Absence of Ground States for a Class of Translation Invariant Models of Non-relativistic QED

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Abstract

We consider translation invariant models of non-relativistic QED with net charge. Under certain natural assumptions we prove that ground states do not exist.

1 Introduction

Over the years there has been much interest in trying to develop an appropriate mathematical framework to describe the interaction of charged particles with the quantized electromagnetic field, see for example [1] and references given therein. Of course relativistic quantum electrodynamics (QED) is a very successful theory but has not been shown to provide a Hilbert space framework for describing the states of charged particles interacting with photons. In spite of this there are certainly prescriptions for getting correct answers to the "right" questions [2].

One of the first questions which arises is perhaps the most elementary: Are there "dressed one-electron states" of fixed momentum which are eigenstates of the appropriate Hamiltonian. These states should of course have an adhering photon cloud. In [3] Faddeev and Kulish gave a suggestion what form such states should take. As far as we know however, this specific suggestion has not been shown to be a rigorous mathematical outcome of a basic theory. The Fadeev-Kulish states do not live in Fock space because of the nature of the photon cloud.

In recent years the simpler mathematical model of non-relativistic QED has been studied by many authors (after some pioneering early work of Fröhlich [4, 5]), see for example [1]. This model suffers from various difficulties but it is hoped that it may serve as a reasonably realistic model for low energies, and a testing ground for understanding the infrared problem. One of the main difficulties is that this model is neither Galilean nor Poincaré covariant. The charged particles are treated non-relativistically while the photons are relativistic. There remains an ultra-violet cutoff in the photon field to produce a well defined theory, but the theory is well defined without an infrared cutoff.

In this work we define our Hamiltonians on the Hilbert space consisting of the Fock space for photons tensored with the usual Hilbert space for the non-relativistic charged particles. It may turn out that there are more suitable Hilbert spaces where the theory makes good physical and mathematical sense and the ground states, which we show do not exist, are resurrected. In any case, it seems to us that there is much work which needs to be done to describe scattering in a Hilbert space framework. Even in the case considered where one-electron ground states do not exist for fixed total momentum an elegant theory may yet exist where a dressed one-electron state is thought of as a limiting situation of a non-normalizable state of fixed energy in the continuum just as a state of fixed momentum is also a limiting concept.

In this paper we consider translation invariant models of non-relativistic QED having a total net charge. The generator of translations defines the operator of total momentum. Translation invariance implies that the Hamiltonian commutes with this operator. We can thus restrict the Hamiltonian to any subspace of fixed total momentum ξ. This restricted Hamiltonian is denoted by $H(\xi)$. For any momentum ξ, $H(\xi)$ is bounded from below. We denote the infimum of its spectrum by $E(\xi)$. One can easily show the the function $E(\cdot)$ is almost everywhere differentiable. In this paper we show that for momenta ξ at which $E(\cdot)$ has a non-vanishing derivative, $H(\xi)$ does not admit a ground state. We do not impose an infrared cutoff, which in fact is the reason for the absence of ground states. The coupling constant is arbitrary, but nonzero.

First we consider an electron (with spin 1/2) coupled to the quantized electromagnetic field. We show that $H(\cdot)$ does not admit a ground state at points where $E(\cdot)$ has a non-vanishing derivative. This model has been previously investigated in [10], where it has been shown that $H(0)$ does have a ground state provided the coupling is sufficiently small. However, the result given there does not prove the absence of ground states for nonzero ξ. Moreover, in [10, 9] it has been shown that $E(\cdot)$ has a non-vanishing derivative for small coupling constant and small but nonzero ξ. Combining this with our result we see that for such small ξ and small coupling, $H(\xi)$ does not have a ground state. The model is introduced and the result is stated in Section 3. The proof of the result is presented in

Section 4.

We then generalize the above result to a positive ion. More specifically, we consider a spinless nucleus with nuclear charge Ze and N electrons each with charge $-e$ where the interaction between the particles includes the Coulomb potential. If $Z \neq N$, we show that $H(\cdot)$ does not admit a ground state at points where $E(\cdot)$ has a non-vanishing derivative. This model has been recently investigated in [6, 8], where it was shown that under natural assumptions $H(\xi)$ does have a ground state provided $N = Z$. In contrast to our result, Coulomb systems without coupling to the quantized electromagnetic field do have positive ions, with fixed nonzero total momentum. In Section 3 we introduce the model describing an ion and state the result. Its proof is presented in Section 4.

In the next section we introduce the Fock space of photons.

2 Fock Space of Photons

The degrees of freedom of the photons are described by a symmetric Fock space, introduced as follows. Let

$$\mathfrak{h} := L^2(\mathbb{Z}_2 \times \mathbb{R}^3) \cong L^2(\mathbb{R}^3; \mathbb{C}^2)$$

denote the Hilbert space of a transversally polarized photon. The variable $\underline{k} = (\lambda, k) \in \mathbb{Z}_2 \times \mathbb{R}^3$ consists of the wave vector k or momentum of the particle and λ describing the polarization. The symmetric Fock space, $\mathcal{F}$, over $\mathfrak{h}$ is defined by

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} S_n(\mathfrak{h}^{\otimes n}) \; ,$$

where S_n denotes the orthogonal projection onto the subspace of totally symmetric tensors. The vacuum is the vector $\Omega := (1, 0, 0, ...) \in \mathcal{F}$. The vector $\psi \in \mathcal{F}$ can be identified with sequences $(\psi_n)_{n=0}^{\infty}$ of n-photon wave functions, $\psi_n(\underline{k}_1, ..., \underline{k}_n) \in L^2((\mathbb{Z}_2 \times \mathbb{R}^3)^n)$, which for $n \geq 1$ are totally symmetric in their n arguments. The Fock space inherits a scalar product from $\mathfrak{h}$, explicitly

$$(\psi, \varphi)_{\mathcal{F}} = \overline{\psi}_0 \varphi_0 + \sum_{n=1}^{\infty} \int \overline{\psi}_n(\underline{k}_1, ..., \underline{k}_n) \varphi_n(\underline{k}_1, ..., \underline{k}_n) d\underline{k}_1 ... d\underline{k}_n \; ,$$

where we used the abbreviation $\int d\underline{k} = \sum_{\lambda=1,2} \int dk$. The number operator N is defined by $(N\psi)_n = n\psi_n$. It is self-adjoint on the domain $D(N) := \{\psi \in \mathcal{F} | N\psi \in \mathcal{F}\}$. For each function $f \in \mathfrak{h}$ one associates an annihilation operator $a(f)$ as follows. For a vector $\psi \in \mathcal{F}$ we define

$$(a(f)\psi)_n(\underline{k}_1, ..., \underline{k}_n) = (n+1)^{1/2} \int \overline{f}(\underline{k}) \psi_{n+1}(\underline{k}, \underline{k}_1, ..., \underline{k}_n) d\underline{k} \quad \forall\, n \geq 0 \ .$$

The domain of $a(f)$ is the set of all ψ such that $a(f)\psi \in \mathcal{F}$. Note that $a(f)\Omega = 0$. The creation operator $a^*(f)$ is defined to be the adjoint of $a(f)$. Note that $a(f)$ is anti-linear, and $a^*(f)$ is linear in f. They are well known to satisfy the canonical commutation relations

$$[a^*(f), a^*(g)] = 0 \quad , \quad [a(f), a(g)] = 0 \quad , \quad [a(f), a^*(g)] = (f, g) \ .$$

where $f, g \in L^2(\mathbb{Z}_2 \times R^3)$ and (f, g) denotes the inner product of $L^2(\mathbb{Z}_2 \times R^3)$. Since $a(f)$ is anti-linear, and $a^*(f)$ is linear in f, we will write

$$a(f) = \int \overline{f}(\underline{k}) a_{\underline{k}} d\underline{k} \quad , \quad a^*(f) = \int f(\underline{k}) a^*_{\underline{k}} d\underline{k} \ ,$$

where the right hand side is merely a different notation for the expression on the left. For a function $f \in L^2(\mathbb{R}^3)$, we will write $a_\lambda(f) := a(f_\lambda)$ and $a^*_\lambda(f) := a^*(f_\lambda)$, where f_λ is the function defined by $f_\lambda(\mu, k) := f(k)\delta_{\lambda,\mu}$. The field energy operator denoted by H_f is given by

$$(H_f \psi)_n(\underline{k}_1, ... \underline{k}_n) = \left(\sum_{i=1}^{n} |k_i| \right) \psi_n(\underline{k}_1, ... \underline{k}_n) \ .$$

It is self-adjoint on its natural domain $D(H_f) := \{\psi \in \mathcal{F} | H_f \psi \in \mathcal{F}\}$. The operator of momentum P_f is given by

$$(P_f \psi)_n(\underline{k}_1, ... \underline{k}_n) = \left(\sum_{i=1}^{n} k_i \right) \psi_n(\underline{k}_1, ... \underline{k}_n) \ .$$

Its components $(P_f)_j$ are each self-adjoint on the domain $D((P_f)_j) := \{\psi \in \mathcal{F} | (P_f)_j \psi \in \mathcal{F}\}$.

3 The Electron: Model and Statement of Result

At first we consider a single free electron interacting with the quantized electromagnetic field. The Hilbert space describing the system composed of an electron and the quantized field is

$$\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} \ .$$

The Hamiltonian is

$$H = \{\sigma \cdot (p + eA(x))\}^2 + H_f \ ,$$

where

$$A(x) = \sum_{\lambda=1,2} \int \frac{\rho(k)}{\sqrt{2|k|}} \left(a_{\lambda,k} e^{ik\cdot x} \varepsilon_{\lambda,k} + a^*_{\lambda,k} e^{-ik\cdot x} \varepsilon_{\lambda,k} \right) dk \ , \tag{1}$$

where the $\varepsilon_{\lambda,k} \in \mathbb{R}^3$ are vectors, depending measurably on $\widehat{k} = k/|k|$, such that $(k/|k|, \varepsilon_{1,k}, \varepsilon_{2,k})$ forms an orthonormal basis; and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where σ_i denotes the i-th Pauli matrix:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

By x we denote the position of the electron and its canonically conjugate momentum by $p = -i\nabla_x$. We have introduced the function

$$\rho(k) = \frac{1}{(2\pi)^{3/2}} \chi_\Lambda(|k|) \ ,$$

where χ_Λ is the characteristic function of the set $[0, \Lambda]$. Since we are interested in the infrared problem we fix the ultraviolet cutoff $0 < \Lambda < \infty$. The Pauli matrices satisfy the commutation relations $[\sigma_1, \sigma_2] = 2i\sigma_3$ and cyclic permutations thereof. Using these commutation relations, we can write the Hamiltonian as

$$H = (p + eA(x))^2 + e\sigma \cdot B(x) + H_f \ ,$$

where

$$B(x) = (\nabla \wedge A)(x) = \sum_{\lambda=1,2} \int \frac{\rho(k)(ik \wedge \varepsilon_{\lambda,k})}{\sqrt{2|k|}} \left(a_{\lambda,k} e^{ik\cdot x} - a^*_{\lambda,k} e^{-ik\cdot x} \right) dk \ .$$

The Hamiltonian is translation invariant and commutes with the generator of translations, i.e., the operator of total momentum

$$P_{\text{tot}} = p + P_f \ .$$

Let F be the Fourier transform in the electron variable x, i.e., on $L^2(\mathbb{R}^3)$,

$$(F\psi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi\cdot x} \psi(x) dx \ .$$

Set

$$W = \exp(ix \cdot P_f) \ .$$

Note $WP_{\text{tot}}W^* = p$ so that in the new representation p is total momentum. We compute

$$WHW^* = \{\sigma \cdot (p - P_f + eA)\}^2 + H_f \ ,$$

where $A := A(0)$. Then the composition $U = FW$ yields the fiber decomposition of the Hamiltonian and the Hilbert space

$$UHU^* = \int_{\mathbb{R}^3}^{\oplus} H(\xi)d\xi \quad , \quad U : \mathcal{H} \to L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes \mathcal{F} d\xi$$

with

$$H(\xi) = \{\sigma \cdot (\xi - P_f + eA)\}^2 + H_f$$

an operator on $\mathbb{C}^2 \otimes \mathcal{F}$. Note that $H(\xi)$ can also be written as

$$H(\xi) = (\xi - P_f + eA)^2 + e\sigma \cdot B + H_f \ ,$$

where $B := B(0)$. The explicit self-adjoint realization of $H(\xi)$ is given by the following Lemma.

Lemma 1. *The operator $H(\xi)$ is self-adjoint on $D(P_f^2 + H_f) = \{\psi \in \mathcal{F} | (P_f^2 + H_f)\psi \in \mathcal{F}\}$ and essentially self-adjoint on any core of $P_f^2 + H_f$.*

For a proof of Lemma 1 see [12]. It is elementary to show that $H(\xi)$ is bounded from below. We write

$$E(\xi) := \inf \sigma(H(\xi)) \ .$$

Proposition 2. *The function $E(\cdot)$ is Lipschitz continuous and hence a.e. differentiable.*

By spherical symmetry $E(\cdot)$ is invariant under rotations. We want to point out that for small e and ξ in a neighborhood of 0, it has been shown that $E(\cdot)$ is twice continuously differentiable with positive Hessian [10, 9]. Using functional integration it has been shown that for any value of e, $E(0) \leq E(\xi)$, [11, 1]. In [4, 1] it is shown that for large ξ, $E(\xi) = |\xi| + O(1)$. It seems probable that for all e and $\xi \neq 0$, $E(\cdot)$ is differentiable with non-vanishing derivative.

Theorem 3. *Let $e \neq 0$. If $E(\cdot)$ is differentiable at ξ and has a nonzero derivative, then $H(\xi)$ does not have a ground state*

We want to relate this to a result obtained in [10]. Where $A = A(0)$ in (1) is replaced by an infrared regularized $A_\sigma(0) = \sum_\lambda \int_{\sigma\le|k|} \rho(k)(2|k|)^{-1/2}(a^*_{\lambda,k}\varepsilon_{\lambda,k} + a_{\lambda,k}\varepsilon_{\lambda,k})dk$. It is shown that if e and ξ are sufficiently small then for any $\sigma > 0$, there exists a normalized ground state ψ_σ. For $\xi = 0$, ψ_σ converges weakly as $\sigma \to 0$ to a nonzero vector. If $\xi \neq 0$ it was shown that $(\Omega, \psi_\sigma) \to 0$ as $\sigma \to 0$. This indicates that if there is a ground state for $\xi \neq 0$ the overlap with the vacuum should be zero, but does not prove this.

4 The Electron: Proof of Results

First we give a proof of the Lipschitz property of the ground state energy. We note that this follows immediately from the inequality $E(k+q) \le E(k) + |q|$, a proof of which is given for example in [4, 1]. However, we will give an independent proof of the Lipschitz property, which easily generalizes to the case of an ion. We write

$$v(\xi) = (\xi - P_f + eA) \ .$$

Proof of Proposition 2. Choose $\varphi_n \in \mathrm{Ran}\chi_{[E(\xi_1),E(\xi_1)+1/n)}(H(\xi_1))$ with $\|\varphi_n\| = 1$. Then

$$\begin{aligned} E(\xi_2) \le (\varphi_n, H(\xi_2)\varphi_n) \qquad &= (\varphi_n, (H(\xi_1) + (\xi_2 - \xi_1)\cdot v(\xi_1))\varphi_n) + (\xi_2 - \xi_1)^2 \\ &\le E(\xi_1) + \frac{1}{n} + (\xi_2 - \xi_1)\cdot(\varphi_n, v(\xi_1)\varphi_n) + (\xi_2 - \xi_1)^2 \ . \end{aligned} \tag{2}$$

We estimate for some finite constants C_0 and C_1,

$$\begin{aligned} |(\varphi_n, v(\xi_1)\varphi_n)|^2 \qquad &\le (\varphi_n, v(\xi_1)^2\varphi_n) \\ &\le (\varphi_n, (v(\xi_1)^2 + e\sigma\cdot B + H_f + C_0)\varphi_n) \\ &\le E(\xi_1) + \frac{1}{n} + C_0 \\ &\le (\Omega, H(\xi_1)\Omega) + C_0 + \frac{1}{n} \\ &\le C_1(1 + |\xi_1|^2) \ , \end{aligned} \tag{3}$$

where in the second inequality we used that $\sigma \cdot B$ is infinitesimally form-bounded with respect to H_f. By (3) and (2),

$$E(\xi_2) - E(\xi_1) \le \frac{1}{n} + |\xi_2 - \xi_1| C_1^{1/2}(|\xi_1| + 1) + (\xi_2 - \xi_1)^2 \ .$$

Since this holds for all $n \ge 1$ and all ξ_1 and ξ_2 we obtain for some finite constant C_2,

$$|E(\xi_2) - E(\xi_1)| \le C_2|\xi_2 - \xi_1|(|\xi_1| + |\xi_2| + 1) \ .$$

This proves the Lipschitz continuity. By Rademacher's Theorem we know that Lipschitz continuity implies a.e. differentiability. $\square$

Before we present the proof of Theorem 3, we need a few Lemmas. For $E(\cdot)$ differentiable at ξ, we fix ξ and consider the following subset of the unit sphere,

$$S_\epsilon := \{\omega \in S^2 | \ \omega \cdot \nabla E(\xi) \leq 1 - \epsilon\} \ .$$

We denote normalized vectors by $\widehat{k} = k/|k|$.

Lemma 4. *Assume that $E(\cdot)$ is differentiable at ξ. For $\widehat{k} \in S_\epsilon$, we have*

$$H(\xi - k) + |k| - E(\xi) \geq \epsilon|k| + o(|k|) \ .$$

Proof. Using that $E(\xi - k)$ is a lower bound for $H(\xi - k)$ and the differentiability of $E(\cdot)$ at ξ, we have

$$H(\xi - k) + |k| - E(\xi) \geq E(\xi - k) - E(\xi) + |k| = -k \cdot \nabla E(\xi) + |k| + o(|k|) \geq \epsilon|k| + o(|k|) \ .$$

$\square$

Let $P_0 = P_0(\xi)$ denote the orthogonal projection onto the kernel of $H(\xi) - E(\xi)$. For $\varphi \in \mathcal{F}$, we set

$$(a_{\underline{k}}\varphi)_n(\underline{k}_1, ..., \underline{k}_n) = (n+1)^{1/2}\varphi_{n+1}(\underline{k}, \underline{k}_1, ..., \underline{k}_n) \ . \tag{4}$$

For a.e. k and all n, the expression in (4) is in $L^2(\mathbb{R}^{3n})$. Its relation to $a(f)$ is outlined in the following Lemma

Lemma 5. *Let $\Omega \subset \mathbb{R}^3$ and $\varphi \in \mathcal{F}$. If $a_{\underline{k}}\varphi \in L^2(\Omega; \mathcal{F})$, then for all $f \in L^2(\Omega; \mathbb{C})$ and $\eta \in \mathcal{F}$*

$$(\eta, a(f)\varphi) = \int \overline{f}(\underline{k})(\eta, a_{\underline{k}}\varphi)d\underline{k} \ .$$

Proof. We have

$$\begin{aligned}
(\eta, a(f)\varphi) \quad &= \sum_{n=0}^{\infty} \int \overline{\eta}_n(\underline{k_1}, ..., \underline{k}_n)(n+1)^{1/2}\overline{f}(\underline{k})\varphi_{n+1}(\underline{k}, \underline{k}_1, ..., \underline{k}_{n+1})d\underline{k}d\underline{k}_1...d\underline{k}_n \\
&= \int \overline{f}(\underline{k}) \left(\sum_{n=0}^{\infty} \int \overline{\eta}_n(\underline{k_1}, ..., \underline{k}_n)(n+1)^{1/2}\varphi_{n+1}(\underline{k}, \underline{k}_1, ..., \underline{k}_{n+1})d\underline{k}_1...d\underline{k}_n \right) d\underline{k} \\
&= \int \overline{f}(\underline{k})(\eta, a_{\underline{k}}\varphi)d\underline{k} \ ,
\end{aligned}$$

where the interchange of the order of integration and summation is justified since

$$\sum_{n=0}^{\infty}\int \left|\overline{\eta}_n(\underline{k_1},...,\underline{k}_n)(n+1)^{1/2}\overline{f}(\underline{k})\varphi_{n+1}(\underline{k},\underline{k}_1,...,\underline{k}_{n+1})\right| d\underline{k}d\underline{k}_1...d\underline{k}_n$$
$$\leq \|f\|\|\eta\|\left(\int_\Omega \|a_{\underline{k}}\varphi\|^2 d\underline{k}\right)^{1/2} < \infty \, .$$

□

Lemma 6. *For each $\varphi \in D(H_f^{1/2})$, the function $k \mapsto a_{\underline{k}}\varphi$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\};\mathcal{F})$.*

Proof. Since $\varphi \in D(H_f^{1/2})$, we conclude that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\int |k||(a_{\underline{k}}\varphi)(\underline{k}_2,...,\underline{k}_{n+1})|^2 d\underline{k}_2...d\underline{k}_{n+1}d\underline{k} \\
&= \sum_{n=0}^{\infty}\int\sum_{j=1}^{n+1}|k_j||\varphi_{n+1}(\underline{k}_1,\underline{k}_2,...,\underline{k}_{n+1})|^2 d\underline{k_1}d\underline{k}_2...d\underline{k}_{n+1} \\
&= \|H_f\varphi\|^2 < \infty \, .
\end{aligned}$$

This implies that $\|a_{\underline{k}}\varphi\|^2$ is integrable over any compact subset of $\mathbb{R}^3 \setminus \{0\}$. □

Lemma 7. *Suppose $E(\cdot)$ is differentiable at ξ and and that ψ is a ground state of $H(\xi)$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $\eta \in \mathcal{F}$,*

$$(\eta, a_{\underline{k}}\psi) = \frac{e\rho(k)}{\sqrt{2|k|}}\left(H(\xi,k)^{-1}\eta, (-2\epsilon_{\underline{k}}\cdot v(\xi) + i(k\wedge\epsilon_{\underline{k}})\cdot\sigma)\,\psi\right) \, , \tag{5}$$

for a.e. k, with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$, where $H(\xi,k) := H(\xi-k)+|k|-E(\xi)$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Let $\varphi \in \mathrm{Ran}(P_{[0,n]}(N))$ be a state having less or equal to n photons and assume each φ_n has compact support. Then for real f,

$$((a^*_\lambda(f)H(\xi,k) - (H(\xi)-E(\xi))a^*_\lambda(f))\varphi,\psi) = ((A^*(f)+R_0^*(f)+R_1^*(f))\varphi,\psi) \, ,$$

with

$$\begin{aligned}
R_0(f) \quad &:= \int f(y)2(y-k)\cdot v(\xi)a_{\lambda,y}\,dy + \int f(y)(k^2-y^2)a_{\lambda,y}\,dy + \int f(y)(|k|-|y|)a_{\lambda,y}\,dy \\
R_1(f) \quad &:= \int f(y)\frac{e\rho(y)}{\sqrt{2|y|}}(k\cdot\varepsilon_{\lambda,y})\,dy \\
A(f) \quad &:= \int f(y)\frac{e\rho(y)}{\sqrt{2|y|}}\left(-2\varepsilon_{\lambda,y}\cdot v(\xi)+i(y\wedge\varepsilon_{\lambda,y})\cdot\sigma\right)\,dy \, .
\end{aligned}$$

Since $\psi \in D(H_f + P_f^2) \subset D(a_\lambda(f))$,

$$(H(\xi,k)\varphi, a_\lambda(f)\psi) = (\varphi, (A(f) + R_0(f) + R_1(f))\psi) \tag{6}$$

Note that this holds for all φ in an operator core for $H(\xi,k)$. For any $\epsilon > 0$, there exists by Lemma 4 a $\delta > 0$ such that for all k with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$, $H(\xi,k)$ has a bounded inverse. This and equation (6) imply that in fact for all such k and all $\eta \in \mathcal{F}$,

$$(\eta, a_\lambda(f)\psi) = (H(\xi,k)^{-1}\eta, (A(f) + R_0(f) + R_1(f))\psi) \ . \tag{7}$$

Now fix $\eta \in \mathcal{F}$. For k, with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$, we choose a δ–sequence centered at k. Explicitly, we choose a nonnegative function $g \in C_0^\infty(\mathbb{R}^3)$ with $\int g(y)dy = 1$ and support in the unit ball. We set $f_{k,m}(y) := m^3 g(m(y-k))$. By Lemmas 5 and 6 it follows that the left hand side of (7) yields, $\lim_{m\to\infty}(\eta, a_\lambda(f_{m,k})\psi) = (\eta, a_{\lambda,k}\psi)$ a.e. k. The term $(H(\xi,k)^{-1}\eta, A(f_{m,k})\psi)$ converges to the right hand side of (5). Below we will show that the terms $(H(\xi,k)^{-1}\eta, R_0(f_{m,k})\psi)$ and $(H(\xi,k)^{-1}\eta, R_1(f_{m,k})\psi)$ vanish as m tends to infinity for a.e. k. The expression containing R_1 vanishes since $k \cdot \epsilon_{\lambda,k} = 0$. To show that the expression involving R_0 vanishes we will only consider one term. The other terms will follow similarly. We set $\phi_l := v_l(\xi)H(k,\xi)^{-1}\eta$ and estimate

$$\begin{aligned}
R_{0,1}(f_{m,k}) &:= |(H(k,\xi)^{-1}\eta, \int f_{k,m}(y)2(y-k)\cdot v(\xi)a_{\lambda,y}\psi)dy| \\
&\le \sum_{l=1}^{3} \left| (v_l(\xi)H(k,\xi)^{-1}\eta, \int f_{k,m}(y)2(y-k)_l a_{\lambda,y}\psi)dy \right| \\
&\le \sum_{l=1}^{3}\sum_{n=0}^{\infty} \int \left| \overline{\phi}_l(\underline{k}_1,...,\underline{k}_n) \int f_{k,m}(y)2(y-k)_l(n+1)^{1/2}\psi_{n+1}(\lambda,y,\underline{k}_1,...,\underline{k}_n)dy \right| d\underline{k}_1...d\underline{k}_n \\
&\le \|\phi\| \int f_{k,m}(y)2|y-k|h_\lambda(y)dy \ ,
\end{aligned}$$

where

$$h_\lambda(y) = \left(\sum_{n=0}^{\infty} \int (n+1)\left|\psi_{n+1}(\lambda,y,\underline{k}_1,...,\underline{k}_n)\right|^2 d\underline{k}_1...d\underline{k}_n \right)^{1/2}$$

and

$$\|\phi\|^2 = \sum_{l=1}^{3} \|\phi_l\|^2 \ .$$

Since ψ is in $D(H_f^{1/2})$,

$$\int \left||y|^{1/2}h_\lambda(y)\right|^2 dy \le \sum_{n=0}^{\infty}\sum_{\mu=1,2} \int (n+1)|y|\left|\psi_{n+1}(\mu,y,\underline{k}_1,...,\underline{k}_n)\right|^2 dyd\underline{k}_1...d\underline{k}_n = (\psi, H_f\psi) \ .$$

Thus $h_\lambda \in L^1_{\rm loc}(\mathbb{R}^3 \setminus \{0\})$. Therefore a.e. point is a Lebesgue point of h_λ. At such points k,

$$\int f_{k,m}(y) h_\lambda(y) dy \to h_\lambda(k) \ ,$$

by Lebesgue's differentiation theorem, see for example [14] Theorem 1.25. Thus $R_{0,1}(f_{k,m})$ tends to zero as $m \to \infty$, a.e. k. ☐

Lemma 8. *If $E(\cdot)$ is differentiable at ξ, then*

$$P_0 2v(\xi) P_0 = \nabla E(\xi) P_0 \ .$$

Proof. Suppose $\psi \in {\rm Ran} P_0$, with $\|\psi\| = 1$, then

$$E(\xi + k) - E(\xi) \leq (\psi, (H(\xi + k) - H(\xi))\psi) = 2k \cdot (\psi, v(\xi)\psi) + |k|^2 \ .$$

This implies

$$k \cdot \nabla E(\xi) \leq 2k \cdot (\psi, v(\xi)\psi) + o(|k|) \quad , \quad |k| \to 0 \ .$$

Since k can have any direction we conclude that

$$\nabla E(\xi) = 2(\psi, v(\xi)\psi) \ .$$

Since this holds for any $\psi \in {\rm Ran} P_0$ the claim follows by polarization. ☐

We set

$$Q(k) = |k|(H(\xi - k) + |k| - E(\xi))^{-1} \ ,$$

whenever this exists. And for $|k| > 0$, we set

$$Q_0(k) = |k|(H(\xi) + |k| - E(\xi))^{-1} \ .$$

By the spectral theorem

$$P_0 = P_0(\xi) = s - \lim_{|k| \to 0} Q_0(k) \ .$$

Lemma 9. *Let $E(\cdot)$ be differentiable at ξ. Given $\epsilon > 0$, then*

$$w - \lim_{\widehat{k} \in S_\epsilon, |k| \to 0} \Big(Q(k) - (1 - \widehat{k} \cdot \nabla E(\xi))^{-1} P_0 \Big) = 0 \ .$$

Proof. Fix ξ

<u>Step 1:</u> $v(\xi)Q_0(k)$ is uniformly bounded for small $|k|$.

Since B is $H_f^{1/2}$ operator bounded, we see that there exists a finite constant C_0 such that

$$v(\xi)^2 \leq H(\xi) + C_0 \leq (H(\xi) + |k| - E(\xi)) + (E(\xi) + C_0) \, . \tag{8}$$

On the other hand

$$\sum_{l=1}^{3} (v(\xi)Q_0(k))_l^* (v(\xi)Q_0(k))_l = \frac{|k|}{H(\xi) - E(\xi) + |k|} v(\xi)^2 \frac{|k|}{H(\xi) - E(\xi) + |k|} \, .$$

By inequality (8) we see that the right hand side is uniformly bounded for small $|k|$. This shows Step 1.

<u>Step 2:</u> We have $s - \lim_{|k|\to 0} v(\xi)Q_0(k) = v(\xi)P_0$.

By the resolvent identity

$$\begin{aligned} & v(\xi)\frac{|k|}{H(\xi) - E(\xi) + |k|} \\ & \quad = v(\xi)\frac{|k|}{H(\xi) - E(\xi) + |k| + 1} - v(\xi)\frac{1}{H(\xi) - E(\xi) + |k| + 1}\frac{|k|}{H(\xi) - E(\xi) + |k|} \, . \end{aligned} \tag{9}$$

Again using the resolvent identity and an argument similar to the one in Step 1,

$$\begin{aligned} & \left\| v(\xi)\frac{1}{H(\xi) - E(\xi) + |k| + 1} - v(\xi)\frac{1}{H(\xi) - E(\xi) + 1} \right\| \\ & \quad = \left\| v(\xi)\frac{1}{H(\xi) - E(\xi) + 1} |k| \frac{1}{H(\xi) - E(\xi) + |k| + 1} \right\| \\ & \quad \xrightarrow{|k|\to 0} 0 \, . \end{aligned}$$

This implies that the first term on the right hand side in (9) converges in norm to zero and the second term converges strongly to $v(\xi)P_0$.

<u>Step 3:</u> Uniformly for $\widehat{k} \in S_\epsilon$,

$$P_0 Q(k) P_0 - \left(P_0 2\widehat{k} \cdot v(\xi) P_0\right)(P_0 Q(k) P_0) \xrightarrow{w} P_0 \quad \text{and} \quad Q(k) - P_0 Q(k) P_0 \xrightarrow{w} 0 \, .$$

Using the second resolvent identity twice we obtain for small $|k|$ and $\widehat{k} \in S_\epsilon$,

$$
\begin{aligned}
Q(k) &= Q_0(k) + Q_0(k)(2\widehat{k} \cdot v(\xi) - |k|)Q(k) && (10)\\
&= Q_0(k) + Q_0(k)(2\widehat{k} \cdot v(\xi) - |k|)Q_0(k)\\
&\quad + Q_0(k)(2\widehat{k} \cdot v(\xi) - |k|)Q(k)(2\widehat{k} \cdot v(\xi) - |k|)Q_0(k) && (11)
\end{aligned}
$$

Now using (11) and the results of Step 1 and Step 2, we find

$$Q(k)(1 - P_0) \xrightarrow{w} 0 \quad , \quad (1 - P_0)Q(k) \xrightarrow{w} 0 \, ,$$

where the limit is uniform for $\widehat{k} \in S_\epsilon$. It follows that

$$Q(k) - P_0 Q(k) P_0 \xrightarrow{w} 0 \, ,$$

uniformly for $\widehat{k} \in S_\epsilon$. Now this and (10) show Step 3.

The claim of the Lemma is now an immediate consequence of Lemma 8 and Step 3. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Suppose $H(\xi)$ has a ground state ψ with $\|\psi\| = 1$. We want to show this leads to a contradiction. We choose an $\eta \in D((N+1)^{1/2})$ such that $(\eta, \psi) \neq 0$. Choose ϵ with $0 < \epsilon < 1$ and $\delta > 0$ sufficiently small. Then by Lemma 7 for a.e. k with $\widehat{k} \in S_\epsilon$ and $|k| < \delta$,

$$
\begin{aligned}
(\eta, a_{\lambda,k}\psi) &= \big(\eta, H(\xi,k)^{-1}(2|k|)^{-1/2}\rho(k)\left(-2\varepsilon_{\lambda,k} \cdot v(\xi) + i(\varepsilon_{\lambda,k} \wedge k) \cdot \sigma\right)\psi\big)\\
&= \frac{e\rho(k)}{\sqrt{2}|k|^{3/2}} \left(\eta, Q(k)\left(-2\varepsilon_{\lambda,k} \cdot v(\xi) + i(\varepsilon_{\lambda,k} \wedge k) \cdot \sigma\right)\psi\right) \, .
\end{aligned}
$$

Now uniformly for $\widehat{k} \in S_\epsilon$,

$$
\begin{aligned}
\left(\eta, Q(k)\left(-2\varepsilon_{\lambda,k} \cdot v(\xi) + i(\varepsilon_{\lambda,k} \wedge k) \cdot \sigma\right)\psi\right) \xrightarrow{|k| \to 0} \quad & -(1 - \widehat{k} \cdot \nabla E)^{-1}\varepsilon_{\lambda,k} \cdot (P_0\eta, 2v(\xi)\psi)\\
&= -(\varepsilon_{\lambda,k} \cdot \nabla E)(1 - \widehat{k} \cdot \nabla E)^{-1}(\eta, \psi) \, ,
\end{aligned}
$$

where in the last step we used Lemma 8. Thus we find for a.e. k with $|k| < \delta$ and $\widehat{k} \in S_\epsilon$,

$$|(\eta, a_{\lambda,k}\psi)| \geq \frac{1}{2} \frac{|e\rho(k)|}{\sqrt{2}|k|^{3/2}} (1 - \widehat{k} \cdot \nabla E)^{-1} \left|(\eta, \psi)(\varepsilon_{\lambda,k} \cdot \nabla E)\right| \, .$$

We introduce the set

$$K := \{\omega \in S^2 | -\frac{1}{2}|\nabla E| \le \omega \cdot \nabla E \le 0\} \subset S_\epsilon \ .$$

Then there exists a positive constant $c_0 > 0$ such that for all $\widehat{k} \in K$,

$$\sum_{\lambda=1,2} |(\varepsilon_{\lambda,k} \cdot \nabla E)|^2 \ge c_0 > 0 \ .$$

Therefore, there exists a $c_1 > 0$ such that for a.e. small k with $\widehat{k} \in K$, we have

$$\frac{|e\rho(k)c_1|^2}{|k|^3} \le \sum_{\lambda=1,2} |(\eta, a_{\lambda,k}\psi)|^2 \le \|(1+N)^{1/2}\eta\|^2 \left(\sum_{\lambda=1,2} \sum_{n=0}^{\infty} \int |\psi_{n+1}(\lambda, k, \underline{k}_1, ..., \underline{k}_n)|^2 d\underline{k}_1 ... d\underline{k}_n \right) \ .$$

Integrating over the set of all k with $\widehat{k} \in K$ and $|k| \le \delta$, we see this is inconsistent with ψ being in $\mathcal{F}$. Thus $H(\xi)$ does not have a ground state. □

5 Positive Ion: Model and Statement of Results

We consider an atom consisting of a spinless nucleus of mass m_0 and charge Ze and N spin 1/2 electrons having charge $-e$ and mass 1. The energy of this system is described by the operator

$$H = \frac{1}{2m_0}(p_0 - ZeA(x_0))^2 + \sum_{j=1}^{N} \frac{1}{2}\{\sigma_j \cdot (p_j + eA(x_j))\}^2 + H_f + V(x_0, ..., x_N) \ ,$$

acting on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \left(\bigwedge_{j=1}^{N} L^2(\mathbb{R}^3; \mathbb{C}^2) \right) \otimes \mathcal{F} \ ,$$

where $p_0 = -i\nabla_0$ acts on the first factor and $p_j = -i\nabla_j$ and σ_j, the three-vector of Pauli matrices, act on the j-th factor of the antisymmetric tensor product. We take the spin of the nucleus to be zero only to simplify notation. We will make the following assumptions about the potential V:

$$V(x_0, ..., x_N) = \sum_{0 \le i < j \le N} V_{ij}(x_i - x_j) \ .$$

Each V_{ij} is infinitesimally bounded with respect to the Laplacian in three dimensions, which we denote by $-\Delta$, i.e., there exists for any $a > 0$ a finite constant b such that for all f in the domain of $-\Delta$,

$$\|V_{ij} f\| \leq a\| - \Delta f\| + b\|f\| \ .$$

The Hamiltonian is translation invariant and therefore commutes with the generator of translations, i.e., the operator of total momentum

$$P_{\text{tot}} = \sum_{j=0}^{N} p_j + P_f \ .$$

Let F be the Fourier transform in the variable x_0, i.e., on $L^2(\mathbb{R}^3)$,

$$(F\psi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi\cdot x_0} \psi(x_0) dx_0 \ .$$

Let

$$W = \exp(ix_0 \cdot (P_f + \sum_{j=1}^{N} p_j)) \ .$$

Note that $WP_{\text{tot}}W^* = p_0$ so that in a new representation, p_0 is the total momentum. Then the composition $U = FW$ yields the decomposition of the Hamiltonian

$$UHU^* = \int_{\mathbb{R}^3}^{\oplus} H(\xi) d\xi \ ,$$

with

$$H(\xi) = \frac{1}{2m_0}(\xi - \sum_{j=1}^{N} p_j - P_f - ZeA(0))^2 + \frac{1}{2}\sum_{j=1}^{N} \{\sigma_j \cdot (p_j + eA(x_j))\}^2 + H_f + \widetilde{V}$$

acting on $\left(\bigwedge^N L^2(\mathbb{R}^3;\mathbb{C}^2)\right) \otimes \mathcal{F}$ and where we have set $\widetilde{V} = V|_{x_0=0}$. Let us cite the following Theorem [6, 13].

Theorem 10. *The operator $H(\xi)$ is self-adjoint on*

$$\bigcap_{j=1}^{N} D(p_j^2) \cap D(P_f^2 + H_f)$$

and essentially self-adjoint on any core of $\sum_{j=1}^{N} p_j^2 + P_f^2 + H_f$.

It is easy to show that for every ξ the operator $H(\xi)$ is bounded below. Let $E(\xi) = \inf \sigma(H(\xi))$ be the infimum of the spectrum. We prove the following results.

Proposition 11. *The function $E(\cdot)$ is Lipschitz continuous and hence almost everywhere differentiable.*

Theorem 12. *Suppose $N \neq Z$ and $e \neq 0$. If $E(\cdot)$ is differentiable at ξ with non-vanishing derivative then $H(\xi)$ does not have a ground state.*

6 Positive Ion: Proof of Result

First we show the following lemma.

Lemma 13. *$|\widetilde{V}|$ is relatively form bounded with respect to $H(\xi)$.*

Proof. By Theorem 10, we know that $H(\xi)$ is self-adjoint on the domain of $P_f^2 + \sum_{j=1}^N p_j^2 + H_f$. Therefore there exist finite constants c_0 and c_1 such that

$$P_f^2 + \sum_{j=1}^N p_j^2 + H_f \leq c_0 H(\xi) + c_1 \ .$$

By assumption $\widetilde{V}$ is infinitesimally small with respect to $\sum_{j=1}^N p_j^2$. Therefore, $|\widetilde{V}|$ is infinitesimally form bounded with respect to $\sum_{j=1}^N p_j^2$. Hence for any $a > 0$ there exists a finite b such that

$$|\widetilde{V}| \leq a \sum_{j=1}^N p_j^2 + b \leq a \left(P_f^2 + \sum_{j=1}^N p_j^2 + H_f \right) + b \leq a c_1 H(\xi) + a c_2 + b \ .$$

□

The proof of Proposition 11 is omitted since it is a minor modification of the proof of Proposition 2, but note that Lemma 13 is needed to obtain the corresponding bound (3). We will prove Theorem 12 using a sequence of Lemmas. For notational convenience we set

$$v(\xi) = \xi - \sum_{j=1}^N p_j - P_f - ZeA(0) \ .$$

Recall the definitions $S_\epsilon := \{\omega \in S^2 |\ \omega \cdot \nabla E(\xi) \leq 1 - \epsilon\}$ and $\widehat{k} := k/|k|$, which are the same as in Section 4.

Lemma 14. *Assume that $E(\cdot)$ is differentiable in ξ. Given $\epsilon > 0$, then for $\widehat{k} \in S_\epsilon$, we have*

$$H(\xi - k) + |k| - E(\xi) \geq \epsilon |k| + o(|k|) \ .$$

The proof of Lemma 14 is the same as the proof of Lemma 4.

Lemma 15. *Let $\mathcal{H}_0$ be any Hilbert space. Let $\Omega \subset \mathbb{R}^3$ and $\varphi \in \mathcal{H}_0 \otimes \mathcal{F}$. If $a_{\underline{k}}\varphi \in L^2(\Omega; \mathcal{H}_0 \otimes \mathcal{F})$. Then for all $f \in L^2(\Omega; \mathbb{C})$ and $\eta \in \mathcal{H}_0 \otimes \mathcal{F}$*

$$(\eta, a(f)\varphi) = \int \overline{f}(\underline{k})(\eta, a_{\underline{k}}\varphi) d\underline{k} \ .$$

The proof of this Lemma is analogous to the proof of Lemma 5. We merely have to replace the inner product of $\mathbb{C}$ by the inner product of $\mathcal{H}_0$. Likewise, one generalizes the proof of Lemma 6 to prove the next lemma. Anticipating our application we set henceforth $\mathcal{H}_0 := \left(\bigwedge^N L^2(\mathbb{R}^3; \mathbb{C}^2) \right)$.

Lemma 16. *Let $\varphi \in D(H_f^{1/2})$. The function $k \mapsto a_{\underline{k}}\varphi$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}; \mathcal{H}_0 \otimes \mathcal{F})$*

Lemma 17. *Suppose $E(\cdot)$ is differentiable at ξ and that ψ is a ground state of $H(\xi)$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $\eta \in \mathcal{H}_0 \otimes \mathcal{F}$,*

$$\begin{aligned} &(\eta, a_{\lambda,k}\psi) \\ &\quad = \frac{e\rho(k)}{\sqrt{2|k|}} \left(H(\xi,k)^{-1}\eta, \left(\frac{Z}{m_0} v(\xi) - \sum_{j=1}^N e^{-ik\cdot x_j} \left(\frac{1}{2} ik \wedge \sigma_j + p_j + eA(x_j)\right) \right) \cdot \varepsilon_{\lambda,k}\psi \right) \ , \end{aligned} \tag{12}$$

for a.e. k, with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$, where $H(\xi,k) := H(\xi - k) + |k| - E(\xi)$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Let $\varphi \in \mathrm{Ran}(P_{[0,n]}(N))$ be a state having less or equal to n photons and assume φ_n is smooth and has compact support. Then a straightforward calculation using the canonical commutation relations, yields for f real,

$$((a_\lambda^*(f)H(\xi,k) - (H(\xi) - E(\xi)a_\lambda^*(f))\varphi, \psi) = ((A^*(f) + R_0^*(f) + R_1^*(f))\varphi, \psi) \ ,$$

with

$$
\begin{aligned}
R_0(f) \quad &:= \int (|k|-|y|) f(y) a_{\lambda,y}\, dy + m_0^{-1} \int f(y)(y-k)\cdot v(\xi) a_{\lambda,y}\, dy \\
&+(2m_0)^{-1} \int f(y)(k^2-y^2) a_{\lambda,y}\, dy \\
R_1(f) \quad &:= -\frac{Z}{2m_0} e \int \frac{\rho(y)}{\sqrt{2|y|}} f(y) k\cdot \varepsilon_{\lambda,y}\, dy \\
A(f) \quad &:= -\sum_{j=1}^{N} e \int \frac{\rho(y)}{\sqrt{2|y|}} e^{-iy\cdot x_j} f(y) \varepsilon_{\lambda,y}\cdot (p_j + eA(x_j))\, dy \\
&+\frac{Z}{m_0} e \int \frac{\rho(y)}{\sqrt{2|y|}} f(y) \varepsilon_{\lambda,y}\cdot v(\xi)\, dy \\
&+\frac{1}{2}\sum_{j=1}^{N} e \int \frac{\rho(y)}{\sqrt{2|y|}} e^{-iy\cdot x_j} f(y)(ik\wedge \varepsilon_{\lambda,y})\cdot \sigma_j\, dy\ .
\end{aligned}
$$

Since $\psi \in \bigcap_{j=1}^{N} D(p_j^2) \cap D(P_f^2 + H_f) \subset D(a_\lambda(f))$,

$$
(H(\xi,k)\varphi, a_\lambda(f)\psi) = (\varphi, (A(f)+R_0(f)+R_1(f))\psi)\ .
$$

Note that this holds for all φ in an operator core for $H(\xi,k)$. For $\epsilon > 0$, there exists by Lemma 14 a $\delta > 0$ such that for all k with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$, $H(\xi,k)$ has a bounded inverse. Thus we conclude by density that for all such k and all $\eta \in \mathcal{H}_0 \otimes \mathcal{F}$,

$$
(\eta, a_\lambda(f)\psi) = (H(\xi,k)^{-1}\eta, (A(f)+R_0(f)+R_1(f))\psi)\ . \tag{13}
$$

Now fix $\eta \in \mathcal{H}_0 \otimes \mathcal{F}$. For k, with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$, we choose a δ-sequence, $f_{m,k}$, centered at k as in the proof of Lemma 7. We insert $f_{m,k}$ for f in equation (13). As $m \to \infty$, it follows by Lemmas 15 and 16 that the left hand side of (13) converges to the left hand side of (12) for a.e. k. In the same limit the term involving A converges to the right hand side of (12). As demonstrated in the proof of Lemma 7 the terms involving R_0 and R_1 vanish as m tends to infinity for a.e. k. This implies the assertion of the Lemma. □

The next lemma would follow easily from the formal commutation relation

$$
[H(\xi), ix_j] = -\frac{1}{m_0} v(\xi) + p_j + eA(x_j)\ .
$$

Lemma 18. *Let P_0 be the projection onto the kernel of $H(\xi) - E(\xi)$. Then for all integer j with $1 \leq j \leq n$,*

$$P_0 \frac{1}{m_0} v(\xi) P_0 = P_0(p_j + eA(x_j))P_0 \ .$$

Proof. Fix a $j \in \{1, 2, ..., n\}$. Let $\chi \in C^\infty(\mathbb{R}_+; [0,1])$ with $\chi\restriction[0,1] = 1$ and $\chi\restriction[2,\infty) = 0$. We set $\chi_n(x_j) = \chi(|x_j|/n)$. Let $\psi \in \mathrm{Ran} P_0$, then for all n

$$\begin{aligned}
0 \quad &= \langle\psi, H(\xi) i\chi_n(x_j)x_j\psi\rangle - \langle\psi, i\chi_n(x_j)x_j H(\xi)\psi\rangle \\
&= \langle\chi_n(x_j)\psi, (-\frac{1}{m_0}v(\xi) + p_j + eA(x_j))\psi\rangle \\
&+ \mathrm{Re}\langle\psi, \frac{1}{n}(\nabla\chi)(|x_j|/n)x_j \cdot (-\frac{1}{m_0}v(\xi) + p_j + eA(x_j))\psi\rangle \\
&- i\left(\psi, \left(\frac{1}{2m_0} + \frac{1}{2}\right)\frac{1}{n}(\nabla\chi)(|x_j|/n)\psi\right) \\
&\overset{n\to\infty}{\longrightarrow} \langle\psi, (-\frac{1}{m_0}v(\xi) + p_j + eA(x_j))\psi\rangle \ .
\end{aligned}$$

The limit as n tends to infinity follows from dominated convergence. By polarization this yields the claim. $\square$

The proof of the next lemma is the same as the proof of Lemma 8.

Lemma 19. *Let P_0 be the projection onto the the kernel of $H(\xi) - E(\xi)$. If $E(\cdot)$ is differentiable at ξ, then*

$$P_0 \frac{1}{m_0} v(\xi) P_0 = \nabla E(\xi) P_0 \ .$$

We set

$$Q(k) = |k|(H(\xi - k) + |k| - E(\xi))^{-1} \ ,$$

whenever this exists. And for $|k| > 0$, we set

$$Q_0(k) = |k|(H(\xi) + |k| - E(\xi))^{-1} \ .$$

Let P_0 be the orthogonal projection onto the kernel of $H(\xi) - E(\xi)$. By the spectral Theorem

$$P_0 = P_0(\xi) = s - \lim_{|k|\to 0} Q_0(k) \ .$$

Lemma 20. *Let $E(\cdot)$ be differentiable at ξ. Given $\epsilon > 0$. Then for $\widehat{k} = k/|k|$,*

$$w - \lim_{\widehat{k}\in S_\epsilon, |k|\to 0} \big(Q(k) - (1 - \omega\nabla E(\xi))^{-1}P_0\big) = 0 \ .$$

The proof follows the steps of Lemma 9, where Step 1 uses Lemma 13. We now present the proof of Theorem 12.

Proof of Theorem 12. Suppose $H(\xi)$ has a ground state ψ with $\|\psi\| = 1$. We want to show that this leads to a contradiction. Choose ϵ with $0 < \epsilon < 1$, and choose $\eta \in D((N+1)^{1/2})$ with $(\eta, \psi) \neq 0$. By Lemma 17 there exits a $\delta > 0$ such that for a.e. k, with $0 < |k| < \delta$ and $\widehat{k} \in S_\epsilon$,

$$\begin{aligned}(\eta, a_{\lambda,k}\psi) \quad &= \frac{e\rho(k)}{\sqrt{2}|k|^{3/2}} \left[\left(\eta, Q(k) \frac{1}{2} \sum_{j=1}^N e^{-ik\cdot x_j} (ik \wedge \varepsilon_{\lambda,k}) \cdot \sigma_j \psi \right) \right. \\ &\left. + \varepsilon_{\lambda,k} \cdot \left\{ \left(\eta, Q(k) \frac{Z}{m_0} v(\xi) \psi \right) + \left(\eta, Q(k) \sum_{j=1}^N (-e^{-ik\cdot x_j})(p_j + eA(x_j))\psi \right) \right\} \right] .\end{aligned}$$

Since $Q(k)$ is uniformly bounded on S_ϵ for small $|k|$,

$$\left(\eta, Q(k) \sum_{j=1}^N e^{-ik\cdot x_j} i(k \wedge \varepsilon_{\lambda,k}) \cdot \sigma_j \psi \right) \stackrel{|k|\to 0}{\longrightarrow} 0 \quad ,$$

uniformly for $k \in S_\epsilon$. Using Lemma 9, we find uniformly for $\widehat{k} \in S_\epsilon$ as $|k| \to 0$,

$$\begin{aligned}\left(\eta, Q(k) \frac{Z}{m_0} v(\xi) \psi \right) \longrightarrow \quad &(1 - \widehat{k} \cdot \nabla E)^{-1} \left(P_0\eta, \frac{Z}{m_0} v(\xi)\psi \right) \\ &= Z(\nabla E)(1 - \widehat{k} \cdot \nabla E)^{-1} (\eta, \psi) \ ,\end{aligned}$$

where we used Lemma 19. Again by Lemma 9 and using that $e^{-ik\cdot x_j}$ converges in the strong operator topology to 1, we find uniformly for $k \in S_\epsilon$ as $|k| \to 0$,

$$\begin{aligned}\left(\eta, Q(k) \sum_{j=1}^N (-e^{-ik\cdot x_j})(p_j + eA(x_j))\psi \right) \longrightarrow \quad &(1 - \widehat{k} \cdot \nabla E)^{-1} \left(P_0\eta, -\sum_{j=1}^N (p_j + eA(x_j))\psi \right) \\ &= (1 - \widehat{k} \cdot \nabla E)^{-1} \left(P_0\eta, \frac{-N}{m_0} v(\xi)\psi \right) \\ &= -\frac{N}{m_0} (\nabla E)(1 - \widehat{k} \cdot \nabla E)^{-1} (\eta, \psi)\end{aligned}$$

where in the second line we used Lemma 18 and in the last again Lemma 19. Collecting the above estimates we conclude that for small $|k|$ uniformly for $\widehat{k} \in S_\epsilon$,

$$|(\eta, a_{\lambda,k}\psi)| \geq \frac{1}{2} \frac{e\rho(k)}{\sqrt{2}|k|^{3/2}} (1 - \widehat{k} \cdot \nabla E)^{-1} \, |\varepsilon_{\lambda,k} \cdot \nabla E| \, |Z - N| |(\eta, \psi)| \ .$$

We introduce the set

$$K := \{\omega \in S^2 | -\frac{1}{2}|\nabla E| \leq \omega \cdot \nabla E \leq 0\} \subset S_\epsilon \ .$$

Then, since by assumption $\nabla E \neq 0$, there exists a positive constant $c_0 > 0$ such that for all $\widehat{k} \in K$,

$$\sum_{\lambda=1,2} |\varepsilon_{\lambda,k} \cdot \nabla E|^2 \geq c_0 > 0 \ .$$

By this and $N \neq Z$, there exists a $c_1 > 0$ such that for all small k with $\widehat{k} \in K$, we find

$$\begin{aligned} \frac{|\rho(k)c_1|^2}{|k|^3} & \leq \sum_{\lambda=1,2} |(\eta, a_{k,\lambda}\psi)|^2 \\ & \leq \|(1+N)^{1/2}\eta\|^2 \left(\sum_{\lambda=1,2} \sum_{n=0}^{\infty} \int \|\psi_{n+1}(\lambda, k, \underline{k}_1, ..., \underline{k}_n)\|^2 d\underline{k}_1 ... d\underline{k}_n \right) \ , \end{aligned}$$

where in the last inequality we used Cauchy-Schwarz. This is inconsistent with ψ being in $\mathcal{F} \otimes \mathcal{H}_0$. Thus $H(\xi)$ does not have a ground state. $\square$

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