UNIFORM CONVERGENCE OF THE LIE-DYSON EXPANSION WITH RESPECT TO THE PLANCK CONSTANT

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Abstract. We prove that the Lie-Dyson expansion for the Heisenberg observables has a nonzero convergence radius in the variable ϵt which does not depend on the Planck constant $\hbar$. Here the quantum evolution $U_{\hbar,\epsilon}(t)$ is generated by the Schrödinger operator defined by the maximal action in $L^2(\mathbb{R}^n)$ of $-\hbar^2\Delta+\mathcal{Q}+\epsilon V$; $\mathcal{Q}$ is a positive definite quadratic form on $\mathbb{R}^n$; the observables and V belong to a suitable class of pseudodifferential operators with analytic symbols. It is furthermore proved that, up to an error of order ϵ, the time required for an exchange of energy between the unperturbed oscillator modes is exponentially long time independently of $\hbar$.

1. Introduction and statement of results

Consider in $L^2(\mathbb{R}^n)$ the Schrödinger operator family

$$H(\epsilon)=H_0+\epsilon V, \qquad H_0=-\frac{\hbar^2}{2m}\Delta+Q. \tag{1.1}$$

Here the Q is the maximal multiplication operator by a non-negative quadratic form $\mathcal{Q}(x):\mathbb{R}^n\to\mathbb{R}$, V is a (semiclassical) pseudodifferential operator of order 0 and $\epsilon\in\mathbb{R}$. Denote $H^2(\mathbb{R}^n)$ the usual Sobolev space, and $\hat{H}^2(\mathbb{R}^n)$ its conjugate by Fourier transform,

$$\hat{f}(s)=h^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-i\langle x,s\rangle/\hbar}\,dx$$

namely,

$$\hat{H}^2(\mathbb{R}^n):=\{\hat{f}\ \mid \int_{\mathbb{R}^n}(1+|s|^2)^2|\hat{f}(s)|^2\,ds\ <\ +\infty\}; \tag{1.2}$$

Under the above conditions H_0, defined on the domain $H^2(\mathbb{R}^n)\cap\hat{H}^2(\mathbb{R}^n)$ is a self-adjoint operator. Hence the unitary group $U_0(t):=e^{i\hbar^{-1}H_0t}$, $t\in\mathbb{R}$, exists. Moreover, V is relatively bounded with respect to H_0; therefore the operator family $H(\epsilon):\epsilon\in\mathbb{R}$ defined on $D(H_0)$ is also self-adjoint so that the corresponding unitary group $U_\epsilon(t):=e^{i\hbar^{-1}H_\epsilon t}$, $t\in\mathbb{R}$, exists. Given a continuous quantum observable, represented by a continuous self-adjoint operator G on $L^2(\mathbb{R}^n)$, consider the corresponding Heisenberg observables under the free and the full evolution, respectively:

$$G_0(t) := U_0(t)GU_0(-t); \qquad G_\epsilon(t) := U_\epsilon(t)GU_\epsilon(-t); \tag{1.3}$$

Perturbation theory in ϵ yields a formal recurrent procedure to construct the full Heisenberg evolution $G_\epsilon(t)$ in terms of the free evolution $G_0(t)$. The result, known as Dyson's expansion, can be briefly obtained as follows: consider the Heisenberg equation

$$\dot{G}_\epsilon(t) = i[H_\epsilon, G_\epsilon(t)]/\hbar = i[H_0, G_\epsilon(t)]/\hbar + i\epsilon[V, G_\epsilon(t)]/\hbar \tag{1.4}$$

and look for a solution under the form

$$G_\epsilon(t) = G_0(t) + \epsilon G_1(t) + \epsilon^2 G_2(t) + \dots; \quad G_0(0) = G \tag{1.5}$$

Inserting this in (1.4) and equating the coefficients of the same powers of ϵ in both sides we obtain the recurrent equations:

$$\dot{G}_k(t) = i[H_0, G_k(t)]/\hbar + i[V, G_{k-1}(t)]/\hbar, \quad k = 1, 2, \dots \tag{1.6}$$

Whence

$$\begin{gathered}G_k(t) = \frac{i}{\hbar}\int_0^t U_0(t-s)[V, G_{k-1}(s)]U_0(s-t)\, ds = \\ \left(\frac{i}{\hbar}\right)^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} U(t_1,\dots,t_k)[V,\dots[V,G_0(t_k)]\dots]U(-t_1,\dots,-t_k)\, dt_k \cdots dt_1\end{gathered} \tag{1.7}$$

$$U(t_1,\dots,t_k) := U_0(t-t_1)\cdots U_0(t_{k-1}-t_k)$$

Since U is unitary, if V and G are bounded operators we immediately get

$$\|G_k(t)\| \le \left(\frac{2\|V\|}{\hbar}\right)^k \|G\| \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} dt_k \cdots dt_1 = \|G\| \left(2\frac{\|V\|}{\hbar}\right)^k \frac{|t|^k}{k!} \tag{1.8}$$

Hence the series (1.5) converges for all t, but not uniformly with respect to the Planck constant $\hbar$. The uniformity is interesting because for $\hbar = 0$ the classical evolution of the observables is formally recovered.

Assume indeed, as usual, G to be the (Weyl) quantization of the classical observable $\mathcal{G}(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n};\mathbb{R})$:

$$(Gu)(x) = \frac{1}{h^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{G}(\frac{x+y}{2},\xi)e^{i\langle(x-y),\xi\rangle/\hbar}u(y)\, dyd\xi, \quad u(x) \in \mathcal{S}(\mathbb{R}^n) \tag{1.9}$$

or, equivalently, that G admits the following distribution kernel:

$$\mathcal{K}_G(x,y;\hbar) = \frac{1}{h^n}\int_{\mathbb{R}^n} \mathcal{G}(\frac{x+y}{2},\xi)e^{i\langle(x-y),\xi\rangle/\hbar}\,d\xi$$

Formula (1.9) defines G as a (semiclassical) pseudodifferential operator of symbol $\mathcal{G}(x,\xi)$. In particular, the semiclassical symbol of H_ϵ is the classical Hamiltonian function $\mathcal{H}_\epsilon : \mathbb{R}^{2n} \to \mathbb{R}$ defined as

$$\mathcal{H}(x,\xi) := \mathcal{H}_0(x,\xi) + \epsilon\mathcal{V}(x,\xi), \quad \mathcal{H}_0(x,\xi) := \frac{1}{2}\xi^2 + \mathcal{Q}(x) \tag{1.10}$$

where $\mathcal{V}(x,\xi)$ is the Weyl symbol of V. In general we denote $\sigma_A(x,\xi;\hbar)$ the Weyl symbol of a continuous observable A. Recall now that under our assumptions the following well known results hold (see e.g.[Ro]):

(1) Let $\widehat{\sigma}_A \in L^2(\mathbb{R}^{2n})$. Then A is a continuous operator in $L^2(\mathbb{R}^n)$ and

$$\|A\| := \|A\|_{L^2\to L^2} \le \|\widehat{\sigma}_A\|_{L^1} \tag{1.11}$$

(2) Denote $\mathcal{G}_\epsilon(\hbar,t) := \mathcal{G}_\epsilon(x,\xi;\hbar,t)$ the symbol of $G_\epsilon(t)$. Then $\mathcal{G}_\epsilon(\hbar,t)$ admits the semiclassical expansion:

$$\mathcal{G}_\epsilon(x,\xi;\hbar,t) = \mathcal{G}_\epsilon(x,\xi;t) + \hbar\mathcal{G}_\epsilon^{(1)}(x,\xi;\hbar,t) + \hbar^2\mathcal{G}_\epsilon^{(2)}(x,\xi;\hbar,t) + \dots \tag{1.12}$$

where (semiclassical Egorov theorem: see e.g.[Ro], IV.3)

$$\mathcal{G}_\epsilon(x,\xi;t) = \mathcal{G}\circ\Phi_\epsilon^t(x,\xi) \tag{1.13}$$

and $(x,\xi) \mapsto \Phi_\epsilon^t(x,\xi)$ the classical Hamiltonian flow generated by $\mathcal{H}_\epsilon(x,\xi)$;

(3) If the two continuous observables F and G have (Weyl) symbols $\mathcal{F}$ and $\mathcal{G}$ then:

$$\sigma^W(FG) = \mathcal{F}\#\mathcal{G}, \qquad \sigma^W(i[F,G]/\hbar) = \{\mathcal{F},\mathcal{G}\}_M \tag{1.14}$$

Here $\{\mathcal{F},\mathcal{G}\}_M := \mathcal{F}\#\mathcal{G} - \mathcal{G}\#\mathcal{F}$ is the Moyal bracket of $\mathcal{F}$ and $\mathcal{G}$, and $\mathcal{F}\#\mathcal{G}$ is the composition of the symbols $\mathcal{F}$ and $\mathcal{G}$. Explicitly, in Fourier space: (see e.g. [Fo],§3.4):

$$\widehat{\mathcal{F}\#\mathcal{G}}(s) = \frac{1}{\hbar}\int_{\mathbb{R}^{2n}} \widehat{\mathcal{F}}(s^1)\widehat{\mathcal{G}}(s-s^1)e^{\pi i\hbar(s-s^1)\wedge s^1/2}\,ds^1 \tag{1.15}$$

$$\{\mathcal{F},\mathcal{G}\}_M^\wedge(s) = \frac{2}{\hbar}\int_{\mathbb{R}^{2n}} \widehat{\mathcal{F}}(s^1)\widehat{\mathcal{G}}(s-s^1)\sin\left[\hbar(s-s^1)\wedge s^1/2\right]ds^1, \tag{1.16}$$

where, given to vectors $s=(u,v)$ and $s^1=(u^1,v^1)$, $s\wedge s^1 := \langle u,v_1\rangle - \langle v,u_1\rangle$.

(4) If either $\mathcal{F}$ or $\mathcal{G}$ is quadratic, then $\{\mathcal{F},\mathcal{G}\}_M = \{\mathcal{F},\mathcal{G}\}$.

Then the Heisenberg equation (1.4), written in terms of symbols, becomes

$$\dot{\mathcal{G}}_\epsilon(t,\hbar) = \{\mathcal{H},\mathcal{G}_\epsilon(t,\hbar)\}_M = \{\mathcal{H}_0,\mathcal{G}_\epsilon(t,\hbar)\} + \epsilon\{\mathcal{V},\mathcal{G}_\epsilon(t,\hbar)\}_M \tag{1.17}$$

because $\mathcal{H}_0$ is a quadratic form in (x,ξ). Hence we can immediately write the recurrent equations for the symbols $\mathcal{G}_k(x,\xi;\hbar,t)$ of $G_k(t)$:

$$\dot{\mathcal{G}}_k(\cdot;\hbar,t) = \{\mathcal{H}_0,\mathcal{G}_k(\cdot;\hbar,t)\} + \{\mathcal{V},\mathcal{G}_{k-1}(\cdot;\hbar,t)\}_M \tag{1.18}$$

The solution is

$$\begin{gathered}\mathcal{G}_k(x,\xi;\hbar,t) = \int_0^t \{\mathcal{V},\mathcal{G}_{k-1}(\cdot;t)\}_M \circ \Phi_0^{t-s}(x,\xi)\,ds = \\ \int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} \{\mathcal{V},\ldots\{\mathcal{V},\mathcal{G}_0(\cdot;t_k)\}_M \circ \Phi_0^{t_k-t_{k-1}}(x,\xi)\ldots\}_M \circ \Phi_0^{t-t_1}(x,\xi)\,dt_k\cdots dt_1\end{gathered} \tag{1.19}$$

$$\mathcal{G}_0(x,\xi;\hbar,t) = \mathcal{G}_0(x,\xi;t) := \mathcal{G}\circ\Phi_0^t(x,\xi)$$

whose solution is $\mathcal{G}_\epsilon(x,\xi;t)$ defined by (1.13) above. Hence for $\hbar = 0$ we recover the Lie expansion around the flow $\Phi_0^t(x,\xi)$ of $\mathcal{H}_0$. We set:

$$\mathcal{G}_\epsilon(x,\xi;t) = \mathcal{G}_0(x,\xi;t) + \epsilon\mathcal{G}_1(x,\xi;t) + \epsilon^2\mathcal{G}_2(x,\xi;t) + \ldots \tag{1.20}$$

where $\mathcal{G}_0(x,\xi;t) := \mathcal{G}\circ\Phi_0^t(x,\xi)$. The recurrent equations now read

$$\dot{\mathcal{G}}_k(\cdot;t) = \{\mathcal{H}_0,\mathcal{G}_k(\cdot;t)\} + \{\mathcal{V},\mathcal{G}_{k-1}(\cdot;t)\}, \quad k = 1,2,\ldots \tag{1.21}$$

and formula (1.19) then yields:

$$\begin{gathered}\mathcal{G}_k(\cdot;t) = \int_0^t \{\mathcal{V},\mathcal{G}_{k-1}(\cdot;t)\} \circ \Phi_0^{t-s}(x,\xi)\,ds = \\ \int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} \{\mathcal{V},\ldots\{\mathcal{V},\mathcal{G}_0(\cdot;t_k)\} \circ \Phi_0^{t_k-t_{k-1}}(x,\xi)\}\ldots\} \circ \Phi_0^{t-t_1}(x,\xi)\,dt_k\cdots dt_1\end{gathered} \tag{1.22}$$

We are now in position to formulate our uniform convergence statement.

Theorem 1.1. *Let:* $\mathcal{A}_\sigma := \{f \in L^2(\mathbb{R}^{2n}) \,|\, \|\hat{f}\|_\sigma < \infty\}, \|\hat{f}\|_\sigma := \int_{\mathbb{R}^{2n}} |\hat{f}(s)|e^{\sigma|s|}\,ds < \infty\}$. *Let* $\mathcal{G}\in\mathcal{A}_\sigma$, $\mathcal{V}\in\mathcal{A}_\sigma$ *for some* $\sigma > 0$. *Then there exists* $\Gamma(\sigma) > 0$ *independent of* $\hbar$ *such that*

$$\|\mathcal{G}_k(\hbar;t)\|_{\sigma/2} \le \Gamma(\sigma)^k|t|^k \tag{1.23}$$

Remarks

(1) Note that the norm $\|\hat{f}\|_\sigma$ is equivalent to the norm
$$\|\hat{f}\|_\sigma^{\mathcal{H}_0} := \int_{\mathbb{R}^{2n}} |\hat{f}(s)| e^{\sigma\sqrt{\mathcal{H}_0(s)}}\, ds.$$
(2) By (1.11), the estimate (1.23) shows that the Lie-Dyson expansion converges (in $B(L^2(\mathbb{R}^n))$ uniformly with respect to $\hbar$ for $|\epsilon t| < \Gamma(\sigma)^{-1}$.
(3) It is well known that, given $\epsilon > 0$, the Lie expansion yields a good approximation to the perturbed dynamics only up to a time t such that ϵt is small. The above statement shows that this assertion holds uniformly with respect to $\hbar$.

For the energy of the individual oscillator modes the quantum normal form constructed in [BGP1] under diophantine conditions on the frequencies allows us to obtain a stability result valid for a time scale exponentially long in the perturbation strength ϵ, uniformly with respect to $\hbar$, up to an error of order ϵ also independent of $\hbar$. Assume without loss of generality

$$\Phi_0^t(x,\xi) := \begin{cases} x_k(t) = x_k \cos\omega_k t + \dfrac{\xi_k}{\omega_k}\sin\omega_k t \\ \xi_k(t) = -\omega_k x_k \sin\omega_k t + \xi_k \cos\omega_k t \end{cases} \qquad k = 1,\dots,n \tag{1.24}$$

and denote

$$z := (x,\xi); \quad \Phi_0^\theta(z) := \{\Phi_0^t(x,\xi) : \omega_k t = \theta_k, k = 1,\dots,n\}. \tag{1.25}$$

Given $g \in L^1(\mathbb{R}^{2n})$, set:

$$\tilde{g}_k(z) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} g(\Phi_0^\theta(z)) e^{-i\langle k,\theta\rangle}\, d\theta, \quad k \in \mathbb{Z}^n \tag{1.26}$$

Hence, if $g \in C^1 \cap L^1$, one has pointwise

$$g(\Phi_0^\theta(z)) = \sum_{k\in\mathbb{Z}^n} \tilde{g}_k(z) e^{i\langle k,\theta\rangle}$$

Whence, for $\theta = 0$:

$$g(z) = \sum_{k\in\mathbb{Z}^n} \tilde{g}_k(z). \tag{1.27}$$

Given $\rho > 0$, $\sigma > 0$, define the norm

$$\|g\|_{\rho,\sigma} := \sum_{k\in\mathbb{Z}^n} e^{\rho|k|} \|\tilde{g}_k(s)\|_\sigma. \tag{1.28}$$

Remark that

$$\|g\|_\sigma \le \|g\|_{\rho,\sigma} \tag{1.29}$$

and denote

$$\mathcal{A}_{\rho,\sigma} := \{g : \mathbb{R}^{2n} \to \mathbb{C} \mid \|g\|_{\rho,\sigma} < +\infty\} \tag{1.30}$$

The following result is proven in [BGP1], Proposition 3.1.

Proposition 1.1. *Let the frequencies $\omega := \omega_k : k = 1, \dots, n$ fulfill the diophantine condition:*

$$|\langle \omega, \nu \rangle| \ge \gamma |\nu|^{-\tau}, \quad \forall\, \nu \in \mathbb{Z}^n \setminus \{0\} \tag{1.31}$$

for some $\tau > n-1$ and some $\gamma > 0$. Let $\mathcal{V} \in \mathcal{A}_{\rho,\sigma}$ for some $\rho > 0$, $\sigma > 0$. Then there exists $\epsilon^* > 0$ and, for $|\epsilon| < \epsilon^*$, a family $T(\epsilon)$ of unitary maps in L^2 such that

$$S(\epsilon) := T(\epsilon) H_\epsilon T(\epsilon)^* = H_0 + \epsilon Z(\epsilon) + R(\epsilon). \tag{1.32}$$

Here the following properties hold:

(1) Let $\mathcal{J}_k, k = 1, \dots, n$ be the the energy of the $k-$ oscillator mode:

$$\mathcal{J}_k := \frac{1}{2}(\xi_k^2 + \omega_k^2 x_k^2); \qquad \mathcal{H}_0(x,\xi) = \sum_{k=1}^n \mathcal{J}_k(x,\xi)$$

and J_k be the corresponding k-th quantized oscillator mode

$$J_k = -\frac{\hbar^2}{2}\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2; \quad H_0 = \sum_{k=1}^n J_k$$

Then Z depends only on $(J_1, \dots, J_n)$ and $\hbar$ or, equivalently, $\mathcal{Z}$ depends only on $(\mathcal{J}_1, \dots, \mathcal{J}_n)$ and $\hbar$, so that $[H_0, Z(\epsilon)] = 0$.

(2) $S(\epsilon)$, $T(\epsilon)$, $Z(\epsilon)$ and $R(\epsilon)$ are semiclassical pseudodifferential operators in L^2. In terms of their symbols $\Sigma_\epsilon(x,\xi;\hbar)$, $\mathcal{T}_\epsilon(x,\xi;\hbar)$, $\mathcal{Z}_\epsilon(x,\xi;\hbar)$, $\mathcal{R}_\epsilon(x,\xi;\hbar)$, (1.32) becomes

$$\Sigma_\epsilon(x,\xi;\hbar) = \mathcal{H}_0(x,\xi) + \epsilon \mathcal{Z}_\epsilon(x,\xi;\hbar) + \mathcal{R}_\epsilon(x,\xi;\hbar), \;\; \{\mathcal{H}_0, \mathcal{Z}_\epsilon\}_M = 0. \tag{1.33}$$

(3) There are $C_1(\rho,\sigma,\tau,\epsilon^*) > 0, C_1(\rho,\sigma,\tau,\epsilon^*)$ independent of $\hbar$ such that

$$\|Z(\epsilon)\| \le \|\mathcal{Z}\|_{\rho/2,\sigma/2} \le C_1; \quad \|R(\epsilon)\| \le \|\mathcal{R}\|_{\rho/2,\sigma/2} \le C_1 \exp\left[-C_2 \epsilon^{-1/(\tau+2)}\right] \tag{1.34}$$

Remark Formula (1.32) represents the quantum normal form of H_ϵ with a remainder of order $\exp\left[-C_2\epsilon^{-1/(\tau+2)}\right]$ uniform with respect to $\hbar$. For $\hbar = 0$ it reduces to the corresponding classical Birkhoff normal form with remainder. Then we have:

Theorem 1.2. *Let $\mathcal{V} \in \mathcal{A}_{\rho,\sigma}$ for some $\rho > 0$, $\sigma > 0$. Let J denote any one of the operators $J_k, k = 1, \ldots, n$. Then there are $D_1(\rho, \sigma, \tau, \epsilon^*) > 0$, $D_2(\rho, \sigma, \tau, \epsilon^*) > 0$ independent of $\hbar$ such that, for any $0 < d < \rho/2$:*

$$\|[J_\epsilon(t) - J]\| \leq \|[J_\epsilon(t) - J]\|_{\rho/2-d,\sigma/2} \leq D_1 \exp[-C_2\epsilon^{-1/(\tau+2)}]|t| + D_2|\epsilon|, \quad |\epsilon| < \epsilon^*.$$

Remark. Let $\psi_\nu(\hbar) : \nu \in \mathbb{Z}^n$ be the eigenvector corresponding to the simple eigenvalue $E_\nu(\hbar) := \sum_{k=1}^{n} \omega_k(\nu_k + \frac{1}{2})\hbar$. Then Theorem 1.2 entails:

$$|\langle\psi_\nu(\hbar), J_\epsilon(t)\psi_\nu(\hbar)\rangle - (\omega\nu + 1/2)\hbar| \leq D_1 \exp[-C_2\epsilon^{-1/(\tau+2)}]|t| + D_2|\epsilon|, \quad |\epsilon| < \epsilon^* \tag{1.35}$$

At the classical limit $\nu_k \to \infty$, $\hbar \to 0$, $\nu_k\hbar \to A_k$, A_k the action of the $k-$th mode, we have (see [DEGH]) $\langle\psi_\nu(\hbar), J_\epsilon(t)\psi_\nu(\hbar)\rangle \to \mathcal{J}_k \circ \Phi^t_\epsilon(x,\xi)$, $(\omega\nu_k + 1/2)\hbar \to \mathcal{J}_k(x,\xi)$. Therefore (1.35) entails that the absence of energy exchange between the different energy modes for an exponentially long time (see e.g. [BGG], [BGP2], [BP]) is stable under the process of quantization, up to an error of order ϵ independent of time.

2. Proof of the statements

To prove the estimate (1.23) we work in the Fourier representation. First we prove the invariance of $\mathcal{A}_\sigma$ under the action of the flow $\Phi_o^t(x,\xi)$:

Lemma 2.1. *Let $\mathcal{G}(x,\xi) \in A_\sigma$. Then $\mathcal{G} \circ \Phi_o^t(x,\xi) \in A_\sigma$ $\forall t \in \mathbb{R}$.*

Proof. Let $z := (x,\xi) \in \mathbb{R}^{2n}$. Since $\mathcal{H}_0(x,\xi)$ is a positive-definite quadratic form we can assume without loss of generality that the map $z \mapsto \Phi_0^t(z) : \mathbb{R}^{2n} \leftrightarrow \mathbb{R}^{2n}$ leaves the $\|\hat{f}\|_\sigma^{\mathcal{H}_0}$ invariant for all $t \in \mathbb{R}$. Now:

$$(\mathcal{G}\circ\Phi_0^t)^\wedge(s) = \int_{\mathbb{R}^{2n}} (\mathcal{G}\circ\Phi_0^t)(z)e^{-i\langle s,z\rangle}\, dz = \int_{\mathbb{R}^{2n}} \mathcal{G}(z)e^{-i\langle s,\Phi_0^{-t}(z)\rangle}\, dz = \hat{\mathcal{G}}(\Phi_0^t(s)).$$

Therefore $\|\hat{\mathcal{G}}(\Phi_0^t(s))\|_\sigma^{\mathcal{H}_0} = \|\hat{\mathcal{G}}(s)\|_\sigma^{\mathcal{H}_0}$ $\forall t \in \mathbb{R}$. Since $\|\hat{f}\|_\sigma$ and $\|\hat{f}\|_\sigma^{\mathcal{H}_0}$ are equivalent, this proves the Lemma.

Our second preliminary result is an estimate on the Moyal brackets worked out in [BGP1], and reproduced here for convenience of the reader.

Lemma 2.2. *Let $g \in \mathcal{A}_\sigma$, $g' \in \mathcal{A}_\sigma$. Then:*

$$\|\{g,g'\}_M\|_{\sigma-\delta-\eta} \le \frac{1}{e^2\eta(\delta+\eta)}\|g\|_\sigma\, \|g'\|_{\sigma-\delta}, \quad 0<\eta<\sigma-\delta \tag{2.36}$$

Proof Since $(s-s^1)\wedge s^1 = s\wedge s^1$, and $|s\wedge s^1| \le |s|\cdot|s^1|$, by definition of $\mathcal{A}_\sigma$-norm and (1.16) we get:

$$\begin{aligned}
&\|(\{g,g'\}_M)\|_{\sigma-\delta} = \\
&\frac{2}{\hbar}\int_{\mathbb{R}^{2n}} e^{(\sigma-\delta)|s|}\, ds \int_{\mathbb{R}^{2n}} |\hat{g}(s^1)\hat{g}'(s-s^1)\sin\left(\hbar(s-s^1)\wedge s^1\right)/2|\, ds^1 \\
&\le \frac{2}{\hbar}\int_{\mathbb{R}^{2n}} ds \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta)(|s|+|s^1|)}|\hat{g}(s)\hat{g}'(s^1)\sin\left(\hbar(s\wedge s^1\right)/2|\, ds^1 \\
&\le \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta)|s|}|\hat{g}(s)|\, ds \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta)|s^1|}|\hat{g}'(s^1)s\wedge s^1|\, ds^1 \\
&\le \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta)|s|}|\hat{g}(s)||s|\, ds \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta)|s^1|}|\hat{g}'(s^1)s||s^1||\, ds^1
\end{aligned}$$

whence the assertion because $xe^{-\delta x} \le \dfrac{1}{e\delta}$, $\forall\, x>0$, $\forall \delta > 0$.

Corollary 2.1. *Under the above assumptions on g and g':*

$$\|\{g,g'\}_M\circ\Phi_0^t(x,\xi)\|_{\sigma-\delta-\eta} \le \frac{1}{e^2\eta(\delta+\eta)}\|g\|_\sigma\, \|g'\|_{\sigma-\delta}, \quad 0<\eta<\sigma-\delta,\ \forall\, t\in\mathbb{R} \tag{2.37}$$

Proof By Lemma 2.1 both $g\circ\Phi^t_0(x,\xi)$ and $g'\circ\Phi^t_0(x,\xi)$ belong to $\mathcal{A}_\sigma$ $\forall\, t\in\mathbb{R}$ whenever g and g' do. Then the assertion follows from (2.36), and this proves the Corollary.

Iterating k times exactly the same arguments we get:

Corollary 2.2. *Let $g\in\mathcal{A}_\sigma$, $\mathcal{V}\in\mathcal{A}_\sigma$, $0<k\eta<\sigma-\delta$. Then:*

$$\begin{aligned}&\|\{\mathcal{V},\{\dots,\{\mathcal{V},g\}_M\circ\Phi^t_0(x,\xi)\dots,\}_M\circ\Phi^t_0(x,\xi)\|_{\sigma-\delta-k\eta} \\ &\le\frac{1}{e^2\eta^k(\delta+\eta)^k}\|\mathcal{V}\|^k_\sigma\,\|g\|_{\sigma-\delta},\qquad\forall\, t\in\mathbb{R}\end{aligned}\tag{2.38}$$

Proof of Theorem 1.1. Consider formula (1.19). By Corollary 2.2 above we have:

$$\begin{aligned}&\|\{\mathcal{V},\dots\{\mathcal{V},\mathcal{G}_0(\cdot;t_k)\}_M\circ\Phi_0^{t_k-t_{k-1}}(x,\xi)\dots\}_M\circ\Phi_0^{t-t_1}(x,\xi)\|_{\sigma-\delta-k\eta}\\ &\le\frac{1}{e^2\eta^k(\delta+\eta)^k}\|\mathcal{V}\|^k_\sigma\,\|\mathcal{G}_0\|_{\sigma-\delta},\qquad\forall\, t\in\mathbb{R}\end{aligned}$$

Now fix $0<\delta<\sigma/2$ and set:

$$\eta=\frac{\delta}{2k}$$

Therefore:

$$\begin{aligned}&\|\{\mathcal{V},\dots\{\mathcal{V},\mathcal{G}_0(\cdot;t_k)\}_M\circ\Phi_0^{t_k-t_{k-1}}(x,\xi)\dots\}_M\circ\Phi_0^{t-t_1}(x,\xi)\|_{\sigma-2\delta}\\ &\le e^{-2}2^k\delta^{2k}(1+\frac{1}{2k})^k k^k\|\mathcal{V}\|^k_\sigma\,\|\mathcal{G}_0\|_{\sigma-\delta}\le H\Gamma(\sigma)^k k! \end{aligned}\tag{2.39}$$

$$H:=\frac{1}{e^2\sqrt{2\pi}}\|\mathcal{G}_0\|_{\sigma-\delta};\quad \Gamma(\sigma):=3e\sigma^2\|\mathcal{V}\|_\sigma\tag{2.40}$$

Here we have used the inequalities $\delta<\sigma$, $(1+\frac{1}{2k})<\frac{3}{2}$, $k=1,\dots$, and the Stirling formula. Hence, again by (1.19):

$$\|\mathcal{G}_k(x,\xi;\hbar,t)\|_{\sigma-2\delta}\le H\Gamma(\sigma)^k k!\int_0^t\int_0^{t_1}\dots\int_0^{t_{k-1}}\,dt_k\cdots\,dt_1\le H\Gamma(\sigma)^k|t|^k,\ k=1,2\dots$$

The assertion of the Theorem is now proved if we choose $\delta<\sigma/4$.

Let us turn now to Theorem 1.2. We first prove an auxiliary result.

Lemma 2.3. *Let $\mathcal{B}\in\mathcal{A}_{\rho,\sigma}$ for some $\rho>0,\sigma>0$, and B the corresponding operator in $L^2(\mathbb{R}^n)$ defined by its Weyl quantization. Then there is $K>0$ independent of $\hbar$ such that, for any $0<d<\rho$:*

$$\|[B,J]/\hbar\|\le\|\{\mathcal{B},J\}\|_{\rho-d,\sigma}\le\frac{K}{d}\|\mathcal{B}\|_{\rho,\sigma}$$

Proof. We have:

$$\frac{1}{\hbar}\|[B,J]\| \leq \|\{\mathcal{B},\mathcal{J}\}_M\|_{\rho,\sigma} = \|\{\mathcal{B},\mathcal{J}\}\|_{\rho,\sigma}$$

because $\mathcal{J}(x_1,\xi_1)$ is quadratic. Here we have set, without loss,

$$\mathcal{J} = \mathcal{J}(x_1,\xi_1) = \frac{1}{2}(\xi_1^2 + \omega_1^2 x_1^2).$$

Explictly:

$$\{\mathcal{B},\mathcal{J}\} = \nabla_{\xi_1}\mathcal{B}\cdot\omega_1 x_1 - \nabla_{x_1}\mathcal{B}\cdot\xi_1$$

whence, in Fourier space

$$\{\mathcal{B},\mathcal{J}\}^\wedge(s) = \int_{\mathbb{R}^{2n}} (s_2\omega_1\partial_{s_1} - s_1\partial_{s_2})\widehat{\mathcal{B}}(s)\,ds.$$

To estimate $\|\{\mathcal{B},\mathcal{J}\}\|_{\rho,\sigma}$ we apply the arguments of [BGP1], Lemma 3.3. Denote $\phi_0^\theta(z) := \{\phi_0^t(z) : \omega_1 t = \theta\}$ the flow of $\mathcal{J}$. Here $\phi_0^t(z) = \Phi_0^t(x_1,\xi_1)$. Then we can write, on account of (1.27)

$$\{\mathcal{B},\mathcal{J}\}(z) = \left.\frac{d}{dt}\right|_{t=0}\mathcal{B}(\phi_0^t(z)) = \sum_{k\in\mathbb{Z}^n}\omega_1 k_1\tilde{\mathcal{B}}_k(z)$$

Therefore

$$\|\{\mathcal{B},\mathcal{J}\}(z)\|_{\rho,\sigma} \leq \sum_{k\in\mathbb{Z}^n}|\omega_1 k_1|e^{|\rho k_1|}\|\tilde{\mathcal{B}}_k\|_\sigma$$

Whence, as in Lemma 3.8 of [BGP1]:

$$\|\{\mathcal{B},\mathcal{J}\}(z)\|_{\rho/2-d,\sigma/2} = \sum_{k\in\mathbb{Z}^n}|\omega_1 k_1|e^{(|\rho|/2-d)k_1|}\|\tilde{\mathcal{B}}_k\|_{\sigma/2} \leq$$

$$\leq \|\mathcal{B}\|_{\rho/2,\sigma/2}\sup_{k_1}|k_1|e^{-d|k_1|} = \frac{\omega_1\|\mathcal{B}\|_{\rho/2,\sigma/2}}{d}, \quad 0<d<\rho.$$

Taking $K := \omega_1$ this proves the Lemma.

Proof of Theorem 1.2. Introduce the Heisenberg observable corresponding to J along the evolution generated by $S(\epsilon)$ defined by (1.32):

$$J_\epsilon^S(t) = e^{iS(\epsilon)t/\hbar}Je^{-iS(\epsilon)t/\hbar}$$

so that

$$\dot{J}_\epsilon^S(t) = \frac{i}{\hbar}e^{iS(\epsilon)t}[S(\epsilon),J]e^{-iS(\epsilon)t} = \frac{i}{\hbar}e^{iS(\epsilon)t}[R(\epsilon),J]e^{-iS(\epsilon)t} \tag{2.41}$$

because $[H_0,J] = [Z(\epsilon),J] = 0$. Hence, upon integration of (2.41):

$$\|J_\epsilon^S(t) - J\|_{\rho/2-d,\sigma/2} \leq \int_0^t \|\{\mathcal{R}(\epsilon),\mathcal{J}\}(z)\|_{\rho/2-d,\sigma/2}\,dt \leq$$

because $\|[R(\epsilon),J]/i\hbar\| \leq \|\{\mathcal{R}(\epsilon),\mathcal{J}\}_M\|_{\rho/2-d,\sigma/2} \leq \|\{\mathcal{R}(\epsilon),\mathcal{J}\}\|_{\rho/2-d,\sigma/2}$ since $\mathcal{J}$ is quadratic. Then, on account of (1.34), the above Lemma yields:

$$\|J^S_\epsilon(t)-J\|_{\rho/2-d,\sigma/2} \leq D\exp\left[-C_2\epsilon^{-1/(\tau+2)}\right]|t|,\ D:=\frac{C_1\omega_1}{d} \tag{2.42}$$

We have now to estimate the difference

$$\begin{aligned}&\|J_\epsilon(t)-J\| = \|T(\epsilon)[e^{iH_\epsilon t/\hbar}J_\epsilon e^{-iH_\epsilon t/\hbar}-J]T(\epsilon)^{-1}\| = \\ &\|e^{iS_\epsilon t/\hbar}T(\epsilon)J_\epsilon T(\epsilon)^{-1}e^{-iS_\epsilon t/\hbar}-T(\epsilon)JT(\epsilon)^{-1}\|\end{aligned}$$

By [BGP1], Proposition 3.2, we can write $T(\epsilon)=e^{iW(\epsilon)/\hbar}$, where $W(\epsilon)$ is, for $|\epsilon|<\epsilon^*$, a bounded self-adjoint semiclassical pseudodifferential operator with symbol $\mathcal{W}_\epsilon(x,\xi;\hbar)$ such that

$$\|W(\epsilon)\| \leq \|\mathcal{W}_\epsilon(x,\xi;\hbar)\|_{\rho/2,\sigma/2} \leq E|\epsilon| \tag{2.43}$$

for some $E>0$ independent of $\hbar$. Hence, by Lemma 2.3 and (1.29), we get:

$$\|[W(\epsilon),J]/i\hbar\| \leq \|\{\mathcal{W}_\epsilon,\mathcal{J}\}\|_{\sigma/2} \leq \|\{\mathcal{W}_\epsilon,\mathcal{J}\}\|_{\rho/2-d,\sigma/2} \leq \frac{E}{d},\ 0<d<\rho/2$$

and Corollary 2.2 for $t=0$ yields the estimate:

$$\begin{aligned}&\|[W(\epsilon),[\ldots[W(\epsilon),J]\ldots]/(i\hbar)^k\| \leq \|\{\mathcal{W}_\epsilon,\{\ldots\{\mathcal{W}_\epsilon,J\}_M\ldots\}_M\|\rho/2-d,\sigma/2 \leq \\ &\leq \frac{1}{e^2\eta^k(\delta+\eta)^k}\|\mathcal{W}_\epsilon\|^k_{\rho/2-d,\sigma/2}\|\{\mathcal{W}_\epsilon,\mathcal{J}\}\|_{\rho/2-d,\sigma/2-\delta},\quad 0<k\eta<\sigma/2-\delta\end{aligned}$$

As in the proof of Theorem 1.1, fix $0<\delta<\sigma/4$ and set:

$$\eta:=\frac{\delta}{4k}$$

Then

$$\|[W(\epsilon),[\ldots[W(\epsilon),J]\ldots]/(i\hbar)^k\| \leq L\Theta(\sigma)^k k! \tag{2.44}$$

$$L:=\frac{1}{e^2\sqrt{2\pi}d},\quad \Theta(\sigma):=5e\sigma^2E|\epsilon| \tag{2.45}$$

Therefore the commutator expansion

$$T(\epsilon)JT(\epsilon)^{-1}=J+\sum_{k=1}^\infty\frac{[W(\epsilon),[\ldots[W(\epsilon),J]\ldots]}{(i\hbar)^k k!}$$

has a nonzero convergence radius, uniformly with respect to $\hbar$. We can therefore write

$$T(\epsilon)JT(\epsilon)^{-1}=J+\epsilon M(\epsilon)$$

where $\|M(\epsilon)\|$ is bounded uniformly with respect to $\hbar$. Hence:

$$\|e^{iS_\epsilon t/\hbar}T(\epsilon)J_\epsilon T(\epsilon)^{-1}e^{-iS_\epsilon t/\hbar} - T(\epsilon)JT(\epsilon)^{-1}\| =$$

$$\|e^{iS_\epsilon t/\hbar}Je^{-iS_\epsilon t/\hbar} - J + \epsilon e^{iS_\epsilon t/\hbar}M(\epsilon)e^{-iS_\epsilon t/\hbar} - \epsilon M(\epsilon)\|$$

$$\leq \|J_\epsilon(t) - J\| + |\epsilon|\|e^{iS_\epsilon t/\hbar}M(\epsilon)e^{-iS_\epsilon t/\hbar} - M(\epsilon)\|$$

Setting $2\|M(\epsilon)\| = D_2$ this estimate concludes the proof of Theorem 1.2.

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