# A Sufficient Condition for the Stochastic inequality

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October 16, 2006

**Abstract**

We will propose a new sufficient condition for the stochastic inequality. This condition differs in an essential way from the usual lattice condition. We also show an example, which satisfies the new condition, but does not satisfy the lattice condition.

## 1 Introduction

Let $E$ be a finite set, let $\Omega = \{0,1\}^E$ be a configuration space, and let $\mathcal{F}$ be the collection of all subsets of $\Omega$. Let $\mu_1$ and $\mu_2$ be strictly positive probability measures on $(\Omega, \mathcal{F})$, that is $\mu_1(\omega) > 0$ and $\mu_2(\omega) > 0$ for arbitrary $\omega \in \Omega$. Holley showed that if these measures satisfy the lattice condition

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2), \quad \omega_1, \omega_2 \in \Omega, \tag{1}$$

then they satisfy the stochastic inequality

$$\mu_1(X) \leq \mu_2(X) \text{ for an arbitrary increasing function } X, \tag{2}$$

where $\omega_1 \vee \omega_2 = (\omega_1 \vee \omega_2(e) : e \in E)$, $\omega_1 \vee \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\}$ and $\omega_1 \wedge \omega_2 = (\omega_1 \wedge \omega_2(e) : e \in E)$, $\omega_1 \wedge \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\}$, and we denote $\mu(X)$ as

$$\mu(X) = \sum_{\omega \in \Omega} X(\omega)\mu(\omega). \tag{3}$$

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We note that the lattice condition (1) is equivalent to the following relation (see [1]):

$$\begin{aligned}\mu_2(\omega(e) = 1|\omega(f) = \zeta(f) \text{ for all } f \in E \setminus \{e\}) \\ \geq \mu_1(\omega(e) = 1|\omega(f) = \xi(f) \text{ for all } f \in E \setminus \{e\})\end{aligned} \tag{4}$$

for all $e \in E$, and all pairs $\xi$, $\zeta \in \Omega$ satisfying $\xi \leq \zeta$. We emphasize that the inequality (4) is essential for proving the stochastic inequality (see [1], [3] and [4]). In this paper, we propose another sufficient condition, which implies the stochastic inequality. In Section 2, we give its exact statement. In Section 3, we give the proof of the theorem. In section 4, we give an example, which satisfies the new sufficient condition (this example also satisfies the stochastic inequality), but it does not satisfy the lattice condition (1).

# 2 Statement of the Main Result

In this section, we propose a new sufficient condition for the stochastic inequality. The statement is as follows:

**Theorem 2.1** Let $\mu_1$ and $\mu_2$ be probability measures on the finite space $(\Omega, \mathcal{F})$, which satisfy

$$\mu_2(\omega^e) + \mu_1(\omega_e) \geq \mu_1(\omega^e) + \mu_2(\omega_e), \quad \omega \in \Omega, e \in E, \tag{5}$$

and, whenever $\omega_1$, $\omega_2 \in \Omega$ satisfy

$$\{\mu_2(\omega_1) - \mu_1(\omega_1)\}\{\mu_2(\omega_2) - \mu_1(\omega_2)\} \leq 0,$$

$$\begin{aligned}\{\mu_2(\omega_1 \vee \omega_2) - \mu_1(\omega_1 \vee \omega_2)\}\{\mu_2(\omega_1 \wedge \omega_2) - \mu_1(\omega_1 \wedge \omega_2)\} \\ \leq \{\mu_2(\omega_1) - \mu_1(\omega_1)\}\{\mu_2(\omega_2) - \mu_1(\omega_2)\}.\end{aligned} \tag{6}$$

Then they satisfy the stochastic inequality:

$$\mu_1(X) \leq \mu_2(X) \text{ for increasing function } X : \Omega \to \mathbf{R}. \tag{7}$$

We remark that $\mu_1(X) \leq \mu_2(X)$ if and only if there exists a coupling $\kappa$ of $\mu_1$ and $\mu_2$ satisfying $\kappa(S) = 1$, where $S = \{(\omega_1, \omega_2) \in \Omega^2 | \omega_1 \leq \omega_2\}$ (see [1]). To construct a coupling, $\mu_1$ and $\mu_2$ may need to satisfy strict positivity and the lattice condition (1), equivalently (4). We emphasize that our sufficient condition assumes neither strict positivity nor the lattice condition (1).

# 3 Proof of Theorem 2.1

## 3.1 Preparation

In this section, we give a proof of Theorem 2.1. To prove Theorem 2.1, we shall prepare the following notations. Let $m$ be the signed measure on $(\Omega, \mathcal{F})$, given by

$$m(\omega) = \mu_2(\omega) - \mu_1(\omega). \tag{8}$$

We note that this measure satisfies $m(\Omega) = 0$ and is increasing, that is, $m(\eta) \geq m(\pi)$ for arbitrary $\eta \geq \pi$ in $\Omega$, because of the condition (5) . Let $P$, $N$ be subsets of $\Omega$ such that

$$\begin{aligned} P &= \{\omega \in \Omega : m(\omega) \geq 0\}, && (9) \\ N &= \{\omega \in \Omega : m(\omega) < 0\}. && (10) \end{aligned}$$

We know that

$$m(P) = -m(N). \tag{11}$$

We define probability measures $m_+$, $m_-$ as follows:

$$\begin{aligned} m_+(\omega) &= \frac{1}{M}\{m(\omega) \vee 0\}, \quad \omega \in \Omega, && (12) \\ m_-(\omega) &= \frac{1}{M}\{-m(\omega) \vee 0\}, \quad \omega \in \Omega, && (13) \end{aligned}$$

where $M = m(P) = -m(N)$. We remark that $m_+$ is increasing, while $m_-$ is decreasing, that is, $m_-(\eta) \leq m_-(\pi)$ for arbitrary $\eta \geq \pi$ in $\Omega$. We also note that $m_+(\omega) > 0$ when $\omega \in P$, and $m_-(\omega) > 0$ when $\omega \in N$. To prove Theorem 2.1, it will be enough to show the following proposition.

**Proposition 3.1** Let $m_+$ be an increasing probability measure on $(\Omega, \mathcal{F})$, and $m_-$ be a decreasing probability measure on $(\Omega, \mathcal{F})$. If $m_+$ and $m_-$ satisfy the lattice condition

$$m_+(\omega_1 \vee \omega_2) m_-(\omega_1 \wedge \omega_2) \geq m_-(\omega_1) m_+(\omega_2), \tag{14}$$

then we have

$$m_-(X) \leq m_+(X) \text{ for increasing functions } X : \Omega \to \mathbf{R}. \tag{15}$$

To prove Proposition 3.1, we prepare probability measures as follows:

$$\begin{aligned} m_+^\epsilon &= \frac{1}{M + |\Omega|\epsilon}\{m_+(\omega) + \epsilon\}, \quad \omega \in \Omega, && (16) \\ m_-^\epsilon &= \frac{1}{M + |\Omega|\epsilon}\{m_-(\omega) + \epsilon\}, \quad \omega \in \Omega, && (17) \end{aligned}$$

where $\epsilon$ is a positive real. We note that $m_+^\epsilon$ and $m_-^\epsilon$ are strictly positive probability measures, and satisfy the following proposition.

**Proposition 3.2** The pair $m_+^\epsilon$ and $m_-^\epsilon$ satisfy the lattice condition

$$m_+^\epsilon(\omega_1 \vee \omega_2)m_-^\epsilon(\omega_1 \wedge \omega_2) \geq m_-^\epsilon(\omega_1)m_+^\epsilon(\omega_2), \quad \omega_1, \omega_2 \in \Omega. \tag{18}$$

*Proof*: Let $\omega_1, \omega_2 \in \Omega$. It is sufficient to show

$$m_+(\omega_1 \vee \omega_2) - m_+(\omega_2) \geq 0, \;\; m_-(\omega_1 \wedge \omega_2) - m_-(\omega_1) \geq 0. \tag{19}$$

But this is obvious by the monotonicity properties of $m_+$ and $m_-$. □

Then, by Holley's theorem [3], we obtain that

$$m_+^\epsilon(X) \geq m_-^\epsilon(X). \tag{20}$$

So, we obtain the following inequality

$$m_+(X) \geq m_-(X). \tag{21}$$

Proposition 3.1 follows.

*Proof of Theorem 2.1*: Assume that the probability measures $\mu_1(\omega)$, $\mu_2(\omega)$ satisfy inequalities (5) and (6). Then, the probability measure $m_+(\omega)$ is increasing, and $m_-(\omega)$ is decreasing, and they satisfy the lattice condition (14). For an arbitrary increasing function $X$, we obtain

$$m_+(X) \geq m_-(X). \tag{22}$$

So it follows that

$$\mu_2(X) - \mu_1(X) = m(X) \geq M\{m_+(X) - m_-(X)\} \geq 0. \tag{23}$$

□

# 4 Example

In this section, we will provide an example of $\mu_1$ and $\mu_2$, which does not satisfy the lattice condition (1), but it satisfies inequalities (5) and (6). We remark that this example also satisfies the following inequalities ([1] and [2]):

$$\mu_2(\omega^e)\mu_1(\omega_e) \geq \mu_1(\omega^e)\mu_2(\omega_e), \qquad \omega \in \Omega, e \in E, \tag{24}$$

$$\mu_2(\omega^{ef})\mu_1(\omega_{ef}) \geq \mu_1(\omega_f^e)\mu_2(\omega_e^f), \qquad \omega \in \Omega, e, f \in E. \tag{25}$$

Let $\mu_1$ and $\mu_2$ be probability measures on $\Omega = \{0,1\}^3$ defined by

$$\begin{aligned} &\mu_2(111) = \frac{1}{8}, \quad \mu_2(000) = \frac{1}{8} - \epsilon, \\ &\mu_2(110) = \mu_2(101) = \mu_2(011) = \frac{1}{8} + \frac{\epsilon}{3}, \\ &\mu_2(100) = \mu_2(001) = \mu_2(010) = \frac{1}{8}, \end{aligned} \tag{26}$$

and define $\mu_1$ symmetrically by $\mu_1(xyz) = \mu_2([1-x][1-y][1-z])$.

**Claim** : The pair $\mu_1$, $\mu_2$ of strictly positive probability measures as defined above satisfies the inequalities (5) and (6), however, $\mu_1$ and $\mu_2$ do not satisfy the lattice condition (1).

*Proof*: It is easy to check the required properties of $\mu_1$ and $\mu_2$. For example, we take $\omega_1 = (001)$, and $\omega_2 = (110)$, then, we have the following inequality:

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) < \mu_1(\omega_1)\mu_2(\omega_2).$$

$\square$

# References


[1] G. Grimmett. The Random-Cluster Model, Springer, 2006.

[2] G. Grimmett. Private communication.

[3] R. Holley. Remarks on the FKG Inequalities, Commun. math. Phys. 36, 227-231 (1974).

[4] C. J. Preston. A Generalization of the FKG Inequalities, Commun. math. Phys. 36. 233-241 (1974).