Cluster expansion for abstract polymer models. New bounds from an old approach

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We revisit the classical approach to cluster expansions, based on tree graphs, and establish a new convergence condition that improves those by Kotecký-Preiss and Dobrushin, as we show in some examples. The two ingredients of our approach are: (i) a careful consideration of the Penrose identity for truncated functions, and (ii) the use of iterated transformations to bound tree-graph expansions.

1. Introduction

Cluster expansions, originally developed to express thermodynamic potentials as power series in activities, are at the heart of important perturbative arguments in statistical mechanics and other branches of mathematical physics. The classical approach to obtain convergence conditions was based on combinatorial considerations [10, 20], which were greatly simplified through the use of tree-graph bounds [4, 2]. A completely new inductive approach originated in the work of Kotecký and Preiss [8], later refined by Dobrushin [5, 6] and many others [12, 1, 11, 19, 21, 18]. This later approach is mathematically very appealing and, in its more elegant version [5, 18], it even disposes of any reference to power series, becoming, in Dobrushin's words, a "no-cluster-expansion" approach. The combinatorial approach, however, kept its adepts who reformulated it in a very clear and compact way [13] and showed how it can lead to bounds at least as good as those given by Kotecký and Preiss [15].

In this paper, we revisit the classical combinatorial approach and point out that it can be used, in a rather simple and natural way, to produce improved bounds on the convergence region and the sum of the expansion. Our approach has two ingredients. First, we exploit an *identity*, due to Oliver Penrose [14], relating the coefficients of the expansion to a family of trees determined by compatibility constraints. (As a matter of fact, we learnt this identity from the nice exposition in [13, Section 3].) Successive approximations are obtained by considering larger families of trees that neglect some of the constraints. If only the very basic constraint is kept (links in the tree must relate incompatible objects), the Kotecky-Preiss condition emerges. To the next order of precision (branches must end in different objects) Dobrushin's condition is found. By refining this last constraint (branches' ends must be mutually compatible rather than just different) we obtain a new convergence condition which leads to improvements in several well-studied cases. In particular, for polymers on a graph —for which compatibility means non-intersection— our criterion yields the *original* polymer condition due to Gruber and Kunz [7, formula (42)]. This somehow forgotten condition —which is better than the ones usually applied— was obtained in the very paper that introduced the polymer formalism, through the use of Kirkwood-Salzburg equations.

Our second ingredient is a strategy to sum tree-graph expansions that is complementary to the classical one. The latter is based on an inductive "defoliation" of tree diagrams, which are summed "from the leaves in" with the help of the convergence condition. Here, we show instead that tree expansions are generated by successive applications of a transformation defined by the convergence condition. Besides leading to an improved convergence criterion, this point of view presents, in our opinion, several advantageous features. On the conceptual side, it shows a direct link between the convergence of tree expansions and inequalities involving the functions found in Kotecký-Preiss and Dobrushin (and our) conditions: The inequalities ensure that the iterative procedure lead to a finite expansion. From a more practical point of view, it is easy to see that finite iterations of the transformations yield progressively sharper bounds on the tree expansions. Thus, our approach produces, for each convergence condition, an associated sequence of upper bounds for the pinned free energy. In particular the majorizing tree expansions are shown to be fix points of the corresponding transformations. All this information is absent in previous treatments.

Finally, regarding future work, our approach leaves ample room for extensions and improvements. To emphasize this fact, we state a general result (Proposition 7) showing how bounds on truncated functions translate into convergence criteria and associated results. To establish our new criterion we used the Penrose identity in the most natural and immediate way. Improvements should come from the incorporation of additional tree conditions contained in the Penrose identity or, for specific models, through a more accurate description of the compatibility constraints. Also, as emphasized in [18] and reviewed in Section 4.1, there is a generalized Penrose identity which allows the use of trees other than Penrose's to characterize truncated functions. These alternative choices may turn out to be of interest in particular settings.

Penrose identity, in its original or generalized form —and thus our approach— is valid only for hard-core interactions (incompatibilities). The extension of our treatment to polymer systems subjected to softer interactions is another direction for further research.

2. Set up and previous results

We adopt the following abstract polymer setting. The starting point is an unoriented graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ —the *interaction graph*— on a countable vertex set. The vertices $\gamma \in \mathcal{P}$ are called *polymers* for historical reasons [7]. The name is misleading; Dobrushin [6] proposes to call them *animals*, but the traditional name holds on. The edge set corresponds to an *incompatibility relation*: Two polymers γ, γ' are incompatible if $\{\gamma, \gamma'\} \in \mathcal{E}$, in which case we write $\gamma \nsim \gamma'$. Otherwise they are *compatible* and we write $\gamma \sim \gamma'$. (Unfortunately, this notation —well established within the mathematical-physics community— is the opposite to that adopted in graph theory.) The set of edges is arbitrary, except for the assumption that it contains all pairs of the form $\{\gamma, \gamma\}$, that is, *every polymer is assumed to be incompatible with itself.* In particular vertices can be of infinite degree (each polymer can be incompatible with infinitely many other polymers). This happens, for instance, for graphs associated to gases of low-temperature contours or "defects".

The physical information of each polymer model is given by the incompatibility relation and a family of *activities* $\boldsymbol{z} = \{z_\gamma\}_{\gamma \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$. For each *finite* family $\Lambda \subset \mathcal{P}$, these ingredients define probability weights on the set of subsets of Λ:

$$\mathrm{Prob}_\Lambda\big(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}\big) \;=\; \frac{1}{\Xi_\Lambda(\boldsymbol{z})}\, z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}} \tag{2.1}$$

for $n \geq 1$ and $\mathrm{Prob}_\Lambda(\emptyset) = 1/\Xi_\Lambda$, where

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}} \,. \tag{2.2}$$

In physical terms, the measure (2.1) corresponds to the grand-canonical ensemble of a polymer gas with activities $\boldsymbol{z}$ and hard-core interaction defined by the incompatibility relation. The abstract formalism makes it equivalent to a lattice gas on the graph $\mathcal{G}$ with self- and nearest-neighbor hard-core repulsion. The normalization constant (2.2) is the grand-canonical partition function in the "volume" Λ. Cluster expansions allow the control of the measures (2.1) uniformly in Λ and absolutely in the activities. [Thus, the control extends to the unphysical region of non-positive (complex) activities, where the expressions on the right-hand side of (2.1) do not define probability measures.] The basic cluster expansion is the *formal* power series ("F") of the logarithm of the partition function, which takes the form (Mayer expansion, see e.g. [17])

$$\log \Xi_\Lambda(\boldsymbol{z}) \ \stackrel{\mathrm{F}}{=}\ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\subset\Lambda^n} \phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n} \tag{2.3}$$

with

$$\phi^T(\gamma_1,\ldots,\gamma_n) \ = \ \begin{cases} 1 & n=1 \\[2ex] \displaystyle\sum_{\substack{G\subset\mathcal{G}_{\{\gamma_1,\ldots,\gamma_n\}} \\ G \text{ conn. spann.}}} (-1)^{|E(G)|} & n\geq 2\,,\ \mathcal{G}_{\{\gamma_1,\ldots,\gamma_n\}} \text{ connected} \\[2ex] 0 & n\geq 2\,,\ \mathcal{G}_{\{\gamma_1,\ldots,\gamma_n\}} \text{ not connected} \end{cases} \tag{2.4}$$

where $\mathcal{G}_{\{\gamma_1,\ldots,\gamma_n\}}$ is the graph of vertices $\{1,\ldots,n\}$ and edges $\big\{\{i.j\} : \gamma_i \nsim \gamma_j, 0 \leq i,j \leq n\big\}$ and G ranges over all its connected spanning subgraphs; here $E(G)$ is the edge set of G. The functions ϕ^T are the *truncated functions* of order n (also called *Ursell functions*). The families $\{\gamma_1,\ldots,\gamma_n\}$ such that $\mathcal{G}_{\{\gamma_1,\ldots,\gamma_n\}}$ is connected are the *clusters*.

A telescoping argument shows that the properties of the measures (2.1) are determined by the one-polymer ratios ("pinned" expansions)

$$\Big[\log \frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\Big](\boldsymbol{z}) \ \stackrel{\mathrm{F}}{=}\ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\subset\Lambda^n \\ \exists i:\, \gamma_i=\gamma_0}} \phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n} \tag{2.5}$$

for each $\gamma_0 \in \Lambda$. A more efficient alternative is to consider instead the formal series

$$\Big[\frac{\partial}{\partial z_{\gamma_0}} \log \Xi_\Lambda\Big](\boldsymbol{z}) \ \stackrel{\mathrm{F}}{=}\ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\subset\Lambda^n} \phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n} \,. \tag{2.6}$$

This leads to the (infinite volume) formal power series

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \ :=\ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \big|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\big|\, \rho_{\gamma_1}\ldots\rho_{\gamma_n} \,, \tag{2.7}$$

for $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$ —in which ϕ^T is replaced by $\big|\phi^T\big|$— and its finite-volume versions $\Pi^\Lambda_{\gamma_0}$ obtained by restricting the sum to polymers in Λ. The finiteness of the positive-term series (2.7) for a certain $\boldsymbol{\rho}$

implies the absolute convergence of (2.3), (2.5) and (2.6), uniformly in Λ for $|\boldsymbol{z}| \leq \boldsymbol{\rho}$. This leads to the control of the measures (2.1) and their $\Lambda \to \mathcal{P}$ limit [6]. [Throughout this paper, operations and relations involving boldface symbols should be understood componentwisely, for instance $\boldsymbol{\rho} \leq \boldsymbol{\mu}$ indicates $\rho_\gamma \leq \mu_\gamma$, $\gamma \in \mathcal{P}$; $-\boldsymbol{z} = \{-z_\gamma\}_{\gamma\in\mathcal{P}}$; $\boldsymbol{\rho}\,\boldsymbol{\Pi} = \{\rho_\gamma \Pi_\gamma\}_{\gamma\in\mathcal{P}}$; $|\boldsymbol{z}| = \{|z_\gamma|\}_{\gamma\in\mathcal{P}}$, etc.]

The truncated functions satisfy the alternating-sign property

$$\phi^T(\gamma_0, \gamma_1, \ldots, \gamma_n) \;=\; (-1)^n \left|\phi^T(\gamma_0, \gamma_1, \ldots, \gamma_n)\right| \;. \tag{2.8}$$

(This is a well known result, that appears, for instance, in [17, Theorem 4.5.3] where it is attributed to Groeneveld [9]. Other proofs can be found in [11, 18] and in Proposition 5 below). Thus, (2.6) and the Λ-restriction of (2.7) are related in the form

$$\Pi^\Lambda_{\gamma_0}(\boldsymbol{\rho}) \;\stackrel{\mathrm{F}}{=}\; \Big[\frac{\partial}{\partial z_{\gamma_0}} \log \Xi_\Lambda\Big](-\boldsymbol{\rho}) \qquad (\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}). \tag{2.9}$$

In the sequel we focus on the convergence of the series (2.7) for positive activities. Its convergence allows the removal of the label "F" in all precedent identities, and it implies the inequalities

$$\left|\Big[\frac{\partial}{\partial z_{\gamma_0}} \log \Xi_\Lambda\Big](\boldsymbol{z})\right| \;\leq\; \Big[\frac{\partial}{\partial z_{\gamma_0}} \log \Xi_\Lambda\Big](-|\boldsymbol{z}|) \;=\; \Pi^\Lambda_{\gamma_0}(|\boldsymbol{z}|) \leq \Pi_{\gamma_0}(|\boldsymbol{z}|)\;. \tag{2.10}$$

and

$$\left|\Big[|\log \frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\Big](\boldsymbol{z})\right| \;\leq\; -\Big[\log \frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\Big](-|\boldsymbol{z}|) \;\leq\; |z_{\gamma_0}|\,\Pi^\Lambda_{\gamma_0}(|\boldsymbol{z}|) \leq\; |z_{\gamma_0}|\,\Pi_{\gamma_0}(|\boldsymbol{z}|) \tag{2.11}$$

A rather detailed study of different properties of these objects can be found in [18].

In the present general setting, two benchmark convergence conditions were published in 1986 [8] and 1996 [5]. For comparison purposes it is useful to write them in the following form. Suppose that for some $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$ there exists $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that

$$\rho_{\gamma_0}\; \exp\Big[\sum_{\gamma\nsim\gamma_0} \mu_\gamma\Big] \;\leq\; \mu_{\gamma_0} \qquad \text{(Kotecký-Preiss)} \tag{2.12}$$

or

$$\rho_{\gamma_0} \prod_{\gamma\nsim\gamma_0} \big(1+\mu_\gamma\big) \;\leq\; \mu_{\gamma_0} \qquad \text{(Dobrushin)} \tag{2.13}$$

for each $\gamma_0 \in \mathcal{P}$. [Please note that the sum and product over γ here include γ_0, which is always incompatible with itself.] Then the power series (2.7) converges for such $\boldsymbol{\rho}$ and, moreover,

$$\rho_{\gamma_0}\,\Pi_{\gamma_0}(\boldsymbol{\rho}) \;\leq\; \mu_{\gamma_0} \tag{2.14}$$

for each $\gamma_0 \in \mathcal{P}$.

The reader may be more familiar with the following forms of these conditions. The change of variables $\mu_\gamma = \rho_\gamma\,\mathrm{e}^{a_\gamma}$ shows that condition (2.12) is equivalent to the existence of $\boldsymbol{a} \in [0,\infty)^{\mathcal{P}}$ such that

$$\sum_{\gamma:\gamma\nsim\gamma_0} \rho_\gamma\,\mathrm{e}^{a_\gamma} \;\leq\; a_{\gamma_0} \qquad \text{(Kotecký-Preiss)} \tag{2.15}$$

for each $\gamma_0 \in \mathcal{P}$, and (2.14) becomes $\boldsymbol{\Pi} \leq \mathrm{e}^{\boldsymbol{a}}$. The substitution $\mu_\gamma = \mathrm{e}^{\alpha_\gamma} - 1$, on the other hand, makes (2.13) equivalent to the existence of $\boldsymbol{a} \in [0,\infty)^{\mathcal{P}}$ such that

$$\rho_{\gamma_0} \;\leq\; \Big(\mathrm{e}^{\alpha_{\gamma_0}} - 1\Big)\; \exp\Big(-\sum_{\gamma:\gamma\nsim\gamma_0} \alpha_\gamma\Big) \qquad \text{(Dobrushin)} \tag{2.16}$$

for each $\gamma_0 \in \mathcal{P}$.

The inequality

$$\prod_{\gamma \not\sim \gamma_0} \left(1 + \mu_\gamma\right) \;\leq\; \exp\Bigl[\sum_{\gamma \not\sim \gamma_0} \mu_\gamma\Bigr] \tag{2.17}$$

shows that Dobrushin condition is an improvement over Kotecký-Preiss'. Nevertheless, the latter is particularly suited for some applications (see, for instance, [19]) and, furthermore, can be extended to polymers with soft self- and two-body interactions. By contrast, the Dobrushin condition can be extended to systems with soft two-body interaction [19] but requires hard-core self-interaction. Looking to inequality (2.17) we see that the difference between both criteria lies in factors μ_γ at powers higher than two, which are absent in the left-hand-side. A quick illustration of the consequences of this fact is provided by polymers subjected only to self-exclusion (each polymer is compatible with everybody else, except itself). In this case $\Xi_\Lambda = \prod_{\gamma \in \Lambda}(1 + z_\gamma)$ and

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;=\; \sum_{n \geq 0} \rho_{\gamma_0}^n \;\stackrel{\mathrm{F}}{=}\; \frac{1}{1 - \rho_{\gamma_0}}\;. \tag{2.18}$$

The Kotecký-Preiss condition requires the existence of $\boldsymbol{\mu} > \mathbf{0}$ such that $\rho_{\gamma_0}\, \mathrm{e}^{\mu_{\gamma_0}} \;\leq\; \mu_{\gamma_0}$ for each $\gamma_o \in \mathcal{P}$, and this yields a radius of convergence for Π_{γ_0} equal to $\sup_{\mu_{\gamma_0}} \mu_{\gamma_0}\, \mathrm{e}^{-\mu_{\gamma_0}} = \mathrm{e}^{-1}$. Dobrushin condition, on the other hand, provides the sharp estimate $\sup_{\mu_{\gamma_0}}\, \mu_{\gamma_0}/(1 + \mu_{\gamma_0}) = 1$.

3. Results

3.1 New convergence criteria

Our new criterion involves the grand-canonical partition functions $\Xi_{\mathcal{N}^*_{\gamma_0}}$, associated to the polymer families $\mathcal{N}^*_{\gamma_0} = \{\gamma \in \mathcal{P} : \gamma \not\sim \gamma_0\}$, $\gamma_0 \in \mathcal{P}$ ($\mathcal{G}$-neighborhood of γ_0, *including* γ_0). These functions, defined in (2.2), can also be written in the form

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n \\ \gamma_0 \not\sim \gamma_i\,,\, \gamma_i \sim \gamma_j\,,\, 1 \leq i,j \leq n}} \mu_{\gamma_1} \mu_{\gamma_2} \ldots \mu_{\gamma_n} \tag{3.1}$$

because compatible polymers are different. Here is the practitioner's version of our criterion (a more detailed statement is given in Theorem 4 below).

Theorem 1. *Let $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$. If there exists a $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that*

$$\rho_{\gamma_0}\, \Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) \;\leq\; \mu_{\gamma_0}\;, \quad \forall \gamma_0 \in \mathcal{P}\;, \tag{3.2}$$

then the series $\Pi_{\gamma_0}(\boldsymbol{\rho})$ [defined in (2.7)] converges for such $\boldsymbol{\rho}$ and satisfies $\rho_{\gamma_0}\, \Pi_{\gamma_0}(\boldsymbol{\rho}) \leq \mu_{\gamma_0}$.

The inequality

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) \;\leq\; \prod_{\gamma \not\sim \gamma_0} \left(1 + \mu_\gamma\right) \tag{3.3}$$

shows that condition (3.2) is an improvement over Dobrushin's condition —which in turns is an improvement over Kotecký-Preiss' condition. The improvement comes from the fact that only monomials involving *mutually compatible* polymers are allowed in the left-hand side. Such improvement comes, therefore, from two sources:

(I1) In $\Xi_{\mathcal{N}^*_{\gamma_0}}$ there are no monomials involving triangle diagrams in $\mathcal{G}$, namely pairs of neighbors of γ_0 that are themselves neighbors.

(I2) In $\Xi_{\mathcal{N}^*_{\gamma_0}}$, the only monomial containing μ_{γ_0} is μ_{γ_0} itself, because γ_0 is incompatible with all other polymers in $\mathcal{N}^*_{\gamma_0}$.

Improvement (I2) is present whichever the graph $\mathcal{G}$, and makes inequality (3.3) strict except for the non-interacting example discussed circa (2.18). The terms corresponding to (I1) and (I2) can be neatly separated by writing

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\rho}) \;=\; \rho_{\gamma_0} + \Xi_{\mathcal{N}_{\gamma_0}}(\boldsymbol{\rho}) \tag{3.4}$$

where $\mathcal{N}_{\gamma_0} = \mathcal{N}^*_{\gamma_0} \setminus \{\gamma_0\}$ ($\Xi_\emptyset := 1$). Using a bound similar to (3.3) but for $\Xi_{\mathcal{N}_{\gamma_0}}$ we obtain another criterion —halfway between ours and Dobrushin's— which may be useful in some settings.

Corollary 2. *Let $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$. If there exists a $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that*

$$\rho_{\gamma_0}\Big[\mu_{\gamma_0} + \prod_{\substack{\gamma \nsim \gamma_0 \\ \gamma \neq \gamma_0}} (1+\mu_\gamma)\Big] \;\le\; \mu_{\gamma_0}\;, \qquad \textit{(improved Dobrushin)} \tag{3.5}$$

for all $\gamma_0 \in \mathcal{P}$, then the series $\Pi_{\gamma_0}(\rho)$ converges for such $\boldsymbol{\rho}$ and satisfies $\rho_{\gamma_0}\,\Pi_{\gamma_0}(\rho) \le \mu_{\gamma_0}$.

Our condition (3.2) coincides with (3.5) for triangle-free graphs $\mathcal{G}$ (ex. trees, $\mathbb{Z}^d$), and it is maximally better for complete ("triangle-full") graphs. This and other examples will be analyzed below. Summing up, available convergence conditions are of the form

$$\rho_{\gamma_0}\,\varphi_{\gamma_0}(\boldsymbol{\mu}) \;\le\; \mu_{\gamma_0} \tag{3.6}$$

with

$$\varphi_{\gamma_0}(\boldsymbol{\mu}) \;=\; \begin{cases} \exp\Big[\sum_{\gamma \nsim \gamma_0} \mu_\gamma\Big] & \text{(Koteck\'y-Preiss)} \\ \prod_{\gamma \nsim \gamma_0} (1+\mu_\gamma) & \text{(Dobrushin)} \\ \mu_{\gamma_0} + \prod_{\substack{\gamma \nsim \gamma_0 \\ \gamma \neq \gamma_0}} (1+\mu_\gamma) & \text{(improved Dobrushin)} \\ \Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) & \text{(ours)} \end{cases} \tag{3.7}$$

Each condition is strictly weaker than the preceding one except for the facts that the improved Dobrushin condition coincides with Dobrushin's if the polymers are non-interacting (only self-excluding) and with our condition if $\mathcal{G}$ does not include any triangle diagram. The corresponding criteria yield information on two issues: (i) regions of convergence, and (ii) upper bounds on each Π_{γ_0}.

Regarding the first issue, it is known that the region of absolute convergence of cluster expansions has the properties of being a "down-region" —convergence for $\boldsymbol{\rho}$ entails convergence for $\widetilde{\boldsymbol{\rho}} \le \boldsymbol{\rho}$— and log-convex. The latter means that if the series converges for $\boldsymbol{\rho}$ and $\widetilde{\boldsymbol{\rho}}$ then it converges for $\boldsymbol{\rho}^\lambda\,\widetilde{\boldsymbol{\rho}}^{1-\lambda}$ for $0 \le \lambda \le 1$ [18]. It is reassuring to verify that these properties also hold for the regions of validity of conditions (3.6)/(3.7). Indeed, the "down" character is obvious, and the log-convexity property is a consequence of the following proposition.

Proposition 3. *Suppose $0 \le \lambda \le 1$ and let us denote*

$$\mathcal{R}_{\rm CD} \;=\; \Big\{(\boldsymbol{\rho},\boldsymbol{\mu}) \in [0,\infty)^\infty \times [0,\infty)^\infty \;\Big|\; \textit{condition } {\rm CD} \textit{ is satisfied}\Big\}\;, \tag{3.8}$$

where "CD" stand for each of the conditions in (3.6)/(3.7). Then,

$$(\boldsymbol{\rho},\boldsymbol{\mu})\,,\,(\widetilde{\boldsymbol{\rho}},\widetilde{\boldsymbol{\mu}}) \in \mathcal{R}_{\rm CD} \quad\Longrightarrow\quad \Big(\boldsymbol{\rho}^\lambda\,\widetilde{\boldsymbol{\rho}}^{1-\lambda}\,,\,\boldsymbol{\mu}^\lambda\,\widetilde{\boldsymbol{\mu}}^{1-\lambda}\Big) \in \mathcal{R}_{\rm CD}\;. \tag{3.9}$$

Proof. Given the form (3.6) of the conditions, we see that it is enough to prove that

$$\varphi_{\gamma_0}(\boldsymbol{\mu})^\lambda\, \varphi_{\gamma_0}(\widetilde{\boldsymbol{\mu}})^{1-\lambda} \;\ge\; \varphi_{\gamma_0}(\boldsymbol{\mu}^\lambda\, \widetilde{\boldsymbol{\mu}}^{1-\lambda}) \tag{3.10}$$

for each of the functions φ_{γ_0} in (3.7). For the last three functions this is a consequence of Hölder inequality in the form

$$\Bigl(\sum_{i=1}^n a_i\Bigr)^\lambda \Bigl(\sum_{i=1}^n b_i\Bigr)^{1-\lambda} \;\ge\; \sum_{i=1}^n a_i^\lambda\, b_i^{1-\lambda} \tag{3.11}$$

($a_i, b_i \ge 0$, $i = 1, \ldots, n$). For the Kotecký-Preiss function, (3.10) is a consequence of the inequality $\lambda a + (1-\lambda) b \ge a^\lambda b^{1-\lambda}$, valid for $a, b \ge 0$ (this is an elementary inequality, see [16, p. 112]). ∎

Our results on the second issue (upper bound on $\boldsymbol{\Pi}$) are contained in the following strengthening of Theorem 1. Its formulation relies on the map iterates used in Section 4.2 to sum tree-graph expansions. For each fixed $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$ let us consider the map $\boldsymbol{T_\rho} : [0,\infty)^{\mathcal{P}} \longrightarrow [0,\infty]^{\mathcal{P}}$ defined by

$$\boldsymbol{T_\rho}(\boldsymbol{\mu}) \;:=\; \boldsymbol{\rho}\, \boldsymbol{\varphi}(\boldsymbol{\mu}) \tag{3.12}$$

where $\boldsymbol{\varphi}$ is any of the functions defined in (3.7). Denote $\boldsymbol{T_\rho^n} = \boldsymbol{T_\rho}(\boldsymbol{T_\rho^{n-1}})$ the successive compositions of $\boldsymbol{T_\rho}$ with itself.

Theorem 4. *Let $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$ be fixed and let $\boldsymbol{T_\rho}$ be a map of the form (3.12)/(3.7). Assume there exists $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ satisfying (3.6), that is,*

$$\boldsymbol{T_\rho}(\boldsymbol{\mu}) \;\le\; \boldsymbol{\mu}\,. \tag{3.13}$$

Then:

(i) *There exists $\boldsymbol{\rho}^* \in [0,\infty)^{\mathcal{P}}$ such that $\boldsymbol{T_\rho^n}(\boldsymbol{\rho}) \nearrow \boldsymbol{\rho}^*$ and $T(\boldsymbol{\rho}^*) = \boldsymbol{\rho}^*$.*

(ii) *For each $n \in \mathbb{N}$,*

$$\boldsymbol{\rho}\,\boldsymbol{\Pi} \;\le\; \boldsymbol{\rho}^* \;\le\; \boldsymbol{T_\rho^{n+1}}(\boldsymbol{\mu}) \;\le\; \boldsymbol{T_\rho^n}(\boldsymbol{\mu}) \;\le\; \boldsymbol{\mu}\,. \tag{3.14}$$

The deepest statement in this theorem is the first inequality in (3.14). The rest of the theorem follows from the fact that for all choices (3.7) of $\boldsymbol{\varphi}$ the map $\boldsymbol{T_\rho}$ is monotonicity-preserving and satisfies $\boldsymbol{\rho} \;\le\; \boldsymbol{T_\rho}(\boldsymbol{\rho}) \;\le\; \boldsymbol{T_\rho}(\boldsymbol{\mu}) \;\le\; \boldsymbol{\mu}$.

3.2 Comparison with previous criteria

To test our criterion we compare the estimates of the regions of convergence provided by the criteria (3.6)–(3.7) for two families of benchmark examples.

Polymer graphs with bounded maximum degree. These are examples where $\mathcal{G}$ has maximum degree $\Delta < \infty$. We shall suppose that all polymers have equal activity $\rho_\gamma \equiv \rho$ for all $\gamma \in \mathcal{G}$, and therefore we search for equally constant functions $\mu_\gamma \equiv \mu$. The preceding criteria take the form $\rho \le \mu/\varphi(\mu)$ for appropriate functions φ, and the maximization of the right-hand side with respect to μ yields the best lower bounds of the radius of convergence of (2.7) [and hence of (2.3)].

In Table 1 we summarize both convergence criteria and best estimates on the convergence radii obtained with Kotecký-Preiss, Dobrushin and improved Dobrushin conditions. The only feature of the graph $\mathcal{G}$ relevant for these criteria is the maximal degree Δ of the vertices. Therefore they provide the sharpest results for graphs which lack of any other feature and whose vertices have all degree Δ. These are the regular trees with branching rate $\Delta - 1$. This fact —trees supply a worst-case condition that can be used whenever we ignore, or decide to ignore, any topological information on the graph— has been emphasized in [19] (see, also, Remark 6). For regular trees, the weak Dobrushin condition coincides with ours, and there is a further, optimal condition, due to Scott and Sokal [18], which we have included in the last line of the table. This condition is derived through a sequence of volume-dependent Dobrushin conditions. It would be interesting to see whether a similar strategy could be developed within our approach.

Condition	Criterion	R_Δ	R_6
Kotecký-Preiss	$\rho \leq \mu\, e^{-(\Delta+1)\mu}$	$\dfrac{1}{(\Delta+1)\, e}$	0.0525
Dobrushin	$\rho \leq \dfrac{\mu}{(1+\mu)^{\Delta+1}}$	$\begin{cases} \dfrac{\Delta^\Delta}{(\Delta+1)^{\Delta+1}} & \Delta \geq 1 \\ 1\ (*) & \Delta = 0 \end{cases}$	0.0566
improved Dobrushin =(3.2) for $(\Delta-1)$-reg. tree	$\rho \leq \dfrac{\mu}{\mu+(1+\mu)^\Delta}$	$\begin{cases} \left[1+\dfrac{\Delta^\Delta}{(\Delta-1)^{\Delta-1}}\right]^{-1} & \Delta \geq 2 \\ (\Delta+1)^{-1}\ (*) & \Delta = 0,1 \end{cases}$	0.0628
Scott-Sokal	[18, Theorem 5.6]	$\dfrac{(\Delta-1)^{(\Delta-1)}}{\Delta^\Delta}\ (*)$	0.067

TABLE 1. Convergence criteria and lower bounds (R_Δ) on the radius of convergence when $\mathcal{G}$ is a graph with maximal degree Δ. A star indicates that the value is exact for the $(\Delta-1)$-regular tree

In Table 2 we show the improved results obtained from the application of our criteria to some popular examples. The values of R in the first two lines are to be compared with the values for R_6 in Table 1, and that of the complete graph with the values of R_Δ. The source of these improvements is, of course, the sensitivity of our new criterium to triangle diagrams. In particular, our criterion gives the exact value of the radius of convergence for the complete graph, for which $\Pi = [1-(\Delta+1)\mu]^{-1}$.

Model	Criterion	R
Domino in $\mathbb{Z}^2$	$\rho \leq \dfrac{\mu}{1+7\mu+9\mu^2}$	0.0769
Triangular lattice	$\rho \leq \dfrac{\mu}{1+7\mu+8\mu^2+2\mu^3}$	$4R^3+8R^2=1\,,\ R\approx 0{,}078$
$(\Delta+1)$-complete graph	$\rho \leq \dfrac{\mu}{1+(\Delta+1)\mu}$	$(\Delta+1)^{-1}\ (*)$

TABLE 2. Convergence criteria and lower bounds (R) on the radius of convergence obtained with condition (3.2) for some graphs $\mathcal{G}$ of finite degree. A star indicates an exact value

Polymers on a graph. This is the general example of cluster expansions for graphs with vertices of infinite degree. Applications include contour ensembles of low-temperature phases, geometrical objects of high-temperature expansions, random sets of the Fortuin-Kasteleyn representation of the Potts model, ... The general setup for these models is a polymer family formed by the finite parts of a given set $\mathbb{V}$ with incompatibility defined by overlapping. (Usually, $\mathbb{V}$ is formed by the vertices of a graph with respect to which polymers form connected sets.)

For these systems it is useful and traditional to pass to exponential weight functions $a(\gamma)$ defined by $\mu_\gamma = \rho_\gamma\, \mathrm{e}^{a(\gamma)}$. Condition (3.2) becomes

$$1 + \sum_{n\geq 1} \sum_{\substack{\{\gamma_1,\ldots,\gamma_n\}\subset\mathcal{P}\\ \gamma_0\cap\gamma_i\neq\emptyset\,,\,\gamma_i\cap\gamma_j=\emptyset\,,\,1\leq i,j\leq n}} \prod_{i=1}^n \rho_{\gamma_i}\, \mathrm{e}^{a(\gamma_i)} \;\leq\; \mathrm{e}^{a(\gamma_0)} \tag{3.15}$$

From the constraint in the sum we only keep the fact that each of the polymers $\gamma_1,\ldots,\gamma_n$ must intersect *different* points in γ_0 (otherwise they would overlap). This implies: (i) $n \leq |\gamma_0|$, and (ii) there are n different points in γ_0 touched by $\gamma_1\cup\cdots\cup\gamma_n$. These points can be chosen in $\binom{|\gamma_0|}{n}$ ways. Hence, the left-hand side of (3.15) is less or equal than

$$1 + \sum_{n=1}^{|\gamma_0|} \binom{|\gamma_0|}{n} \Bigg[\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, \mathrm{e}^{a(\gamma)}\Bigg]^n \;=\; \Bigg[1+\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, \mathrm{e}^{a(\gamma)}\Bigg]^{|\gamma_0|}\;,$$

which leads us to the following sufficient condition for (3.15):

$$\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, \mathrm{e}^{a(\gamma)} \;\leq\; \mathrm{e}^{a(\gamma_0)/|\gamma_0|} - 1 \tag{3.16}$$

This condition entails the finiteness of $\mathbf{\Pi}$:

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;\leq\; \mathrm{e}^{a(\gamma_0)}\;. \tag{3.17}$$

In practice, the function $a(\gamma)$ is chosen of the form $a(\gamma) = a\,|\gamma|$, with a a positive constant. This choice, which in many cases is the expected optimal asymptotic behavior of $a(\gamma)$ for large polymers, simplifies the procedure reducing it to the determination of the single constant a. Our emphasis in a general dependence is not just mathematical finesse. As dominants contributions come from the smallest polymers, a dependence of $a(\gamma)$ dealing more accurately with them would improve precision. Also, the criteria are usually presented in the slightly weaker form obtained by replacing the supremum over $x\in\gamma_0$ by a supremum over $x\in\mathbb{V}$. In this form, a condition like (3.16) is, in fact, present in the seminal paper by Gruber and Kunz [7] [formula (42) with normalization $\phi(x) = 1$ and parametrization $\xi_0 = e^a$.] Table 3 lists the different conditions with the preceding usual choices.

Kotecký-Preiss	Dobrushin	Gruber-Kunz
$\sup_x \sum_{\gamma\in\mathcal{P}:\gamma\ni x} \rho_\gamma\, \mathrm{e}^{a\|\gamma\|} \;\leq\; a$	$\sup_x \prod_{\gamma\in\mathcal{P}:\gamma\ni x} \Big[1+\rho_\gamma\, \mathrm{e}^{a\|\gamma\|}\Big] \;\leq\; \mathrm{e}^a$	$\sup_x \sum_{\gamma\in\mathcal{P}:\gamma\ni x} \rho_\gamma\, \mathrm{e}^{a\|\gamma\|} \;\leq\; \mathrm{e}^a - 1$

TABLE 3. Convergence conditions for general polymer models. Our condition (3.16) with $a(\gamma) = a\,|\gamma|$ coincides with that by Gruber and Kunz.

4. Proofs

The argument has two distinct parts. First, we use the Penrose tree identity for the truncated functions to turn (2.7) into a sum over trees —a *tree-graph expansion*. In the second part, we control this expansion through a natural iterative procedure defined by the functions (3.7).

4.1 Partitionability and the Penrose identity

Formula (2.4) involves a huge number of cancellations. Penrose [14] realized that they can be optimally handled through what is now known as the property of *partitionability* of the family of connected spanning subgraphs. While his original argument involved a particular partition scheme, it works equally well for any other choice, as emphasized in [18]. For the sake of completeness, and due to its potential use for extensions and alternative versions of our criterion, we start by reproducing this simple but deep argument. Our exposition is based on [18, Section 2.2].

Let us consider a finite graph $\mathbb{G} = (\mathbb{U}, \mathbb{E})$ and denote $\mathcal{C}_\mathbb{G}$ the set of all connected spanning subgraphs of $\mathbb{G}$ and $\mathcal{T}_\mathbb{G}$ the family of trees belonging to $\mathcal{C}_\mathbb{G}$. Further, we consider $\mathcal{C}_\mathbb{G}$ partial ordered by bond inclusion:

$$G \leq \widetilde{G} \quad\Longleftrightarrow\quad E(G) \subset E(\widetilde{G})\ . \tag{4.1}$$

If $G \leq \widetilde{G}$, let us denote $[G, \widetilde{G}]$ the set of $\widehat{G} \in \mathcal{C}_\mathbb{G}$ such that $G \leq \widehat{G} \leq \widetilde{G}$. Let us call a *partition scheme* for the family $\mathcal{C}_\mathbb{G}$ to any map $R : \mathcal{T}_\mathbb{G} \to \mathcal{C}_\mathbb{G}$ such that

(i) $E\big(R(T)\big) \supset E(T)$, and
(ii) $\mathcal{C}_\mathbb{G}$ is the disjoint union of the sets $[T, R(T)]$, $T \in \mathcal{T}_\mathbb{G}$.

A number of such partition schemes are by now available (see references in [18, Section 2.2]). The one proposed by Penrose is constructed in the following way: Let us fix an enumeration $v_0, v_1, \ldots, v_n$ for the vertices of $\mathbb{G}$, and for each $\tau \in \mathcal{T}_\mathbb{G}$ (thought as a tree rooted in v_0), let $d(i)$ be the tree distance of the vertex v_i to v_0. Penrose scheme associates to τ the graph $R_{\rm Pen}(\tau)$ formed by adding (only once) to τ all edges $\{v_i, v_j\} \in \mathbb{E} \setminus E(\tau)$ such that either:

(p1) $d(i) = d(j)$ (edges between vertices of the same generation), or
(p2) $d(i) = d(j) - 1$ and $i < j$ (edges connecting to predecessors with smaller index).

For a partition scheme R, let us denote

$$\mathcal{T}_R \;:=\; \Big\{\tau \in \mathcal{T}_\mathbb{G} \;\Big|\; R(\tau) = \tau\Big\} \tag{4.2}$$

(set of *R-trees*). In particular, $\mathcal{T}_{R_{\rm Pen}}$ is the set of *Penrose trees*. The following is the generalized version of Penrose identity.

Proposition 5.

$$\sum_{G\in\mathcal{C}_\mathbb{G}} (-1)^{|E(G)|} \;=\; (-1)^{|\mathbb{V}|-1}\big|\mathcal{T}_R\big|\ . \tag{4.3}$$

for any partition scheme R.

Proof. For any numbers x_e, $e \in \mathbb{E}$, we have

$$\begin{aligned}\sum_{G\in\mathcal{C}_\mathbb{G}} \prod_{e\in E(G)} x_e \;&=\; \sum_{T\in\mathcal{T}_\mathbb{G}} \prod_{e\in E(T)} x_e \sum_{\mathcal{F}\subset E(R(T))\setminus E(T)} \prod_{e\in\mathcal{F}} x_e \\ &=\; \sum_{T\in\mathcal{T}_\mathbb{G}} \prod_{e\in E(T)} x_e \prod_{e\in E(R(T))\setminus E(T)} (1 + x_e)\ .\end{aligned} \tag{4.4}$$

The first equality is due to property (ii) of partition schemes. If $x_e = -1$, the last factor kills the contributions of all trees with $E(R(T)) \setminus E(T) \neq \emptyset$. Furthermore, for any tree $\bigl|E(T)\bigr| = |\mathbb{V}| - 1$.■

We see that the hard-core condition is crucial for the identity. For polymer models with soft repulsion, only $|1 + x_e| \leq 1$ is guaranteed, and this leads to the inequality

$$\Bigl|\sum_{G\in\mathcal{C}_{\mathbb{G}}} \prod_{e\in E(G)} x_e\Bigr| \;\leq\; \sum_{T\in\mathcal{T}_{\mathbb{G}}} \prod_{e\in E(T)} |x_e| \;\leq\; |\mathcal{T}_{\mathbb{G}}| \;. \tag{4.5}$$

This much weaker inequality is the one used in traditional treatments of the tree expansion [4, 2].

The previous proposition applied to the Penrose scheme implies

$$\bigl|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\bigr| \;=\; \sum_{T\in\mathcal{T}_n^0} \mathbb{1}_{\{T\in\mathcal{T}_{\rm Pen}(\gamma_0,\gamma_1,\ldots,\gamma_n)\}} \;, \tag{4.6}$$

where $\mathcal{T}_n^0$ is the set of (labeled) trees with vertices $\{0, 1, \ldots, n\}$ rooted in 0, and $\mathcal{T}_{\rm Pen}(\gamma_0,\gamma_1,\ldots,\gamma_n)$ denotes the Penrose trees on the graph $\mathcal{G}_{\{\gamma_0,\gamma_1,\ldots,\gamma_n\}}$ (with the canonical enumeration of vertices). Similar formulas are valid replacing "Pen" by any partition scheme R.

Remark 6. As the number of Penrose trees grows with the disappearance of triangle diagrams, the value of $\mathbf{\Pi}$ (resp. the region of convergence of the cluster expansion) for a given graph $\mathcal{G}$ is bounded above by (resp. contains) that of a tree where each vertex has a degree larger or equal than that at $\mathcal{G}$. Furthermore, the latter is bounded above (resp. contains) that of a homogeneous tree with branching rates equal to the maximal rate.

4.2 Trees and convergence

Replacing (4.6) into (2.7) we obtain a sum in terms of trees. Traditionally, such expansions have been inductively summed *alla* Cammarota [4], namely "from the leaves in". Conditions of the type (3.6) guarantee the reproducibility of the inductive hypothesis. Here we present a complementary approach, based in generating the expansion through repeated application of a nonlinear map $\boldsymbol{T_\rho}$. Conditions (3.6) prevent the successive partial sums to diverge.

The end product of this section is the following proposition. Each $\tau \in \mathcal{T}_n^0$ is uniquely defined by the branching factor s_i of each vertex i and the labels $i_1, \ldots, i_{s_i}$ of its descendants.

Proposition 7. *Let $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ be a polymer system and assume there exist functions $c_n : \mathcal{P}^{n+1} \to [0,\infty)$, for $n \in \mathbb{N}$, invariant under permutations of the last n arguments such that*

$$\bigl|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\bigr| \;\leq\; \sum_{\tau\in\mathcal{T}_n^0} \prod_{i=0}^{n} c_{s_i}(\gamma_i,\gamma_{i_1},\ldots,\gamma_{i_{s_i}}) \;. \tag{4.7}$$

Consider the function $\boldsymbol{\varphi} : [0,\infty)^{\mathcal{P}} \to [0,\infty]^{\mathcal{P}}$ defined by

$$\varphi_{\gamma_0}(\boldsymbol{\mu}) \;=\; 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} c_n(\gamma_0,\gamma_1,\ldots,\gamma_n)\, \mu_{\gamma_1}\cdots\mu_{\gamma_n} \tag{4.8}$$

for each $\gamma_0 \in \mathcal{P}$. Assume that, for a given $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$ there exists $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that

$$\rho_{\gamma_0}\, \varphi_{\gamma_0}(\boldsymbol{\mu}) \;\leq\; \mu_{\gamma_0} \tag{4.9}$$

for each $\gamma_0 \in \mathcal{P}$. Then,

(a) *The cluster expansion (2.3) for the system $\mathcal{G}$ converges absolutely and uniformly in Λ and in the activities $\boldsymbol{z}$ with $|\boldsymbol{z}| \le \boldsymbol{\rho}$.*

(b) *Furthermore, if $\boldsymbol{T}_{\boldsymbol{\rho}} = \boldsymbol{\rho}\,\boldsymbol{\varphi}$ is the map defined as in (3.12) but with $\boldsymbol{\varphi}$ given by (4.8), then*

  (i) *There exist $\boldsymbol{\rho}^*, \boldsymbol{T}_{\boldsymbol{\rho}}^{\infty}(\boldsymbol{\mu}) \in [0,\infty)^{\mathcal{P}}$ such that*

$$\boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\rho}) \underset{n\to\infty}{\nearrow} \boldsymbol{\rho}^* \quad , \quad \boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu}) \underset{n\to\infty}{\searrow} \boldsymbol{T}_{\boldsymbol{\rho}}^{\infty}(\boldsymbol{\mu})\,. \tag{4.10}$$

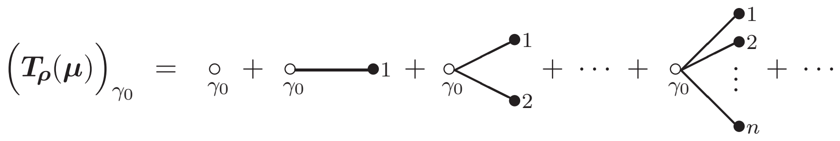
  (ii) $T(\boldsymbol{\rho}^*) = \boldsymbol{\rho}^*$.

  (iii) *For each $n \in \mathbb{N}$,*

$$\boldsymbol{\rho}\,\boldsymbol{\Pi} \;\le\; \boldsymbol{\rho}^* \;\le\; \boldsymbol{T}_{\boldsymbol{\rho}}^{\infty}(\boldsymbol{\mu}) \;\le\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+1}(\boldsymbol{\mu}) \;\le\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu}) \;\le\; \boldsymbol{\mu}\,. \tag{4.11}$$

The proof requires only elementary manipulations which, however, require some previous considerations to introduce the necessary notation.

It is useful to visualize the maps (3.12) in the diagrammatic form



The sum is over all single-generation rooted trees. In each tree, open circles represents a factor ρ, bullets a factor μ and vertices other than the root must be summed over all possible polymers γ. At each vertex with n descendants, a "vertex function" $c_n/n!$ acts, having as arguments the ordered $n+1$-tuple formed by the polymer at the vertex, the polymer at the top offspring, the polymer at the next offspring from the top,..., in that order. With this representation, the iteration $T^2(\boldsymbol{\mu})$ corresponds to replacing each of the bullets by each one of the diagrams of the expansion for T. This leads to rooted trees of up to two generations, with open circles at first-generation vertices and bullets at second-generation ones. In particular, all single-generation trees have only open circles. Notice that the two drawings of Figure 1 appear in two different terms of the expansion, and hence should be counted as *different* diagrams. More generally, the k-th iteration of T involves all possible rooted tree diagrams, counting as different those obtained by permutations of non-identical branches. We shall call these diagrams *planar rooted trees.* In each term of the expansion, vertices of generation k are occupied by bullets and all the others by open circles.

Formally, the definition of planar rooted trees is determined by a labeling choice which we fix as follows. There is a special vertex, labeled 0 (the root), placed, say, at the leftmost position of the drawing. From it there emerge s_0 branches ending at the first-generation vertices. The value $s_0 = 0$ describes the trivial tree with the root as its only vertex. Otherwise these vertices are drawn along a vertical line at the right of the root and labeled $(0,1), \ldots (0, s_0)$ with the second subscript increasing from the top to the bottom of the line. The construction continues rightwards: Each of the vertices $(0,i)$, gives rise to a family of second-generation vertices $(0,i,1), \ldots (0,i,s_{(0,i)})$ and so on. The vertex v is of generation ℓ if its label has the form $v = (0, i_1, \ldots, i_\ell)$ with $1 \le i_j \le s_{(0,i_1,\ldots,i_{j-1})}$, $1 \le j \le \ell$ ($i_0 \equiv 0$). The sequence of such branching factors $s_{(0,i_1,\ldots,i_\ell)} \in \mathbb{N} \cup \{0\}$ define the planar rooted tree. Let us denote $\mathbb{T}^{0,k}$ the set of trees with maximal generation number k; $\mathbb{T}^{0,0}$ being the trivial tree. Figure 1 shows two different trees of $\mathbb{T}^{0,2}$. We enumerate the vertices following the generation number and the "top to bottom" order in case of equal generation. [This amounts to declaring $(0, i_1, \ldots, i_\ell) < (0, i'_1, \ldots, i'_{\ell'})$ if $\ell < \ell'$ and using lexicographic order if $\ell = \ell'$].

FIGURE 1. Planar rooted trees defined by (a) $s_0 = s_{(0,1)} = 2$ and $s_{(0,2)} = s_{(0,1,1)} = s_{(0,1,2)} = 0$; (b) $s_0 = s_{(0,2)} = 2$ and $s_{(0,1)} = s_{(0,2,1)} = s_{(0,2,2)} = 0$

A straightforward inductive argument shows that

$$\Big(\boldsymbol{T}_{\boldsymbol{\rho}}^k(\boldsymbol{\mu})\Big)_{\gamma_0} \;=\; \rho_{\gamma_0}\Big[\sum_{\ell=0}^{k-1} R_{\gamma_0}^{(\ell)}(\boldsymbol{\rho}) + R_{\gamma_0}^{(k)}(\boldsymbol{\rho},\boldsymbol{\mu})\Big] \tag{4.12}$$

with

$$R_{\gamma_0}^{(\ell)}(\boldsymbol{\rho}) \;=\; \sum_{t\in\mathbb{T}^{0,\ell}} \;\sum_{(\gamma_{v_1},\dots,\gamma_{v_{|V_t|}})\in\mathcal{P}^{|V_t|}} \;\prod_{i=0}^{|V_t|} \frac{1}{s_{v_i}!}\, c_{s_{v_i}}\big(\gamma_{v_i},\gamma_{(v_i,1)},\dots,\gamma_{(v_i,s_{v_i})}\big)\,\rho_{\gamma_{(v_i,1)}}\cdots\rho_{\gamma_{(v_i,s_{v_i})}} \tag{4.13}$$

and $R_{\gamma_0}^{(k)}(\boldsymbol{\rho},\boldsymbol{\mu})$ has a similar expression but with the activities of the vertex of the k-th generation weighted by $\boldsymbol{\mu}$. In this expression V_t denotes the set of non-root vertices of t and we agree that $c_0(\gamma_v)\equiv 1$ and $\prod_\emptyset \equiv 1$. We are interested in the $k\to\infty$ limit of (4.12). Let us denote $\mathbb{T}^0 = \cup_\ell \mathbb{T}^{0,\ell}$. These considerations make almost immediate the proof of the following lemma which, together with a simple combinatorial argument, proves Proposition 7.

Proposition 8. *For some fixed $\boldsymbol{\rho}\in[0,\infty)^{\mathcal{P}}$ let $\boldsymbol{T}_{\boldsymbol{\rho}}$ be a map of the form (3.12)/(4.8) and assume there exists $\boldsymbol{\mu}\in[0,\infty)^{\mathcal{P}}$ such that $\boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\mu}) \le \boldsymbol{\mu}$. Then $\boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\rho}) \nearrow \boldsymbol{\rho}^*\in[0,\infty)^{\mathcal{P}}$ as $n\to\infty$, with*

$$\rho_{\gamma_0}^* \;:=\; \rho_{\gamma_0}\sum_{t\in\mathbb{T}^0} \;\sum_{(\gamma_{v_1},\dots,\gamma_{v_{|V_t|}})\in\mathcal{P}^{|V_t|}} \;\prod_{i=0}^{|V_t|} \frac{1}{s_{v_i}!}\, c_{s_{v_i}}\big(\gamma_{v_i},\gamma_{(v_i,1)},\dots,\gamma_{(v_i,s_{v_i})}\big)\,\rho_{\gamma_{(v_i,1)}}\cdots\rho_{\gamma_{(v_i,s_{v_i})}} \tag{4.14}$$

for each $\gamma_0\in\mathcal{P}$. Furthermore,

(i) *$T(\boldsymbol{\rho}^*) = \boldsymbol{\rho}^*$.*
(ii) *There exists $\boldsymbol{T}_{\boldsymbol{\rho}}^\infty(\boldsymbol{\mu})\in[0,\infty)^{\mathcal{P}}$ such that $\boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\mu}) \searrow \boldsymbol{T}_{\boldsymbol{\rho}}^\infty(\boldsymbol{\mu})$ as $n\to\infty$.*
(iii) *For all $\ell,n\in\mathbb{N}$,*

$$\boldsymbol{\mu} \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\mu}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+1}(\boldsymbol{\mu}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^\infty(\boldsymbol{\mu}) \;\ge\; \boldsymbol{\rho}^* \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{\ell+1}(\boldsymbol{\rho}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{\ell}(\boldsymbol{\rho}) \;\ge\; \boldsymbol{\rho}\,. \tag{4.15}$$

Proof. The map $\boldsymbol{T}_{\boldsymbol{\rho}}$ is obviously monotinicity preserving in the coordinatewise partial order of $[0,\infty]^{\mathcal{P}}$ and

$$\boldsymbol{\mu} \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\mu}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\rho}) \;\ge\; \boldsymbol{\rho}\,. \tag{4.16}$$

[The first inequality is by hypothesis, the second one by monotonicity and the third one is immediate from the definition of $\boldsymbol{T}_{\boldsymbol{\rho}}$.] Therefore, by induction,

$$\boldsymbol{\mu} \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\mu}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+1}(\boldsymbol{\mu}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+\ell+1}(\boldsymbol{\mu}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+\ell+1}(\boldsymbol{\rho}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{\ell+1}(\boldsymbol{\rho}) \;\ge\; \boldsymbol{T}_{\boldsymbol{\rho}}^{\ell}(\boldsymbol{\rho}) \;\ge\; \boldsymbol{\rho} \tag{4.17}$$

for all $\ell, n \in \mathbb{N}$. This shows that, for each $\gamma \in \mathcal{G}$, the series $\big(\boldsymbol{T}_{\boldsymbol{\rho}}^{\ell}(\boldsymbol{\rho})\big)_{\ell}$ is increasing and bounded above while $\big(\boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu})\big)_{n}$ is decreasing and bounded below. Thus, the limits $\boldsymbol{\rho}^* := \sup_\ell \boldsymbol{T}_{\boldsymbol{\rho}}^{\ell}(\boldsymbol{\rho})$ and $\boldsymbol{T}_{\boldsymbol{\rho}}^{\infty}(\boldsymbol{\mu}) := \inf_\ell \boldsymbol{T}_{\boldsymbol{\rho}}^{\ell}(\boldsymbol{\mu})$ exist and are finite and, by letting alternatingly $\ell \to \infty$ and $n \to \infty$ in (4.11), we obtain the inequalities (4.15). The fact that $\boldsymbol{T}_{\boldsymbol{\rho}}^{\infty}(\boldsymbol{\rho}) = \boldsymbol{\rho}^*$ is immediate from expression (4.12). Finally,

$$\boldsymbol{\rho}^* \;=\; \lim_{n\to\infty} \boldsymbol{T}_{\boldsymbol{\rho}}\big(\boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\rho})\big) \;=\; \boldsymbol{T}_{\boldsymbol{\rho}}\Big(\lim_{n\to\infty} \boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\rho})\Big) \;=\; \boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\rho}^*) \tag{4.18}$$

where the middle identity is by monotone convergence. ■

We notice that $T_{\gamma_0}^{\infty}(\boldsymbol{\mu}) \;=\; \rho_{\gamma_0}^* + \lim_k R_{\gamma_0}^{(k)}(\boldsymbol{\mu})$. The last limit is in fact an infimum because $\boldsymbol{R}^{(k)}(\boldsymbol{\rho},\boldsymbol{\mu}) \le \boldsymbol{R}^{(k-1)}\big(\boldsymbol{\rho},\boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\mu})\big) \le \boldsymbol{R}^{(k-1)}(\boldsymbol{\rho},\boldsymbol{\mu})$.

Proof of Proposition 7. The sum in (4.14) can be written in the form

$$\rho_{\gamma_0}^* \;=\; \rho_{\gamma_0} \sum_{t\in\mathbb{T}^0} W_{\gamma_0}(t)\;. \tag{4.19}$$

The symmetry of the vertex functions $c_n(\gamma_0,\gamma_1,\ldots,g_n)$ implies that the weights $\boldsymbol{W}(t)$ that are invariant under permutations of the branches of the planar tree t. That is, they depend only on the underlying labeled tree τ obtained by neglecting the order of the vertices. Formally, if $\mathcal{T}_n^0$ is the set of rooted trees on $\{0,1,\ldots,n\}$ (=labelled trees of $n+1$ vertices), there is a map $\mathcal{T}_n^0 \ni \tau \mapsto t_\tau \in \mathbb{T}_n^0$ where t_τ is the planar tree obtained by drawing branches starting on the root according to the order given by the labels of the first offspring, and continuing in this way for branches within branches. This map is many-to-one, in fact, the cardinality of the preimage of a tree t (=number of ways of labelling the $|V_t|$ non-root vertices of a planar rooted tree with $|V_t|$ distinct labels consistently with the rule “from high to low”) is

$$\beta_t \;=\; \frac{|V_t|!}{\prod_{i=0}^{|V_t|} s_{v_i}!} \tag{4.20}$$

(see e.g. theorem 145B in [3]). Thus, we can replace the sum in (4.19) by a sum over trees τ on the set $\mathcal{T}^0 = \cup_n \mathcal{T}_n^0$ of rooted trees:

$$\rho_{\gamma_0}^* \;=\; \rho_{\gamma_0} \sum_{\tau\in\mathcal{T}^0} \frac{W_{\gamma_0}(t_\tau)}{\beta_{t_\tau}}\;. \tag{4.21}$$

If we expand $\boldsymbol{W}$ and permute the sum over trees with the sum over polymer sequences (allowed operation for a series of positive terms), we obtain

$$\rho_{\gamma_0}^* \;=\; \rho_{\gamma_0} \sum_{n\ge 0} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \Big[\sum_{\tau\in\mathcal{T}_n^0} \prod_{i=0}^{n} c_{s_i}(\gamma_i,\gamma_{i_1},\ldots,\gamma_{i_{s_i}})\Big]\, \rho_{\gamma_1}\cdots\rho_{\gamma_n}\;. \tag{4.22}$$

Comparing this expression with (2.7), we see immediately that hypothesis (4.7) implies that $\boldsymbol{\rho}\,\boldsymbol{\Pi} \le \boldsymbol{\rho}^*$. The remaining statements are a consequence of Proposition 8. ■

4.2 Proof of Theorem 4

We just have to show that the different convergence conditions can be written in the form (4.8) for vertex functions c_s satisfying (4.7). Theorem 4 then follows from Proposition 7.

We use Penrose identity (4.6) to obtain a bound of the form (4.7). For this, we keep only the vertex constraints of a Penrose tree τ: The descendants of a given vertex may not be linked by an edge in the initial graph $\mathcal{G}$. Otherwise [by condition (p1) in Sectionn 4.1], the graph $R_{\rm Pen}(\tau)$ would include such an edge and would, therefore, differ from τ. That is, we consider the larger family of trees such that

$$\text{If } \{i,i_1\} \text{ and } \{i,i_2\} \text{ are edges of } \tau, \text{ then } \gamma_{i_1}\sim\gamma_{i_2} \tag{4.23}$$

In this way we obtain bounds of the form (4.7) with

$$c_n(\gamma_0,\gamma_1,\ldots,\gamma_n) \;=\; \prod_{i=1}^n \mathbb{1}_{\{\gamma_0\not\sim\gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i\sim\gamma_j\}}\;, \tag{4.24}$$

and Proposition 7 applies with

$$\varphi_{\gamma_0}(\boldsymbol{\mu}) \;=\; 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\not\sim\gamma_i\,,\,\gamma_i\sim\gamma_j\,,\,1\leq i,j\leq n}} \mu_{\gamma_1}\ldots\mu_{\gamma_n}\;,=\;\Xi_{\mathcal{P}_{\gamma_0}}(\boldsymbol{\mu})\;. \tag{4.25}$$

This proves the criterion of Theorem 1.

If we replace in (4.23) the condition $\gamma_i\not\sim\gamma_j$ by the weaker requirement $\gamma_i\neq\gamma_j$ we obtain

$$c_n^{\rm Dob}(\gamma_0,\gamma_1,\ldots,\gamma_n) \;=\; \prod_{i=1}^n \mathbb{1}_{\{\gamma_0\not\sim\gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i\neq\gamma_j\}}\;, \tag{4.26}$$

and

$$\varphi^{\rm Dob}_{\gamma_0}(\boldsymbol{\mu}) \;=\; 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\not\sim\gamma_i\,,\,\gamma_i\neq\gamma_j\,,\,1\leq i,j\leq n}} \mu_{\gamma_1}\ldots\mu_{\gamma_n}\;=\;\prod_{\gamma\not\sim\gamma_0}(1+\mu_\gamma)\;, \tag{4.27}$$

which corresponds to Dobrushin condition. The improved Dobrushin condition is obtained by strengthening (4.26) through the further requirement that $\gamma_i\neq\gamma_0$ for $i=1,\ldots,n$ and $n\geq 2$.

Finally, if requirement (4.23) is ignored altogether,

$$c_n^{\rm KP}(\gamma_0,\gamma_1,\ldots,\gamma_n) \;=\; \prod_{i=1}^n \mathbb{1}_{\{\gamma_0\not\sim\gamma_i\}}\;, \tag{4.28}$$

and

$$\varphi^{\rm KP}_{\gamma_0}(\boldsymbol{\mu}) \;=\; 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\not\sim\gamma_i\,,\,1\leq i\leq n}} \mu_{\gamma_1}\ldots\mu_{\gamma_n}\;=\;\exp\Big[\sum_{\gamma\not\sim\gamma_0}\mu_\gamma\Big] \tag{4.29}$$

yields the criterion of Kotecký and Preiss. ■

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References

[1] A. Bovier and M. Zahradnk (2000): A simple inductive approach to the problem of convergence of cluster expansions of polymer models. *J. Statist. Phys.* **100**, 765–78.
[2] D. C. Brydges (1984): A short cluster in cluster expansions. In *Critical Phenomena, Random Systems, Gauge Theories*, Osterwalder, K. and Stora, R. (eds.), Elsevier, 129–83.
[3] J. C. Butcher (1987): *The Numerical Analysis of Ordinary Differential Equations. Runge-Kutta and General Linear Methods*, Wiley-Interscience Publication.
[4] C. Cammarota (1982): Decay of correlations for infinite range interactions in unbounded spin systems. *Comm. Math. Phys.* **85**, 517–28.
[5] R. L. Dobrushin (1996): Estimates of semiinvariants for the Ising model at low temperatures. *Topics in Statistics and Theoretical Physics*, Amer. Math. Soc. Transl. (2), **177**, 59–81.
[6] R. L. Dobrushin (1996a): Perturbation methods of the theory of Gibbsian fields. In *Ecole d'Eté de Probabilités de Saint-Flour XXIV – 1994*, Springer-Verlag (Lecture Notes in Mathematics **1648**), Berlin–Heidelberg–New York, 1–66.
[7] C. Gruber and H. Kunz (1971): General properties of polymer systems. *Comm. Math. Phys.* **22**, 133–61.
[8] R. Kotecký and D. Preiss (1986): Cluster expansion for abstract polymer models. *Commun. Math. Phys.*, **103**, 491–498.
[9] J. Groeneveld (1962): Two theorems on classical many-particle systems. *Phys. Letters*, **3**, 50–1.
[10] V. A. Malyshev (1980): Cluster expansions in lattice models of statistical physics and quantum theory of fields. *Russian Mathematical Surveys*, **35**, 1–62.
[11] S. Miracle-Solé (2000): On the convergence of cluster expansions. *Physica A* **279**, 244–9.
[12] F. R. Nardi, E. Olivieri and M. Zahradnk (1999): On the Ising model with strongly anisotropic external field. *J. Stat. Phys.* **97**, 87–144.
[13] Ch.-E. Pfister (1991): Large deviation and phase separation in the two-dimensional Ising model. *Helv. Phys. Acta* **64**, 953-1054.
[14] O. Penrose (1967): Convergence of fugacity expansions for classical systems. In *Statistical mechanics: foundations and applications*, A. Bak (ed.), Benjamin, New York.
[15] A. Procacci, B. Scoppola (1999): Polymer gas approach to N-body lattice systems. *J. Statist. Phys.* **96**, 49–68.
[16] H. L. Royden (1968): *Real Analysis* (2nd. ed.). Macmillan Publishing Co., New York.
[17] D. Ruelle (1969): *Statistical mechanics: Rigorous results*. W. A. Benjamin, Inc., New York-Amsterdam.
[18] A. Scott and A. D. Sokal (2005): The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma. *J. Stat. Phys.* **118**, 1151–261.
[19] A. Sokal (2001): Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions. *Combin. Probab. Comput.* **10**, 41-77.
[20] E. Seiler (1982): *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics **159**, Springer-Verlag, Berlin–Heidelberg–New York.
[21] D. Ueltschi (2004): Cluster expansions and correlation functions. *Mosc. Math. J.* **4**, 511–22, 536.