ON L^P-ESTIMATES FOR THE TIME DEPENDENT SCHRÖDINGER OPERATOR ON L^2.

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Abstract. Let L denote the time-dependent Schrödinger operator in n space variables. We consider a variety of Lebesgue norms for functions u on $\mathbb{R}^{n+1}$, and prove or disprove estimates for such norms of u in terms of the L^2 norms of u and Lu. The results have implications for self-adjointness of operators of the form $L+V$ where V is a multiplication operator. The proofs are based mainly on Strichartz-type inequalities.

Introduction.

Let $(x,t)\in\mathbb{R}^{n+1}$ where $n\geq 1$. The Schrödinger equation $\frac{\partial u}{\partial t}=i\triangle_x u$ has been much studied using spectral properties of the self-adjoint operator $\triangle_x$. When a multiplication operator (potential) V is added it becomes important to determine whether $\triangle_x+V$ is a self-adjoint operator, and there is a vast literature on this question (see e.g. [9]).

One can also, however, regard the operator $L=-i\frac{\partial}{\partial t}-\triangle_x$ as a self-adjoint operator on $L^2(\mathbb{R}^{n+1})$, and that is the point of view taken in this paper. We ask what can be said about the domain of L, more specifically we ask which L^q spaces, and more generally mixed $L^q_t(L^r_x)$ space, a function u must belong to, given that u is in the domain of L (i.e. u and Lu both belong to $L^2(\mathbb{R}^{n+1})$). We answer this question and, using the Kato-Rellich theorem, deduce sufficient conditions on V for $L+V$ to be self-adjoint.

Our approach is based on the fact that any sufficiently well-behaved function u on $\mathbb{R}^{n+1}$ can be regarded as the solution of the initial value problem (IVP)

$$\begin{cases} -iu_t-\triangle_x u=g(x,t), \\ u(x,\alpha)=f(x) \end{cases} \tag{1}$$

where $\alpha\in\mathbb{R}$, $f(x)=u(x,\alpha)$ and $g=Lu$.

To apply this we will use estimates for u based on given bounds for f and g. A number of such estimates are known and generally called Strichartz inequalities, after [12] which obtained such an L^q bound for u. This has since been generalised to give inequalities for mixed norms [13, 4]. The specific inequalities we use concern the case $g=0$ of (1) and give bounds for u in terms of $\|f\|_{L^2(\mathbb{R}^n)}$ - see (6) below. The precise range of mixed $L^\infty_t(L^2_x)$ norms for which the bound (6) holds is known as a result of [13, 4] and the counterexample in [6].

In Section 1 we prove a special case of our main theorem, namely a bound for u in $L^\infty_t(L^2_x)$, which does not require Strichartz estimates, only elementary arguments using the Fourier transform. The main theorem, giving $L^q_t(L^r_x)$ bounds

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for the largest possible set of (q,r) pairs, is proved in Section 2. In fact we prove a somewhat stronger bound, in a smaller space $\mathcal{L}_{2,q,r}$ defined below. The fact that the set of pairs (q,r) covered by Theorem 1 is the largest possible is shown in Section 3.

Some results on a similar question for the wave operator can be found in [7]. For Strichartz-type inequalities for the wave operator see e.g. [11, 12, 2, 3, 4].

We assume notions and definitions about the Fourier Transform and unbounded operators and for a reference one may consult [8], [5] or [10]. We also use on several occasions the well-known Duhamel principle for the Schrödinger equation (see e.g. [1]).

Notation. The symbol $\hat{u}$ stands for the Fourier transform of u in the space (x) variable while the inverse Fourier transform will be denoted either by $\mathcal{F}^{-1}u$ or $\check{u}$.

We denote by $C_0^\infty(\mathbb{R}^{n+1})$ the space of infinitely differentiable functions with compact support.

We denote by $\mathbb{R}^+$ the set of all positive real numbers together with $+\infty$.

For $1\leq p\leq\infty$, $\|\cdot\|_p$ is the usual L^p-norm whereas $\|\cdot\|_{L_t^p(L_x^q)}$ stands for the mixed spacetime Lebesgue norm defined as follows

$$\|u\|_{L_t^q(L_x^r)}=\left(\int_{\mathbb{R}}\|u(t)\|_{L_x^r}^q dt\right)^{1/q}.$$

We also define some modified mixed norms. First we define, for any integer k,

$$\|u\|_{L_{t,k}^q(L_x^r)}=\left(\int_k^{k+1}\|u(t)\|_{L_x^r}^q dt\right)^{1/q}.$$

and then

$$\|u\|_{\mathcal{L}_{p,q,r}}=\left(\sum_{k\in\mathbb{Z}}\|u\|_{L_{t,k}^q(L_x^r)}^p\right)^{1/p}$$

We note that $\|u\|_{\mathcal{L}_{p,q_1,r}}\geq\|u\|_{\mathcal{L}_{p,q_2,r}}$ if $q_1\geq q_2$, and that $\|u\|_{L_t^q(L_x^r)}\leq\|u\|_{\mathcal{L}_{p,q,r}}$ if $q\geq p$.

Finally we define

$$M_L^n=\{f\in L^2(\mathbb{R}^{n+1}):Lf\in L^2(\mathbb{R}^{n+1})\}$$

where L is defined as in the abstract and where the derivative is taken in the distributional sense. We note that $M_L^n=\mathcal{D}(L)$, the domain of L, and also that $C_0^\infty(\mathbb{R}^{n+1})$ is dense in M_L^n in the graph norm $\|u\|_{L^2(\mathbb{R}_{n+1})}+\|Lu\|_{L^2(\mathbb{R}_{n+1})}$.

1. $L_t^\infty(L_x^2)$ Estimates.

Before stating the first result, we are going to prepare the ground for it. Take the Fourier transform of the IVP (1) in the space variable to get

$$\begin{cases}-i\hat{u}_t+\eta^2\hat{u}=\hat{g}(\eta,t),\\ \hat{u}(\eta,\alpha)=\hat{f}(\eta)\end{cases}$$

which has the following solution (valid for all $t\in\mathbb{R}$):

$$\hat{u}(\eta,t)=\hat{f}(\eta)e^{-i\eta^2 t}+i\int_\alpha^t e^{-i\eta^2(t-s)}\hat{g}(\eta,s)ds \tag{2}$$

where $\eta \in \mathbb{R}^n$.

Duhamel's principle gives an alternative way of writing the part of the solution depending on g. Taking the case $f = 0$, the solution of (1) can be written as

$$u(x,t) = i\int_\alpha^t u_s(x,t)ds. \tag{3}$$

where u_s is the solution of

$$\begin{cases} Lu_s = 0, & t > s, \\ u_s(x,s) = g(x,s). & \end{cases}$$

Now we state a result which we can prove using (2). In the next section we prove a more general result using Strichartz inequalities and Duhamel's principle (3).

Proposition 1. *For all $a > 0$, there exists $b > 0$ such that*

$$\|u\|_{\mathcal{L}_{2,\infty,2}} \leq a\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + b\|u\|^2_{L^2(\mathbb{R}^{n+1})}$$

for all $u \in M^n_L$.

Proof. We prove the result for $u \in C_0^\infty(\mathbb{R})$ and a density argument allows us to deduce it for $u \in M^n_L$.

We use the fact that any such u is, for any $\alpha \in \mathbb{R}$, the unique solution of (1), where $f(x) = u(x,\alpha)$ and $g = Lu$, and therefore satisfies (2).

Let $k \in \mathbb{Z}$ and let t and α be such that $k \leq t \leq k+1$ and $k \leq \alpha \leq k+1$. Squaring (2), integrating with respect to η in $\mathbb{R}^n$, and using Cauchy-Schwarz (and the fact that $|t-\alpha| \leq 1$) we obtain

$$\|\hat{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq 2\int_{\mathbb{R}^n} |\hat{u}(\eta,\alpha)|^2 d\eta + 2\int_{\mathbb{R}^n}\int_\alpha^t |\hat{g}(\eta,s)|^2 ds d\eta. \tag{4}$$

Now integrating against α in $[k,k+1]$ allows us to say that

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq 2\int_k^{k+1}\int_{\mathbb{R}^n} |\hat{u}(\eta,\alpha)|^2 d\eta d\alpha + 2\int_k^{k+1}\int_{\mathbb{R}^n} |\hat{g}(\eta,s)|^2 d\eta ds$$

Now take the essential supremum of both sides in t over $[k,k+1]$ then sum in k over $\mathbb{Z}$ to get (recalling that $g = Lu$)

$$\sum_{k=-\infty}^{\infty} \operatorname*{ess\,sup}_{k \leq t \leq k+1} \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq 2\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + 2\|u\|^2_{L^2(\mathbb{R}^{n+1})}.$$

Finally to get a arbitrarily small constant in the Lu term we use a scaling argument: let m be a positive integer and let $v(x,t) = u(mx, m^2t)$. Then we find $\|v\|_{L^2(\mathbb{R}^{n+1})} = m^{-1-n/2}\|u\|_{L^2(\mathbb{R}^{n+1})}$ and $\|Lv\|_{L^2(\mathbb{R}^{n+1})} = m^{1-n/2}\|Lu\|_{L^2(\mathbb{R}^{n+1})}$. Also $\|v(\cdot,t)\|_{L^2(\mathbb{R}^n)} = m^{-n/2}\|u(\cdot,m^2t)\|_{L^2(\mathbb{R}^n)}$ and so

$$\sup_{k \leq t \leq k+1} \|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = m^{-n} \sup_{m^2k \leq t \leq m^2(k+1)} \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \sum_{j=m^2k}^{m^2(k+1)-1} \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

and then summing over k and gives

$$\begin{aligned}\|v\|^2_{\mathcal{L}_{2,\infty,2}} &\leq m^{-n}\|u\|_{\mathcal{L}_{2,\infty,2}} \leq m^{-n}(2\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + 2\|u\|^2_{L^2(\mathbb{R}^{n+1})}) \\ &\leq 2m^{-2}\|Lv\|^2_{L^2(\mathbb{R}^{n+1})} + 2m^2\|v\|^2_{L^2(\mathbb{R}^{n+1})}\end{aligned}$$

and choosing m so that $2m^{-2} < a$ completes the proof. □

Now we recall the Kato-Rellich theorem which states that if L is a self-adjoint operator on a Hilbert space and V is a symmetric operator defined on $\mathcal{D}(L)$, and if there are positive constants $a < 1$ and b such that $\|Vu\| \leq a\|Lu\| + b\|u\|$ for all $u \in \mathcal{D}(L)$ then $L+V$ is self-adjoint on $\mathcal{D}(L)$. See [9].

Corollary 1. *Let V be a real-valued function in $\mathcal{L}_{\infty,2,\infty}$. Then $L+V$ is self-adjoint on $\mathcal{D}(L) = M_L^n$.*

Proof. One easily checks that

$$\|Vu\|_{L^2(\mathbb{R}^{n+1})} \leq \|V\|_{\mathcal{L}_{\infty,2,\infty}} \|u\|_{\mathcal{L}_{2,\infty,2}}$$

Choose $a < \|V\|_{\mathcal{L}_{\infty,2,\infty}}$ and then Proposition 1 shows that $L+V$ satisfies the hypothesis of the Kato-Rellich theorem. □

In particular, it follows that $L+V$ is self-adjoint whenever $V \in L_t^2(L_x^\infty)$.

2. $L_t^q(L_x^r)$ Estimates.

Now we come to the main theorem in this paper, which depends on the following Strichartz-type inequality. Suppose $n \geq 1$ and q and r are positive real numbers (possibly infinite) such that $q \geq 2$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{5}$$

When $n = 2$ we exclude the case $q = 2$, $r = \infty$. Then there is a constant C such that if $f \in L^2(\mathbb{R}^n)$ and $g = 0$, the solution u of (1) satisfies

$$\|u\|_{L_t^q(L_x^r)} \leq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{6}$$

This result can be found in [13] for $q > 2$; the more difficult 'end-point' case where $q = 2$, $n \geq 3$ is treated in [4]. That (6) fails in the exceptional case $n = 2, q = 2, r = \infty$ is shown in [6].

For $n \geq 1$ we define a region $\Omega_n \in \mathbb{R}^+ \times \mathbb{R}^+$ as follows: for $n \neq 2$,

$$\Omega_n = \{(q,r) \in \mathbb{R}^+ \times \mathbb{R}^+ : \ \frac{2}{q} + \frac{n}{r} \geq \frac{n}{2}, \ q \geq 2, \ r \geq 2\} \tag{7}$$

and for $n = 2$, Ω_2 is defined in by the same expression, but omitting the point $(2, \infty)$.

The sets Ω_n are probably most easily visualised in the $(\frac{1}{q}, \frac{1}{r})$-plane. Then Ω_1 is a quadrilateral with vertices $(\frac{1}{4}, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ and for $n \geq 2$ Ω_n is a triangle with vertices $(\frac{1}{2}, \frac{n-2}{2n}), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$, the point$(\frac{1}{2}, 0)$ being excluded in the case $n = 2$.

Theorem 1. *Let $n \geq 1$, and let $(q,r) \in \Omega_n$. Then for all $a > 0$, there exists $b > 0$ such that*

$$\|u\|_{\mathcal{L}_{2,q,r}} \leq a\|Lu\|_{L^2(\mathbb{R}^{n+1})} + b\|u\|_{L^2(\mathbb{R}^{n+1})} \tag{8}$$

for all $u \in M_L^n$.

Proof. By the inclusion $\mathcal{L}_{2,q_1,r} \subseteq \mathcal{L}_{2,q_2,r}$ when $q_1 \geq q_2$ it suffices to treat the case where $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, for which (6) holds.

Let $k \in \mathbb{Z}$ and let $\alpha \in [k, k+1]$. As in te proof of Proposition 1 we use the fact that u is the solution of (1) with $f = u(\cdot, \alpha)$ and $g = Lu$. Now we split u into two parts $u = u_1 + u_2$ where u_1, u_2 are the solutions of

$$\begin{cases} Lu_1 = g, \\ u_1(x,\alpha) = 0, \end{cases} \begin{cases} Lu_2 = 0, \\ u_2(x,\alpha) = f. \end{cases}$$

The estimate for u_2 is deduced from 6:

$$\|u_2\|_{L^q_t(L^r_x)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \leq C\|u(\cdot,\alpha)\|_{L^2(\mathbb{R}^n)}. \tag{9}$$

For u_1 we apply (3) to obtain

$$u_1(x,t) = i\int_\alpha^t u_s(x,t)ds. \tag{10}$$

from which we deduce

$$\|u_1(\cdot,t)\|_{L^r(\mathbb{R}^n)} \leq \int_k^{k+1} \|u_s(\cdot,t)\|_{L^r(\mathbb{R}^n)} ds$$

for $t \in [k, k+1]$, and hence

$$\begin{aligned} \|u_1\|_{L^q_{t,k}(L^r_x)} &\leq \int_k^{k+1} \|u_s\|_{L^q_t(L^r_x)} ds \leq C\int_k^{k+1} \|g(\cdot,s)\|_{L^2(\mathbb{R}^n)} ds \\ &\leq C\|g\|_{L^2(\mathbb{R}^n\times[k,k+1])}. \end{aligned}$$

Combining this with (9) we have

$$\|u\|^2_{L^q_{t,k}(L^r_x)} \leq 2C^2\|u(\cdot,\alpha)\|^2_{L^2(\mathbb{R}^n)} + 2C^2\|Lu\|^2_{L^2(\mathbb{R}^n\times[k,k+1])}.$$

Integrating w.r.t. α from k to $k+1$ gives

$$\|u\|^2_{L^q_{t,k}(L^r_x)} \leq 2C^2\|u\|^2_{L^2(\mathbb{R}^n\times[k,k+1])} + 2C^2\|Lu\|^2_{L^2(\mathbb{R}^n\times[k,k+1])}.$$

Summing over k then gives

$$\|u\|^2_{\mathcal{L}_{2,q,r}} \leq 2C^2\|u\|_{L^2(\mathbb{R}^{n+1})} + 2C^2\|Lu\|_{L^2(\mathbb{R}^{n+1})}$$

and the proof is completed by a similar scaling argument to that used in Proposition 1. □

Using the inclusion $\mathcal{L}_{2,q,r} \subseteq L^q_2(L^r_x)$ for $q \geq 2$ we deduce

Corollary 2. *Let $n \geq 1$, and let $(q,r) \in \Omega_n$. Then for all $a > 0$, there exists $b > 0$ such that*

$$\|u\|_{L^q_t(L^r_x)} \leq a\|Lu\|_{L^2(\mathbb{R}^{n+1})} + b\|u\|_{L^2(\mathbb{R}^{n+1})} \tag{11}$$

for all $u \in M^n_L$.

In particular we get such a bound for $\|u\|_{L^q(\mathbb{R}^{n+1})}$ whenever $2 \leq q \leq (2n+4)/n$.

By applying the Kato-Rellich theorem we can deduce a generalistion of Corollary 1 from Theorem 1. We first define

$$\Omega^*_n = \{(p,s) \in \mathbb{R}^+ \times \mathbb{R}^+ :\ \frac{2}{p} + \frac{n}{s} \leq 1,\ p \geq 2,\ s \geq 2\} \tag{12}$$

for $n \neq 2$, and for $n = 2$, Ω_2 is defined in by the same expression, but omitting the point $(\infty, 2)$.

Corollary 3. *Let $n \geq 1$ and let $(p, s) \in \Omega_n^*$. Let V be a real-valued function belonging to $\mathcal{L}_{\infty,p,s}$. Then $L + V$ is self-adjoint on M_L^n.*

Proof. Let $q = \frac{2p}{p-2}$ and $r = \frac{2s}{s-2}$. Then $(q, r) \in \Omega_n$ and the conclusion (8) of Theorem 1 applies. Now we have

$$\begin{aligned}\int_k^{k+1} \|Vu(.,t)\|^2_{L^2(\mathbb{R}^n)} &\leq \int_k^{k+1} \|u(\cdot,t)\|^2_{L^r(\mathbb{R}^n)} \|V(\cdot,t)\|^2_{L^s(\mathbb{R}^n)} \\ &\leq \|u\|^2_{L^q_{t,k}(L^r_x)} \|V\|^2_{L^p_{t,k}(L^s_x)}\end{aligned}$$

and summation over k gives

$$\|Vu\|_{L^2(\mathbb{R}^{n+1})} \leq \|u\|_{\mathcal{L}_{2,q,r}} \|V\|_{\mathcal{L}_{\infty,p,s}}$$

Then using (8) the result follows in the same way as Corollary 1. $\square$

It follows from Corollary 3 that $L + V$ is self-adjoint whenever $V \in L^p_t(L^s_x)$ for $(p, s) \in \Omega_n^*$. Taking the case $s = p$ we find that $L+V$ is self-adjoint if $V \in L^p(\mathbb{R}^{n+1})$ for some $p \geq n + 2$.

3. Counterexamples

Now we show that Theorem 1 is sharp, as far as the allowed set of q, r is concerned.

Proposition 2. *Let $n \geq 1$ and let q and r be positive real numbers, possibly infinite, such that $(q, r) \notin \Omega_n$. Then there are no constants a and b such that (11) holds for all $u \in M_L^n$.*

Proof. For (q, r) to fail to be in Ω_n one of the following three possibilities must occur: (i) $q < 2$ or $r < 2$; (ii) $\frac{2}{q} + \frac{n}{r} < \frac{n}{2}$; (iii) $n = 2$, $q = 2$ and $r = \infty$. We consider these cases in turn.

(i) If $q < 2$, choose a sequence $(\beta_k)_{k\in\mathbb{Z}}$ which is in l^2 but not in l^q. Let $\phi(x, t)$ be a smooth function of compact support on $\mathbb{R}^{n+1}$ which vanishes for t outside $[0, 1]$, and let $u(x, t) = \sum_{k\in\mathbb{Z}} \beta_k\phi(x, t - k)$. Then $u \in M_L^n$ but $u \notin L^q_t(L^r_x)$ for any r.

The case $r < 2$ can be treated similarly. We chose a sequence β_k which is in l^2 but not l^r, and a smooth ϕ which vanishes for x_1 outside $[0, 1]$, then set $u(x, t) = \sum_{k\in\mathbb{Z}} \beta_k\phi(x - ke_1, t)$ where e_1 is the unit vector $(1, 0, \cdots, 0)$ in $\mathbb{R}^n$. Then $u \in M_L^n$ but $u \notin L^q_t(L^r_x)$ for any q.

(ii) In this case we use the scaling argument which shows the Strichartz estimates fail, together with a cutoff to ensure u and Lu are in L^2.

We start with a non-zero $f \in L^2(\mathbb{R}^n)$, and let u be the solution of (1) with $\alpha = 0$ and $g = 0$. (An explicit example would be $f(x) = e^{-|x|^2}$ and then $u(x, t) = (1 + 4it)^{-n/2}e^{-|x|^2/(1+4it)}$). Choose a smooth function ϕ on $\mathbb{R}$ such that $\phi(0) \neq 0$ and such that ϕ and ϕ' are in L^2. Then for $\lambda > 0$ define

$$v_\lambda(x, t) = \lambda^{n/2}u(\lambda x, \lambda^2 t)\phi(t)$$

Then (using $Lu = 0$) we find $Lv(x,t) = -i\lambda^{n/2}u(\lambda x, \lambda^2 t)\phi'(t)$. We calculate $\|v_\lambda\|_{L^2(\mathbb{R}^{n+1})} = \|f\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2}$ and $\|Lv_\lambda\|_{L^2(\mathbb{R}^{n+1})} = \|f\|_{L^2(\mathbb{R}^n)}\|\phi'\|_{L^2}$. Also

$$\|v_\lambda\|_{L^q_t(L^r_x)} = \lambda^\beta \left\{ \int_{\mathbb{R}} \|u(.,t)\|^q_{L^r(\mathbb{R}^n)} |\phi(\lambda^{-2}t)|^q dt \right\}^{1/q}$$

where $\beta = \frac{n}{2} - \frac{n}{r} - \frac{2}{q} > 0$. So $\lambda^{-\beta}\|v_\lambda\|_{L^q_t(L^r_x)} \to |\phi(0)|\|u\|_{L^q_t(L^r_x)}$ (note that the norm on the right may be infinite) and hence $\|v_\lambda\|_{L^q_t(L^r_x)}$ tends to ∞ as $\lambda \to \infty$, completing the proof.

(iii) This exceptional case we treat in a similar fashion to (ii), but we need the result from [6] that the Strichartz inequality fails in this case. We start by fixing a smooth function ϕ on $\mathbb{R}$ such that $\phi = 1$ on $[-1,1]$ and ϕ and ϕ' are in L^2.

Now let $M > 0$ be given and we use [6] to find $f \in L^2(\mathbb{R}^2)$ with $\|f\|_{L^2(\mathbb{R}^2)} = 1$ such that the solution u of (1) with $\alpha = 0$ and $g = 0$ satisfies $\|u\|_{L^2_t(L^\infty_x)} > M$. Then we can find $R > 0$ so that $\int_{-R}^R \|u(\cdot,t)\|^2_{L^\infty(\mathbb{R}^2)}dt > M^2$. Let $\lambda = R^{1/2}$ and define $v(x,t) = \lambda^{n/2}u(\lambda x, \lambda^2 t)\phi(t)$. Then $\|v\|_{L^2(\mathbb{R}^3)} = \|\phi\|_{L^2}$, $\|Lv\|_{L^2(\mathbb{R}^3)} = \|\phi'\|_{L^2}$ and

$$\|v\|^2_{L^2_t(L^\infty_x)} \geq \int_{-1}^1 \|v(\cdot,t)\|^2_{L^\infty(\mathbb{R}^2)}dt > M^2$$

which completes the proof, since M is arbitrary. □

We remark that [6] also gives an example of $f \in L^2(\mathbb{R}^2)$ such that $u \notin L^2_t(BMO_x)$ and the argument of part (iii) can then be applied to show that no inequality

$$\|u\|_{L^2_t(BMO_x)} \leq a\|Lu\|_{L^2(\mathbb{R}^3)} + b\|u\|_{L^2(\mathbb{R}^3)}$$

can hold.

4. Question

We saw as a result of Corollary 3 that if $(p,s) \in \Omega^*$ the $L+V$ is self-adjoint on M^n_L whenever $V \in L^p_t(L^s_x)$. One can ask whether this can be extended to a larger range of (p,s) with $p,s \geq 2$. If one asks that $L+V$ is defined on M^n_L then we would require a bound $\|Vu\|_{L^2(\mathbb{R}^{n+1})} \leq a\|Lu\|_{L^2(\mathbb{R}^{n+1})} + b\|u\|$ to hold for all $u \in M^n_L$, and if such a bound is to hold for all $V \in L^p_t(L^s_x)$ then in fact we require (11) to hold for $q = \frac{2p}{p-2}$ and $r = \frac{2s}{s-2}$, which we know cannot hold unless $(p,s) \in \Omega^*$.

One can instead ask for $L+V$, defined on say $C^\infty_0(\mathbb{R}^{n+1}$, to be essentially self-adjoint. This is equivalent to saying that the only (distribution) solution in $L^2(\mathbb{R}^{n+1})$ of the PDE

$$-iu_t - \triangle_x u + Vu = \pm iu$$

is $u = 0$. See e.g. [8]

We do not know if there are any values of (p,s) not in Ω^*_n such that this holds for all $V \in L^p_t(L^s_x)$. The analogous question for the Laplacian is extensively discussed in [9].

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