ESTIMATES FOR THE OPTIMAL CONSTANTS IN MULTIPOLAR HARDY INEQUALITIES FOR SCHRÖDINGER AND DIRAC OPERATORS

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Abstract. By expanding squares, we prove several Hardy inequalities with two critical singularities and constants which explicitly depend upon the distance between the two singularities. These inequalities involve the L^2 norm. Such results are generalized to an arbitrary number of singularities and compared with standard results given by the IMS method. The generalized version of Hardy inequalities with several singularities is equivalent to some spectral information on a Schrödinger operator involving a potential with several inverse square singularities. We also give a generalized Hardy inequality for Dirac operators in the case of a potential having several singularities of Coulomb type, which are critical for Dirac operators.

Introduction

For $N \geq 3$, the simplest form of Hardy's inequality is easily obtained by the "expansion of the square" method as follows: for any function $u \in H^1(\mathbb{R}^N)$,

$$0 \leq \int_{\mathbb{R}^N} \left|\nabla u + \alpha\,\frac{x}{|x|^2}\,u\right|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \left[\alpha^2 - (N-2)\,\alpha\right] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx\;,$$

which shows for $\alpha = (N-2)/2$ that, for all $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx\;,$$

and it is well known that the constant $(N-2)^2/4$ is optimal. With two singularities located at $\pm y \in \mathbb{R}^N$, from the above inequality we get without effort that

$$\textstyle\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{(N-2)^2}{8} \int_{\mathbb{R}^N} \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2}\right) |u|^2\,dx = \frac{1}{2} \sum_{\pm} \int_{\mathbb{R}^N} |\nabla u_\pm|^2 - \frac{(N-2)^2}{4} \frac{|u_\pm|^2}{|x|^2}\,dx \geq 0$$

where $u_\pm(\cdot) = u(\cdot \pm y)$. For a given function u with compact support and $d := |y|$ large enough, it is however clear that the constant $(N-2)^2/8$ can be replaced by $(N-2)^2/4$. To improve upon $(N-2)^2/8$ for general functions and in presence of two singularities, one has to break the scaling invariance by introducing a new scale. This can be done by adding a lower order term. One of the goals in this paper is to obtain estimates for the best constant $\lambda = \lambda(\mu, d)$, that is, the smallest positive constant, in the inequality

$$\mu \int_{\mathbb{R}^N} \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2}\right) |u|^2\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \lambda \int_{\mathbb{R}^N} |u|^2\,dx \quad \forall\, u \in H^1(\mathbb{R}^N)\;,$$

for any $\mu \in (0, (N-2)^2/4]$ and any $y \in \mathbb{R}^N$ with $d := |y| > 0$.

The above result can of course be reinterpreted as a lower estimate on the spectrum of the Schrödinger operator

$$-\Delta - \mu \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2}\right)$$

since for a fixed $\mu \in (0, (N-2)^2/4)$, $-\lambda$ is the bottom of the spectrum of this operator. Such an operator is indeed semi-bounded from below, as we shall see below. By a simple rescaling argument, it is easy to see that $\lambda(\mu, d)/d^2$ is independent of d. A similar fact also holds in the case of more than two singularities. In such a case, after division by the correct scaling parameter, the best constant does not depend on the distances between the singularities anymore, but still depends on the relative position of each of them, *i.e.*, on the geometric pattern of the singularity points.

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It is certainly out of the scope of this introduction to present all various forms of Hardy's inequality. One can for instance refer to [45] for an up-to-date introduction. Multipolar potentials have recently been studied in [28], in a different context. Huge efforts have been done over the past years to improve on Hardy's inequality, but mostly in the case of a single singularity: see [22, 3, 4, 15, 23]. Also see [21, 20, 19] and references therein for some existence results of ground states for linear and nonlinear Schrödinger equations with multipolar inverse square singular potentials, and [19] for questions related to the self-adjointness of the operators. Among other results, it is proved in these papers that the operator $-\Delta - \sum_{k=1}^M \mu_k\,|x-y_k|^{-2}$ is positive if $\sum_{k=1}^M \mu_k^+ < (N-2)^2/4$, where $\{y_k\,:\,k=1,\,2\,,\dots\,M\}$ is any set of disjoint poles. Reciprocally, if $\sum_{k=1}^M \mu_k^+ > (N-2)^2/4$, there exists a configuration of poles such that $-\Delta - \sum_{k=1}^M \mu_k\,|x-y_k|^{-2}$ is not positive. Moreover, there exists a configuration of poles such that this operator is positive if and only if $\mu_k^+ < (N-2)^2/4$ for all k and $\sum_{k=1}^M \mu_k^+ < (N-2)^2/4$. Felli, Marchini and Terracini also consider in [21] modified potentials which are sums of inverse square potentials restricted to compact supports around a given number of singularities. In this case, they discuss again the positivity of the operator and, in some cases, also give estimates for the lowest eigenvalue of the corresponding Schödinger operator.

Our purpose in the case of the Schrödinger operators is to give an as good as possible estimate for the lower bound of the spectrum of the operator $-\Delta - \sum_{k=1}^M \frac{\mu}{|x-y_k|^2}$, $\mu \in (0,(N-2)^2/4]$, $M \geq 2$, $y_1\,,\dots\,y_M \in \mathbb{R}^N$. We will use two different methods, the so-called IMS method, see Section 1, and the "expansion of the squares" method, see Section 2, which has already been introduced above in the case of a single singularity.

From a mathematical point of view, inverse square potentials are interesting because of their criticality. There are various motivations for applications in physics, see for instance a list of topics in [21, 19], and related nonlinear problems in PDEs. On the other hand, the $1/|x|^2$ term appears in the linearization of critical nonlinear PDEs, see for instance [5, 43, 7], and plays a crucial role for understanding the asymptotic behavior of branches of solutions in some bifurcation problems.

The second topic of this paper is concerned with Hardy inequalities for Dirac equations in the framework of relativistic quantum mechanics. In such a setting, the Coulomb potential is critical. Hardy inequalities have been studied long ago for simplified relativistic equations coupled with multipolar potentials of quantum chemistry for instance in [9, 34], motivated by the question of the stability of matter. A key estimate in such a case is a Hardy inequality which was derived by Kato in [30]:

$$(-\Delta+1)^{1/2} \geq \frac{2}{\pi}\,\frac{1}{|x|}\;,$$

in dimension $N=3$. A refined relativistic operator was introduced by Brown and Ravenhall and optimal corresponding inequalities have been derived in [6, 24]. Also see [26, 2] for some consequences. In the case of the Dirac operator coupled to Coulomb interactions, Hardy type inequalities are much more recent, see [13, 12]. For a pure Coulomb singularity these inequalities read as follows. For all $\nu \in (0,1]$, for all $\phi \in L^2(\mathbb{R}^3,\mathbb{C}^2)$, with $S(\nu) := \sqrt{1-\nu^2}$,

$$\int_{\mathbb{R}^3} \frac{|\vec\sigma\cdot\nabla\,\phi|^2}{1+S(\nu)+\frac{\nu}{|x|}}\,dx + \big[1-S(\nu)\big]\int_{\mathbb{R}^3}|\phi|^2 dx \geq \int_{\mathbb{R}^3}\frac{\nu}{|x|}\,|\phi|^2\,dx\;.$$

Here some of the terms can be infinite and the matrices σ_i, $i=1,2,3$ are the so-called Pauli matrices. See Section 3 for more precisions and explanations on these matrices and the above inequality. In particular, exactly as in the case of the Schrödinger operator, the constant $S(\nu)$ is the smallest eigenvalue of $H-\nu/|x|$, where H is the Dirac operator and $\nu \in (0,1)$. We postpone the precise explanation of the relation of the optimal value of $S(\nu)$ and the spectrum of H to Section 3, as well as some comments on the self-adjointness of the operators. For a general introduction to the Dirac equation we refer to [44].

Scaling properties explain why inverse square potentials and Coulomb potentials are critical respectively for the Schrödinger and the Dirac operators. It is clear that $1/|x|^2$ has the same scaling law as $-\Delta$. As for Dirac's operator, the principal part of H scales like $1/|x|$, which can also be seen in the above inequality at the level of the operator $\phi \mapsto -(\vec\sigma\cdot\nabla)(|x|\,\vec\sigma\cdot\nabla\,\phi)$. Some further details will be given at the beginning of Section 3. More strikingly, one can also prove that the standard Hardy inequality for the Schrödinger operator is a consequence of the Hardy-like inequality for the Dirac operator coupled to one Coulomb singularity. For the convenience of the reader, we reproduce here some computations which already appeared in [12]. By taking the limit as $\nu \to 1$, we get

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma} \cdot \nabla \phi|^2}{1+\frac{1}{|x|}}\, dx + \int_{\mathbb{R}^3} |\phi|^2\, dx \geq \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|}\, dx \quad \forall\, \phi \in H^1(\mathbb{R}^3, \mathbb{C}^2)\ .$$

If we replace $\phi(\cdot)$ by $\varepsilon^{-1}\phi(\varepsilon^{-1}\cdot)$ and take the limit as $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^3} |x|\, |\vec{\sigma} \cdot \nabla \phi|^2\, dx \geq \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|}\, dx\ .$$

By taking $\phi = (f, 0)$ with f purely real, we end up with

$$\int_{\mathbb{R}^3} |x|\, |\nabla f|^2\, dx \geq \int_{\mathbb{R}^3} \frac{|f|^2}{|x|}\, dx$$

for all $f \in H^1(\mathbb{R}^3, \mathbb{C})$, which is itself equivalent to

$$\int_{\mathbb{R}^3} |\nabla u|^2\, dx \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2}\, dx \quad \forall\, u \in H^1(\mathbb{R}^3, \mathbb{C})\ ,$$

as shown by considering $u = \sqrt{|x|}\, f$. We observe that $(N-2)^2/4 = 1/4$ if $N = 3$, so that we recover the optimal constant, and even corrective terms at all orders. See [22, 12] for more details and further references.

A major difference between Dirac and Schrödinger operators coupled respectively to Coulomb type interactions and to multiple inverse square singular potentials lies in the structure of the continuous spectrum. While the Schrödinger operators are semi-bounded from below, thus allowing only for some positive continuous spectrum, Dirac operators have continuous spectrum everywhere except in a gap. There is therefore a natural limitation of the coupling constant, called the *coupling constant threshold,* which has been studied in [32] for smooth potentials and in [11] in the case of one Coulomb singularity and a constant magnetic field.

Our purpose in the case of Dirac operators is to give estimates of the lowest energy level for a Dirac-Coulomb operator in terms of the interdistance between the M singularities, or equivalently to determine estimates of the coupling threshold. Such a coupling threshold can be seen from two points of view:

(i) Either we fix the configuration of the singularities and look for estimates of the largest coupling constant ν for which the lowest eigenvalue is above -1, the upper bound of the negative continuous spectrum.

(ii) Or we fix the coupling constant ν to some value in $(0, 1)$ and look for estimates that guarantee that the lowest eigenvalue of the operator is above -1. This determines a minimal distance between singularities if $\nu M > 1$.

Both approaches are equivalent and amount to finding estimates for the optimal constant in Hardy type inequalities for Dirac operators with multiple Coulomb singularities.

In the Schrödinger and in the Dirac case, we investigate the asymptotics $d \to +\infty$ when the singularities are asymptotically far from each other, and also the limit $d \to 0$ corresponding to all singularities merged in a single point (further assumptions are however needed for the Dirac operator).

From a physical point of view, the multipolar Dirac-Coulomb operator describes the state of one charged particle in a molecule with M fixed nuclei, in the Born-Oppenheimer approximation. In atomic units, $\nu = \alpha\, Z$, Z is the nuclear charge number and α is the Sommerfeld fine-structure constant, whose value is $1/137.037\ldots$ Hence the condition $\nu < 1$ means that the nuclear charge cannot be larger than $1/\alpha$. Finding the lowest energy level in the gap of the Dirac-Coulomb operator is a basic question for studying stability, computing energy levels and getting estimates for related nonlinear models.

To our knowledge, when there is more than one singularity, no explicit estimates for the lowest eigenvalue of Dirac operators have been derived yet. In the case of a crystal, which is a slightly different setting, Hardy inequalities for Dirac type operators are currently under investigation, see [8]. Estimates could be deduced from Kato's inequality or from the inequality stated in [6, 24] for the Brown-Ravenhall Hamiltonian and their extensions to the multipolar case, but they would anyway not cover the whole range of the coupling constant, as we do here.

In [31], Klaus studied conditions under which the Dirac operator coupled to a potential with several Coulomb singularities is self-adjoint. The case of two singularities when they are far apart from each other has been studied in [25]. Also see [10] for a more general approach of double well Hamiltonians, and [46] for a semi-classical analysis in the case of potentials with multiple wells.

The paper is organized as follows. In Section 1 we use the well-known IMS method to derive some lower estimates on the lowest eigenvalue of the operator $-\Delta-\sum_{k=1}^{M}\frac{\mu}{|x-y_k|^2}$, $\mu\in(0,(N-2)^2/4]$, $M\geq 2$ and $y_1,\ldots y_M\in\mathbb{R}^N$. This method consists in localizing the wave functions around the singularities by using a well chosen partition of $\mathbb{R}^N$. In some cases the geometric pattern defined by the singularity centers allows for better estimates than the general ones.

In Section 2 we expand some well chosen squared quantities and integrate by parts to prove another type of estimates, which in some cases improve those obtained by the IMS method. The idea here is to show that for some function $Q(x,y_1,\ldots y_M)$, the operator $-\Delta-\sum_{k=1}^{M}\frac{\mu}{|x-y_k|^2}+Q$ is nonnegative. The constant λ is then defined as the infimum of Q. Of course, this procedure cannot provide the best constant λ, since it is based on a pointwise estimate of Q.

Section 3 deals with Hardy-type inequalities for Dirac operator with Coulomb singularities. Due to the non homogeneity of the Dirac operator, the best constant heavily depends on the interdistance between the singularities, and not only on the geometric pattern defined by them. In this case we are only able to use the IMS method, which gives good results only for large interdistances, that is, for very distant singularities. For nearby singularities, we introduce a slightly modified version of the Hardy inequality.

When dealing with the Schrödinger operator, we always consider dimensions $N\geq 3$. The Dirac case is studied only in the physical space with $N=3$. Also note that in the case of Dirac operators, the essential spectrum is not bounded from below, and so, when we speak of the lowest eigenvalue, we always mean the lowest eigenvalue in the gap of the essential spectrum. All estimates can be worked out for complex valued functions and spinors, but since we do not consider evolution equations, results will be stated for real valued functions whenever possible, and without further notice. For simplicity, we have assumed that coupling constants are the same for all singularities. The extension to the case when they differ can be worked out by the same methods, but is left to the reader.

1. Hardy inequalities for the Schrödinger operator and the IMS method

The results of this section rely on the so-called "IMS" (for Ismagilov, Morgan, Morgan-Simon, Sigal, see [35, 41]) truncation method. Our goal is to obtain estimates for the lowest eigenvalue of $-\Delta-\mu\,V_M(x)$, where $V_M(x)=\sum_{k=1}^{M}\frac{1}{|x-y_k|^2}$, $M\geq 2$ and $y_1,\ldots y_M$ are M points of $\mathbb{R}^N$. Notice that the Hamiltonian is essentially self-adjoint if $\mu\leq(N-2)^2/4-1$, otherwise one has to use the corresponding Friedrichs extension. See [29, 19] for more details. Define d by

$$d:=\min_{1\leq j\neq k\leq M}|y_k-y_j|/2\,.$$

Theorem 1. *Consider $\mu\in(0,(N-2)^2/4]$. For any $M\geq 2$, there is a nonnegative constant $K_M<\pi^2$ such that, for any $u\in H^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{K_M+(M+1)\,\mu}{d^2}\int_{\mathbb{R}^N}|u|^2\,dx\geq\mu\int_{\mathbb{R}^N}V_M(x)\,|u|^2\,dx\,.$$

A *partition of unity* in $\mathbb{R}^N$ is a finite set $\{J_k\}_{k=1}^{M+1}$ of real valued functions $J_k\in W^{1,\infty}(\mathbb{R}^N)$, such that $\sum_{k=1}^{M+1}J_k^2=1$. Under these conditions, we have:

(*a*) $\sum_{k=1}^{M+1}J_k\,\partial_aJ_k=0$ for any $a=1,\ldots N$,

(*b*) $J_{M+1}:=\sqrt{1-\sum_{k=1}^{M}J_k^2}$,

(*c*) $\sum_{k=1}^{M+1}|\nabla J_k|^2\in L^\infty(\mathbb{R}^N)$.

If we additionally require that

$$\Omega_k\cap\Omega_l=\emptyset\quad\text{for any }k,\,l=1,\ldots M,\ k\neq l\,,\tag{1}$$

where $\Omega_k:=\mathrm{Int}\,(\mathrm{supp}(J_k))$, $k=1,\ldots M$, then we obtain an explicit formula for the sum of the gradients:

$$(d)\qquad\sum_{k=1}^{M+1}|\nabla J_k|^2=\sum_{k=1}^{M}|\nabla J_k|^2+\sum_{k=1}^{M}\frac{J_k^2}{1-J_k^2}\,|\nabla J_k|^2=\sum_{k=1}^{M}\frac{|\nabla J_k|^2}{1-J_k^2}\,,$$

where we have computed

$$\sum_{a=1}^{N}|J_{M+1}\,\partial_a J_{M+1}|^2=\sum_{a=1}^{N}\Big|\sum_{k=1}^{M}J_k\,\partial_a J_k\Big|^2=\sum_{a=1}^{N}\sum_{k=1}^{M}|J_k\,\partial_a J_k|^2\implies|\nabla J_{M+1}|^2=\sum_{k=1}^{M}\frac{J_k^2}{1-J_k^2}\,|\nabla J_k|^2.$$

In the sequel we will always use partitions of unity that satisfy (1).

Note that to avoid a singularity for the gradient of J_{M+1} at the points where $1-J_k^2=0$, from (d) we shall assume that the additional constraint $|\nabla J_k|^2=F(x)(1-J_k^2)$, for $k=1,\dots M$ and for some $F\in L^\infty(\mathbb{R}^N)$. The functions sinus and cosinus, for example, satisfy this last requirement.

Partitions of unity can be used to localize the Schrödinger operator. Here is a statement taken from [41], with its proof.

Lemma 2. *Let* $(J_k)_{k=1}^{M+1}$ *be a partition of unity satisfying* (1)*. For any* $u\in H^1(\mathbb{R}^N)$ *and any* $V\in L^1_{\rm loc}(\mathbb{R}^N)$*,*

$$\int_{\mathbb{R}^N}\Big(|\nabla u|^2-V\,|u|^2\Big)\,dx=\sum_{k=1}^{M+1}\int_{\mathbb{R}^N}\Big(|\nabla(J_k\,u)|^2-V\,|J_k\,u|^2\Big)\,dx-\int_{\mathbb{R}^N}\left(\sum_{k=1}^{M+1}|\nabla J_k|^2\right)|u|^2\,dx\;.$$

Proof. On the one hand,

$$\int_{\mathbb{R}^N}V\left(\sum_{k=1}^{M+1}|J_k\,u|^2\right)dx=\int_{\mathbb{R}^N}V\left(\sum_{k=1}^{M+1}|J_k|^2\right)|u|^2\,dx=\int_{\mathbb{R}^N}V\,|u|^2\,dx\;.$$

On the other hand,

$$\sum_{k=1}^{M+1}|\nabla(J_k\,u)|^2=|\nabla u|^2+\left(\sum_{k=1}^{M+1}|\nabla J_k|^2\right)|u|^2+\left(\sum_{k=1}^{M+1}J_k\,\nabla J_k\right)\nabla(|u|^2)\;.$$

Combined with Property (a), this proves the result. □

Thanks to Lemma 2 and Property (d), we can write

$$Q[u]:=\int_{\mathbb{R}^N}\Big(|\nabla u|^2-\mu\,V_M\,|u|^2\Big)\,dx=\sum_{k=1}^{M}Q[J_k\,u]+\mathcal{R}_M\;,$$

where

$$\begin{aligned}\mathcal{R}_M&=\int_{\mathbb{R}^N}|\nabla(J_{M+1}\,u)|^2\,dx-\mu\int_{\mathbb{R}^N}V_M\,|J_{M+1}\,u|^2\,dx-\sum_{k=1}^{M+1}\int_{\mathbb{R}^N}|\nabla J_k|^2\,|u|^2\,dx\\&=\int_{\mathbb{R}^N}|\nabla(J_{M+1}\,u)|^2\,dx-\mu\int_{\mathbb{R}^N}V_M\,\Big(1-\textstyle\sum_{k=1}^{M}J_k^2\Big)\,|u|^2\,dx-\displaystyle\sum_{k=1}^{M}\int_{\mathbb{R}^N}\frac{|\nabla J_k|^2}{1-J_k^2}\,|u|^2\,dx\;,\end{aligned}$$

$$\mathcal{R}_M\geq-\sum_{k=1}^{M}\int_{\mathbb{R}^N}\frac{|\nabla J_k|^2}{1-J_k^2}\,|u|^2\,dx-\mu\int_{\mathbb{R}^N}V_M\,\Big(1-\textstyle\sum_{k=1}^{M}J_k^2\Big)\,|u|^2\,dx\;.$$

Here we successively used Properties (b) and (d) and $|\nabla(J_{M+1}\,u)|^2\geq 0$. Using the fact that $\Omega_j\cap\Omega_k=\emptyset$ for any $j,\,k=1,\dots M$, $j\neq k$, then the above inequality can be rewritten as

$$\mathcal{R}_M\geq-\sum_{k=1}^{M}\int_{\Omega_k}\left[\frac{|\nabla J_k|^2}{1-J_k^2}+\mu\,(1-J_k^2)\,V_M(x)\right]|u|^2\,dx-\int_{\Upsilon}\mu\,V_M(x)\,|u|^2\,dx\;,$$

with $\Upsilon:=\mathbb{R}^N\setminus\cup_{k=1}^{M}\overline{\Omega}_k$. Consider now the case $M=2$, $y_1=y$, $y_2=-y$ and assume that $0<d\leq|y|$.

Lemma 3. *There is a partition of unity* $\{J_k\}_{k=1}^3$ *satisfying* (1) *with* $J_1\equiv 1$ *on* $B(y,d/2)$*,* $J_1\equiv 0$ *on* $B(y,d)^c$*,* $J_2(x)=J_1(-x)$ *for any* $x\in\mathbb{R}^N$*,* $0<d\leq|y|$*, such that, for any* $\mu>0$*, there exists a constant* $K_2\in[0,\pi^2)$ *for which, almost everywhere for all* $x\in\Omega:=\mathrm{supp}(J_1)\cup\mathrm{supp}(J_2)$*, we have*

$$\sum_{k=1}^{3}|\nabla J_k|^2+\mu\,J_3^2\,V_2(x)=\sum_{k=1,2}\frac{|\nabla J_k|^2}{1-J_k^2}+\mu\,J_3^2\,V_2(x)\leq\frac{K_2+2\,\mu}{d^2}\,.$$

In the proof of this result, we use first a partition of unity defined as follows. Let

$$J(r) := \begin{cases} 1 & \text{if} \quad r \leq 1/2 \\ \sin(\pi\, r) & \text{if} \quad 1/2 \leq r \leq 1 \\ 0 & \text{if} \quad r \geq 1 \end{cases} \tag{2}$$

and define $J_1(x) := J(|x-y|/d)$, $J_2(x) := J(|x+y|/d)$, and $J_3 := \sqrt{1-J_1^2-J_2^2}$.

Proof. The lower bound $K_2 \geq 0$ immediately follows by evaluating the l.h.s. at $x=0$. As for the upper bound, consider the above partition of unity. Thus defined, $\{J_1, J_2, J_3\}$ is a partition of unity of $\mathbb{R}^N$ satisfying (1). Let $\theta := |y|/d \geq 1$ and $\Omega_+ := \{x \in \Omega\,,\ x\cdot y > 0\}$. By Property (d), using the symmetry with respect to the hyperplane $\{x \in \mathbb{R}^N\ :\ x \cdot y = 0\}$, we get

$$\begin{aligned}
\sup_{\Omega}\left[\sum_{k=1}^{3} |\nabla J_k|^2 + \mu\, J_3^2\, V_2\right] &= \sup_{\Omega_+}\left[\frac{|\nabla J_1|^2}{1-J_1^2} + \mu\left(1-J_1^2\right) V_2\right] \\
&= \frac{1}{d^2} \sup_{t\in(1/2,1)}\left[\frac{|J'(t)|^2}{1-|J(t)|^2} + \mu\big(1-|J(t)|^2\big)\left(\frac{1}{(s+\theta)^2}+\frac{1}{(s-\theta)^2}\right)_{|s=\theta-t}\right] \\
&= \frac{1}{d^2} \sup_{t\in(1/2,1)}\left[\frac{|J'(t)|^2}{1-|J(t)|^2} + \mu\big(1-|J(t)|^2\big)\left(\frac{1}{(2\theta-t)^2}+\frac{1}{t^2}\right)\right] \\
&\leq \frac{1}{d^2} \sup_{t\in(1/2,1)}\left[\frac{|J'(t)|^2}{1-|J(t)|^2} + \mu\big(1-|J(t)|^2\big)\left(\frac{1}{(2-t)^2}+\frac{1}{t^2}\right)\right] \\
&= \frac{\pi^2}{d^2} + \frac{\mu}{d^2} \max_{1/2\leq t\leq 1}\left[\cos^2(\pi\, t)\left(\frac{1}{(2-t)^2}+\frac{1}{t^2}\right)\right] = \frac{\pi^2+2\,\mu}{d^2}\ .
\end{aligned}$$

The choice (2) is not optimal and can be improved by taking

$$J(r) := \begin{cases} 1 & \text{if} \quad r \leq 1/2 \\ g(r) & \text{if} \quad 1/2 \leq r \leq 1 \\ 0 & \text{if} \quad r \geq 1\,, \end{cases}$$

and g as the solution to the ODE

$$\frac{|g'(t)|^2}{1-g^2(t)} = A - \mu\left(1-g^2(t)\right)\left(\frac{1}{(2\theta-t)^2}+\frac{1}{t^2}\right)$$

with $g(1/2)=1$, and by adjusting $A>0$ such that $g(1)=0$ and $g(t)>0$ for any $t \in (1/2,1)$. Hence we can slightly improve the value of the constant K_2, but this value now depends on μ. ☐

Consider the case $M=2$, $V=V_2$. With the notations of Lemma 3, we observe that

$$\sum_{k=1}^{2} Q[J_k\, u] \geq -\frac{\mu}{d^2}\int_{\Omega} |u|^2\, dx$$

using the Hardy inequality for one singularity, for $\mu \leq (N-2)^2/4$. As a consequence, we obtain

$$\mathcal{R}_2 \geq -\frac{K_2+2\,\mu}{d^2}\int_{\Omega}|u|^2\, dx - \frac{2\,\mu}{d^2}\int_{\mathbb{R}^N\setminus\Omega}|u|^2\, dx\ ,$$

since $|x-y_k| \geq d$ on $\mathbb{R}^N\setminus\Omega$ for $k=1,\,2$. This proves that

$$\int_{\mathbb{R}^N}|\nabla u|^2\, dx + \frac{K_2+3\,\mu}{d^2}\int_{\mathbb{R}^N}|u|^2\, dx \geq \mu\int_{\mathbb{R}^N} V_2(x)\,|u|^2\, dx\ .$$

Theorem 1 is a generalization of this inequality to $M \geq 3$.

Proof of Theorem 1. Consider a partition of unity $\{J_k\}_{k=1}^{M+1}$ satisfying (1) such that $J_k(x) = J(|x-y_k|/d)$ for all $x\in\mathbb{R}^N$, $1\leq k\leq M$, with J as in Lemma 3 and recall that $\Omega_k := B(y_k, d)$ so that $\overline{\Omega}_k$ is the support of J_k, $k=1\,,\ldots M$. Let $\Upsilon := \mathbb{R}^N \setminus \cup_{k=1}^{M}\overline{\Omega}_k$. For every $k=1\,,\ldots M$, the Hardy inequality for one singularity and the estimate $|x-y_l|\geq d$ on $\overline{\Omega}_k$ for $l\neq k$, give

$$\int_{\mathbb{R}^N}|\nabla(J_k\, u)|^2\, dx\ -\ \mu\int_{\mathbb{R}^N}\left[\frac{1}{|x-y_k|^2}+\sum_{l\neq k}\frac{1}{|x-y_l|^2}\right]|J_k\, u|^2\, dx \geq -\,\frac{\mu\,(M-1)}{d^2}\int_{\Omega_k}|J_k\, u|^2\, dx$$

and thus provide a lower bound for $Q[J_k\, u]$:

$$Q[J_k\, u] \geq -\,\frac{\mu\,(M-1)}{d^2}\int_{\Omega_k} |J_k\, u|^2\, dx\;.$$

Now, observe that $V_M \leq M/d^2$ on Υ. For every $k = 1,\, 2\,,\ldots,\, M$, we can apply Lemma 3 on Ω_k with $(y_k, y_l) = (-y, y)$ up to a change of coordinates, for some $y_l \neq y_k$, and for all $j \neq k,\, l$, bound $|x-y_j|^{-2}$ by $1/d^2$. Hence we get

$$\mathcal{R}_M \geq -\sum_{k=1}^{M}\int_{\Omega_k}\left[\frac{K_2+2\,\mu}{d^2}+\frac{\mu\,(M-2)}{d^2}\,(1-J_k^2)\right]|u|^2 - \frac{\mu\, M}{d^2}\int_\Upsilon |u|^2\, dx\;.$$

Collecting the terms, that is $(K_2+2\,\mu)+\mu\,(M-2)\,(1-J_k^2)+\mu\,(M-1)\,J_k^2 \leq K_2+\mu\,(M+1)$, we obtain

$$\sum_{k=1}^{M} Q[J_k\, u] + \mathcal{R}_M \geq -\,\frac{K_2+\mu\,(M+1)}{d^2}\sum_{k=1}^{M}\int_{\Omega_k}|u|^2\, dx - \frac{\mu\, M}{d^2}\int_\Upsilon |u|^2\, dx\;,$$

from which the result follows. □

Remarks. There are many possibilities for improving the result of Theorem 1 for $M \geq 2$ and $\mu < (N-2)^2/4$.

(1) One can notice that when applying the Hardy inequality for one singularity, we can write

$$\int_{\mathbb{R}^N} |\nabla(J_k\, u)|^2\, dx \;-\; \mu\int_{\mathbb{R}^N}\frac{|J_k\, u|^2}{|x-y_k|^2}\, dx \geq \big((N-2)^2/4-\mu\big)\int_{\mathbb{R}^N}\frac{|J_k\, u|^2}{|x-y_k|^2}\, dx$$

using the optimal constant in the inequality. Consequently,

$$\begin{aligned} Q[J_k\, u] &\geq \min_{x\in\overline{\Omega}_k}\left[\left(\frac{(N-2)^2}{4}-\mu\right)\frac{1}{|x-y_k|^2} - \mu\sum_{\substack{l=1\,,\ldots\, M\\ l\neq k}}\frac{1}{|x-y_l|^2}\right]\int_{\mathbb{R}^N}|J_k\, u|^2\, dx\\ &\geq \frac{1}{d^2}\left(\frac{(N-2)^2}{4}-\mu\, M\right)\int_{\mathbb{R}^N}|J_k\, u|^2\, dx\;. \end{aligned}$$

On the other hand, we know that

$$\mathcal{R}_M \geq -\frac{1}{d^2}\sum_{k=1}^{M}\int_{\Omega_k}\left[(K_2+2\,\mu)+\mu\,(M-2)\,(1-J_k^2)\right]|u|^2\, dx - \frac{\mu\, M}{d^2}\int_\Upsilon |u|^2\, dx\;.$$

Collecting these estimates, with $\tilde\mu := 2\,\mu-(N-2)^2/4 < \mu$, this proves that

$$\int_{\mathbb{R}^N}|\nabla u|^2\, dx + \frac{1}{d^2}\,\left[(K_2+\tilde\mu)_+ + \mu\, M\right]\int_{\mathbb{R}^N}|u|^2\, dx \geq \mu\int_{\mathbb{R}^N} V_M(x)\,|u|^2\, dx$$

for any $u \in H^1(\mathbb{R}^N)$.

(2) The estimate $|x-y_l| \geq d$ on $\overline{\Omega}_k$ for $l \neq k$ is certainly extremely rough for large values of M, due to volume filling effects.

(3) Other partitions of unity can be considered. For a given set of poles $y_1,\, y_2\,,\ldots\, y_M \in \mathbb{R}^N$, define for instance the corresponding Voronoi cells Γ_k by

$$\Gamma_k := \left\{x \in \mathbb{R}^N\;:\; |x-y_k| \leq |x-y_l|\,,\;\forall\, l \neq k\right\}$$

and let $d_k := \mathrm{dist}(y_k, \partial\Gamma_k)$. Then the truncation functions J_k can be defined such that $J_k(x) = J(\mathrm{dist}(x,\Gamma_k^c)/d_k)$, $1 \leq k \leq M$, for some continuous truncation function J with $J \equiv 1$ in $[1/2, \infty)$, $0 \leq J \leq 1$ and $J(0) = 0$. For any $k = 1\,,\ldots\, M$, J_k is supported in the closure of Γ_k.

(4) In some cases, it is also possible to introduce an artificial singularity in the component Υ, in order to control V_M on Υ by an additional Hardy inequality.

We now illustrate points (3) and (4) in the simple case $M=2$, $y_1=y$ and $y_2=-y$. Consider the partition of the unity $\{J_k\}_{k=1}^3$ such that

$$\begin{aligned} J_1(x) &= \begin{cases} 1 & \text{if } |x-y|\le|x|\ , \\ \sin(\pi(x\cdot y)/d^2) & \text{if } 0\le x\cdot y\le d^2/2\ , \\ 0 & \text{otherwise}\ , \end{cases} \\ J_2(x) &= J_1(-x)\ , \\ J_3(x) &= \sqrt{1-J_1(x)^2-J_2(x)^2}\ . \end{aligned} \tag{3}$$

Proposition 4. *Let $M=2$ and assume that $\mu\in(0,(N-2)^2/4]$. Then for any $u\in H^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{\pi^2+\mu}{d^2}\int_{\mathbb{R}^N}|u|^2\,dx\ge\mu\int_{\mathbb{R}^N}V_2(x)\,|u|^2\,dx\ .$$

Proof. Up to a translation, we can work with $y_1=y$ and $y_2=-y$, that is with the potential $\mu\,V_2(x)=\frac{\mu}{|x+y|}+\frac{\mu}{|x-y|}$. Property (d) implies

$$\sum_{k=1}^3|\nabla J_k|^2=\frac{|\nabla J_1|^2}{1-J_1^2}+\frac{|\nabla J_2|^2}{1-J_2^2}\le\frac{\pi^2}{d^2}\,\mathbb{I}_{\mathrm{supp}(\nabla J_3)}\ ,$$

where $\mathbb{I}_\Omega$ is the characteristic function of the set $\Omega=\Omega_1\cup\Omega_2$. Our aim is to estimate $Q[J_k\,u]$ from below. Observing that $V_2(x)\le|x-y_k|^{-2}+d^{-2}$ for any $x\in\Omega_k$, $k=1$, 2, we derive

$$Q[J_k\,u]\ge\int_{\mathbb{R}^N}|\nabla(J_k\,u)|^2\,dx-\mu\int_{\mathbb{R}^N}\left(\frac{1}{|x-y_k|^2}+\frac{1}{d^2}\right)|J_k\,u|^2\,dx\quad\text{for } k=1\,,\ 2\ .$$

By Hardy's inequality, we get

$$Q[J_k\,u]\ge-\frac{\mu}{d^2}\int_{\mathbb{R}^N}|J_k\,u|^2\,dx\quad\text{for } k=1\,,\ 2\ .$$

Further, for any $x\in\mathrm{supp}(J_3)$,

$$\frac{1}{|x|^2}-\frac{1}{|x-y|^2}-\frac{1}{|x+y|^2}\ge\frac{1}{d^2}\min_{s\in[0,1/2]}\left(\frac{1}{s^2}-\frac{1}{|s-1|^2}-1\right)\ge-\frac{1}{d^2}\ .$$

Hence we obtain

$$\mathcal{R}_2\ge\int_{\mathbb{R}^N}|\nabla(J_3\,u)|^2\,dx-\mu\int_{\mathbb{R}^N}\left(\frac{1}{|x|^2}+\frac{1}{d^2}\right)|J_3\,u|^2\,dx-\frac{\pi^2}{d^2}\int_{\mathbb{R}^N}\mathbb{I}_{\mathrm{supp}(\nabla J_3)}\,|u|^2\,dx\,.$$

Applying again the Hardy's inequality, it results that

$$\mathcal{R}_2\ge-\frac{\mu}{d^2}\int_{\mathbb{R}^N}|J_3\,u|^2\,dx-\frac{\pi^2}{d^2}\int_{\mathbb{R}^N}|u|^2\,dx$$

and the proof is complete. □

2. "Expansion of the square" and Hardy inequalities for the Schrödinger operator

The estimates of Theorem 1 are not very good when $M\mu$ is close to $(N-2)^2/4$. The goal of this section is to remedy to this problem by the "expansion of the square" method, already used in the introduction in the case of a single singularity. We start with an elementary computation in the case of two singularities, see Lemma 5, (see Lemma 8 in case $M\ge2$). Then we show how the optimal constant in the multipolar Hardy inequality can be estimated. The result is twofold: we establish some qualitative properties of the best constant, see Lemmas 6 and 9, and then explicitly estimate it in Theorems 7 and 10. This is done first in the case of two singularities, and then extended to $M\ge2$ singularities.

Lemma 5. *For any $u\in H^1(\mathbb{R}^N)$, for any $y\in\mathbb{R}^N$,*

$$\frac{(N-2)^2}{8}\int_{\mathbb{R}^N}|u|^2\left(\frac{1}{|x-y|^2}+\frac{1}{|x+y|^2}\right)\left(1+\frac{|y|^2}{|x|^2+|y|^2}\right)dx\le\int_{\mathbb{R}^N}|\nabla u|^2\,dx\ .$$

When $y=0$ we recover the standard Hardy inequality with one singularity.

Proof. Assume that $u \in \mathcal{D}(\mathbb{R}^N)$ and $\alpha > 0$. We compute

$$\begin{aligned} 0 \;&\leq\; \int_{\mathbb{R}^N} \left| \nabla u + \alpha\,\frac{x-y}{|x-y|^2}\,u + \alpha\,\frac{x+y}{|x+y|^2}\,u \right|^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \big[\alpha^2 - (N-2)\,\alpha\big] \int_{\mathbb{R}^N} |u|^2 \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) dx \qquad (4) \\ &\quad + 2\,\alpha^2 \int_{\mathbb{R}^N} |u|^2\,\frac{(x-y)\cdot(x+y)}{|x-y|^2\,|x+y|^2}\,dx \end{aligned}$$

where we have used an integration by parts. From the parallelogram law, $|x-y|^2 + |x+y|^2 = 2\,|x|^2 + 2\,|y|^2$, we get

$$\begin{aligned} \frac{(x-y)\cdot(x+y)}{|x-y|^2\,|x+y|^2} = \frac{|x|^2 - |y|^2}{|x-y|^2\,|x+y|^2} &= \frac{1}{|x-y|^2\,|x+y|^2} \Big(\frac{|x-y|^2}{2} + \frac{|x+y|^2}{2} - 2\,|y|^2 \Big) \\ &= \frac{1}{2\,|x+y|^2} + \frac{1}{2\,|x-y|^2} - \frac{2\,|y|^2}{|x-y|^2\,|x+y|^2}\;. \end{aligned}$$

Hence

$$0 \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \big[2\alpha^2 - (N-2)\alpha\big] \int_{\mathbb{R}^N} |u|^2 \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) dx - 4\,\alpha^2 \int_{\mathbb{R}^N} |u|^2 \frac{|y|^2}{|x-y|^2\,|x+y|^2}\,dx\;.$$

We also have

$$2\,\alpha^2 - (N-2)\,\alpha \geq -\frac{1}{8}\,(N-2)^2\;,$$

with equality if and only if $\alpha = (N-2)/4$. By choosing this value and writing

$$\frac{2\,|y|^2}{|x-y|^2\,|x+y|^2} = \Big(\frac{1}{|x+y|^2} + \frac{1}{|x-y|^2} \Big) \frac{|y|^2}{|x|^2 + |y|^2}\;,$$

we get the result. By density, we extend the inequality to any $u \in H^1(\mathbb{R}^N)$. □

With $y = 0$, the inequality in Lemma 5 covers the optimal case with only one singularity. For a given u, the optimal case is also recovered in the limit $d = |y| \to +\infty$. The factor $\frac{|y|^2}{|x|^2+|y|^2}$ indeed converges to 1 for any $x \in \mathbb{R}^N$ as $d \to +\infty$. Hence, replacing u by $u(\cdot - y)$, we recover

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx\;.$$

The inequality in Lemma 5 is therefore an optimal interpolation inequality, with a weight, which interpolates between the case $y = 0$ and the limit $d \to +\infty$. It is however not very useful in the sense that for a fixed value of $|y|$, one cannot obtain improved values (uniformly with respect to u) of the constant $(N-2)^2/8$, as a simple scaling argument shows.

To do better than this we look for estimates of the positive constant $L(\mu, N)$ such that the following inequality

$$\mu \int_{\mathbb{R}^N} |u|^2 \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{L(\mu, N)}{d^2} \int_{\mathbb{R}^N} |u|^2\,dx$$

holds for any $u \in H^1(\mathbb{R}^N)$ and any $y \in \mathbb{R}^N$, $d := |y| > 0$. Let $V_2(x) := \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2}$ and define

$$L(\mu, N) := d^2 \sup_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\mu \int_{\mathbb{R}^N} V_2\,|u|^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} |u|^2\,dx}\;,$$

so that $-L(\mu, N)/d^2$ is the first eigenvalue of the operator $-\Delta - \mu\,V_2$.

Lemma 6. *Let $N \geq 3$ and let $\mu \in \mathbb{R}$, $y \in \mathbb{R}^N \setminus \{0\}$ and $d := |y|$. With the above definition, the map $\mu \mapsto L(\mu, N)$ is nondecreasing, independent of d and satisfies:*

(i) *If $\mu \leq (N-2)^2/8$, $L(\mu, N) = 0$.*
(ii) *If $(N-2)^2/8 < \mu \leq (N-2)^2/4$, $L(\mu, N)$ takes finite positive values.*
(iii) *$L(\mu, N) = +\infty$, for all $\mu > (N-2)^2/4$.*

Proof. As a function of μ, $L(\mu,N)$ is nondecreasing by its definition, using the fact that V_2 is positive. (i) is a consequence of Lemma 5.

By Theorem 1, $L(\mu,N)$ takes finite nonnegative values in case (ii). Let us prove that $L(\mu,N)$ has to be positive. If it were not the case for some $\mu\in((N-2)^2/8<\mu\le(N-2)^2/4)$, by applying the inequality to $u_\varepsilon(x)=u(x/\varepsilon)$, we would be able to write

$$\mu\int_{\mathbb{R}^N}|u|^2\left(\frac{1}{|x-\varepsilon\,y|^2}+\frac{1}{|x+\varepsilon\,y|^2}\right)dx\le\int_{\mathbb{R}^N}|\nabla u|^2\,dx\;.$$

Letting $\varepsilon\to0$, this would prove that Hardy's inequality with only one singularity holds for some $\mu>(N-2)^2/4$, a contradiction with the fact that $(N-2)^2/4$ is optimal.

Concerning (iii), if the inequality were true for some $\mu>(N-2)^2/4$, we could also consider $u_\lambda(x):=u(\lambda\,(x-y))$. Taking the limit $\lambda\to\infty$, this would prove that Hardy's inequality holds with constant μ, again a contradiction with the optimality on $(N-2)^2/4$. The independence of $L(\mu,N)$ in terms of d is also a consequence of the scaling properties of the inequality. By considering $u(\cdot/d)$, the problem can indeed be reduced to the case $d=1$ without loss of generality. ☐

The "expansion of the square" method goes beyond these estimates. We shall establish in Theorem 7 an explicit expression of a nondecreasing function $\mu\mapsto K(\mu,N)$ such that

$$L(\mu,N)\le K(\mu,N)\quad\forall\,\mu\in\mathbb{R}$$

and

(i) If $\mu\le(N-2)^2/8$, $K(\mu,N)=0$.
(ii) If $(N-2)^2/8<\mu<(N-2)^2/4$, $K(\mu,N)$ takes finite positive values.

Assume that $u\in\mathcal{D}(\mathbb{R}^N)$, $y\in\mathbb{R}^N$. In (4), we want to estimate pointwise the last term by a combination of the bipolar potential and a constant. For this purpose, we choose any $\beta\in(0,1)$ and look for an optimal pointwise upper estimate of

$$F(x):=2\,\frac{|x|^2-|y|^2}{|x-y|^2\,|x+y|^2}-\beta\left(\frac{1}{|x-y|^2}+\frac{1}{|x+y|^2}\right)\;.$$

Let $r:=|x|$, $|y|=d$ and $\cos\theta=x\cdot y/(r\,d)$. We can rewrite F as

$$F(x)=\frac{2\,(|x|^2-|y|^2)-2\,\beta\,(|x|^2+|y|^2)}{|x-y|^2\,|x+y|^2}\;,$$

and since

$$|x-y|^2\,|x+y|^2=(r^2+d^2)^2-4\,r^2\,d^2\,\cos^2\theta$$

is nonnegative, we get

$$F(x)=2\,\frac{(1-\beta)\,r^2-(1+\beta)\,d^2}{(r^2+d^2)^2-4\,r^2\,d^2\,\cos^2\theta}$$

and F achieves its maximum for $\cos\theta=1$:

$$F(x)\le2\,\frac{(1-\beta)\,r^2-(1+\beta)\,d^2}{(r^2-d^2)^2}=:f(r)$$

under the condition

$$r\ge\sqrt{\frac{1+\beta}{1-\beta}}\,d>d\;.$$

Using

$$(r^2-d^2)^3\,f'(r)=4\,r\,[d^2\,(1+3\,\beta)-(1-\beta)\,r^2\,]\;,$$

we see that f realizes its maximum on $\left(\sqrt{\frac{1+\beta}{1-\beta}}\,d,\infty\right)$ at $r=\sqrt{\frac{1+3\,\beta}{1-\beta}}\,d$ and get

$$f(r)\le f\left(\sqrt{\frac{1+3\,\beta}{1-\beta}}\,d\right)=\frac{(1-\beta)^2}{4\,\beta\,d^2}\;.$$

This shows the pointwise inequality

$$2\frac{(x-y)\cdot(x+y)}{|x-y|^2\,|x+y|^2}\le\beta\left(\frac{1}{|x-y|^2}+\frac{1}{|x+y|^2}\right)+\frac{(1-\beta)^2}{4\,\beta}\,\frac{1}{|y|^2}\quad\forall\,(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\;.$$

Recall that $V_2(x)=\frac{1}{|x-y|^2}+\frac{1}{|x+y|^2}$ and inject the above estimate in

$$0\le\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\big[\alpha^2-(N-2)\,\alpha\big]\int_{\mathbb{R}^N}V_2\,|u|^2\,dx+\,2\,\alpha^2\int_{\mathbb{R}^N}|u|^2\,\frac{(x-y)\cdot(x+y)}{|x-y|^2\,|x+y|^2}\,dx\;.$$

We thus get

$$\big[(N-2)\,\alpha-\alpha^2\,(1+\beta)\big]\int_{\mathbb{R}^N}V_2\,|u|^2\,dx\le\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\gamma\,\alpha^2\int_{\mathbb{R}^N}|u|^2\,dx\;,$$

where

$$\gamma=\frac{(1-\beta)^2}{4\,\beta}\,\frac{1}{d^2}\;.$$

Let us fix

$$\mu=(N-2)\,\alpha-\alpha^2\,(1+\beta)\in\left(0,\frac{(N-2)^2}{4}\right)\;,$$

or equivalently

$$\beta=\frac{(N-2)\,\alpha-\alpha^2-\mu}{\alpha^2}\in(0,1)\;.$$

From the conditions $\mu>0$, $\beta\in(0,1)$ we deduce that

$$t:=\frac{N-2}{\sqrt{\mu}}\in(2,\infty)\;\;.$$

In terms of α and μ, we have

$$\mu\int_{\mathbb{R}^N}V_2\,|u|^2\,dx\le\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\lambda\int_{\mathbb{R}^N}|u|^2\,dx$$

with

$$\lambda=\frac{1}{4\,d^2}\,\frac{\big[(N-2)\,\alpha-2\,\alpha^2-\mu\big]^2}{(N-2)\,\alpha-\alpha^2-\mu}=\frac{\mu}{4\,d^2}\,\frac{\big[t\,a-2\,a^2-1\big]^2}{t\,a-a^2-1}\;,$$

where $a=\frac{\alpha}{\sqrt{\mu}}$. Recall that t, a and β are related by

$$1=t\,a-a^2\,(1+\beta)\;,\quad\beta\in(0,1)\;,$$

which means that at t fixed, a has to be taken such that

$$(1+\beta)\,a^2-t\,a+1=0\;.$$

Solving the equation, we get

$$a=a_\beta^\pm=\frac{t\pm\sqrt{t^2-4\,(1+\beta)}}{2\,(1+\beta)}$$

for some $\beta\in(0,1)$. The admissible domain $\mathcal{D}$ for a is therefore given by

$$\begin{array}{lll} a_0^-(t)<a<a_0^+(t) & & \text{if}\quad 2<t<2\sqrt{2}\;,\\ a_0^-(t)<a<a_1^-(t)\quad\text{or}\quad a_1^+(t)<a<a_0^+(t) & & \text{if}\quad t\ge 2\sqrt{2}\;. \end{array}$$

Notice that on such a domain, $2\,a^2-t\,a+1>0$. The function

$$a\mapsto\frac{\big(t\,a-2\,a^2-1\big)^2}{t\,a-a^2-1}=:g(a)$$

achieves its infimum on $\mathcal{D}$ at $a=\frac{1}{4}\,\big(t\pm\sqrt{t^2-8}\big)$, that is for $\beta=1$. This is admissible only for $t\ge 2\sqrt{2}$. Since

$$-\,\big(a^2-t\,a+1\big)^2\,g'(a)=\big(2\,a^2-t\,a+1\big)\,h(a)$$

with $h(a):=4\,a^3-6\,t\,a^2+(t^2+6)\,a-t$, the sign of g' is therefore the same as the one of $-h$ on $\mathcal{D}$. For any given $t>2$, h changes sign at three different values $\bar a_1(t)<\bar a_2(t)<\bar a_3(t)$. For $t>2$, the curves $t\mapsto\bar a_1(t)$ and $t\mapsto\bar a_3(t)$ do not intersect $\overline{\mathcal{D}}$, while the curve $t\mapsto\bar a_2(t)$ intersects $\overline{\mathcal{D}}$ for $2<t<2\sqrt{2}$. The minimum of g on $\overline{\mathcal{D}}$ is therefore achieved for $a=\bar a_2(t)$, $2<t\le 2\sqrt{2}$, and is 0 for $t\ge 2\sqrt{2}$.

The curve $t\mapsto\bar a_2(t)$ is explicitly given for $t>2$ by

$$\bar a_2(t)=\frac{t}{2}-2^{-4/3}\,3^{-2/3}\,\rho^{1/3}(t)\,\mathsf{Re}\,\Big((1+i\,\sqrt{3})\,e^{i\theta(t)/3}\Big)$$

where

$$\begin{aligned} &A(t) = 9\,t\,(t^2-4)\ ,\\ &B(t) = \sqrt{3\,(5\,t^6 - 72\,t^4 + 432\,t^2 - 864)}\ ,\\ &\rho(t)\,e^{i\theta(t)} = A(t) + i\,B(t)\ ,\\ &\rho(t) = 96\,(3-t^2)^{3/2}\ . \end{aligned}$$

Notice that the curves $t \mapsto \bar a_1(t)$ and $t \mapsto \bar a_3(t)$ can also be computed

$$\bar a_1(t) = \frac{t}{2} - 2^{-4/3}\,3^{-2/3}\,\rho^{1/3}(t)\,\mathsf{Re}\left((1 - i\,\sqrt{3})\,e^{i\theta(t)/3}\right)\ ,$$

$$\bar a_3(t) = \frac{t}{2} + 2^{-1/3}\,3^{-2/3}\,\rho^{1/3}(t)\,\cos(\theta(t)/3)\ .$$

Define the function κ by

$$\begin{aligned} \kappa(t) &= 0 && \text{if}\quad t \geq 2\sqrt{2} \iff \mu \leq (N-2)^2/8\ ,\\ \kappa(t) &= g(\bar a_2(t)) && \text{if}\quad 2 < t < 2\sqrt{2} \iff (N-2)^2/8 < \mu < (N-2)^2/4\ . \end{aligned}$$

We have therefore proved the following result.

Theorem 7. *With the above notations, let* $K(\mu,N) := \frac{1}{4}\,\kappa\left(\frac{N-2}{\sqrt{\mu}}\right)$. *If* $\mu \in (0, (N-2)^2/4)$, *then*

$$\mu \int_{\mathbb{R}^N} |u|^2 \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2}\right) dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{K(\mu,N)}{d^2} \int_{\mathbb{R}^N} |u|^2\,dx$$

holds for any $u \in H^1(\mathbb{R}^N)$ *and any* $y \in \mathbb{R}^N$, $d := |y| > 0$.

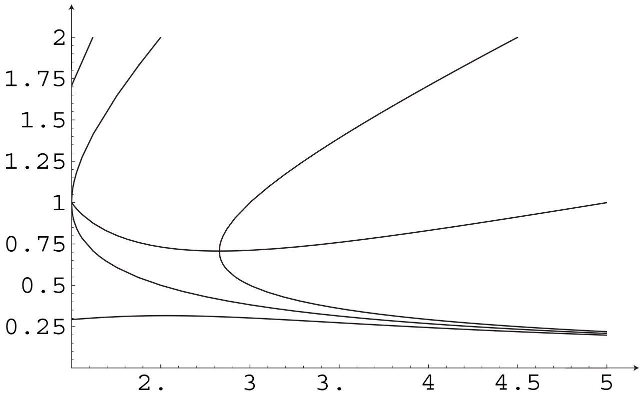

FIGURE 1. Curves $t \mapsto a_0^{\pm}(t)$, $a_1^{\pm}(t)$, $\bar a_1(t)$, $\bar a_2(t)$, $\bar a_3(t)$.

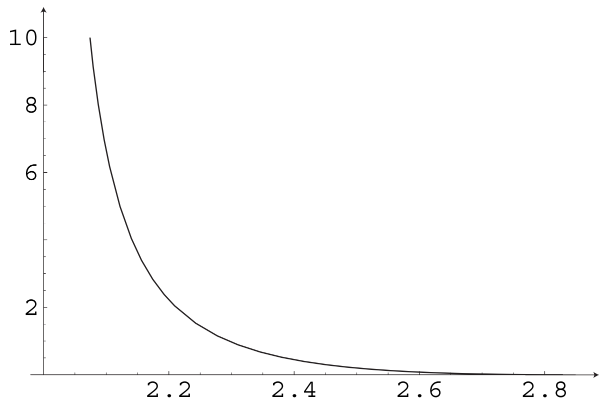

FIGURE 2. The curve $t \mapsto g(\bar a_2(t)) =: \kappa(t)$, $2 < t < 2\sqrt{2} \approx 2.82843\ldots$

Notice that the estimate of Theorem 7 is optimal as long as one uses only a local estimate of the potential. Next we prove an inequality in the case of a potential with $M \geq 2$ singularities. As a first step, we state the M-poles version of Lemma 5.

Lemma 8. *For any $u \in H^1(\mathbb{R}^N)$ and any set of poles $y_1, y_2, \dots y_M \in \mathbb{R}^N$, $M \geq 2$,*

$$\frac{(N-2)^2}{4M}\int_{\mathbb{R}^N}\sum_{k=1}^M \frac{|u|^2}{|x-y_k|^2}\,dx + \frac{(N-2)^2}{8\,M^2}\int_{\mathbb{R}^N}\sum_{\substack{j,k=1\\ j\neq k}}^M \frac{|y_j-y_k|^2}{|x-y_j|^2\,|x-y_k|^2}\,|u|^2\,dx \leq \int_{\mathbb{R}^N}|\nabla u|^2\,dx\,.$$

When $y_k = y_l$ for all k, l, one recovers the standard Hardy inequality with one singularity.

Proof. The proof is similar to the one of Lemma 5. Assume that $u \in \mathcal{D}(\mathbb{R}^N)$ and $\alpha > 0$. We compute

$$\begin{aligned} 0 \;&\leq\; \int_{\mathbb{R}^N}\left|\nabla u + \alpha\sum_{k=1}^M \frac{x-y_k}{|x-y_k|^2}\,u\right|^2 dx \\ &= \int_{\mathbb{R}^N}|\nabla u|^2\,dx + \left[\alpha^2-(N-2)\,\alpha\right]\int_{\mathbb{R}^N}|u|^2\left(\sum_{k=1}^M\frac{1}{|x-y_k|^2}\right)dx \\ &\quad + \alpha^2\int_{\mathbb{R}^N}|u|^2\sum_{\substack{j,k=1\\ j\neq k}}^M \frac{(x-y_k)\cdot(x-y_j)}{|x-y_k|^2\,|x-y_j|^2}\,dx\,, \end{aligned} \tag{5}$$

where we have applied an integration by parts. To rewrite the mixed term

$$\sum_{\substack{j,k=1\\ j\neq k}}^M \frac{(x-y_k)\cdot(x-y_j)}{|x-y_k|^2\,|x-y_j|^2} = \sum_{\substack{j,k=1\\ j\neq k}}^M \frac{|x|^2 - x\cdot y_k - x\cdot y_j + y_j\cdot y_k}{|x-y_k|^2\,|x-y_j|^2}\,,$$

we use the identity

$$\frac{|x|^2}{2}+\frac{|x|^2}{2} - x\cdot y_k - x\cdot y_j + y_j\cdot y_k = \frac{|x-y_j|^2}{2}+\frac{|x-y_k|^2}{2}-\frac{|y_k-y_j|^2}{2}\,,$$

and note that

$$\sum_{\substack{j,k=1\\ j\neq k}}^M \left(\frac{1}{|x-y_k|^2}+\frac{1}{|x-y_j|^2}\right) = 2\,(M-1)\sum_{k=1}^M\frac{1}{|x-y_k|^2}\,.$$

As a consequence,

$$0 \leq \int_{\mathbb{R}^N}|\nabla u|^2\,dx + \left[M\alpha^2-(N-2)\alpha\right]\int_{\mathbb{R}^N}\sum_{k=1}^M\frac{|u|^2}{|x-y_k|^2}\,dx - \frac{\alpha^2}{2}\int_{\mathbb{R}^N}\sum_{\substack{j,k=1\\ j\neq k}}^M \frac{|y_k-y_j|^2}{|x-y_k|^2\,|x-y_j|^2}\,|u|^2\,dx$$

and we conclude by using $M\alpha^2-(N-2)\,\alpha \geq -(N-2)^2/(4M)$, with equality if and only if $\alpha = (N-2)/(2M)$. By density we extend the proof to any $u \in H^1(\mathbb{R}^N)$. □

Define the characteristic length scale D by

$$\frac{1}{D^2} \;:=\; 2\sum_{1\leq j\neq k\leq M}\frac{1}{|y_j-y_k|^2}$$

and consider the potential $V_M := \sum_{j=1}^M \frac{1}{|x-y_j|^2}$. Notice that in case $M=2$, $D^2 = d^2$. With these notations, we can now state the following result in the case of more than two singularities.

Lemma 9. *Let $N \geq 3$, $\mu \in \mathbb{R}$ and consider a set of distinct points of $\mathbb{R}^N$, $y_1, \dots y_M$, $M \geq 2$. With D as above, there exists a constant $K(\mu, N, M) \geq 0$ such that*

$$\mu\int_{\mathbb{R}^N}V_M\,|u|^2\,dx \leq \int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{K(\mu,N,M)}{D^2}\int_{\mathbb{R}^N}|u|^2\,dx$$

and the map $\mu \mapsto K(\mu, N, M)$ *is nondecreasing and satisfies:*

(i) *If* $\mu \leq (N-2)^2/(4M)$, $K(\mu, N, M) = 0$.

(ii) *If* $(N-2)^2/(4M) < \mu \leq (N-2)^2/4$, $K(\mu, N, M)$ *takes finite positive values.*

If $\mu > (N-2)^2/4$, *then* $\mu \int_{\mathbb{R}^N} V_M\,|u|^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx$ *can be taken arbitrarily large for a well chosen* $u \in H^1(\mathbb{R}^N)$ *with* $\|u\|_{L^2(\mathbb{R}^N)} = 1$.

The proof of this lemma is straighforward and follows the same lines as the one of Lemma 6. The best constant

$$L := \sup_{u \in H^1(\mathbb{R}^N),\, \|u\|_{L^2(\mathbb{R}^N)}=1} \left(\mu \int_{\mathbb{R}^N} V_M\,|u|^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx \right)$$

a priori depends on the detail of the configuration of the singularities $\{y_1\,, \ldots\, y_M\}$, up to translations and rotations. Lemma 9 only gives an upper bound for L. We are now going to give an explicit expression for $K(\mu, N, M)$, which extends to $M \geq 2$ the result of Theorem 7. See Theorem 10 below for the precise statement.

Let us consider the square expansion (5) and look for an optimal pointwise upper estimate of

$$2\,\frac{(x-y_k)\cdot(x-y_j)}{|x-y_k|^2\,|x-y_j|^2} - \beta \left(\frac{1}{|x-y_k|^2} + \frac{1}{|x-y_j|^2} \right) .$$

If we notice that $x - y_k = (x - (y_j + y_k)/2) - (y_k - (y_j + y_k)/2) = (x - (y_j + y_k)/2) - y$ with $y := (y_k - y_j)/2$ and $x - y_j = (x - (y_j + y_k)/2) - (y_j - (y_j + y_k)/2) = (x - (y_j + y_k)/2) + y$, up to a translation of $(y_j + y_k)/2$, what we have to estimate is

$$2\,\frac{(x-y)\cdot(x+y)}{|x-y|^2\,|x+y|^2} - \beta \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) .$$

and exactly as in the proof of Lemma 6, we get

$$2\,\frac{(x-y)\cdot(x+y)}{|x-y|^2\,|x+y|^2} \leq \beta \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) + \frac{(1-\beta)^2}{4\,\beta}\,\frac{1}{|y|^2} \quad \text{a.e. in } \mathbb{R}^N .$$

Returning to the original coordinates we find

$$\frac{(x-y_k)\cdot(x-y_j)}{|x-y_k|^2\,|x-y_j|^2} \leq \beta \left(\frac{1}{|x-y_k|^2} + \frac{1}{|x-y_j|^2} \right) + \frac{(1-\beta)^2}{\beta}\,\frac{1}{|y_j-y_k|^2}$$

$$\begin{aligned}
\sum_{1 \leq j \neq k \leq M} \frac{(x-y_k)\cdot(x-y_j)}{|x-y_k|^2\,|x-y_j|^2} \quad &\leq \quad \sum_{1 \leq j \neq k \leq M} \left[\beta \left(\frac{1}{2\,|x-y_k|^2} + \frac{1}{2\,|x-y_j|^2} \right) + \frac{(1-\beta)^2}{2\,\beta\,|y_j-y_k|^2} \right] \\
&= (M-1)\,\beta \sum_{k=1}^{M} \frac{1}{|x-y_k|^2} + \frac{(1-\beta)^2}{\beta} \sum_{1 \leq j \neq k \leq M} \frac{1}{2\,|y_j-y_k|^2} \\
&= \beta_M \sum_{k=1}^{M} \frac{1}{|x-y_k|^2} + \gamma_M \;,
\end{aligned}$$

where we set $\beta_M := (M-1)\,\beta$, $\quad \gamma_M := \beta^{-1}\,(1-\beta)^2/(4\,D^2)$, where $D^{-2} = 2 \sum_{1 \leq j \neq k \leq M} |y_j - y_k|^{-2}$. From the previous estimate and (5) we obtain

$$\big[(N-2)\,\alpha - \alpha^2\,(1+\beta_M) \big] \int_{\mathbb{R}^N} |u|^2 \left(\sum_{k=1}^{M} \frac{1}{|x-y_k|^2} \right) dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \gamma_M\,\alpha^2 \int_{\mathbb{R}^N} |u|^2\,dx \;.$$

Now, as in the case with 2 poles, we set $(N-2)\,\alpha - \alpha^2\,(1+\beta_M) =: \mu$ to get

$$\mu \int_{\mathbb{R}^N} |u|^2 \left(\sum_{k=1}^{M} \frac{1}{|x-y_k|^2} \right) dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \lambda_M \int_{\mathbb{R}^N} |u|^2\,dx \,,$$

with

$$\lambda_M = \frac{1}{4\,(M-1)}\,\frac{\big[(N-2)\,\alpha - M\alpha^2 - \mu \big]^2}{(N-2)\,\alpha - \alpha^2 - \mu}\,\frac{1}{D^2} = \frac{1}{4\,(M-1)}\,\frac{\big(t\,a - M\,a^2 - 1 \big)^2}{t\,a - a^2 - 1}\,\frac{1}{D^2} \;,$$

where $a = \frac{\alpha}{\sqrt{\mu}}$, $t = \frac{N-2}{\sqrt{\mu}}$. Recall that t, a, M and β are related by

$$1 = t\,a - a^2\,(1 + \beta\,(M-1)) \;, \quad \beta \in (0,1) \;,$$

which means that at t fixed, a has to be taken such that

$$(1+\beta\,(M-1))\,a^2 - t\,a + 1 = 0\ ,$$

i.e., for some $\beta \in (0,1)$,

$$a = a_\beta^\pm = \frac{t \pm \sqrt{t^2 - 4\,(1+\beta\,(M-1))}}{2\,(1+\beta\,(M-1))}\ .$$

As in the case of 2 poles we investigate the function $g_M(a) = \frac{(t\,a - M\,a^2-1)^2}{t\,a-a^2-1}$ and see that its infimum is equal to 0 for $t \geq 2\sqrt{M}$, that is, for $\mu \in \big(0, (N-2)^2/(4\,M)\big)$, while for $\mu \in \big((N-2)^2/(4\,M), (N-2)^2/4\big)$, the infimum of g_M in its interval of definition is equal to the value of g_M at the second root $\bar{a}_{2,M} = \bar{a}_{2,M}(t)$ of the function

$$h_M(a) := 2\,M\,a^3 - 3\,M\,t\,a^2 + (t^2 + 4\,M - 2)\,a - t$$

in the interval $\big(\big(t - \sqrt{t^2-4}\,\big)/4, \big(t + \sqrt{t^2-4}\,\big)/4\big)$. Define the function κ_M by

$$\begin{aligned} \kappa_M(t) &= 0 && \text{if} \quad t \geq 2\sqrt{M} \iff \mu \leq (N-2)^2/(4\,M)\ , \\ \kappa(t) &= g_M(\bar{a}_{2,M}(t)) && \text{if} \quad 2 < t < 2\sqrt{M} \iff (N-2)^2/(4\,M) < \mu < (N-2)^2/4\ . \end{aligned}$$

The result of this computation can be summarized into the following statement.

Theorem 10. *With the above notations, let* $K(\mu, N, M) := \frac{1}{4\,(M-1)}\,\kappa_M\left(\frac{N-2}{\sqrt{\mu}}\right)$. *If* $\mu \in (0, (N-2)^2/4)$, *then*

$$\mu \int_{\mathbb{R}^N} V_M\,|u|^2\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{K(\mu,N,M)}{D^2} \int_{\mathbb{R}^N} |u|^2\,dx$$

holds for any $u \in H^1(\mathbb{R}^N)$ *and any* $y_1, \ldots y_M$ *in* $\mathbb{R}^N$.

3. Hardy inequalities for the Dirac operator

3.1. **The Dirac-Coulomb operator.** In this section we deal with Hardy-like inequalities for the Dirac operator with the multipolar Coulomb potential $\nu\,W_M$, with $\nu \in (0,1)$ and $W_M(x) = \sum_{k=1}^M \frac{1}{|x-y_k|}$. Coulomb singularities are fixed at $y_1, \ldots y_M \in \mathbb{R}^3$. The Dirac-Coulomb operator takes the form

$$H_{\nu,W_M} := H - \nu\,W_M(x)\,\mathbb{I}_4\ ,$$

$$H := -i\,\vec{\alpha}\cdot\nabla + \beta \quad \text{where} \quad \beta := \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \quad \alpha_k := \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},\ k = 1\,,\ 2\,,\ 3\,.$$

Here $\mathbb{I}_n$ the identity matrix in $(\mathbb{C}^3)^n$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

If W is a bounded function which tends to 0 at infinity, one can easily prove that the operator $H - W$ with domain $H^1(\mathbb{R}^3)$ is self-adjoint. If W has singularities, as it is the case for instance if $W = \nu\,W_M$, one is interested in defining self-adjoint extensions of $(H-W)\restriction C_0^\infty(\mathbb{R}^3)$. The method used to do this depends on the singularity. Let us for instance consider Coulomb potentials $\nu\,W_1(x) = \nu/|x|$, $\nu > 0$. Then for $\nu \in (0, \pi/2]$ one can use the pseudo-Friedrich extension method to define an extension which satisfies

$$\mathcal{D}(H_{\nu,W_1}) \subset \mathcal{D}(|H|^{1/2}) = H^{1/2}(\mathbb{R}^3)\,.$$

This result is obtained by using Kato's inequality : $|H| \geq \frac{2}{\pi\,|x|}$. Actually one can prove that H_{ν,W_1} is essentially self-adjoint if $\nu < \sqrt{3}/2$ ([40]).

When the singularities are stronger, that is for $\nu \geq \sqrt{3}/2$, other methods need to be used. Various works have dealt with this issue, and it appears that for potentials W which have a singularity at the origin, the condition

$$\sup_{x \neq 0} |x|\,W(x) < 1\,, \tag{6}$$

is sufficient to define a distinguished extension of $(H-W)\restriction C_0^\infty(\mathbb{R}^3)$. This has been done by various methods : see [47, 48, 49] in the case of semibounded potentials W and [39] without this assumption. The extension $\widetilde{T}$ is then defined as $\widetilde{T} := T^*\restriction D(T^*) \cap D(|x|^{-1/2})$. On the other hand, under Assumption (6), Nenciu proved in [38] the existence of a unique extension $\widetilde{T}$ with domain contained in $H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$. Finally, in the case of semibounded potentials satisfying (6), Klaus and Wüst proved in [33] that the aforementioned methods lead to the same self-adjoint extension.

As in the case of the Schrödinger operator, Hardy inequalities for Dirac operators provide us with lower estimates for some eigenvalues. If $\psi=\binom{\phi}{\chi}$ is an eigenfunction of $H_{\nu,W}$ with eigenvalue λ, the eigenvalue equation can be rewritten as

$$\begin{aligned} K\chi+\phi-W\,\phi&=\lambda\,\phi\;,\\ K\phi-\chi-W\,\chi&=\lambda\,\chi\;,\end{aligned}$$

where $K:=-i\,\vec{\sigma}\cdot\nabla$. We can eliminate the lower component χ. The equation for ϕ is

$$K\left(\frac{K\phi}{1+\lambda+W}\right)+\phi-W\,\phi=\lambda\,\phi$$

and so, ϕ appears as a critical point of $\phi\mapsto\int_{\mathbb{R}^3}\frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+\lambda+W}\,dx+(1-\lambda)\int_{\mathbb{R}^3}|\phi|^2\,dx-\int_{\mathbb{R}^3}W\,|\phi|^2\,dx$, with critical value 0. Such a functional is monotone decreasing as a function of λ. Hence if $\Lambda=\lambda_1$ is the smallest eigenvalue of $H_{\nu,W}$ in $(-1,1)$, then

$$\int_{\mathbb{R}^3}\frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+\Lambda+W}\,dx+(1-\Lambda)\int_{\mathbb{R}^3}|\phi|^2\,dx\geq\int_{\mathbb{R}^3}W\,|\phi|^2\,dx \tag{7}$$

holds with $\Lambda=\lambda_1$. Reciprocally, for a large class of potentials W, any eigenvalue of $H_{\nu,W}$ in $(-1,1)$ can be characterized as a min-max of the Rayleigh quotient $(H_{\nu,W}\,\psi,\psi)\,/\|\psi\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$ where $\psi=\binom{\phi}{\chi}$ is decomposed into an upper component ϕ and a lower one, χ. Under appropriate conditions, see [13, 14],

$$\lambda\int_{\mathbb{R}^3}|\phi|^2\,dx=\max_{\chi}\frac{(H_{\nu,W}\psi,\psi)}{\|\psi\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}}=\int_{\mathbb{R}^3}\frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+\lambda+W}\,dx+\int_{\mathbb{R}^3}|\phi|^2\,dx\geq\int_{\mathbb{R}^3}W\,|\phi|^2\,dx\;,$$

which implicitly determines $\lambda=\lambda[\phi]$. The minimization step is then reduced to establish that

$$\lambda_1=\min_{\phi\neq0}\lambda[\phi]\;.$$

This completes the proof, at least at a formal level, that finding the lowest eigenvalue of $H_{\nu,W}$ in the gap $(-1,1)$ is equivalent to getting the best constant in Inequality (7). It was proved in [13, 14] that for a large class of potentials with at most one singularity, the first eigenvalue of $H-W$ in the spectral gap $(-1,1)$ is the largest constant Λ for which (7) holds. For instance, in the case of the radial Coulomb potential, $-\nu/|x|$, for all $\nu\in(0,1)$,

$$\int_{\mathbb{R}^3}\frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+S(\nu)+\frac{\nu}{|x|}}\,dx+\big[1-S(\nu)\big]\int_{\mathbb{R}^3}|\phi|^2\,dx\geq\int_{\mathbb{R}^3}\frac{\nu}{|x|}\,|\phi|^2\,dx\;, \tag{8}$$

where $S(\nu):=\sqrt{1-\nu^2}$ is the best possible constant. Passing to the limit $\nu\to1_-$, Inequality (8) also holds for $\nu=1$.

In this section we show that inequalities like (7) can be proved for the multipolar potentials $\nu\,W_M$ and provide us with lower estimates for the eigenvalues of $H-\nu\,W_M$ in the interval $(-1,1)$. We shall use the IMS method to localize the integrals and to reduce the problem to locally radial potentials.

3.2. **A priori estimates.**

Consider as in Section 1 a partition of unity defined by (2). The operator H_{ν,W_M} acts on the 4-spinors $\psi=\binom{\phi}{\chi}\in H^1(\mathbb{R}^3,\mathbb{C}^4)$. If we write $\phi=\binom{\phi_1}{\phi_2}$ with $\phi_1,\,\phi_2\in H^1(\mathbb{R}^3,\mathbb{C})$, the expressions $|\phi|^2$, $|\nabla\phi|^2$, $|\vec{\sigma}\cdot\nabla\,\phi|^2$ denote respectively the quantities $|\phi_1|^2+|\phi_2|^2$, $\sum_{k=1}^3|\partial_k\phi_1|^2+|\partial_k\phi_2|^2$ and $|\partial_3\phi_1+\partial_1\phi_2-i\,\partial_2\phi_2|^2+|\partial_1\phi_1+i\,\partial_2\phi_1-\partial_3\phi_2|^2$ where the notation ∂_k denotes ∂/∂_{x_k}, $k=1,\,2,\,3$. Denoting by $L:=-i\,\nabla\wedge x$ the angular momentum operator, we recall that

$$\vec{\sigma}\cdot\nabla=\Big(\vec{\sigma}\cdot\frac{x}{|x|}\Big)\Big(\partial_r-\frac{1}{r}\,\vec{\sigma}\cdot L\Big)\,, \tag{9}$$

with $r=|x|$ and $\partial_r=\frac{x}{r}\cdot\nabla$. The spectrum of $\vec{\sigma}\cdot L+1$ is the discrete set $\{l\in\mathbb{Z}\,:\,l\neq0\}$. See [12] and [44] for more details. If P_l is the spectral projection onto the eigenspace of $H^1(\mathbb{R}^3,\mathbb{C}^2)$ corresponding to the eigenvalue l of $\vec{\sigma}\cdot L+1$, then, in analogy with Lemma 5 of [12], we can prove that

$$P_k\,(\vec{\sigma}\cdot\nabla)^2\,P_l=P_l\,(\vec{\sigma}\cdot\nabla)^2\,P_k\equiv0\quad\text{in } H^1(\mathbb{R}^3,\mathbb{C}^2)\quad\forall\,k\,,\;l\in\mathbb{Z}\setminus\{0\}\,,\;k\neq l\;.$$

This implies that $\langle P_k\, \phi, P_l\, \phi\rangle_{L^2(\mathbb{R}^3, m\, dx)} = 0$, for $k \neq l$ and for each measurable radial function $m = m(|x|)$. Hence the L^2-norm of $\vec{\sigma}\cdot\nabla\phi$ can be written in terms of the above-mentioned spectral decomposition as

$$\|(\vec{\sigma}\cdot\nabla)\,\phi\|^2_{L^2(\mathbb{R}^3,\, m\,dx)} = \sum_{l\in\mathbb{Z}\setminus\{0\}} \|(\vec{\sigma}\cdot\nabla)\,P_l\phi\|^2_{L^2(\mathbb{R}^3,\, m\,dx)} = \sum_{l\in\mathbb{Z}\setminus\{0\}} \|(\partial_r - \tfrac{(l-1)}{r})\,P_l\phi\,\|^2_{L^2(\mathbb{R}^3,\, m\,dx)}\,, \tag{10}$$

where we have used $|\vec{\sigma}\cdot x/|x|\,| = 1$.

We now decompose the term $|\vec{\sigma}\cdot\nabla\phi|^2$ by using the partition of unity defined in Section 1 and take advantage of the fact that the function J is scalar and takes real values.

Lemma 11. *Let* $\{J_k\}_{k=1}^{M+1}$ *be any partition of unity such that* $\mathrm{Int}\,(\mathrm{supp}(J_k)) \cap \mathrm{Int}\,(\mathrm{supp}(J_l)) = \emptyset$ *if* $1 \le k \neq l \le M$*. Then*

$$|\vec{\sigma}\cdot\nabla\,\phi|^2 = \sum_{k=1}^{M+1} |\vec{\sigma}\cdot\nabla\,(J_k\phi)|^2 - \sum_{k=1}^{M+1} |\nabla J_k|^2\,|\phi|^2\,.$$

Proof. We first denote by J an arbitrary element of the partition of unity. By applying the definition of $|\vec{\sigma}\cdot\nabla\,(J\phi)|^2$ and grouping the terms with J^2, $\sum_{a=1}^3 |\partial_a J|^2$ and $J\,\partial_a J$, we obtain the following:

$$|\vec{\sigma}\cdot\nabla\,(J\phi)|^2 = J^2\,|\vec{\sigma}\cdot\nabla\phi|^2 + |\nabla J|^2|\phi|^2 + f_1(\phi)\,(J\,\partial_1 J) + f_2(\phi)\,(J\,\partial_2 J) + f_3(\phi)\,(J\,\partial_3 J)\,,$$

with f_1, f_2, f_3 the real valued functions

$$\begin{aligned} f_1(\phi) &:= 2\,\mathsf{Re}\,(\langle\phi_2, \partial_1\phi_2 - i\,\partial_2\phi_2 + \partial_3\phi_1\rangle) + 2\,\mathsf{Re}\,(\langle\phi_1, \partial_1\phi_1 + i\,\partial_2\phi_1 - \partial_3\phi_2\rangle)\,,\\ f_2(\phi) &:= 2\,\mathsf{Re}\,(\langle -i\,\phi_2, \partial_1\phi_2 - i\,\partial_2\phi_2 + \partial_3\phi_1\rangle) + 2\,\mathsf{Re}\,(\langle i\,\phi_1, \partial_1\phi_1 + i\,\partial_2\phi_1 - \partial_3\phi_2\rangle)\,,\\ f_3(\phi) &:= 2\,\mathsf{Re}\,(\langle\phi_1, \partial_1\phi_2 - i\,\partial_2\phi_2 + \partial_3\phi_1\rangle) + 2\,\mathsf{Re}\,(\langle -\phi_2, \partial_1\phi_1 + i\,\partial_2\phi_1 - \partial_3\phi_2\rangle)\,, \end{aligned}$$

where $\langle a, b\rangle = \overline{a}\,b$ and $\mathsf{Re}(z)$ denotes the real part of the complex number z. To obtain the result we have to sum over all the indexes $k = 1, \dots M+1$ and remember that $\sum_{k=1}^{M+1} J_k\,\partial_a J_k = 0$ for $a = 1,\,2,\,3$.

□

The next result is a local estimate on domains where the potential W admits a radial dominant potential.

Lemma 12. *Let* $W \ge 0$ *and* $J \in W^{1,\infty}(\mathbb{R}^3)$ *be such that for some* $z \in \mathbb{R}^3$, $C_W \in \mathbb{R}^+$ *and* $\nu \in (0,1)$,

$$W(x) \le \frac{\nu}{|x - z|} + C_W \quad \text{on} \quad \mathrm{supp}(J)\,. \tag{11}$$

Then, for any $\phi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ *and any* $\lambda > -1$,

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\,(J\phi)|^2}{1 + \lambda + W(x)}\,dx \ge \int_{\mathbb{R}^3} W\,|J\phi|^2\,dx - C_1 \int_{\mathbb{R}^3} |J\phi|^2\,dx$$

where $C_1 = C_1(\lambda, C_W) := (1 + \lambda + C_W)\,\big[1 - S(\nu)\big]^2/\nu^2 + C_W$*.*

Proof. The proof is divided into three steps.

Step 1: Spectral decomposition. From (11), we know that

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\,(J\phi)|^2}{1 + \lambda + W(x)}\,dx \ge \int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\varphi|^2}{1 + \lambda + C_W + \frac{\nu}{r}}\,dx$$

for $\varphi(x) := (J\phi)(x+z)$. In order to apply (10) to the last integral, we write φ as a linear combination of eigenfunctions $\varphi_l := P_l\,\varphi$ of the operator $\vec{\sigma}\cdot L + 1$ with corresponding eigenvalues $l \in \mathbb{Z}\setminus\{0\}$. Set $g(r) := 1/(b\,r + \nu)$, $b = 1 + \lambda + C_W$, and use (9) to get

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\varphi_l|^2}{1 + \lambda + C_W + \frac{\nu}{r}}\,dx = \int_{\mathbb{R}^3} \frac{r\,|\vec{\sigma}\cdot\nabla\,\varphi_l|^2}{b\,r + \nu}\,dx\\ &= \int_{\mathbb{R}^3} g(r)\,r\,\Big|\partial_r\varphi_l - (l-1)\,\frac{1}{r}\,\varphi_l\Big|^2\,dx\\ &= \int_{\mathbb{R}^3} g(r)\,r\,|\partial_r\varphi_l|^2\,dx + \int_{\mathbb{R}^3} \frac{(l-1)^2}{r}\,g(r)\,|\varphi_l|^2\,dx + (l-1)\int_{\mathbb{R}^3} \Big(g'(r) + \frac{2\,g(r)}{r}\Big)\,|\varphi_l|^2\,dx\\ &= \int_{\mathbb{R}^3} g(r)\,r\,|\partial_r\varphi_l|^2\,dx + \int_{\mathbb{R}^3} \Big[(l^2 - 1)\,\frac{g(r)}{r} + (l-1)\,g'(r)\Big]\,|\varphi_l|^2\,dx\,, \end{aligned}$$

where we have integrated by parts in the variable r. Since the last integral is non-negative for every $l \in \mathbb{Z}\setminus\{0\}$, we obtain

$$\int_{\mathbb{R}^3} \frac{r\,|\vec{\sigma}\cdot\nabla\varphi_l|^2}{b\,r+\nu}\,dx \geq \int_{\mathbb{R}^3} g(r)\,r\,|\partial_r\varphi_l|^2\,dx\;.$$

Step 2. Completing the square. Let $h \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a measurable radial function, a.e. differentiable, and consider the following square expansion,

$$\begin{aligned} 0 \;&\leq\; \int_{\mathbb{R}^3} \Big|\sqrt{r\,g(r)}\,\partial_r\varphi_l + \frac{h(r)}{\sqrt{r\,g(r)}}\,\varphi_l\Big|^2\,dx \\ &=\; \int_{\mathbb{R}^3} r\,g(r)\,|\partial_r\varphi_l|^2\,dx - \int_{\mathbb{R}^3}\Big(h'(r) + \frac{2\,h(r)}{r} - (b\,r+\nu)\,\frac{h^2(r)}{r}\Big)\,|\varphi_l|^2\,dx\;. \end{aligned}$$

If $h(r) = \frac{1-S(\nu)}{\nu}$, it holds

$$\int_{\mathbb{R}^3} g(r)\,r\,|\partial_r\varphi_l|^2\,dx \geq \int_{\mathbb{R}^3}\Big(\frac{\nu}{r} - \frac{b\,\big[1-S(\nu)\big]^2}{\nu^2}\Big)\,|\varphi_l|^2\,dx\;.$$

Hence we have shown that for every $l \in \mathbb{Z}\setminus\{0\}$, for every $b>0$, the eigenfunction φ_l satisfies the inequality

$$\int_{\mathbb{R}^3} \frac{r\,|\vec{\sigma}\cdot\nabla\varphi_l|^2}{b\,r+\nu}\,dx \geq \int_{\mathbb{R}^3} \frac{\nu}{r}\,|\varphi_l|^2\,dx - b\,\frac{[1-S(\nu)]^2}{\nu^2}\int_{\mathbb{R}^3}|\varphi_l|^2\,dx\;.$$

Step 3. Recomposition. A sum over $l \in \mathbb{Z}\setminus\{0\}$ gives

$$\int_{\mathbb{R}^3} \frac{r\,|\vec{\sigma}\cdot\nabla\varphi|^2}{b\,r+\nu}\,dx \geq \int_{\mathbb{R}^3} \frac{\nu}{r}\,|\varphi|^2\,dx - \frac{b\,[1-S(\nu)]^2}{\nu^2}\int_{\mathbb{R}^3}|\varphi|^2\,dx\;.$$

Now we return to the original coordinates with $b = 1+\lambda+C_W$, we add and subtract $C_W\int_{\mathbb{R}^3}|J\phi|^2\,dx$ to the right hand side. From a last application of (11) we get the result with $C_1 = b\,[1-S(\nu)]^2/\nu^2 + C_W$. ☐

3.3. **Case d large.** We consider an arbitrary configuration of poles $y_k \in \mathbb{R}^3$, and define the following quantities :

$$a := \frac{\nu\,M}{d}\;,\quad b := \frac{\big[1-S(\nu)\big]^2}{\nu^2}\;,\quad c := 2\,\frac{1-S(\nu)}{\nu^2}\;,$$

$$d^*(\nu) = \frac{1}{2}\,M\,\nu\,c + \pi\,\sqrt{c}\,,$$

$$\lambda^*(d,\nu,M) := \frac{1}{c}\Big[1+\sqrt{c\,(a^2\nu^{-2}-\pi^2\,d^{-2}-a)+1-a^2\,\nu^{-2}}\Big] - 1 - \frac{a}{2}\;,$$

with $S(\nu) = \sqrt{1-\nu^2}$ and $d := \min_{i\neq k}|y_i - y_j|/2$.

Theorem 13. *With the above notations, for all $\nu \in (0,1)$, $M \geq 2$, if $d \geq d^*(\nu)$, then for all $\phi \in H^1(\mathbb{R}^3,\mathbb{C}^2)$ we have*

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+\lambda^*+\nu\,W_M}dx + (1-\lambda^*)\int_{\mathbb{R}^3}|\phi|^2\,dx \geq \nu\int_{\mathbb{R}^3} W_M\,|\phi|^2\,dx\;.$$

We observe that $\lambda^* > -1$ for every $d \geq d^*(\nu)$ and that $\lambda^* = 0$ for

$$d = \bar{d} := \frac{1}{c-2}\,\Big(-\frac{1}{2}\,M\,\nu\,c - \sqrt{M^2\,(c-1)-\pi^2\,(c-2)}\Big)\;.$$

Proof. Let $\{J_k\}_{k=1}^{M+1}$ be a partition of unity supported on balls defined as in Section 1, namely $J_k(x) = J(|x-y_k|/d)$ for all $x \in \mathbb{R}^3$, $1 \leq k \leq M$, with J as in (2) and $J_{M+1} = \sqrt{\sum_{k=1}^M J_k^2}$. For any $k = 1,\dots M$, the function J_k is supported in $\overline{B(y_k,d)}$. Note that

$$\nu\,W_M(x) \leq \begin{cases} \dfrac{\nu}{|x-y_k|} + \dfrac{M-1}{d}\,\nu & \text{on } \operatorname{supp}(J_k)\quad \forall\,k = 1,\dots M\;, \\[2ex] \dfrac{\nu}{|x-y_1|} + \dfrac{M}{d}\,\nu & \text{on } \operatorname{supp}(J_{M+1})\;. \end{cases}$$

On $\operatorname{supp}(J_{M+1})$ we observe indeed that

(1) Either $x \in \operatorname{supp}(J_{M+1})\setminus\cup_{k=2}^M \operatorname{supp}(J_k)$: $\nu\,W_M(x) \leq \nu/|x-y_1| + \nu\,(M-1)/d$.

(2) Or $x \in \operatorname{supp}(J_{M+1}) \cap \operatorname{supp}(J_k)$ for some $k = 2\,, \dots M$: $\operatorname{dist}(x, y_k) \geq d/2$ and $\operatorname{dist}(x, y_j) \geq d$ for $j \neq k$.

We can then apply Lemma 12 to $\int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\,(J_k\phi)|^2}{1+\lambda+\nu\,W_M}\,dx$ with

$$C_1 = \Big(1 + \lambda + \frac{M-1}{d}\,\nu\Big)\,\frac{\big[1 - S(\nu)\big]^2}{\nu^2} + \frac{\nu\,(M-1)}{d} =: K_1(\lambda, \nu, d, M)\;,$$

for $k = 1\,, \dots M$, and

$$C_1 = \Big(1 + \lambda + \frac{\nu M}{d}\Big)\,\frac{\big[1 - S(\nu)\big]^2}{\nu^2} + \frac{\nu\,M}{d} =: K_2(\lambda, \nu, d, M)$$

for $k = M + 1$. From this and Lemma 11, it follows that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+\lambda+\nu\,W_M}\,dx &= \sum_{k=1}^{M+1}\int_{\mathbb{R}^3}\frac{|\vec{\sigma}\cdot\nabla\,(J_k\phi)|^2}{1+\lambda+\nu\,W_M} - \int_{\mathbb{R}^3}\sum_{k=1}^{M+1}|\nabla J_k|^2\,\frac{|\phi|^2}{1+\lambda+\nu\,W_M}\,dx\\ &\geq \int_{\mathbb{R}^3}\nu\,W_M|\phi|^2\,dx - K_1\sum_{k=1}^{M}\int_{\mathbb{R}^3}|J_k\phi|^2\,dx - K_2\int_{\mathbb{R}^3}|J_{M+1}\phi|^2\,dx - \int_{\mathbb{R}^3}\sum_{k=1}^{M+1}|\nabla J_k|^2\,\frac{|\phi|^2}{1+\lambda}\,dx\;.\end{aligned}$$

Property (d) of the partition of unity, see Section 1, gives

$$\sum_{k=1}^{M+1}|\nabla J_k|^2 \leq \frac{\pi^2}{d^2}\,\mathbb{I}_{\operatorname{supp}(|\nabla J_{M+1}|)}\;,$$

thus implying

$$\int_{\mathbb{R}^3}\frac{|\vec{\sigma}\cdot\nabla\,\phi|^2}{1+\lambda+\nu\,W_M}\,dx \geq \int_{\mathbb{R}^3}\nu\,W_M|\phi|^2\,dx - K_2\int_{\mathbb{R}^3}|\phi|^2\,dx - \int_{\mathbb{R}^3}\frac{\pi^2}{d^2\,(1+\lambda)}\,|\phi|^2\,dx\;, \tag{12}$$

since $K_2 > K_1$. To get λ^*, we now choose the largest possible λ such that

$$K_2(\lambda, \nu, d, M) + \frac{\pi^2}{d^2\,(1+\lambda)} \leq 1 - \lambda\;,$$

that is the largest root of

$$(1+b)\,\lambda^2 + \big[a + (2+a)\,b\big]\,\lambda + (1+a)\,b + a + \frac{\pi^2}{d^2} - 1 = 0\;. \tag{13}$$

The value $d^*(\nu)$ is the minimal d for which the discriminant of the second order equation (13) is nonnegative. For such a d, one can check that $\lambda^* > -1$. □

In the case $M = 2$, a better result can be achieved by considering the partition of unity $\{J_k\}_{k=1}^3$ defined by (3), *i.e.* such that

$$J_1(x) = \begin{cases} 1 & \text{if } |x-y| \leq |x|\;,\\ \sin\!\big(\pi\big(x\cdot y\big)/d^2\big) & \text{if } 0 \leq x\cdot y \leq d^2/2\;,\\ 0 & \text{otherwise}\;,\end{cases}$$

$J_2(x) := J_1(-x)$ for any $x \in \mathbb{R}^3$, and $J_3 := \sqrt{1 - J_1^2 - J_2^2}$

Corollary 14. *If $M = 2$, the results of Theorem 13 hold with $a = \nu/d$ and $d^*(\nu) = \frac{\nu\,c}{2} + \pi\,\sqrt{c}$.*

Proof. From Property (d) of Section 1 we derive

$$\sum_{k=1}^{3}|\nabla J_k|^2 = \frac{|\nabla J_1|^2}{1 - J_1^2} + \frac{|\nabla J_2|^2}{1 - J_2^2} \leq \frac{\pi^2}{d^2}\,\mathbb{I}_{\operatorname{supp}(\nabla J_3)}\;. \tag{14}$$

Moreover, we have the estimate

$$W_2(x) \leq \begin{cases} \frac{1}{|x-y|} + \frac{1}{d} & \text{on } \operatorname{supp}(J_1)\;,\\ \frac{1}{|x+y|} + \frac{1}{d} & \text{on } \operatorname{supp}(J_2)\;,\\ \frac{1}{|x|} + \frac{1}{d} & \text{on } \operatorname{supp}(J_3)\;.\end{cases} \tag{15}$$

The first inequality holds because $|x+y| \geq d$ on $\operatorname{supp}(J_1) = \{x \in \mathbb{R}^3 \,:\, |x-y| \leq |x+y|\}$. The second inequality is similar. The last one can be obtained as in the proof of Proposition 4. The proof is then the same as the one of Theorem 13, with $K_2(\lambda, \nu, d, 2)$ replaced by $K_2(\lambda, \nu, d, 1)$. □

In the case $M > 2$, if one wants to improve on Theorem 13, one has to make a geometric assumption, which is always satisfied for $M = 2$.

Corollary 15. *Let $M \geq 2$. Assume that there exists a partition of unity $\{J_k\}_{k=1}^{M+1}$ such that for some $z_1, \dots z_M, z_{M+1} \in \mathbb{R}^3$,*

$$W_M(x) \leq \frac{1}{|x - z_k|} + \frac{(M-1)}{d} \quad on\, \mathrm{supp}(J_k) \quad \forall\, k = 1, \dots M+1\,,$$

and $\sum_{k=1}^{M+1} |\nabla J_k|^2 \leq \pi^2/d^2$. Then the results of Theorem 13 hold with $a = (M-1)\,\nu/d$ and $d^*(\nu) = \nu\, c\,(M-1)/2 + \pi\,\sqrt{c}$.

Proof. One proceeds as in the proof of Theorem 13, with K_2 replaced by $K_2(\lambda, \nu, d, M-1) = K_1(\lambda, \nu, d, M)$. ☐

Theorem 13 has several consequences. Assume first that $b_M = M$.

Corollary 16 (Asymptotics for λ^*)**.** *For $\nu \to 1$, we find $\bar{d} \to +\infty$ and $d^*(\nu) \to b_M + \pi\sqrt{2}$, while for $\nu \to 0$, $\bar{d} \to \pi$ and $d^*(\nu) \to \pi$. Moreover, for fixed $\nu \in (0,1)$, $M \geq 2$,*

$$\lambda^* = \sqrt{1-\nu^2} - b_M\, \frac{\nu}{d} + O\left(\frac{1}{d^2}\right) \quad as\; d \to \infty\,.$$

Under the assumption of Corollaries 14 ($M = 2$) and 15, we can take $b_M = M - 1$. We also observe that λ^* provides a lower bound for the lowest eigenvalue in the spectral gap of H_{ν, W_M}. As $d \to +\infty$, λ^* converges to $S(\nu) = \sqrt{1-\nu^2}$ and one recovers the Hardy-like inequality (8) for the 1-pole potential.

3.4. **Further estimates.** One can actually use Lemma 12 to obtain slightly different inequalities which are not Hardy inequalities for Dirac operators as (7). The difference lies in the fact that the coefficient in front of the L^2 term is taken bigger than in (7). Such inequalities are valid for all $d > 0$ and are useful to obtain asymptotics both in the cases $d \to +\infty$ and $d \to 0$, for ν small enough.

With the same notations as in Corollary 16, let

$$\Lambda(d, \nu) := \frac{2\,[1 - S(\nu)]}{\nu\, d}\, b_M + \frac{\pi^2}{d^2\,(1 + S(\nu))}\,.$$

By taking $\lambda = S(\nu)$ instead of λ^* in (12), we obtain the following result. Note that for $d \to \infty$, one recovers the Hardy-like inequality (8) for 1-pole.

Corollary 17. *For all $\phi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ and $\nu \in (0,1)$,*

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma} \cdot \nabla \phi|^2}{1 + S(\nu) + \nu\, W_M(x)}\, dx + \big[1 - S(\nu)\big] \int_{\mathbb{R}^3} |\phi|^2\, dx + \Lambda(d, \nu) \int_{\mathbb{R}^3} |\phi|^2\, dx \geq \nu \int_{\mathbb{R}^3} W_M\, |\phi|^2\, dx\,.$$

Motivated by the asymptotics corresponding to $d = |y_2 - y_1| \to 0$ in case $M = 2$, we can prove another inequality for the bipolar Coulomb potential, which also holds for any $d > 0$. Let $W_2(x) = \frac{1}{|x-y|} + \frac{1}{|x+y|}$ with $y = (d, 0, 0) \in \mathbb{R}^3$, $d > 0$ and define $\Omega(d) := \{x \in \mathbb{R}^3 \,:\, |(x \cdot y)| \leq d^2/2\}$.

Theorem 18. *With the above notations, for any $\nu \in (0, 1/2)$ and any $\phi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$,*

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma} \cdot \nabla \phi|^2}{1 + S(2\nu) + \nu\, W_2}\, dx + \big[1 - S(2\nu)\big] \int_{\mathbb{R}^3} |\phi|^2\, dx + \frac{\pi^2}{d^2\,\big[1 + S(2\nu)\big]} \int_{\Omega(d)} |\phi|^2\, dx \geq \nu \int_{\mathbb{R}^3} W_2\, |\phi|^2\, dx\,.$$

Proof. We consider $(J_k)_{k=1}^3$ the partition of unity (3). From Lemma 11 we get

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma} \cdot \nabla \phi|^2}{1 + \lambda + \nu\, W_2(x)}\, dx = \sum_{k=1}^{3} \int_{\mathbb{R}^3} \frac{|\vec{\sigma} \cdot \nabla\, (J_k \phi)|^2}{1 + \lambda + \nu\, W_2(x)}\, dx - \int_{\mathbb{R}^3} \sum_{k=1}^{3} |\nabla J_k|^2\, \frac{|\phi|^2}{1 + \lambda + \nu\, W_2(x)}\, dx\,,$$

and observe that

$$W_2(x) \leq \begin{cases} \frac{2}{|x-y|} & \text{on} \quad \mathrm{supp}(J_1)\,, \\ \frac{2}{|x+y|} & \text{on} \quad \mathrm{supp}(J_2)\,, \\ \frac{2}{|x|} & \text{on} \quad \mathrm{supp}(J_3)\,. \end{cases}$$

We apply Lemma 12 to $(J_k\, u)$, $k = 1, 2, 3$, with $C_W = 0$ and ν replaced by 2ν:

$$\int_{\mathbb{R}^3} \frac{|\vec{\sigma} \cdot \nabla\, (J_k \phi)|^2}{1 + S(2\nu) + \nu\, W_2}\, dx \geq \nu \int_{\mathbb{R}^3} W_2\, |J_k \phi|^2\, dx - \big[1 - S(2\nu)\big] \int_{\mathbb{R}^3} |J_k \phi|^2\, dx\,.$$

On the other hand, on $\Omega(d) := \{x \in \mathbb{R}^3\,:\,|(x\cdot y)| \le d^2/2\}$, we have the estimate

$$\sum_{k=1}^{3} |\nabla J_k|^2 \,\frac{1}{1+S(2\nu)+\nu\, W_2} \le \frac{\pi^2}{d^2}\,\frac{1}{1+S(2\nu)}\;.$$

Putting the above estimates together we complete the proof. □

We are able to recover the Hardy-like inequality for the Dirac operator with a radial Coulomb potential (8) by taking the limit as $d \to 0$ only under the following technical assumption:

$$\lim_{d\to 0}\,\frac{1}{d^2}\int_{|(x\cdot y)|\le d^2/2} |\phi|^2\,dx = 0\;.$$

A sufficient condition is $\phi(0,x_2,x_3) = 0$ for almost any $(x_2,x_3) \in \mathbb{R}^2$. Namely, choose $\phi = \binom{\phi_1}{\phi_2} \in C^1(\mathbb{R}^3,\mathbb{C}^2)$ and apply Poincaré's inequality. With the notations $y = (d,0,0)$ and $\Omega(d) := [-d/2, d/2]\times\mathbb{R}^2$, we can write

$$|\phi(x_1,x_2,x_3)| \le |\phi(0,x_2,x_3)| + \left|\int_0^{x_1} \partial_1\phi(s,x_2,x_3)\,ds\right|\;,$$

$$|\phi(x)|^2 \le 2\,|\phi(0,x_2,x_3)|^2 + 2\,d\int_{-d/2}^{d/2} |\partial_1\phi(s,x_2,x_3)|^2\,ds\;,$$

where we have used $|a+b|^2 \le 2\,|a|^2 + 2\,|b|^2$ and Cauchy-Schwartz's inequality. Integrating both terms over $\Omega(d)$, it results

$$\int_{\Omega(d)} |\phi|^2\,dx \le 2\,d\int_{\mathbb{R}^2} |\phi(0,x_2,x_3)|^2\,dx_2\,dx_3 + 2\,d^2\int_{\Omega(d)} |\partial_1\phi|^2\,dx\;.$$

We know that $\int_{\Omega(d)} |\partial_1\phi|^2\,dx \le \int_{\Omega(d)} |\nabla\phi|^2\,dx \to 0$ as $d \to 0$, which completes the proof. By density, we can extend the result to all $\phi \in \{\, f \in H^1(\mathbb{R}^3,\mathbb{C}^2)\,:\, f(0,x_2,x_3) = 0 \text{ a.e.}\,\}$.

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