_-)}$.

(ii) $\widetilde{H}$ *is unitarily equivalent to* $-\widetilde{H}$*. In particular,* $\sigma(\widetilde{H})$ *(resp.* $\sigma_{\rm p}(\widetilde{H})$*) is symmetric with respect to the origin.*

(iii) $\mathcal{H} = \mathcal{H}_+(H) \oplus \mathcal{H}_-(H)$*.*

(iv) $P_\mp(H)A_- \subset A_-P_\pm(H)$*.*

(v) A_- *maps* $D(A_-) \cap \mathcal{H}_\pm$ *to* $\mathcal{H}_\mp$*.*

Proof. By Theorem 4.6, A_- is self-adjoint and strongly anticommutes with H. Hence we can apply Lemmas 5.1 and 5.2 to the case $T = H$ and $S = A_-$ and obtain the desired results (the injectivity of H implies that $\ker H = \{0\}$). ■

Corollary 5.4 *Assume (H). Suppose that there exists an operator* $A \in \{H^2\}'_{\rm w}$ *having the following properties:*

(a) $D(H) \subset D(A) \cap D(A_H)$*.*

(b) A_- *is injective.*

Then $\sigma(H)$ *and* $\sigma_{\rm p}(H)$ *are symmetric with respect to the origin.*

Proof. Condition (b) implies that $\overline{\mathrm{Ran}(A_-)} = \mathcal{H}$. Hence $\widetilde{H} = H$. Then Theorem 5.3(ii) gives the desired result. ■

In concluding this section, we give a sufficient condition for A_- to be injective (condition (b) in Corollary 5.4).

Proposition 5.5 *Assume (H) and let* $A \in \{H^2\}'_{\rm w}$ *be self-adjoint.*

(i) Suppose that there exists a self-adjoint operator P *on* $\mathcal{H}$ *having the following properties:*

(a) P *strongly commutes with* H*.*

(b) $AH + HA = 2P$ *on* $D(AH) \cap D(HA) \cap D(P)$*.*

(c) H^{-1} *and* PH^{-1} *are bounded.*

(d) $D(H^2) \subset D(A_H) \cap D(P)$

Then $D(A_-) = D(A)$ with

$$A_- = A - PH^{-1}. \tag{5.5}$$

(ii) In addition to the assumption of part (i), assume the following (f), (g) and (h):

(f) P is injective and strongly commutes with A.

(g) A is injetive with A^{-1} bounded and $D(H) \subset D(A) \cap D(HA)$.

(h) $\{\psi \in D(AP^{-1}) \mid \|AP^{-1}\psi\| = \|H^{-1}\psi\|\} = \{0\}$.

Then A_- is injective. In particular, $\sigma(H)$ and $\sigma_{\mathrm{p}}(H)$ are symmetric with respect to the origin.

Proof. (i) Let $\psi \in D(H^2)$. Then $H^{-1}\psi \in D(AH) \cap D(HA) \cap D(P)$. Hence, by condition (b), we have

$$HAH^{-1}\psi = 2PH^{-1}\psi - A\psi,$$

which implies that $A_-\psi = A\psi - PH^{-1}\psi$. Since $(PH^{-1})^* \supset H^{-1}P$ and $(PH^{-1})^*$ is bounded, it follows that $H^{-1}P$ is closable with domain equal to $D(P)$ and $\overline{H^{-1}P} = (PH^{-1})^*$. By the Kato-Rellich theorem, $B := A - (PH^{-1} + \overline{H^{-1}P})/2$ is self-adjoint and every core of A is a core of B. In particular, $D(H^2)$ is a core of B. It follows from the above equality that $A_-|D(H^2) \subset B$. Hence $A_- = B$, since A_- is essentially self-adjoint on $D(H^2)$. Thus $A_- = A - \frac{1}{2}(PH^{-1} + \overline{H^{-1}P})$. It follows from condition (a) and the boundedness of H^{-1} that $H^{-1}P \subset PH^{-1}$. This implies that $\overline{H^{-1}P} = PH^{-1}$. Thus (5.5) follows.

(ii) To prove the injectivity of A_-, we show that $\ker A_- = \{0\}$. Let $\psi \in \ker A_-$. Then, by (5.5),

$$A\psi = PH^{-1}\psi. \tag{5.6}$$

Hence $A\psi \in D(P^{-1})$ and

$$P^{-1}A\psi = H^{-1}\psi. \tag{5.7}$$

By conditions (f) and (g), we have $A^{-1}P \subset PA^{-1}$, which, together with (5.6), implies that $\psi = PA^{-1}H^{-1}\psi$. This means that $\psi \in D(P^{-1})$ and $P^{-1}\psi = A^{-1}H^{-1}\psi$. Hence $P^{-1}\psi \in D(A)$ and $AP^{-1}\psi = H^{-1}\psi$. By this equation and (5.7), we obtain that $\psi \in D(P^{-1}A) \cap D(AP^{-1})$ and

$$P^{-1}A\psi = AP^{-1}\psi. \tag{5.8}$$

Multiplying (5.7) by H^{-1} from the left and using the strong commutativity of H and P, we have $P^{-1}H^{-1}A\psi = H^{-2}\psi$. Condition (b) implies that $H^{-1}A\psi = -AH^{-1}\psi + 2H^{-1}PH^{-1}\psi$. Hence we obtain that $H^{-2}\psi = P^{-1}AH^{-1}\psi$. By (5.7), the right hand side is equal to $P^{-1}AP^{-1}A\psi$. By this fact and (5.8), we obtain that $\|H^{-1}\psi\|^2 = \|AP^{-1}\psi\|^2$. Hence, by condition (h), $\psi = 0$. Thus $\ker A_- = \{0\}$. The last assertion follows from Corollary 5.4 (note that $D(H) \subset D(A_H)$, since H^{-1} is bounded and $D(H) \subset D(HA)$). ■

6 Generalized weak canonical commutation relation and Heisenberg operators

6.1 A general theory

Let H be a self-adjoint operator on $\mathcal{H}$. For a linear operator X on $\mathcal{H}$, we define a linear operator $X(t)$ ($t \in \mathbb{R}$) by

$$X(t) := e^{itH} X e^{-itH}, \tag{6.1}$$

which is called the Heisenberg operator of X with respect to H.

Remark 6.1 In applications to quantum mechanics, we have in mind the case where H is the Hamiltonian of a quantum system and X is an observable. In this case, $X(t)$ is interpreted as the time development of X. From this point of view too, it is important to study properties of $X(t)$.

In this section we present a class of linear operators X such that $X(t)$ has an explicit representation.

Definition 6.1 Let C be a linear operator on $\mathcal{H}$. We say that a linear operator X on $\mathcal{H}$ is in the set $\mathsf{K}(H, C)$ if $D(X)$ is dense and

$$\langle X^*\psi, H\phi\rangle - \langle H\psi, X\phi\rangle = \langle \psi, iC\phi\rangle, \quad \psi, \phi \in D(X) \cap D(X^*) \cap D(C) \cap D(H). \tag{6.2}$$

Remark 6.2 In a terminology introduced in the previous paper [7], (6.2) means that the triple (X, H, C) obeys the generalized weak canonical commutation relation if X is symmetric and C is a bounded self-adjoint operator on $\mathcal{H}$ ([7, §4]).

We need a lemma.

Lemma 6.2 *Assume (H). Then, for all $t \in \mathbb{R}$, the operator*

$$G_H(t) := \frac{1}{2i} H^{-1}(e^{2itH} - 1) \tag{6.3}$$

is bounded and

$$\|G_H(t)\| \leq |t|. \tag{6.4}$$

Moreover, $G_H(t)$ is strongly differentiable in t and

$$\text{s-}\frac{dG_H(t)}{dt} = e^{2itH}, \tag{6.5}$$

where s-$dG_H(t)/dt$ *means the strong derivative of $G_H(t)$.*

Proof. By the functional calculus and the easily proven estimate

$$|e^{2it\lambda} - 1|^2 \leq 4t^2\lambda^2, \quad \lambda \in \mathbb{R},$$

we see that $D(G_H(t)) = \mathcal{H}$ and $\|G_H(t)\psi\| \leq |t|\|\psi\|$, $\psi \in \mathcal{H}$. This proves the first half of the the lemma. Equation (6.5) follows from the functional calculus. ■

The following theorem defines a class of linear operators X, whose Heisenberg operator $X(t)$ with respect to H has an explicit representation.

Theorem 6.3 *Assume (H). Let $C_\pm$ be self-adjoint operators such that C_+ strongly commutes with H and C_- strongly anticommutes with H, and $C := C_+ + C_-$. Let $X \in \mathsf{K}(H, C)$. Suppose that there exists a dense subspace D such that*

(i) $D \subset D(X) \cap D(X^*) \cap D(H) \cap D(C)$;

(ii) $e^{-itH}D \subset D(X) \cap D(X^*) \cap D(H) \cap D(C)$ for all $t \in \mathbb{R}$;

(iii) for all $\psi \in D$, $Xe^{-itH}\psi$ is strongly continuous in $t \in \mathbb{R}$.

Then, for all $t \in \mathbb{R}$

$$X(t) = X + C_+t + G_H(t)C_- \tag{6.6}$$

on D.

Proof. For each vector $\eta \in \mathcal{H}$, we set $\eta(t) := e^{-itH}\eta$, $t \in \mathbb{R}$. Let $\psi, \phi \in D$ and $f(t) := \langle \psi, X(t)\phi \rangle$, $t \in \mathbb{R}$. Then, it is easy to see that f is differentiable with

$$\frac{df(t)}{dt} = i\{\langle H\psi(t), X\phi(t)\rangle - \langle X^*\psi(t), H\phi(t)\rangle\}.$$

Since $\psi(t)$ and $\phi(t)$ are in $D(H) \cap D(C) \cap D(X) \cap D(X^*)$ by the present assumption, we can compute, using (6.2), the right hand side of the above equation to obtain $df(t)/dt = \left\langle \psi, e^{itH} C e^{-itH} \phi \right\rangle$. Note that $e^{-itH}\phi \in D(C)$ due to the present assumption. Hence $e^{itH} C e^{-itH} \phi = e^{itH} C_+ e^{-itH} \phi + e^{itH} C_- e^{-itH} \phi$. The strong commutativity of C_+ and H implies that $e^{itH} C_+ e^{-itH} = C_+$, while the strong anticommutativity of C_- with H implies that $e^{itH} C_- e^{-itH} = e^{2itH} C_-$. Therefore

$$\frac{df(t)}{dt} = \langle \psi, C_+ \phi \rangle + \left\langle \psi, e^{2itH} C_- \phi \right\rangle .$$

Integrating this equation and using (6.5), we obtain

$$f(t) = \langle \psi, X\phi \rangle + \langle \psi, C_+ t\phi \rangle + \langle \psi, G_H(t) C_- \phi \rangle .$$

Hence (6.6) on D follows. ∎

Corollary 6.4 *Let the same assumption as in Theorem 6.3 be satisfied. Suppose that X is closed, D is a core of X and there exist constants $a_\pm \geq 0$ and $b_\pm \geq 0$ such that*

$$\|C_\pm \psi\| \leq a_\pm \|X\psi\| + b_\pm \|\psi\|, \quad \psi \in D. \tag{6.7}$$

Then, for all $t \in \mathbb{R}$, $e^{-itH} D(X) = D(X)$ and (6.6) holds on $D(X)$. Moreover

$$\|X(t)\psi\| \leq \|X\psi\| + (\|C_+\psi\| + \|C_-\psi\|)|t|, \quad \psi \in D(X), t \in \mathbb{R}. \tag{6.8}$$

Proof. By (6.6) and (6.4), we have for all $\psi \in D$

$$\|X e^{-itH} \psi\| \leq \|X\psi\| + (\|C_+\psi\| + \|C_-\psi\|)|t|. \tag{6.9}$$

Using (6.7), we obtain

$$\|X e^{-itH} \psi\| \leq (1 + (a_+ + a_-)|t|) \, \|X\psi\| + (b_+ + b_-)|t| \, \|\psi\|.$$

By this estimate and the condition that D is a core of X, we can show by a simple limiting argument that $e^{-itH} D(X) \subset D(X)$. Since $t \in \mathbb{R}$ is arbitrary and $\{e^{-itH}\}_{t \in \mathbb{R}}$ is a unitary group, it follows that $e^{-itH} D(X) = D(X)$ for all $t \in \mathbb{R}$. Then, by a limiting argument again, we can extend (6.6) to an equation on $D(X)$. Similarly (6.9) extends to all $\psi \in D(X)$ giving (6.8). ∎

In the context of quantum mechanics, the quantity $\langle \psi, T\phi \rangle$ with T a linear operator on a Hilbert space $\mathcal{H}$ and $\psi \in \mathcal{H}$, $\phi \in D(T)$ is called a matrix element of T and has physical importance.

The following result gives an explicit formula for each element in a subset of matrix elements of $X(t)$.

Corollary 6.5 *Let the same assumption as in Theorem 6.3 be satisfied. Let $\psi, \phi \in \mathcal{H}_+(H) \cap D$ or $\psi, \phi \in \mathcal{H}_-(H) \cap D$. Then*

$$\langle \psi, X(t)\phi \rangle = \langle \psi, X\phi \rangle + \langle \psi, C_+\phi \rangle\, t. \tag{6.10}$$

Proof. By the functional calculus, $G_H(t)^*\mathcal{H}_\pm(H) \subset \mathcal{H}_\pm(H)$. This fact and Lemma 5.2(ii) with $S = C_-$ and $T = H$ imply that $\langle \psi, G_H(t)C_-\phi \rangle = 0$. Hence (6.10) follows. ■

The following theorem describes the asymptotic behavior of matrix elements of $X(t)$ as $t \to \pm\infty$.

We denote by $\mathcal{H}_{\rm ac}(H)$ the subspace of absolute continuity with respect to H.

Theorem 6.6 *Let the same assumption as in Theorem 6.3 be satified. Let $\psi \in D(H^{-1}) \cap D$ and $\phi \in \mathcal{H}_{\rm ac}(H) \cap D$. Then*

$$\langle \psi, X(t)\phi \rangle = \langle \psi, X\phi \rangle - \left\langle H^{-1}\psi, \frac{1}{2i}C_-\phi \right\rangle + \langle \psi, C_+\phi \rangle\, t + o\left(\frac{1}{t}\right) \quad (t \to \pm\infty). \tag{6.11}$$

Proof. Let

$$\delta(t) := \frac{1}{2i}\left\langle H^{-1}\psi, e^{2itH}C_-\phi \right\rangle. \tag{6.12}$$

Then we have by (6.6)

$$\langle \psi, X(t)\phi \rangle = \langle \psi, X\phi \rangle - \left\langle H^{-1}\psi, \frac{1}{2i}C_-\phi \right\rangle + \langle \psi, C_+\phi \rangle\, t + \delta(t).$$

By the strong anticommutativity of C_- and H, we have $E_H(J)C_- \subset C_-E_H(-J)$ for all Borel sets $J \subset \mathbb{R}$. This implies that, if ϕ is in $\mathcal{H}_{\rm ac}(H) \cap D(C_-)$, then $C_-\phi \in \mathcal{H}_{\rm ac}(H)$. Hence $\lim_{t\to\pm\infty} \delta(t) = 0$. ■

6.2 The case $C = A_+ + A_-$ with $A \in \{H^2\}'_{\rm w}$

We now apply the results of the precdeding subsection to the operator theory concerning the pair (A, H) with $A \in \{H^2\}'_{\rm w}$.

Theorem 6.7 *Assume (H) and let $A \in \{H^2\}'_{\mathrm{w}}$ with $D(H) \subset D(A_H) \cap D(A)$ (hence Theorems 4.4 and 4.6 hold). Let $X \in \mathsf{K}(H, A_+ + A_-)$. Suppose that there exists a dense subspace D_0 such that*

(i) $D_0 \subset D(X) \cap D(X^*) \cap D(H)$;

(ii) $e^{-itH} D_0 \subset D(X) \cap D(X^*) \cap D(H)$ for all $t \in \mathbb{R}$;

(iii) for all $\psi \in D_0$, $Xe^{-itH}\psi$ is strongly continuous in $t \in \mathbb{R}$.

Then, for all $t \in \mathbb{R}$,

$$X(t) = X + A_+ t + G_H(t) A_- \tag{6.13}$$

on D_0.

Proof. We can apply Theorem 6.3 with $C_\pm = A_\pm$ to obtain the desired result. ∎

Corollary 6.8 *Let the same assumption as in Theorem 6.7 be satisfied. Suppose that X is closed, D_0 is a core of X and there exist constants $a_\pm \geq 0$ and $b_\pm \geq 0$ such that*

$$\|A_\pm \psi\| \leq a_\pm \|X\psi\| + b_\pm \|\psi\|, \quad \psi \in D_0. \tag{6.14}$$

Then, for all $t \in \mathbb{R}$, $e^{-itH} D(X) = D(X)$ and (6.13) holds on $D(X)$. Moreover

$$\|X(t)\psi\| \leq \|X\psi\| + (\|A_+\psi\| + \|A_-\psi\|)|t|, \quad \psi \in D(X), t \in \mathbb{R}. \tag{6.15}$$

Proof. A simple application of Corollary 6.4. ∎

Corollary 6.9 *Let the same assumption as in Theorem 6.7 be satisfied. Let $\psi, \phi \in \mathcal{H}_+(H) \cap D_0$ or $\psi, \phi \in \mathcal{H}_-(H) \cap D_0$. Then*

$$\langle \psi, X(t)\phi \rangle = \langle \psi, X\phi \rangle + \langle \psi, A_+\phi \rangle \, t. \tag{6.16}$$

Proof. A simple application of Corollary 6.5. ∎

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Theorem 6.10 *Let the same assumption as in Theorem 6.7 be satified. Let $\psi \in D(H^{-1})$ $\cap$ D_0 and $\phi \in \mathcal{H}_{\mathrm{ac}}(H) \cap D_0$. Then*

$$\langle \psi, X(t)\phi \rangle = \langle \psi, X\phi \rangle - \left\langle H^{-1}\psi, \frac{1}{2i} A_- \phi \right\rangle + \langle \psi, A_+\phi \rangle \, t + o\left(\frac{1}{t}\right) \quad (t \to \pm\infty). \tag{6.17}$$

Proof. A simple application of Theorem 6.7. ∎

7 Connection with the generalized weak Weyl relation

In the previous paper [7], we introduced a notion, which we call the generalized weak Weyl relation (GWWR). This is defined as follows: Let T be a symemtric operator on $\mathcal{H}$, H be a self-adjoint operator on $\mathcal{H}$, and $K(t)$ $(t \in \mathbb{R})$ be a bounded self-adjoint operator on $\mathcal{H}$ for all $t \in \mathbb{R}$. We say that the triple (T, H, K) obeys the GWWR if $e^{-itH}D(T) \subset D(T)$ for all $t \in \mathbb{R}$ and

$$Te^{-itH}\psi = e^{-itH}(T + K(t))\psi, \quad \forall \psi \in D(T), \forall t \in \mathbb{R}.$$

In this section we point out that there is a connection of the operator theory developed in the previous sections with the theory of GWWR.

Theorem 7.1 *Suppose that the same assumption as in Theorem 6.7 holds with X being symmetric and that $A_\pm$ and H^{-1} are bounded. Let*

$$K_H(t) := A_+ t + G_H(t)A_-. \tag{7.1}$$

Then $K_H(t)$ is a bounded self-adjoint operator on $\mathcal{H}$ for all $t \in \mathbb{R}$ and (X, H, K_H) obeys the GWWR.

Proof. Under the present assumption, the assumption of Corollary 6.8 is satisfied. Hence (6.13) holds on $D(X)$. Hecne it is sufficient to prove that $K_H(t)$ is a bounded self-adjoint operator. The boundedness of $K_H(T)$ is obvious, since $A_\pm$ are bounded by the present assumption. Using the fact that $A_- e^{-2itH} = e^{2itH}A_-$ and $A_- H^{-1} = -H^{-1}A_-$, which follow from the strong anticommutativity of A_- and H, one can show that $G_H(t)A_-$ is self-adjoint. ∎

Corollary 7.2 *Let the same assumption as in Theorem 7.1 be satisfied. Then, for all $\psi, \phi \in \mathcal{H}$,*

$$\lim_{t\to\pm\infty} \left\langle \psi, e^{-itH}A_+\phi \right\rangle = 0. \tag{7.2}$$

Proof. We have

$$\text{s-}\lim_{t\to\pm\infty} \frac{K_H(t)}{t} = A_+.$$

Hence, applying [7, Corollary 3.3], we obtain (7.2). ∎

Theorem 7.3 *Assume (H) with H^{-1} bounded on $\mathcal{H}$. Suppose that there exist a closed symmetric operator X and a bounded self-adjoint operator $A \in \{H^2\}'_{\rm w}$ such that $X \in \mathsf{K}(H, A)$ and $A_\pm$ are bounded with $\ker A_+ = \{0\}$. Then $\sigma_{\rm p}(H) = \emptyset$.*

Proof. The present assumption imply that the assumption of Theorem 6.7 is satified. Hence Theorem 7.1 holds. Then, a simple application of [7, Corollary 3.4] yields the desired result. ■

8 Applications

Let (A, B, L) be a triple of self-adjoint operators on a Hilbert space on $\mathcal{H}$ obeying the following conditions:

(A.1) A is bounded and injective with A^{-1} bounded.

(A.2) A and B strongly anticommute.

(A.3) L strongly commutes with A and B.

We define an operator H by

$$H := AL + B \tag{8.1}$$

This operator is an abstract form of Dirac operators. A typical example is given as follows.

Example 8.1 Let $\alpha_j\,(j = 1, 2, 3)$ and β be 4×4 Hermitian matrices satisfying the anticommutation relations

$$\begin{aligned}&\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad \{\alpha_j, \beta\} = 0, \quad j, k = 1, 2, 3,\\ &\beta^2 = 1,\end{aligned}$$

where $\{X, Y\} := XY + YX$. Let $m > 0$ be a constant and D_j be the generalized partial differential operator in the variable x_j ($x = (x_1, x_2, x_3) \in \mathbb{R}^3$) acting in $L^2(\mathbb{R}^3)$. Then the free Dirac operator with mass m is defined by

$$H_{\rm D} := \sum_{j=1}^{3} \alpha_j(-iD_j) + m\beta$$

acting in the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)$. It is well known that H is self-adjoint [13, Theorem 1.1]. Let $B_j = \sum_{k\neq j}\alpha_k(-iD_k) + m\beta$ $(j = 1,2,3)$. Then $H_{\rm D} = \alpha_j(-iD_j) + B_j$ and $(\alpha_j, B_j, -iD_j)$ satisfies (A.1)–(A.3) with $A = \alpha_j, B = B_j$ and $L = -iD_j$. The triple $(\alpha_j, B_j, -iD_j)$ satisfies also the conditions assumed in the discussions below.

For other examples, see, e.g., [1, 2, 5, 7].

We first prove the following fact.

Theorem 8.1 *The operator H is self-adjoint and*

$$H^2 = A^2L^2 + B^2. \tag{8.2}$$

Moreover

$$\|H\psi\|^2 = \|AL\psi\|^2 + \|B\psi\|^2, \quad \psi \in D(H) = D(L)\cap D(B). \tag{8.3}$$

To prove this theorem, we need two lemmas.

Lemma 8.2 *Let S be an injective, bounded self-adjoint operator with S^{-1} bounded and T be a self-adjoint operator strongly commuting with S. Then ST is self-adjoint and $ST = TS$.*

Proof. Since $D(ST) = D(T)$, ST is densely defined. By the boundedness of S, we have $(ST)^* = TS$. The strong commutativity of S and T implies that $ST \subset TS$. Hence $(ST)^* \supset ST$. Therefore ST is symmetric. By the strong commutativity of S and T, there exists a unique two dimensional spectral measure E such that $E(J\times K) = E_S(J)E_T(K)$ for all Borel sets $J, K \subset \mathbb{R}$ and $S = \int_{\mathbb{R}^2}\lambda\, dE(\lambda,\mu), T = \int_{\mathbb{R}^2}\mu\, dE(\lambda,\mu)$. The operator $F := \int_{\mathbb{R}^2}\lambda\mu\, dE(\lambda,\mu)$ is self-adjoint with $D(F) = \{\psi\in\mathcal{H} | \int_{\mathbb{R}^2}\lambda^2\mu^2 d\|E(\lambda,\mu)\psi\|^2 < \infty\}$. It is easy to see that $ST \subset F$. By the condition that S^{-1} is bounded, the support of E is included in $[-\|S\|,-\delta]\cup[\delta,\|S\|\,]\times\mathbb{R}$ for some $\delta > 0$. Using this fact, one can show that $D(F)\subset D(T)$. Thus $ST = F$, showing that ST is self-adjoint. ■

Lemma 8.3 *Let S and T be as in Lemma 8.2. Let C be a self-adjoint operator strongly anticommuting with S and strongly commuting with T. Then C strongly anticommutes with ST.*

Proof. For all $t \in \mathbb{R}$, we have $e^{itC}S = Se^{-itC}$ (strong anticommutativity of C and S) and $e^{itC}T \subset Te^{itC}$ (strong commutativity of C and T). Hence it follows that $e^{itC}ST \subset STe^{-itC}$. Thus C strongly anticommutes with ST. ■

Proof of Theorem 8.1

We can apply Lemmas 8.2 and 8.3 with $S = A, T = L$ and $C = B$ to conclude that AL is self-adjoint strongly anticommuting with B. Hecne, by Lemma 4.11, H is self-adjoint and (8.2) holds. ■

Corollary 8.4 $A \in \{H^2\}'_{\rm w}$.

Proof. Since A strongly anticommutes with B, it strongly commutes with $|B|$ [10, Theorem 3.1]. Hence, for all $t \in \mathbb{R}$, we have by (8.2) $e^{itA}H^2 = H^2e^{itA}$, which implies the strong commutativity of A and H^2 and that $AH^2 \subset H^2A$. Thus, as remarked in Introduction, $A \in \{H^2\}'_{\rm w}$. ■

Lemma 8.5 *The operator L strongly commutes with H.*

Proof. One can easily see that, for all $t \in \mathbb{R}$, $e^{itL}H \subset He^{itL}$. This implies the strong commutativity of L and H. ■

Lemma 8.6 $AD(H) \subset D(H)$.

Proof. We have $D(H) = D(L) \cap D(B)$. Since A strongly commutes with L and strongly anticommutes with B, it follows that $AD(L) \subset D(L)$ and $AD(B) \subset D(B)$. Hence $AD(H) \subset D(H)$. ■

By Lemma 8.6, we have

$$D(H) \subset D(HA) \cap D(AH) \cap D(L). \tag{8.4}$$

Lemma 8.7 $AH + HA = 2A^2L$ *on* $D(H)$.

Proof. Let $\psi \in D(H)$. Then $AH\psi = A^2L\psi + AB\psi = A^2L\psi - BA\psi$ and $HA\psi = A^2L\psi + BA\psi$. Hence the desired result follows. ■

In what follows, we assume the following condition too:

Hypothesis (B): $B^2 \geq M^2$ for some constant $M > 0$.

Lemma 8.8 *Assume (B). Then:*

(i) H is injective and H^{-1} is bounded with $\|H^{-1}\| \le M^{-1}$.

(ii) LH^{-1} is bounded with $\|LH^{-1}\| \le \|A^{-1}\|$.

(iii) BH^{-1} is bounded with $\|BH^{-1}\| \le 1$.

(iv) $D(A_H) = \mathcal{H}$ and A_H is bounded.

Proof. (i) By (8.3), we have $\|H\psi\| \ge M\|\psi\|,\ \psi \in D(H)$. This implies that H is injective and H^{-1} is bounded with $\|H^{-1}\| \le M^{-1}$.

(ii) By (8.3), we have $\|L\psi\| \le \|A^{-1}\|\|H\psi\|$, which implies that $\|LH^{-1}\phi\| \le \|A^{-1}\|\|\phi\|$, $\phi \in \mathcal{H}$.

(iii) Similar to part (ii).

(iv) For all $\psi \in \mathcal{H}$, $H^{-1}\psi \in D(H)$. By Lemma 8.6, $AH^{-1}\psi \in D(H)$. Hence $\psi \in D(A_H)$. Thus $D(A_H) = \mathcal{H}$. Morever, by (8.3),

$$\begin{aligned} \|A_H\psi\| &\le \|ALAH^{-1}\psi\| + \|BAH^{-1}\psi\| \\ &\le (\|A\|^2\|LH^{-1}\| + \|A\|\|BH^{-1}\|)\|\psi\|, \end{aligned}$$

where we have used the relation $AB \subset -BA$, which follows from the strong anticommutativity of A and B. Hence A_H is bounded. ∎

In the present example, by Lemma 8.8, $A_\pm$ are bounded and self-adjoint and

$$A = A_+ + A_-. \tag{8.5}$$

Moreover, by the above results, we can apply Theorems 4.4 and 4.6 to conclude that A_+ strongly commutes with H and A_- strongly anticommutes with H. Also we can apply Proposition 5.5 with $P = A^2L$ to obtain

$$A_- = A - A^2LH^{-1}. \tag{8.6}$$

Hence

$$A_+ = A^2LH^{-1}. \tag{8.7}$$

Proposition 8.9 *The operators A_+ and A_- strongly anticommute.*

Proof. Since $A_\pm$ are bounded, it is sufficient to show that $A_-A_+ + A_+A_- = 0$. But this is easily done. An alternative proof is to check (4.14). ■

Theorem 8.10 *Suppose that L is injective. Then $\sigma(H)$ and $\sigma_{\mathrm{p}}(H)$ are symmetric with respect to the origin.*

Proof. We show that condition (h) in Proposition 5.5 holds in the present case ($P = A^2L$). Let $\psi \in D((A^2L)^{-1})$ and $\|A(A^2L)^{-1}\psi\| = \|H^{-1}\psi\|$. Then

$$\left\langle L^{-1}\psi, A^{-2}L^{-1}\psi\right\rangle = \left\langle \psi, (A^2L^2 + B^2)^{-1}\psi\right\rangle \le \left\langle \psi, (A^2L^2 + M^2)^{-1}\psi\right\rangle.$$

Since A and L strongly commute, there exists a two-dimensional spectral measure $E_{A,L}$ such that $E_{A,L}(J \times K) = E_A(J)E_L(K)$ for all Borel sets $J, K \subset \mathbb{R}$ and

$$A = \int_{\mathbb{R}^2} \lambda \, dE_{A,L}(\lambda, \mu), \quad L = \int_{\mathbb{R}^2} \mu \, dE_{A,L}(\lambda, \mu).$$

By the above inequality, we have

$$\int_{\mathbb{R}^2} \left(\frac{1}{\lambda^2\mu^2 + M^2} - \frac{1}{\lambda^2\mu^2} \right) d\|E_{A,L}(\lambda, \mu)\psi\|^2 \ge 0,$$

which implies that

$$\int_{\mathbb{R}^2} \frac{1}{\lambda^2\mu^2(\lambda^2\mu^2 + M^2)} d\|E_{A,L}(\lambda, \mu)\psi\|^2 = 0.$$

Hence $\psi = 0$. Thus we can apply Proposition 5.5 to obtain the desired result. ■

Remark 8.1 Theorem 8.10 is interesting in that it clarifies a general structure for a self-adjoint operator H of the form (8.1) to have the spectral reflection symmetry.

Let X be a closed symmetric operator satisfying the following conditions:

(X.1) $X \in \mathsf{K}(L, I) \cap \mathsf{K}(B, 0)$.

(X.2) $AX \subset XA$.

(X.3) There exists a core D_0 of X such that

(i) $D_0 \subset D(X) \cap D(H)$.

(ii) $e^{-itH}D_0 \subset D(X) \cap D(H)$ for all $t \in \mathbb{R}$.

(iii) for all $\psi \in D_0$, $Xe^{-itH}\psi$ is strongly continuous in $t \in \mathbb{R}$.

Theorem 8.11 *For all $t \in \mathbb{R}$, $e^{-itH}D(X) = D(X)$ and*

$$X(t) = X + A_+ t + G_H(t)A_- \tag{8.8}$$

on $D(X)$. Moreover

$$\|X(t)\psi\| \le \|X\psi\| + (\|A_+\| + \|A_-\|)\|\psi\||t|, \quad \psi \in D(X),\, t \in \mathbb{R}.$$

Proof. Using (X.1) and (X.2), we can show that $X \in \mathsf{K}(H, A)$. Then, applying Corollary 6.8, we obtain the desired results. ∎

Also we have a corollary and a theorem following from Corollary 6.9 and Theorem 6.10 respectively. But we omit writing them down here.

Example 8.2 As for Example 8.1, we have the decomposition

$$\alpha_j = \alpha_{j,+} + \alpha_{j,-}$$

with $\alpha_{j,-} = \alpha_j - (-iD_j)H_{\mathrm{D}}^{-1}$, where $\alpha_{j,+}$ strongly commutes with H_{D} and $\alpha_{j,-}$ strongly anticommutes with H_{D}. In the context of quantum mechanics, the operator $\alpha_{j,+} = (-iD_j)H_{\mathrm{D}}^{-1}$ is called the classical velocity operator and the $\alpha_{j,-}$ describes the so-called *Zitterbewegung* [13, §1.6].

The position operator x_j satisfies all the conditions imposed on X in the above theory. Hence we can apply Theorem 8.11 to obtain an explicit formula for the Heisenberg operator $x_j(t) = e^{itH_{\mathrm{D}}}x_j e^{-itH_{\mathrm{D}}}$, which conincides with the known formula [13, Theorem 1.3].

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