GROUND STATES AND CRITICAL POINTS FOR AUBRY-MATHER THEORY IN STATISTICAL MECHANICS

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ABSTRACT. We consider statistical mechanics systems that are defined on a set with some symmetry properties. Each of the sites has a real order parameter. We assume that the interaction is ferromagnetic as well as symmetric with respect to the action of a group which is assumed to be finitely generated and residually finite.

Given any cocycle of the symmetry group, we prove that there are ground states which satisfy an order condition (known as Birkhoff property) and that are at a bounded distance to the cocycle.

Such ground states are organized into a (possibly singular) lamination.

Furthermore, we show that, given any completely irrational cocycle, either the above lamination consists of a foliation made of a continuous one parameter family of ground states, or, inside any gap of the lamination, there is a well-ordered critical point which is not a ground state.

The classical multidimensional Frenkel-Kontorova model with periodic coefficients is included in this framework as a particular case. In case that the substrate is one-dimensional, these are the beginning results of Aubry-Mather theory, but the present framework applies to systems such as the Bethe lattice and it allows to discuss interactions which are not nearest neighbor.

1. INTRODUCTION

We consider a discrete system of sites. Each of the sites is described by a real valued order parameter. The sites interact among each other. The interaction is, as standard in Statistical Mechanics [Rue99] described by family of energies depending on finite sets of variables. We will be concerned with the existence and qualitative properties of minimizers (ground states) and critical points of (metastable states) of the interaction. In particular, we will look at the interplay between the symmetry and the ferromagnetic properties of the system (the interaction favors sites which align).

The arguments we present depend on very few properties of the problem and, therefore, we have formulated them in a general framework with encompasses many models which have been considered in the literature.

Somewhat informally, the theorems that we will prove are instances of a meta-theorem that asserts that that given any scalar variational problem which

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- Is invariant under translations.
- Is invariant under addition of one integer to the unknown.
- Satisfies some inequalities under comparison.
- Satisfies some mild regularity and compactness properties.

Then

(i) For every "frequency" there are minimizers (ground states) with this frequency. These ground states satisfy some order properties.
(ii) If we fix the frequency, then, either there is a continuum of ground states or there is an equilibrium solution that is not a minimizer.

Rigorous statements will be provided in Theorems 8 and 10.
The arguments we present to prove the results are general and rely only on the ferromagnetic and symmetry properties. To prove the results in (i), we just rely on rearrangements (using an argument developed in [CdlL01]) and to prove the alternative in (ii) we use gradient flows relying on an argument patterned in [dlLV05].
To emphasize the generality of the arguments, we have chosen to follow [CdlL98] and present our results in a rather general set-up. The model that we deal with will be described more precisely in Section 1.1, but we anticipate here a somehow informal exposition. A detailed exposition of the methods of this paper applied to a concrete model is in [dlLV06b].
The *crystal* Λ will be just a countable set endowed with some group action satisfying some invariance properties. This includes as particular case the situation when the crystal is the lattice $\mathbb{Z}^d$ and the group acting on it is just the group of translations. It also includes other models such as the so called *Bethe lattice* (see [Rue99]) and models of layered materials. The generality of the models not only allows us to deal with interesting physical models but also emphasizes what is the heart of the argument. The study of groups acting on graphs and the interaction of the groups with the geometry is very active in the mathematical literature [Ser77, DD89, BL01].
We will assume that each site of the crystal has a one-dimensional order parameter, so that a configuration is a mapping from the crystal to the reals. The fact that the order parameters are one-dimensional will allow us to order them and use rearrangements.
We will assign some energy to the configurations. This is just a formal sum of the energies associated to the interactions of several sites. We do not need to assume that the interaction is pairwise, and, in fact, we may deal with many body interactions.
We will assume that such an interaction is invariant under the translations of the crystal, since this is quite natural for models in statistical mechanics (see, e.g., [Rue99]).
We will furthermore assume that the interaction is invariant under the addition of an integer to all the sites.
Finally, we will make some assumption that indicates that the configurations lower their energy by aligning themselves (i.e., that the crystal has a "ferromagnetic" behavior).
We note that many models that have been studied before are included in our framework as particular cases. A list of concrete examples will be provided in Section 1.4.

We recall that the case when the crystals are lattices as $\mathbb{Z}^d$ is a model for spin waves and for interactions among oscillators (the order parameter represents, in this case, the relative phase of the oscillators).
A particularly interesting case is also given by the Frenkel-Kontorova model, which is used to describe the dislocation or, when $d = 1$, the deposition of a material on a substratum (see, e.g., [BK04]). Similar models appear in phylotaxis (see [Gol92]).
The case $d = 1$ has also been considered extensively in dynamical systems. The equilibrium solutions for models of the above type when the interactions are nearest neighbors can be interpreted as orbits of twist maps of the annulus.
In this context, the existence of well-ordered orbits was the beginning of the Aubry-Mather theory because of [ALD83, Mat82]. Our Theorem 8 is a generalization of the results in [ALD83, Mat82] and our Theorem 10 is a generalization of the main results in [Mat86]. Note that the case when there is a continuous family of minimizers in $d = 1$ corresponds to the existence of invariant circles in twist mappings. Our criterion for the non-existence of invariant circles in terms of the renormalized energy is an analogue of the criterion of [Mat86] of the existence of invariant circles if and only if the Peierls-Nabarro barrier vanishes.
The results in [ALD83, Mat82, Mat84, Mat86] have been generalized in many directions. Among them, we mention that systems with $d = 1$, but many body interactions were considered in [Ang90]. Analogues of [ALD83, Mat82] for lattices in $\mathbb{Z}^d$ were obtained in [Bla89, KdlLR97, CdlL98]. Many numerical conjectures were considered in [Val86].

1.1. **The models considered in this paper.** We now describe the model we are interested in. In Section 1.4, we will see how this abstract framework includes as particular cases several models that have been considered in the literature. Notably, Frenkel-Kontorova models on $\mathbb{Z}^d$ and, for $d = 1$ the monotone recurrences of [Ang90] and the twist mappings.
We consider a discrete set Λ and a potential H_B, associated to the finite subsets B of Λ, which describes the interaction between the particles lying in Λ. We assume that Λ admits an action by a group G, satisfying some mild assumptions, explicitly stated below. We also assume that H_B is invariant under the action of G and that moreover, it satisfies a "ferromagnetic" condition (called "twist" in the dynamical systems model). This ferromagnetic condition means that the interaction favors sets to align. A detailed list of the assumptions made in this paper in is given in what follows. We just note that the standard models in Statistical Mechanics of lattice gases [Rue99, Isr79, Sim93] correspond to taking $\Lambda = \mathbb{Z}^d$, $G = \mathbb{Z}^d$.
We consider the formal sum

$$S(u) = \sum_{B \subset \Lambda} H_B(u), \tag{1}$$

where $u : \Lambda \to \mathbb{R}$ is called a *configuration* on Λ.
We construct configurations u on Λ which are minimal or stationary points for S, which are "well-ordered" and which have a "prescribed rotation number". In our situation,

the ordering of the solution is related to the so called Birkhoff property in the Aubry-Mather theory and the rotation number is a cocycle for the group G (see below for definitions).

Let us now start to introduce our mathematical framework and to formally state our assumptions.

First of all, we recall that a cocycle on G is a map

$$\sigma : G \to \mathbb{R}$$

such that

$$\sigma(\gamma\gamma') = \sigma(\gamma) + \sigma(\gamma') \,. \tag{2}$$

We say that a cocycle σ is *rational* if there is a normal finite index subgroup H of G in such a way $\sigma(H) \subset \mathbb{Z}$. A cocycle which is not rational is said to be *irrational*.

We now give a list of the hypotheses on the group G that we assume. First, we assume that the group G acts on the countable set Λ. If $p \in \Lambda$ and $\gamma \in G$, we denote the action of γ on p by γp. We assume that:

- **G1:** The group G is *finitely generated*.
- **G2:** There is a *finite fundamental domain* for the action, i.e., a finite subset F of Λ which intersects each orbit of G in exactly one point.
- **G3:** The group G acts on Λ *without nontrivial stabilizers*, i.e.: if $\gamma \in G$ is such that there exists $p \in \Lambda$ in such a way that $\gamma p = p$, then $\gamma = 0$.
- **G4:** The group G is *residually finite*, i.e., for each element of G, other than the identity, there is a normal finite index subgroup of G which does not contain it.

Of course, up to identifying a representation class of Λ/G with the corresponding element of F, one can identify Λ/G with F: more explicitly, given $p \in \Lambda$, if, according to **G2**, f is the unique element of f for which $f = \gamma p$ for some $\gamma \in G$, we identify f with $\{gp \,,\ g \in G\} \in \Lambda/G$.

Condition **G4** implies that there is a sequence of nested normal finite index subgroups H_i of G whose fundamental domains exhaust Λ (see Proposition 2 of [CdlL98] for further details on this).

Though perhaps not optimal, these assumptions on G are quite mild, as discussed in [CdlL98].

In the light of **G1**, we may say that G is generated by some elements $g_1^\star, \ldots, g_d^\star$. Then, given a cocycle σ, we define its norm by

$$\|\sigma\| \;:=\; \sup_{1\le k\le d} |\sigma(g_k^\star)| \,. \tag{3}$$

We now give a formal definition of the interaction potential in (1). Let $\mathcal{S}$ denote a collection of nonempty, *finite* subsets of Λ. An interaction potential is a collection of maps

$$H \;=\; \{H_B \,, \quad B \in \mathcal{S}\} \,,$$

where each H_B is a real valued function from the configurations u of Λ satisfying the following properties:

H1: If
$$u(p) = v(p) \text{ for any } p \in B$$
then
$$H_B(u) \; = \; H_B(v)\,.$$
H2: For any finite subset X of Λ, the series
$$H(u,X) \; := \sum_{B\cap X\neq\emptyset} H_B(u) \tag{4}$$
converges.

Notice that, by **H1**, we may regard any function H_B as acting on the space of sequences over the indexes of B
$$\mathbb{R}^B \; = \; \{u(p) \quad | \quad p \in B\}\,.$$
Then, with a slight abuse of notation (which has become standard in the literature), we will sometimes write $u_p := u(p)$, for $p \in \Lambda$ and take derivatives of H_B with respect to u_p (indeed, H_B will be assumed to have at least two derivatives, see **H8** below). We expect that no confusion arises between the p-index in u_p and indices of different types that we use in the paper (such as, sequences of configurations, sometimes denoted by u_n, or configurations associated to cocycles, denoted by u_σ, and the like). Also, by possibly neglecting the sets B in $\mathcal{S}$ for which $H_B \equiv 0$, without loss of generality we will suppose that $H_B \not\equiv 0$ if $B \in \mathcal{S}$.
The assumption in **H2** is clearly satisfied if the interaction is finite range. In fact, the assumption in **H2** is an expression that the interaction decreases fast enough with the range of the interaction. Such assumptions are very common in Statistical Mechanics (see [Rue99]). Later in Assumption **H8** we will make some other assumptions on the convergence of derivatives of the interaction with respect to the points as well as some requirements that the convergence is uniform in some sets of interactions.

We now list some further assumptions required on the interaction potential. For this, we introduce some auxiliary notation about a given configuration u.
First, observe that the action of G on Λ extends naturally to the space of configurations. Namely, if $\gamma \in G$, we define
$$\mathcal{T}_\gamma u(p) \; := \; u(\gamma(p))\,,$$
for any $p \in \Lambda$. Given $\ell \in \mathbb{R}$, we also set
$$\mathcal{R}_\ell u \; := \; u + \ell\,.$$
With this notation, we assume:

H3: The potential $H = \{H_B\}$ is G-*invariant*, i.e. if $B \in \mathcal{S}$, then $\gamma B \in \mathcal{S}$ for any $\gamma \in G$ and
$$H_{\gamma B}(u) \; = \; H_B(\mathcal{T}_\gamma u)$$
for all $\gamma \in G$, all B and all configurations u on Λ.

H4: H has *periodic phase*, i.e.,
$$H_B(\mathcal{R}_s u) = H_B(u)$$
for any configuration u, any $B \in \mathcal{S}$ and any $s \in \mathbb{Z}$.

H5: The action of the group on the potential is *nontrivial*, i.e., we suppose that if $g \in G$ is such that $gB = B$ for some $B \in \mathcal{S}$, then g is the identity.

We also suppose that

H6: H is *coercive*, i.e., we suppose that $H_B(u) \geq 0$ for any $B \in \mathcal{S}$ and
$$H_F(u) \, \geq \, \kappa \max_{i,j\in F} |u_i - u_j|^\theta - K \,,$$
for some κ, K, $\theta > 0$.

We now make an assumption about how the energy behaves with respect to the cocycles of the group.
That is, we suppose that

H7: There exist ϑ, $S > 0$ and $\varsigma : \mathcal{S} \to [0, +\infty)$ in such a way that
$$H_B(\sigma) \, \leq \, \varsigma(B) \left(\|\!|\sigma|\!\|^\vartheta + 1 \right), \tag{5}$$
for any cocycle σ and any $B \in \mathcal{S}$, and
$$\sum_{\substack{B\in\mathcal{S} \\ B\cap X\neq\emptyset}} \varsigma(B) \, \leq \, S \, \sharp X,$$
for any $X \subset \Lambda$.

We recall that the norm $\|\!| \cdot |\!\|$ of a cocycle (which has just has been used in (5) here above) was defined in (3).
We now want to define $\mathcal{O}_\sigma$ to be the space of configurations which are "at a bounded distance" from a given cocycle σ. This concept is formalized in the following way.
First, a cocycle σ defines a configuration u_σ on Λ as follows. Since F is the fundamental domain, for any $p \in \Lambda$ there exists a unique $q \in F$ such that $p = \gamma q$, with $\gamma \in G$. Then, we define
$$u_\sigma(p) = \sigma(\gamma) \,. \tag{6}$$
The above is a bona-fide definition, thanks to **G3** and (2). Also, $u_\sigma(gp) = u_\sigma(p)+\sigma(g)$, that is, u_σ "behaves as a linear function".
Then, if $\kappa, K, S, \theta, \vartheta$ are as in **H6** and **H7**, we set
$$K^\star_\sigma \, := \, \begin{cases} 9 & \text{if } \sharp(\Lambda/G) = 1, \\ 9 + \left(\dfrac{S\,(\|\!|\sigma|\!\|^\vartheta + 1)\,\sharp F + K}{\kappa} \right)^{1/\theta} & \text{if } \sharp(\Lambda/G) \geq 2. \end{cases} \tag{7}$$
Then, given $K \geq 0$, we define $\mathcal{O}^K_\sigma$ to be the space of all configurations u such that
$$\sup_{p\in\Lambda} |u(p) - u_\sigma(p)| \, \leq \, K \,. \tag{8}$$
We also set $\mathcal{O}_\sigma := \mathcal{O}^{K^\star_\sigma}_\sigma$. We will seek our solutions in the space $\mathcal{O}_\sigma$ (Lemma 29 will assure an a-priori bounds for minimizers which confine them into this space).
With the above conventions, we also assume that

H8: For any $K \geq 0$, the interaction potential H is C^2-bounded on $\mathcal{O}_\sigma^K$, that is, for any $B \in \mathcal{S}$, the map H_B is twice differentiable and

$$\begin{aligned} &\sup_{u \in \mathcal{O}_\sigma^K} \sup_{p \in \Lambda} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \left| \frac{\partial H_B}{\partial u_p}(u) \right| \\ &+ \sup_{u \in \mathcal{O}_\sigma^K} \sup_{p \in \Lambda} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \sum_{q \in B} \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(u) \right| \; < \; +\infty \, . \end{aligned} \tag{9}$$

H9: H satisfies the *ferromagnetic* (or *twist*) condition, i.e.

$$\frac{\partial^2 H_B}{\partial u_p \, \partial u_q}(u) \; \leq \; 0 \, ,$$

for any $u \in \mathcal{O}_\sigma^K$, any $K \geq 0$ and any p, q in Λ, with $p \neq q$.

H10: There exist $\delta : \mathcal{S} \to [0, +\infty)$ in such a way that
- $\delta(B) = \delta(gB)$, for any $B \in \mathcal{S}$ and any $g \in G$.
- For any $K \geq 0$ and any finite set $X \subset \Lambda$,

$$\lim_{R \to +\infty} \sup_{u \in \mathcal{O}_\sigma^K} \sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset \\ \delta(B) > R}} H_B(u) \; = \; 0 \, .$$

- Given any $R > 0$ and any finite set $X \subset \Lambda$, there exists a finite set $Y \subset \Lambda$ in such a way that

$$\bigcup_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset \\ \delta(B) \leq R}} B \; \subset \; Y \, . \tag{10}$$

In concrete applications, one may take $\delta(B)$ in **H10** to be diameter of the set B.

Remark 1. The above conditions (namely, **G1**–**G4** on the group G and **H1**–**H10** on the interaction potential H) will be assumed in order to obtain our results and they are, in fact, somewhat weaker than the ones in [CdlL98].

Remark 2. It is important to note that it is not necessary that the group G is the maximal symmetry group of Λ and the interaction.
Notice that if $\tilde{G} \subset G$ is a subgroup and the interaction is invariant under G, then, it is invariant under $\tilde{G}$. On the other hand, it could well happen that the group $\tilde{G}$ has more cocycles than the group G. In particular, if we consider the group generated by the powers of an element, we always have a cocycle.
By considering different subgroups, we can obtain a variety of different ground states associated to the cocycles of the subgroups.

Remark 3. The assumptions in **H2**, **H7**, **H8** and **H10** are satisfied automatically if the interaction has finite range. So, one can consider them as an expression of the fact that the interaction decays fast enough. These assumptions are more or less tailored to the argument that we will develop. We write them at this stage, so that they can be verified for models of interest. In contrast the assumptions statistical mechanics in [Rue99] which involve mostly the size of the interaction, our assumptions involve also the size of derivatives and of second derivatives. This is natural, since we consider the existence of critical points (the equations for critical points involve derivatives of the interaction) and their minimization properties (which is natural involve the second derivative).

Remark 4. We note that the coercivity assumption **H6** could be somewhat weakened. Some uniformity of growth in the distance is needed. The choice of powers is done to make the other assumptions on decay explicit. Weaker conditions on the coercivity would require stronger assumptions on the decay.

Remark 5. Note that we have formulated many of the assumptions in terms of the interaction functions H_B. Note however that there are several choices of interaction functions that lead to the same interaction.
That is, different functions H_B, $\tilde{H}_B$ in such a way that the energy functional is

$$\sum_B H_B(u) = \sum_B \tilde{H}_B(u)$$

for all u, at least formally. One such a way of doing this would be, e.g. to add terms to the 2-body interactions and to cancel them by adding carefully chosen terms to the 3-body and 4-body interactions.
In the same vein, we mention that it is also customary to add terms to the interactions – exact differentials – which keep the same critical points but that change the minimizers. The procedure of changing the interactions but keeping the minimizers or critical points is one of the main tools of the renormalization group techniques. We will not consider those techniques here.

1.2. **An order relation and the Birkhoff property.** To state our main theorem, we need now some further notation concerning the space of configurations. First, we endow the space of configurations with a *partial ordering* by saying that two configurations u and v are such that

$$u \le v$$

if

$$u(p) \le v(p)$$

for any p in Λ. An analogous definition holds for "$<$", "$>$" and "$\geq$" replacing "$\leq$".
Then, generalizing a standard terminology in Aubry-Mather theory,

Definition 6. *Given a cocycle σ of G, we say that a configuration $u \in \mathcal{O}_\sigma$ satisfies the* Birkhoff property *if the following holds: if $\gamma \in G$ and $s \in \mathbb{Z}$ are so that $\sigma(\gamma) \geq s$ (resp., $\sigma(\gamma) \leq s$), then $\mathcal{T}_\gamma u \geq \mathcal{R}_s u$ (resp., $\mathcal{T}_\gamma u \leq \mathcal{R}_s u$). The set of Birkhoff configurations in $\mathcal{O}_\sigma$ will be denoted by $\mathcal{B}_\sigma$.*

Definition 7. *We say that a configuration u is a* solution of our variational equation *(or, in the calculus of variations terminology, a* critical point*) if, for any $p \in \Lambda$,*

$$\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u) = 0 . \tag{11}$$

We also say that u is a ground state *(or, in the terminology of* [Mor24], *a* class-A minimizer*) if*

$$H(u, X) \leq H(v, X)$$

for any finite subset X of Λ and any configuration v such that $u = v$ on $\Lambda - X$ (recall the definition of $H(\cdot, X)$ given in (4)).

Note that the sum in (11) is absolutely convergent, thanks to **H8**.

1.3. **Statement of the results.** The first result we prove here deals with the existence of ground states lying at a bounded distance from any given cocycle:

Theorem 8. *Assume* **G1**–**G4** *and* **H1**–**H10**. *Then, given any cocycle σ, there exists a ground state u of the interaction S such that*

$$|u(p) - u_\sigma(p)| < K^\star_\sigma\,. \tag{12}$$

Moreover, $u \in \mathcal{B}_\sigma$.

We recall that a lamination is a closed family of manifolds – called the leaves of the lamination – that do not intersect.

Definition 9. *We say that a collection of functions is a* possibly singular lamination *when a) the collection is closed b) all the functions are comparable.*
In case that collection is such functions are such $u \geq v$ and $u \not\equiv v$ imply $u_i > v_i \quad \forall i \in \Lambda$, we say that the collection of functions is a lamination

We note that the Birkhoff property of the minimizers produced in Theorem 8 and their translations are all comparable. We see, therefore that the existence of a minimizer implies the existence of a family of comparable functions all of which are minimizers. Using that the pointwise limit of minimizer is also a minimizer, we obtain that, without loss of generality, we can assume that the set of comparable minimizers is closed. Hence the set of translations of the minimizers in Theorem 8 constitute a (possibly singular) lamination. Whether they are a lamination or not depends on whether the models are such that the critical points satisfy a strong maximum principle. The maximum principle, even if interesting on its own right, does not play any role in this paper and we will not discuss it. The papers [CdlL98, dlLV06b] present some conditions that ensure that the models satisfy the maximum principle.

For rational cocycles the graphs of a ground ground state and their translation is discrete set of leaves in $F \times \mathbb{T}^1$. On the other hand, for the irrational case, the leaves can accumulate in complicated ways (see, for instance, [Ban89] for related examples for PDE's). Further discussions about the lamination structures will be provided in Section 3.

In the irrational case, we prove a multiplicity result, which states that either the above lamination has no gaps (i.e., it is a foliation), or one can find solutions of our problem inside any gap:

Theorem 10. *Assume* **G1**–**G4** *and* **H1**–**H10**. *Let σ be a cocycle of G. Let*

$$H := \{g \in G \mid \sigma(g) \in \mathbb{Z}\}\,.$$

Suppose that H contains only the identity. Let $u^{(0)}$, $u^{(1)} \in \mathcal{B}_\sigma$. Assume that the graphs of $u^{(0)}$ and $u^{(1)}$ belong to the (possibly singular) lamination of ground states induced *by Theorem 8. Suppose that* $u^{(1)} \geq u^{(0)}$ *and* $u^{(1)} \not\equiv u^{(0)}$. *Then, there exists* $u \in \mathcal{B}_\sigma - \{u^{(0)}, u^{(1)}\}$ *such that*

$$u^{(1)} \geq u \geq u^{(0)}$$

and which is a solution of the variational equation.

We remark that the hypothesis that H contains only the identity (which we christen by saying that "σ is *completely irrational*") deals somehow with the typical case. For instance, when cocycles $G = \mathbb{Z}^n$ and the cocycles are are just linear functions, such an assumption becomes that the components of the cocycle are rationally independent.

The assumption that the cocycle is irrational is removed in [dlLV06a]. Nevertheless, the paper [dlLV06a] needs the hypothesis that the interaction are finite range (or, in fact, a generalization of it), which is not present in this paper.

Theorem 10 may be seen as a generalization of some results in Aubry-Mather theory (see, in particular, [Mat86]) and it fits into the theory of plane-like (quasi)periodic structures, the recent literature of which is quite rich (see, for instance, [Mos86, Ban88, CdlL01, Val01, RS03, Val04, Tor04, RS04a, RS04b, RS04c, CdlL05, PV05a, PV05b, Bes05]). With respect to the above mentioned theory, the notion of "plane" is replaced here by the more general concept of "level set of a cocycle".

The proof of Theorem 10 is reminiscent of the one in [dlLV05], in which an elliptic PDE problem is considered. As a matter of fact, the proof is somewhat simpler here, since the discrete setting avoids some smoothness and compactness issues that complicate the situation in [dlLV05]. On the other hand, the case considered here becomes conceptually more difficult than the one in [dlLV05] since we allow also long-range interaction and the problems considered in this paper have a non-local nature, a complication which is not present in the PDE case considered in [dlLV05].

1.4. **Some concrete examples of the models considered.** The abstract formulation developed in Section 1.1 makes the proofs more conceptual.

Of course, it makes the results applicable to many models that have been proposed in the literature. In this Section, we will show how very different models fit simultaneously in the framework developed. The fact that one can obtain results for different models at the same time is, in our opinion, part of the appeal of the generality of the methods we present here.

1.4.1. *The Frenkel-Kontorova model/Twist mappings.* This is the most basic model, but which has been the basis of much of the theory.

In this case, we consider $\Lambda = \mathbb{Z}$ and $G = \mathbb{Z}$, acting on Λ by translations.

The variational principle of the Frenkel-Kontorova model is given by

$$S(u) = \sum_i \frac{1}{2}(u_{i+1} - u_i - a)^2 + V(u_i)\,, \tag{13}$$

with $V(x+1) = V(x)$. The energy functional in (13) provides a model for a system with connecting springs of length $a \in \mathbb{R}$ under a periodic substrate potential V. The particles of the system are subject to mutual interactions, given by the term $\frac{1}{2}(u_{i+1} - u_i - a)^2$, as well as interactions with a substratum of period 1, given by V.

The variational principle of (13) corresponds to taking

$$H_B(u) \;=\; \begin{cases} V(u_i) & \text{if } B = \{i\}, \\ \frac{1}{2}(u_{i+1} - u_i - a)^2 & \text{if } B = \{i, i+1\}, \\ 0 & \text{otherwise} \end{cases}$$

in the general formalism.
We note that the equilibrium equations for (13) are

$$u_{i+1} + u_{i-1} - 2u_i - V'(u_i) = 0\,. \tag{14}$$

The assumption that $H_B \geq 0$ in **H6** may thus be obtained by replacing V with $V - \inf V$, which does not change (14).
Equation (14) can be considered as a dynamical system in $\mathbb{R}^2$, since if we know the pair (u_{i-1}, u_i), we can determine the pair (u_i, u_{i+1}).
The particular case of (14) when $V(x) = \lambda(\cos(2\pi x) + 1)$ is called the standard map (see, e.g., [Chi79]).
More generally, we can consider action principles of the form

$$S(u) = \sum_i S(u_i, u_{i+1})\,. \tag{15}$$

The equilibrium equations for action principles of the form (15) are

$$S_1(u_i, u_{i+1}) + S_2(u_{i-1}, u_i) = 0\,, \tag{16}$$

where we use sub-indices to denote derivatives.
Under the assumption that

$$S_{12} \leq -\delta < 0, \tag{17}$$

we see that if we fix u_i and u_{i-1}, then equation (16) determines uniquely u_{i+1} hence, it can be considered as a dynamical system on $\mathbb{R}^2$. The system is area preserving, and, indeed, the variational principle (16) is very common in the presentation of area preserving maps. The function S is often called the generating function.
The assumption in (17) corresponds to the assumption in **H9** of ferromagnetism in the set-up in Section 1.1.
Under the assumption that $S(x+1, y+1) = S(x, y)$, the iteration in (16) can be considered as dynamical systems on the annulus. This correspond to property **H4** in our set-up.
The other properties of our abstract set-up are straightforward to check in this concrete case, in which the cocycles $\sigma : \mathbb{Z} \to \mathbb{R}$ are again the linear functions.
As for the subgroups H_i (namely, the ones whose fundamental domains exhaust Λ, thanks to **G4**), it is natural to take $H_i = F_i\mathbb{Z}$, with F_i an increasing sequence of positive integers. Note that $G/H_i = \{0, 1, \ldots F_i - 1\}$ and $H_{i+1}/H_i = \{0, \ldots, F_{i+1} - F_i\}$. The rational cocycles are the rational numbers with denominators F_i.
The notion of Birkhoff orbits in this case agrees with the usual meaning in dynamical systems, that is, that the projection of the orbit is semi-conjugate to a rotation of the circle.

Then, applying Theorem 8 to this case, we obtain the fundamental theorem of existence of quasi-periodic orbits with any rotation number in [ALD83, Mat82]. Theorem 10, in this particular case, implies the main result in [Mat86].

1.4.2. *Bag models, degenerate twist maps.* One of the interesting features of the present proofs is that we do not need that the twist condition is strict.
In the Physics literature, one can find variants of the Frenkel-Kontorova models in which the interaction vanishes for short distances (these are sometimes called *bag models*) so that the particles can move freely provided that their distance does not increase beyond a threshold.
A model of this type is given by the interaction

$$S(u) = \sum_i F_L(u_i - u_{i+1} - a) + V(u_i) \tag{18}$$

where $V(t+1) = V(t)$ and F_L is a convex function that vanishes when its argument is smaller than L. For example $F_L(t) = (\max(|t| - L, 0))^2$.
Other models are models with anharmonic couplings

$$S(u) = \sum_i |u_i - u_{i+1} - a|^p + V(u_i) \tag{19}$$

where $V(t+1) = V(t)$ and $p > 2$.
Note that these models fail to satisfy the uniform twist condition in open sets. Therefore, the Euler-Lagrange equations for (18) cannot be transformed into a second order map. The Euler-Lagrange equations for (19) can be transformed into a second order map away from singularities, but they are rather singular. So, the examples (18), (19) are examples of models with a Lagrangian description but not a Hamiltonian one.
A very detailed study of the model (19) can be found in [dlLV06b]. The paper [dlLV06b] studies generalizations of (19) to higher dimensional lattices and with interactions which are not nearest neighbor. It also includes results on the maximum principle, which we do not consider here.

1.4.3. *Periodic twist maps, periodic Frenkel-Kontorova models.* We consider modifications of (15) in which the actions depend (periodically) on the index. That is, we consider actions of the form

$$S(u) = \sum_i S(i; u_i, u_{i+1}) \tag{20}$$

The equilibrium equations are

$$S_1(i; u_i, u_{i+1}) + S_2(i-1; u_{i-1}, u_i) = 0$$

Again, under the assumption that $S_{1,2}(i; x, y) \le -\delta < 0$, we see that we can determine u_{i+1} given u_i and u_{i-1}. Hence, this defines a sequence of maps, which are area preserving. Under the assumption $S(i; x+1, y+1) = S(i; x, y)$, these are maps of the annulus.
A particularly interesting case is when the sequence of maps is periodic, that is

$$S(i; x, y) = S(i+N; x, y) .$$

This corresponds to a periodic sequence of twist maps.
We note that the composition of twist maps can fail to be a twist map. The variational theory for compositions of twist maps has been developed in [Mat90, Mat91].
In the context of Frenkel-Kontorova models, we could consider situations in which V and a depend also on i in a periodic fashion. In the Frenkel-Kontorova physical interpretation, the periodic dependence on V corresponds to the substratum on which we deposit the new material being a composite, and the periodic dependence on a corresponds to depositing a periodic material.
These periodic systems can be fit into our set-up by setting $\Lambda = \mathbb{Z}$ and $G = N\mathbb{Z}$, with $N \in \mathbb{N}$ being the period of the medium.
Again, the cocycles $\sigma : \mathbb{Z} \to \mathbb{R}$ are the linear functions and one can take the increasing sequence of subgroups H_i to be $F_i\mathbb{Z}$, with $F_i = Nf_i$ and f_i is a sequence of positive energies tending to $+\infty$.
The notion of Birkhoff functions in this case is related to the homeomorphisms of the circle.

1.4.4. *Monotone recurrences.* The results of these paper apply also to generalizations of the models in Section 1.4.3 for which the interaction is not nearest neighbor, but includes also terms of longer range interaction, provided that they satisfy the twist and the invariance conditions. These models are natural from the physical point of view since in the models of deposition, it is natural to include the possibility that an atom interacts with other models besides the nearest neighbors. Including interactions which are longer range is also natural in the Physical interpretation of spin wave models. From the mathematical point of view, these models appear naturally also in the study of integrable systems (see [Ves91]).
For example, we can consider models of the form

$$S(\{u_n\}_{n\in\mathbb{Z}}) = \sum_{L\in\mathbb{N}} \sum_{k\in\mathbb{Z}} H_L(u_k, \dots, u_{k+L})\,, \tag{21}$$

as long as the interaction terms satisfy $H_L(u_0, \dots, u_L) = H_L(u_0 + 1, \dots, u_L + 1)$ and $\partial_i\partial_j H_L(u_0, \dots, u_L) \le 0$, as well as the regularity assumptions.
One example of systems satisfying the assumptions of this paper is given by the variational principle of the form

$$\begin{aligned} S(\{u_n\}) = \sum_n \frac{1}{2}(u_{n+1} - u_n - a)^2 + \frac{A}{2}(u_{n+2} - u_n - b)^2 \\ + \frac{\epsilon}{4\pi^2}(\cos(2\pi u_n) + 1)\,. \end{aligned} \tag{22}$$

The equilibrium equations corresponding to (22) are:

$$(u_{n+1} - 2u_n + u_{n-1}) + A \cdot (u_{n+2} - 2u_n + u_{n-2}) + \frac{\epsilon}{2\pi}\sin(2\pi u_n) = 0\,. \tag{23}$$

Equation (23) defines a 4-dimensional map for $A \neq 0$.
It is interesting to note that, when A is small, it is natural to think of (23) as a small perturbation of the standard map. Nevertheless, as a dynamical system, it is very different. Even the phase space has different dimensions and there are large coefficients.

It is also interesting to note that the variational methods produce solutions with just one frequency even if the corresponding dynamical system has 2 degrees of freedom.
Models of the type (21) were considered in [Ang90], who called them *monotone recurrences*. The paper [Ang90] used a gradient flow method similar to the one we use in Section 4.2 to construct well-ordered equilibrium solutions. The paper [Ang90] also contained many important ideas on the study of the number of periodic solutions. Our Theorem 8 applied to (21) is Theorem 6.1 in [Ang90].
Recently, [dlL05] developed a KAM theory for quasi-periodic solutions of the equilibria in (21) even in the case that the interaction is infinite range. This is a Lagrangian KAM theory without an straightforward Hamiltonian counterpart. Note that the limit $A \to 0$ is a singular limit in (22).

1.4.5. *Models of spin waves in higher dimensions.* The standard lattice models as in [Rue99] corresponds to taking $\Lambda = \mathbb{Z}^d$ and $G = \mathbb{Z}^d$.
We can think of Λ as describing a standard crystal, which is symmetric under $\mathbb{Z}^d$-translations.
In our case, each site in the crystal has an order parameter characterized by a number. The assumptions on invariance of the interactions under translation are standard and can be found in [Rue99]. The more detailed assumptions about the regularity and the differentiability of the interactions are not considered in [Rue99], which did not take into account the geometric properties of the ground states.
The assumption on invariance under global change of the phases is natural when the order parameter is a phase. For example, when we consider spin waves. In the case of spin waves, it is natural to consider many body interactions as well as interactions that are of unbounded range.
In this case, the cocycles of the group are just the linear functions $\sigma(g) = \sum_{j=1}^d \sigma_j g_j$. The increasing set of subgroups can be taken to be $H_i = (F_i\mathbb{Z})^d$ with F_i an increasing sequence of positive integers going to $+\infty$.
One can also consider materials with some structure, by assuming that the symmetry group of the problem is smaller than the crystal. For example, one can take $G = \prod_{j=1}^d (N_j\mathbb{Z})$. In this case, there is a fundamental cell

$$\{0, \dots, N_1 - 1\} \times \{0, \dots, N_2 - 1\} \times \cdots \times \{0, \dots, N_d - 1\}$$

and the properties of this cell keep on repeating under translations.
As increasing subgroups H_i, it is possible to take $H_i = (F_i^1 N_1\mathbb{Z}) \times \cdots \times (F_i^d N_d\mathbb{Z})$, where the F_i^j are positive integers and, for each fixed j, the sequences F_i^j are increasing in i and tend to $+\infty$. The cocycles are linear functions in this case too.
The existence of quasi-periodic solutions in problems of this kind was shown in [Bla89, Bla90, KdlLR97, CdlL98]. In [dlLV06b], one can find a discussion of models of this type with interactions which are not just nearest neighbor.

1.4.6. *Bethe lattice, Cayley graphs.* The Bethe lattice is a cycle free graph in which every site is connected to k neighbors (see, e.g., [Bet35, Geo88]).
This set has been used in many studies in statistical mechanics and other areas of probability (e.g. percolation, Anderson localization, etc.), since it is a rather homogeneous

set which has a hierarchical structure. Hence, it is very suitable for the study of renormalization groups. Since the balls grow exponentially fast, it is customary to consider it heuristically as an analogue of a Euclidean lattice of infinite dimension. Mathematically, it is also natural to consider it as an analogue of a space with negative curvature. The geometric study of groups acting on graphs is a very active mathematical subject (see [BL01] for a recent survey). We are not interested only on the geometry of the space but also on interacting statistical mechanical models described in this setting. A good introduction to the properties of groups is [LS01]. In particular, Chapter 1 contains properties of the free group.
The group F_ℓ acts on the graphs by right translation. One way to construct invariant interactions is to use a generalization of the Cayley construction, – which applies to any group – which we now outline.
Given a finite set $B \subset F$ we consider a function $H_B : \mathbb{R}^B \to \mathbb{R}$ which satisfy the coerciveness **H3** and the periodic phase **H4** and then, we construct $H_{\gamma B}$ in such a way that **H3** is satisfied. We can repeat the construction a finite number of times and add the resulting functions. Then, the conditions **H8-H10** are satisfied. Of course, if the functions H_B as well as their derivatives decrease fast enough with the size of the set B, it is possible to arrange that the conditions **H8-H10** are satisfied. The standard Bethe lattice model corresponds to taking $H_B = 0$ if $\mathrm{diam} B > 1$ or if $\#B \geq 2$, so that we only consider sets of the form $B = \{i, gi\}$ where g is a generator. or $B = \{i\}$ and we set $H_{\{i,gi\}} = |u_i - u_{gi} - a|^2$, $H_{\{i\}}(u) = V(u_i)$ with $V(t+1) = V(t)$.
The free group is residually finite because it can be considered as a subset of $SL(k, \mathbb{R})$. and $SL(k, \mathbb{R})$ as some well known explicit normal subgroups of finite index which exhaust, namely the *congruence subgroups* (see [Weh73] or Theorem 6.1.9 of [Rob82] for details).
It is easy to see that, for any choice of $(\sigma_1, \ldots, \sigma_\ell) \in \mathbb{R}^k$, we can obtain a cocycle on the free group by setting

$$\sigma(g) = \sum_{j=1}^{k} \sigma_j \#_j(g)\,, \tag{24}$$

where $\#_j(g)$ is the number of times that the generator a_j appears in g minus the time that the generator a_j^{-1} appears.
Even if we will not make any use of this, we point out that the cocycles in (24) are the only cocycles of the free group. This follows from the fact that the abelianization of the free group is $\mathbb{N}^k$ and that any cocycle has to factor through the abelianization (see [Weh73, LS01]).

1.4.7. *Layered materials.* Some models of layered materials can be fit in our framework. To describe L layers of d-dimensional materials, it suffices to take $\Lambda = \{1, \ldots, L\} \times \mathbb{Z}^d$. There are several possibilities for the group G.
If we require that the layers are independent, then it is natural to take $G = \mathbb{Z}^d$ acting by translations in such a way that all the layers are moved simultaneously, in such a way that

$$g(\alpha, k) = (\alpha, k + g)\,.$$

Another, more symmetric possibility, is to take $G = \left(\mathbb{Z}^d\right)^L$ acting by

$$(g_1, \ldots, g_L)(\alpha, k) = (\alpha, k + g_\alpha)\,.$$

In the first case, the interactions are homogeneous in each layer, but the interactions among the layers are independent only when we translate all the layers.

In the second case, the interactions are invariant when we translate each of the layers independently.

The cocycles and the subgroups in this case are very similar to those in the previous cases. Indeed, the subgroups are integer multiples of the lattice and the cocycles are linear functions.

Another interesting variant is to include permutations among the layers (natural for materials composed of alternate layers of different elements) so that in this case, $G = \Pi \times \mathbb{Z}^d$, where Π is a subgroup of the permutations on L elements.

2. Proof of Theorem 8

The proof of Theorem 8 borrows several ideas from the proof of the results in [CdlL01] for minimal surfaces and it is based on the construction of a particular minimizer which enjoys further symmetry properties.

It also makes use of the following result, which can be found in Proposition 1 of [CdlL98].

Proposition 11. *Let G be a residually finite group. Then, given any cocycle σ we can find a sequence $\sigma^{(n)}$ of rational cocycles such that*

$$\lim_{n\to+\infty} \|\sigma^{(n)} - \sigma\| = 0\,.$$

We now introduce the following equivalence relation on subsets of Λ, induced by a given normal finite index subgroup H of G. Given $A, B \subset \Lambda$, we say that $A \sim_H B$ if there exists $h \in H$ in such a way that $A = hB$. The equivalence class of a set B with respect to this relation will be denoted by $[B]_H$ (no confusion should arise with the interaction potential H_B). We also set $\mathcal{S}_H := \{\ [B]_H \text{ with } B \in \mathcal{S}\}$ and

$$\mathcal{C}_{\sigma,H} \;=\; \{u \in \mathcal{O}_\sigma \mid \mathcal{T}_g u = \mathcal{R}_{\sigma(g)} u,\ \text{for any } g \in H\}\,. \tag{25}$$

We now take σ to be a *rational* cocycle (recall the definition given on page 4). Accordingly, we let $H \subset G$ be a normal subgroup of finite index such that $\sigma(H) \subset \mathbb{Z}$.

In this framework, $\mathcal{C}_{\sigma,H}$ is the set of "periodic" configurations with "slope" σ. Clearly, each of the configurations on $\mathcal{C}_{\sigma,H}$ is determined by the values on Λ/H, which is a finite set, since H is finite index.

For any $[B]_H \in \mathcal{S}_H$ and any $u \in \mathcal{C}_{\sigma,H}$, we also define

$$H_{[B]_H}(u) \;:=\; H_B(u)\,.$$

We remark that this is a bona-fide definition: indeed, if $[A]_H = [B]_H$, then $A = hB$ for some $h \in H$ and so

$$H_A(u) = H_{hB}(u) = H_B(\mathcal{T}_h u) = H_B(\mathcal{R}_{\sigma(h)} u) = H_B(u)\,, \tag{26}$$

thanks to **H3** and **H4**.

The idea of the proof of Theorem 8 goes as follows. First, in Section 2.1, we define a cut-off energy for rational cocycles and we prove that the pointwise minimum of periodic minimizers is still a minimizer. This will allow us to define the concept of infimal minimizer in Section 2.2 and to show in Section 2.4 that it has all the symmetries needed for our purposes.
Then, in Section 2.5, we show that the infimal minimizer, which, in principle, is just a minimizer of the 'periodified, cut-off energy functional, is also a ground state for the cut-off energy. This result is far from trivial since being a minimizer for the periodified energy just means that the energy cannot be lowered by periodic modifications (or with some extra work, by compact support perturbations whose support is contained inside the fundamental domain of the period). It could, in principle happen that the energy of a minimizer of the periodic energy could be lowered by perturbations of much longer period. This phenomenon is very common in variational problems and is sometimes called *symmetry breaking.* Excluding symmetry breaking in our case, makes heavy use of the ferromagnetic property and that the configurations we consider are just real valued. See Proposition 22.
In Section 2.6, we will show that the infimal minimizer is Birkhoff and that it stays at a bounded distance from the cocycle σ. The assumption that σ is rational will be dropped in Section 2.7, while the cut-off will be dismissed in Section 2.8.

2.1. **Cut-off interactions for the rational case.** Roughly speaking, we now want to introduce an energy functional which is adapted to dealing with periodic problems. When the configurations we consider are periodic, it is natural to consider an energy functional adapted to these periodic functions. We will mainly impose that, for periodic functions the critical points of the functional will also be critical points for the full functional.
We will need that the functional does not see "does not see too large sets". The concept of a set B to be unpleasantly large is quantified by the quantity $\delta(B)$ in **H10**. Condition **H10** also tells us that these large sets "do not contribute" too much to the energy.
In order to avoid some technical difficulties, we will start considering cut-off finite range interactions in periodic problems and then, pass to the limit of aperiodic limits and to the problem of long range interactions.
A subtle point is that, in order to apply the direct methods of the calculus of variations, we will first look for *minimizers among periodic configurations* rather than *ground states.* It could happen that the minimizers of the periodified energy are not ground states (they could be unstable under perturbations of longer period, because of symmetry breaking). They key step is excluding the symmetry breaking in Proposition 22. This will which will then allow us to "localize" the perturbations inside a large domain and repeat this domain periodically. We note that the proof of Proposition 22 uses heavily the ferromagnetism property and the fact that our configurations are real valued. We will construct a minimizer, *the infimal minimizer* which enjoys some extra geometric property (Birkhoff property), that – besides intrinsic interest – will allow us to prove that the the infimal minimizer does not present symmetry-breaking. In

fact, all minimizers of the periodified energy – even if they do not satisfy the Birkhoff property — also satisfy a version of the no-symmetry-breaking property.

2.1.1. *The reduced interaction.* In order to formalize the above ideas, we fix

$$R > \delta(F) \tag{27}$$

and we define the *reduced energy functional* on $\mathcal{C}_{\sigma,H}$ by

$$S^R_{\sigma,H}(u) := \sum_{\substack{[B]_H \in \mathcal{S}_H \\ \delta(B) \le R}} H_{[B]_H}(u)\,. \tag{28}$$

Again, this is a good definition, since $\delta(B) = \delta(hB)$ for any $h \in H$, due to **H10**. Furthermore:

Lemma 12. *For any finite set $X \subset \Lambda$ and any $R > 0$,*

$$\sharp\{B \in \mathcal{S},\ B \cap X \neq \emptyset,\ \delta(B) \le R\} < +\infty\,. \tag{29}$$

Moreover, the series in (28) *is just the sum of a finite number of terms.*

Proof. By **H10**,

$$\bigcup_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset \\ \delta(B)\le R}} B = \{p_1, \dots, p_\ell\}\,,$$

for suitable $p_1, \dots, p_\ell \in \Lambda$ and so each B in (29) belongs to the set of all subsets of $\{p_1, \dots, p_\ell\}$. Since the latter has at most 2^ℓ elements, the set $\{B \in \mathcal{S},\ B \cap X \neq \emptyset,\ \delta(B) \le R\}$ (and so, by choosing $X := \Lambda/H$, the sum in (28)) has only a finite number of terms. ☐

Corollary 13. *Suppose that $v^{(n)} \in \mathcal{C}_{\sigma,H}$ for any $n \in \mathbb{N}$ and that*

$$\lim_{n\to+\infty} v^{(n)}_p = v_p\,,$$

for any $p \in \Lambda$.
Then, $v \in \mathcal{C}_{\sigma,H}$,

$$\lim_{n\to+\infty} H_{[B]_H}(v^{(n)}) = H_{[B]_H}(v)\,,$$

for any $[B]_H \in \mathcal{S}_H$ so that $\delta(B) \le R$, and

$$\lim_{n\to+\infty} S^R_{\sigma,H}(v^{(n)}) = S^R_{\sigma,H}(v)\,.$$

Proof. The fact that $v \in \mathcal{C}_{\sigma,H}$ is obvious. Also,

$$\begin{aligned} |H_{[B]_H}(v^{(n)}) - H_{[B]_H}(v)| &\le \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_q}(\tau v^{(n)} + (1-\tau)v) \right| d\tau \cdot |v^{(n)}_p - v_p| \\ &\le C \sum_{p\in B} |v^{(n)}_p - v_p|\,, \end{aligned}$$

for some $C > 0$, thanks to **H8**.

Since B is a finite set, we thus conclude that

$$\lim_{n\to+\infty} H_{[B]_H}(v^{(n)}) = H_{[B]_H}(v)\,.$$

Then the claim follows from Lemma 12. $\square$

We now point out an easy variation of a classical result in quotient group theory. To this extent, we use the standard notation for the equivalence classes induced by a normal subgroup $\tilde{H}$ of finite index:

$$[p]_{\tilde{H}} := \{hp\,,\ \ h \in \tilde{H}\}\,.$$

Lemma 14. *Let $\tilde{H} \subset H$ be normal, finite index subgroups of G.*
Given $p \in \Lambda$, let f_p be the unique element of the fundamental domain Λ/H for which $h_p p = f_p$, for some $h_p \in H$.
We define

$$\begin{array}{rcc} \Lambda/\tilde{H} & \longrightarrow & (\Lambda/H)\times(H/\tilde{H}) \\ {[p]_{\tilde{H}}} & \longmapsto & \Big(f_p, [h_p]_{\tilde{H}}\Big)\,. \end{array} \tag{30}$$

Then, the above map is a bijection.
In particular,

$$\sharp(\Lambda/\tilde{H}) \;=\; \sharp(\Lambda/H)\cdot\sharp(H/\tilde{H})\,.$$

Proof. First of all, we need to show that (30) is a bona fide definition. Indeed, if also $\hat{h}_p p = f_p$ for some $\hat{h}_p \in \tilde{H}$, we have that $\hat{h}_p^{-1} h_p p = p$ and therefore, by **G3**, $\hat{h}_p = h_p$. Furthermore, if $[p]_{\tilde{H}} = [q]_{\tilde{H}}$, we have that $h_p\tilde{h}q = h_p p = f_p$, for some $\tilde{h} \in \tilde{H}$, and so the map in (30) sends $[q]_{\tilde{H}}$ to $(f_p, [h_p\tilde{h}]_{\tilde{H}}) = (f_p, [h_p]_{\tilde{H}})$.
Given $f \in \Lambda/H$ and $h \in H$, we have that the point $h^{-1}f \in \Lambda$ is sent by the map in (30) to $(f, [h]_{\tilde{H}})$, thence such map is onto.
We now show that the map in (30) is injective. For this, suppose that $(f_p, [h_p]_{\tilde{H}}) = (f_q, [h_q]_{\tilde{H}}) \in (\Lambda/H)\times(H/\tilde{H})$ for some p, $q \in \Lambda$. Then, $f_p = f_q$ and $h_q = \tilde{h}h_p$ for some $\tilde{h} \in \tilde{H}$. Thence, $h_p p = f_p = f_q = h_q q = \tilde{h}h_p q$. Then, since $\tilde{H}$ is normal, $h^* := h_p^{-1}\tilde{h}h_p \in \tilde{H}$ and $p = h^* q$. So, $[p]_{\tilde{H}} = [q]_{\tilde{H}}$.
Accordingly, the map in (30) is a bijection. $\square$

Remark 15. The result in Lemma 14 may not hold if one drops condition **G3**. As an example, let $G = H$ be the group of the permutations of three elements $\{1,2,3\}$ (or, equivalently, the dihedral group of order 6, or the symmetric group of order 3, in the notation of [Wik06a, Wik06b]).
Let $a : (1,2,3) \mapsto (2,1,3)$ and $b : (1,2,3) \mapsto (1,3,2)$. Then, G is generated by a and b and it has 6 elements: a, b, ab, ba, aba and the identity id.
The fact that G is finite implies that it is residually finite, because the identity is finite index.
The group G acts on the lattice $\Lambda := \{1,2,3\}$ in the obvious way (for instance, $a(1) = 2$, $a(2) = 1$, $a(3) = 3$) and since any element of Λ is sent to any other one by a suitable permutation, it follows that Λ/G contains only one element.

Let $\tilde{H} := \{\mathrm{id}, ab, ba\}$. By inspection, one sees that $\tilde{H}$ is a normal subgroup of G and that $G/\tilde{H} = \{[\mathrm{id}]_{\tilde{H}}, [a]_{\tilde{H}}\}$.
In particular,

$$\sharp\Big(G/\tilde{H}\Big) = 2\,. \tag{31}$$

Furthermore, since each point of Λ reaches any other via either ab or ba, we have that also $\Lambda/\tilde{H}$ contains only one element.
Thus, in this case,

$$\sharp(\Lambda/H)\cdot\sharp(H/\tilde{H}) = 1\cdot 2 \neq 1 = \sharp(\Lambda/\tilde{H})\,,$$

which shows that Lemma 14 does not hold in this case.

2.1.2. *Existence of minimizers for the reduced energy.* Note that $\mathcal{C}_{\sigma,H}$ is finite dimensional (that is, any configuration $u \in \mathcal{C}_{\sigma,H}$ is determined by its values on Λ/H, which is a finite set) and $S^R_{\sigma,H}$ is bounded from below. We also note that, because of **H4** the problem is invariant under addition of 1 to all the sites. Hence, it suffices to consider $u_0 \in \mathbb{T}^1$. In this case, we see that, because of **H6**, the energy goes to $+\infty$ when the other sites tend to $+\infty$. Hence, we can conclude that $S^R_{\sigma,H}$ reaches a minimum on $\mathcal{C}_{\sigma,H}$.
Of course, the minimizers of $S^R_{\sigma,H}$ in $\mathcal{C}_{\sigma,H}$ can be considered also as configurations in $\mathcal{O}_\sigma$. At this stage of the argument, we are not claiming that these minimizers in $\mathcal{C}_{\sigma,H}$ are minimizers of S in $\mathcal{O}_\sigma$.
Unfortunately, the minimizers of the periodic problem produced by the above compactness argument are not very suitable for further developments. Indeed, at the moment, we do not know whether the minimizers now produced are ground states. It could happen that their energy could be lowered by perturbations which have periods longer than the one now considered.

2.1.3. *The infimal minimizer.* The goal of this section is to show that among all the minimizers in $\mathcal{C}_{\sigma,H}$ there is a special one which is indeed a ground state (after removing the cut-off $\delta(B) \leq R$). Moreover, this minimizer will have special geometric properties that will allow us to take the limit of irrational minimizers.
Note that in this construction we will use heavily the ferromagnetism property **H9** which we have not used so far. The technical result which follows will allow us to rearrange minimizers:

Lemma 16. *Let $u, v \in \mathcal{C}_{\sigma,H}$. Denote*

$$u \wedge v = \max\{u, v\} \quad , \quad u \vee v = \min\{u, v\}\ .$$

Then, $u \wedge v, u \vee v \in \mathcal{C}_{\sigma,H}$ and

$$S^R_{\sigma,H}(u \wedge v) + S^R_{\sigma,H}(u \vee v) \leq S^R_{\sigma,H}(u) + S^R_{\sigma,H}(v)\,. \tag{32}$$

Proof. We write

$$u \wedge v = u + \alpha$$
$$u \vee v = u - \beta$$

where α, β are non-negative configurations.

Since $u \wedge v + u \vee v = u + v$, we have that $v = u + \alpha - \beta$.
We note that $(u \wedge v)_{gi} = (u \wedge v)_i$ and $(u \vee v)_{gi} = (u \vee v)_i$ for any $g \in H$. Therefore, $\alpha_{gi} = \alpha_i$ and $\beta_{gi} = \beta_i$, for any $g \in H$. Hence, $u + t\alpha + s\beta \in \mathcal{C}_{\sigma,H}$, for any $-1 \le t, s \le 1$, that yields the first claim in Lemma 16.
We also note that, by construction, we have

$$\operatorname{supp} \alpha \cap \operatorname{supp} \beta = \emptyset\,. \tag{33}$$

Consider the function

$$F(t,s) \;:=\; S^R_{\sigma,H}(u + t\alpha + s\beta)\,.$$

We have the calculus identity (recall indeed that, by **H8**, F is twice differentiable and Lemma 12 allows to interchange the sum with the derivatives)

$$\begin{aligned} & F(1,0) + F(0,-1) - F(0,0) - F(1,-1) \\ & = \int_0^1 \frac{\partial F}{\partial t}(t,0)\,dt - \int_0^1 \frac{\partial F}{\partial t}(t,-1)\,dt \\ & = \int_0^1 \int_{-1}^0 \frac{\partial^2 F}{\partial t \partial s}(t,s)\,ds\,dt\,. \end{aligned} \tag{34}$$

Now we observe that

$$\frac{\partial^2 F}{\partial t \partial s}(t,s) \;=\; \sum_{\substack{[B]_H \in \mathcal{S} \\ \delta(B) \le R \\ p \in \operatorname{supp} \alpha \\ q \in \operatorname{supp} \beta}} \frac{\partial^2 H_B}{\partial u_p \partial u_q}(u + t\alpha + s\beta)\alpha_p \beta_q\,. \tag{35}$$

Taking into account the ferromagnetic property **H9**, the fact that $\alpha_i, \beta_j \ge 0$ and (33), we obtain that (35) is non-positive. Hence, (34) establishes the second conclusion of Lemma 16. □

Corollary 17. *In the hypothesis of Lemma 16, if u_1 and u_2 are minimizers of $S^R_{\sigma,H}$ on $\mathcal{C}_{\sigma,H}$, so are $u_1 \vee u_2$ and $u_1 \wedge u_2$.*

Proof. We argue by contradiction. Suppose, for instance, that

$$S^R_{\sigma,H}(u_1) = S^R_{\sigma,H}(u_2) = \min_{\mathcal{C}_{\sigma,H}} S^R_{\sigma,H} < S^R_{\sigma,H}(u_1 \wedge u_2)\,.$$

It follows from Lemma 16 that

$$\begin{aligned} \min_{\mathcal{C}_{\sigma,H}} S^R_{\sigma,H} \;&\le\; S^R_{\sigma,H}(u_1 \vee u_2) \\ &\le\; S^R_{\sigma,H}(u_1) + S^R_{\sigma,H}(u_2) - S^R_{\sigma,H}(u_1 \wedge u_2) \\ &=\; 2\min_{\mathcal{C}_{\sigma,H}} S^R_{\sigma,H} - S^R_{\sigma,H}(u_1 \wedge u_2) \\ &<\; \min_{\mathcal{C}_{\sigma,H}} S^R_{\sigma,H}\,, \end{aligned}$$

which is an obvious contradiction. □

2.2. **The infimal minimizer.** Using Corollary 17, we can define a particularly interesting minimizer that will have several extra properties.
We define the infimal minimizer of $S^R_{\sigma,H}$ on $\mathcal{C}_{\sigma,H}$ as

$$u^{*,\sigma,H,R} = \bigvee_{\substack{u \text{ is a minimizer of } S^R_{\sigma,H} \text{ in } \mathcal{C}_{\sigma,H} \\ u_{gi} \geq \sigma(g)\,, \quad \forall\, i \in \Lambda/H\,, \quad g \in G}} u\,. \tag{36}$$

Note that, by **H4**,

$$S^R_{\sigma,H}(\mathcal{R}_\ell u) = S^R_{\sigma,H}(u)$$

and so, given any minimizer u, we have that $\mathcal{R}_\ell u$ is a minimizer too, for any $\ell \in \mathbb{Z}$. Furthermore, if u satisfies the constraint on the right hand side of (36), then

$$(\mathcal{R}_\ell u)_{gi} = u_{gi} + \ell \geq \sigma(g) + \ell \geq \sigma(g)$$

for any $\ell \in \mathbb{N}$ and any $i \in \Lambda/H$. Hence, the infimal minimizer carries information about all minimizers in the right hand side of (36).
The constraint on the right hand side of (36) also makes $u^{*,\sigma,H,R}(p) > -\infty$ for any $p \in \Lambda$.
Moreover, the constraint on the right hand side of (36) does not give rise to the empty set. Indeed, let u be any minimizer of $S^R_{\sigma,H}$ in $\mathcal{C}_{\sigma,H}$ and take $\ell \in \mathbb{N}$ such that

$$\ell \geq \max_{\substack{\gamma \in G/H \\ i \in \Lambda/H}} \left(|u_{\gamma i}| + |\sigma(\gamma)| \right)\,. \tag{37}$$

Note that G/H and Λ/H are finite set by construction, thence the definition in (37) makes sense. Then, $\mathcal{R}_\ell u$ is a minimizer too and, if $i \in \Lambda/H$, $g \in G$, we write $g = h\gamma$, with $h \in H$ and $\gamma \in G/H$ to get that

$$(\mathcal{R}_\ell u)_{gi} = u_{h\gamma i} + \ell = u_{\gamma i} + \sigma(h) + \ell = u_{\gamma i} + \sigma(g) - \sigma(\gamma) + \ell \geq \sigma(g)\,,$$

thence the set in the right hand side of (36) is not empty.
We now point out that the infimal minimizer is indeed a minimizer:

Lemma 18. *$u^{*,\sigma,H,R}$ is a minimizer of $S^R_{\sigma,H}$ in $\mathcal{C}_{\sigma,H}$.*

Proof. For any $q \in \Lambda/H$ let $u^{(q,n)}$ be a sequence of minimizers satisfying the constraints in the right hand side of (36) in such a way that

$$\lim_{n \to +\infty} u^{(q,n)}(q) \;=\; u^{*,\sigma,H,R}(q)\,.$$

Define also

$$u^{(n)}(p) := \bigvee_{q \in \Lambda/H} u^{(q,n)}(p)\,, \tag{38}$$

for any $p \in \Lambda/H$. Note that, since Λ/H is a finite set, the minimization performed on the right hand side of (38) is done among a finite number of candidates, thence $u^{(n)}$ is also a minimizer as in the right hand side of (36), because of Corollary 17.
Then, by (36), $u^{*,\sigma,H,R} \leq u^{(n)}$. On the other hand, by construction,

$$\lim_{n \to +\infty} u^{(n)}(p) \leq \lim_{n \to +\infty} u^{(p,n)}(p) = u^{*,\sigma,H,R}(p)\,,$$

for any $p \in \Lambda/H$.
Consequently,

$$\lim_{n\to+\infty} u^{(n)}(p) = u^{*,\sigma,H,R}(p)$$

for any $p \in \Lambda/H$, and so, by Corollary 13,

$$S^R_{\sigma,H}(u^{*,\sigma,H,R}) = \lim_{n\to+\infty} S^R_{\sigma,H}(u^{(n)}) = S^R_{\sigma,H}(u^{(1)}),$$

since all the $u^{(n)}$ are minimizers. Accordingly, $u^{*,\sigma,H,R}$ is a minimizer of $S^R_{\sigma,H}$ in $\mathcal{C}_{\sigma,H}$. □

2.3. **Some group algebra considerations on $S^R_{\sigma,H}$.** Next results will investigate how "doubling the periodicity induced by the subgroups" reflects into "doubling the energy" (a precise statement will be given in Corollary 20).

Lemma 19. *Let $\tilde H \subset H$ be normal, finite index subgroups of G. Let $\gamma_1, \dots, \gamma_\ell \in H$ in such a way that*

$$H/\tilde H = \Big\{[\gamma_1]_{\tilde H}, \dots, [\gamma_\ell]_{\tilde H}\Big\}.$$

Then, given any $B \in \mathcal{S}$ and any $A \in [B]_H$ there exists a unique $j \in \{1, \dots, \ell\}$ in such a way that there exists $\tilde h \in \tilde H$ such that $\tilde h A = \gamma_j B$. Also, if $[\gamma_i B]_{\tilde H} = [\gamma_j B]_{\tilde H}$, then $i = j$.

Proof. Let $h \in H$ such that $A = hB$ and let γ_j be so that $[h]_{\tilde H} = [\gamma_j]_{\tilde H}$. Then $\gamma_j = \tilde h h$, for some $\tilde h \in \tilde H$. Then, $\tilde h A = \gamma_j B$, as desired.
Suppose now that $\tilde h_j A = \gamma_j B$ with $\tilde h_j \in \tilde H$, for, say, $j = j_1, j_2$. Let $h_* := \gamma_{j_1}^{-1} \tilde h_{j_1} \tilde h_{j_2}^{-1} \gamma_{j_2} \in H$. Then, $h_* B = B$ and so, by **H5**, we see that h_* is the identity. Consequently, $[\gamma_{j_1}]_{\tilde H} = [\gamma_{j_2}]_{\tilde H}$ and so $j_1 = j_2$.
Assume now that $[\gamma_i B]_{\tilde H} = [\gamma_j B]_{\tilde H}$. Then, $B = \gamma_i^{-1} \tilde h \gamma_j B$, for some $\tilde h \in \tilde H$ and then, by **H5**, we have that $[\gamma_i]_{\tilde H} = [\gamma_j]_{\tilde H}$. So $i = j$. □

Corollary 20. *Let $\tilde H \subset H$ be normal, finite index subgroups of G and let $u \in \mathcal{C}_{\sigma,H}$. Then,*

$$S^R_{\sigma,\tilde H}(u) = \sharp\Big(H/\tilde H\Big)\, S^R_{\sigma,H}(u).$$

Proof. We exploit Lemma 19 and the invariance of $\delta(\cdot)$ under the action of G (recall **H9**) to write

$$\Big\{[B]_{\tilde H} \in \mathcal{S}_{\tilde H},\ \delta(B) \le R\Big\} = \Big\{[\gamma_j B]_{\tilde H},\ [B]_H \in \mathcal{S}_H,\ \delta(B) \le R,\ j = 1, \dots, \ell\Big\}.$$

Note that, by Lemma 19, the $[\gamma_j B]_{\tilde H}$'s do not repeat themselves and $\ell = \sharp(H/\tilde H)$.
Also, by (26),

$$H_{[\gamma_1 B]_{\tilde H}}(u) + \dots + H_{[\gamma_\ell B]_{\tilde H}}(u) = \ell H_{[B]_H}(u).$$

Therefore,

$$
\begin{aligned}
S^R_{\sigma,\tilde H}(u) &= \sum_{\substack{[B]_{\tilde H}\in\mathcal{S}_{\tilde H}\\ \delta(B)\le R}} H_{[B]_{\tilde H}}(u)\\
&= \sum_{\substack{[B]_{H}\in\mathcal{S}_{H}\\ \delta(B)\le R}} \sum_{j=1}^{\ell} H_{[\gamma_j B]_{\tilde H}}(u)\\
&= \sum_{\substack{[B]_{H}\in\mathcal{S}_{H}\\ \delta(B)\le R}} \sum_{j=1}^{\ell} \Big(\ell\, H_{[B]_H}(u)\Big)\\
&= \ell\, S^R_{\sigma,H}(u)\,,
\end{aligned}
$$

as desired. □

Remark 21. The result in Corollary 20 may not hold if assumption **H5** is removed. For instance, let $G = H$ and $\tilde H$ be as in Remark 15.
Let $\Lambda := \{1, 2, 3, 4\}$. Then, G acts on Λ in the obvious way by fixing the element 4 (for instance, in the notation of Remark 15, we have that $a(1) = 2$, $a(2) = 1$, $a(3) = 3$ and $a(4) = 4$).
Let $\mathcal{S} := \{\{4\}\}$ and let the interaction potentials be $H_{\{4\}}(u) := 1 - \cos u_4$.
Then, $\mathcal{S}_H = \mathcal{S}_{\tilde H} = \{\{4\}\}$ and so, by taking, for instance $\delta(\cdot)$ to be the cardinality of a set and $R \ge 1$, we have that

$$S^R_{\sigma,H}(u) = S^R_{\sigma,\tilde H}(u) = 1 - \cos u_4\,.$$

This and (31) show that Corollary 20 does not hold in this case.

2.4. **No-symmetry breaking.** The infimal minimizer, for a cocycle σ and a fixed subgroup H, is obviously unique, due to (36). This says that it will have all the symmetries of the boundary conditions, in particular there is no-symmetry breaking, according to the following result:

Proposition 22. *Assume that $\tilde H \subset H$ is also a normal subgroup of finite index. Then, $u^{*,\sigma,H,R} = u^{*,\sigma,\tilde H,R}$.*

Proof. We note that, given $k \in H$,

$$\tilde u = \mathcal{R}_{-\sigma(k)}\mathcal{T}_k u^{*,\sigma,\tilde H,R}$$

is also a minimizer for $S^R_{\sigma,H}$ in $\mathcal{C}_{\sigma,\tilde H}$, due to **H3** and **H4**. Also, it satisfies

$$\tilde u_{gi} = u^{*,\sigma,\tilde H,R}_{kgi} - \sigma(k) \ge \sigma(k) + \sigma(g) - \sigma(k) = \sigma(g)$$

for any $g \in G$. Hence, by the definition of infimal minimizer,

$$u^{*,\sigma,\tilde H,R} \le \bigvee_{k\in H} \mathcal{R}_{-\sigma(k)}\mathcal{T}_k u^{*,\sigma,\tilde H,R}\,.$$

Therefore, since the converse inequality is obvious,

$$u^{*,\sigma,\tilde{H},R} = \bigvee_{k\in H} \mathcal{R}_{-\sigma(k)}\mathcal{T}_k u^{*,\sigma,\tilde{H},R} .$$

In particular,

$$u^{*,\sigma,\tilde{H},R} \leq \mathcal{R}_{-\sigma(k)}\mathcal{T}_k u^{*,\sigma,\tilde{H},R} ,$$

for any $k \in H$.
By applying the above inequality also to $k^{-1} \in H$, one gets that

$$\mathcal{R}_{-\sigma(k)}\mathcal{T}_k u^{*,\sigma,\tilde{H},R} = u^{*,\sigma,\tilde{H},R}$$

for all $k \in H$, and therefore

$$u^{*,\sigma,\tilde{H},R} \in \mathcal{C}_{\sigma,H} .$$

Since to any set in $[B]_H$ correspond $\sharp(H/\tilde{H})$ sets in $[B]_{\tilde{H}}$, according to Corollary 20 we have that

$$S^R_{\sigma,\tilde{H}} = \sharp(H/\tilde{H})\, S^R_{\sigma,H} .$$

Thence, we see that

(i) $u^{*,\sigma,\tilde{H},R}$ is a minimizer for $S^R_{\sigma,H}$ on $\mathcal{C}_{\sigma,H}$, because, if $S^R_{\sigma,\tilde{H}}(u^{*,\sigma,\tilde{H},R} + \beta)$ does not decrease choosing $\beta \in \mathcal{C}_{\sigma,\tilde{H}}$, a fortiori it does not decrease choosing $\beta \in \mathcal{C}_{\sigma,H}$.

(ii) $u^{*,\sigma,H,R}$ is a minimizer of $S^R_{\sigma,\tilde{H}}$ on $\mathcal{C}_{\sigma,\tilde{H}}$, because, since we know that the minimum of $S^R_{\sigma,\tilde{H}}$ is reached on a function in $\mathcal{C}_{\sigma,H}$, we see that we can bound $S^R_{\sigma,\tilde{H}}$ by its values on $S^R_{\sigma,H}$.

From (i) we obtain that

$$u^{*,\sigma,H,R} \leq u^{*,\sigma,\tilde{H},R}$$

and from (ii) we get

$$u^{*,\sigma,\tilde{H},R} \leq u^{*,\sigma,H,R} .$$

Hence,

$$u^{*,\sigma,H,R} = u^{*,\sigma,\tilde{H},R} ,$$

as claimed.. □

Thanks to Proposition 22, we can now suppress the index H in $u^{*,\sigma,H,R}$ and denote it just as $u^{*,\sigma,R}$.

2.5. **Ground states for the cut-off energy.** Our next goal is to show that $u^{*,\sigma,R}$ is a ground state for the cut-off energy. This will be obtained by fixing a compactly supported perturbation, considering all the sets which interact with this support and taking a fundamental domain which contains them. In this way, the compactly supported perturbation interacts with $u^{*,\sigma,R}$ as a periodic one, and its energy has to grow, since $u^{*,\sigma,R}$ is energy minimizing among periodic configurations. Note that Proposition 22 plays a crucial role here, since, when we put all the interactions into a large fundamental domain, in general, one may obtain a different minimizer. On the contrary, in our case, Proposition 22 assures that the infimal minimizers remain the same.

Moreover, the fact that the energy has been cut off is also important here, since we want that all the sets interacting with a given finite set may be put inside a *finite* fundamental domain (and, for cut-off interactions, this is assured by **H10**).
The concept that $u^{*,\sigma,R}$ is a ground state for the cut-off energy and the above arguments are formalized in the following way:

Proposition 23. *$u^{*,\sigma,R}$ satisfies the following local minimization property: given any finite set $X \subset \Lambda$ and any configuration β such that $\beta_p = 0$ for any $p \in \Lambda - X$ and $u^{*,\sigma,R} + \beta \in \mathcal{O}_\sigma$, we have that*

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,R}) \leq \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,R}+\beta)\,.$$

Proof. Let the configuration β be supported in the finite set $X \subset \Lambda$. We exploit **H10** and we consider Y as in (10). Then, we can always find a normal finite index subgroup $\tilde H$ such that $X \cup Y \subset \Lambda/\tilde H$ (this is a consequence of **G4**, as discussed on page 4), and, by Proposition 22, $u^{*,\sigma,\tilde H,R} = u^{*,\sigma,H,R} = u^{*,\sigma,R}$ does not really depend on the choice of $\tilde H$.
Now, for any $q \in \Lambda/\tilde H$, we define $w_q := u^{*,\sigma,R}_q + \beta_q$. Also, for any $p \in \Lambda$, we define $v_p := w_q + \sigma(h)$, as long as $p = hq$, with $q \in \Lambda/\tilde H$ and $h \in \tilde H$. Note that the above is a well-posed definition, by **G3**. Moreover, by construction,

$$(\mathcal{T}_g v)_p = v_{ghq} = w_q + \sigma(g) + \sigma(h) = (\mathcal{R}_{\sigma(g)} v)_p\,,$$

for any $g \in \tilde H$, so $v \in \mathcal{C}_{\sigma,\tilde H}$. Therefore,

$$S^R_{\sigma,\tilde H}(u^{*,\sigma,\tilde H,R}) \leq S^R_{\sigma,\tilde H}(v)\,. \tag{39}$$

We claim that

$$\begin{aligned}&\text{If } \delta(B) \leq R,\ B\cap X = \emptyset \text{ and } B \cap (\Lambda/\tilde H) \neq \emptyset,\\ &\text{then } (gB)\cap X = \emptyset \text{ for any } g \in \tilde H.\end{aligned} \tag{40}$$

Indeed, if, by contradiction, $\delta(B) \leq R$, $B \cap X = \emptyset$, $B \cap (\Lambda/\tilde H) \neq \emptyset$ and $(gB)\cap X \neq \emptyset$, we would have that $gB \subset Y$, thanks to (10), and so $(gB) \subset (\Lambda/\tilde H)$. Then, take $q \in B \cap (\Lambda/\tilde H) \neq \emptyset$. It would follow that both q and gq lie in $\Lambda/\tilde H$, thence g should be the identity. But then

$$\emptyset \neq (gB)\cap X = B \cap X = \emptyset\,,$$

and this contradiction proves (40). We now show that

$$\text{If } \delta(B) \leq R,\ B \cap X = \emptyset \text{ and } B\cap(\Lambda/\tilde H) \neq \emptyset, \text{ then } H_B(v) = H_B(u^{*,\sigma,R}). \tag{41}$$

To prove this, let B as above and take any $p \in B$. Let $q \in \Lambda/\tilde H$ in such a way $p = hq$, with $h \in \tilde H$. Then, $q \notin X$, otherwise $(h^{-1}p) \in (h^{-1}B)\cap X$, contrary to (40). Accordingly,

$$\begin{aligned} v_p &= w_q + \sigma(h) = u^{*,\sigma,R}_q + \beta_q + \sigma(h)\\ &= u^{*,\sigma,R}_q + \sigma(h) = u^{*,\sigma,R}_{hq} = u^{*,\sigma,R}_p\,,\end{aligned}$$

since $u^{*,\sigma,R} \in \mathcal{C}_{\sigma,H}$. Then, (41) is a consequence of **H1**.
Now, given $B \subset \Lambda$, we define

$$U_{\tilde{H}}(B) := \bigcup_{h \in \tilde{H}} (hB) \,.$$

Thus,

(42) If $[B]_{\tilde{H}} \in \mathcal{S}_{\tilde{H}}$, $\delta(B) \leq R$ and $U_{\tilde{H}}(B) \cap X = \emptyset$, then $H_{[B]_{\tilde{H}}}(v) = H_{[B]_{\tilde{H}}}(u^{*,\sigma,R})$.

Indeed, take B as above and let $B^* \in [B]_{\tilde{H}}$ such that $B^* \cap (\Lambda/\tilde{H}) \neq \emptyset$. Then, $B^* \cap X \subset U_{\tilde{H}}(B) \cap X = \emptyset$ and so, by (41),

$$H_{[B]_{\tilde{H}}}(v) = H_{B^*}(v) = H_{B^*}(u^{*,\sigma,R}) = H_{[B]_{\tilde{H}}}(u^{*,\sigma,R}) \,,$$

proving (42).
Moreover,

(43) If $\delta(B) \leq R$, $B \cap X \neq \emptyset$ and $B \neq A \in [B]_{\tilde{H}}$, then $A \cap X = \emptyset$.

Indeed, if (43) did not hold, we would have that $A = hB$ for some $h \in \tilde{H}$ and that both A and B lie in $Y \subset (\Lambda/\tilde{H})$. Then, h must be the identity and so $A = B$, in contradiction with the assumption, which gives (43).
As a consequence of (43),

$$\sum_{\substack{[B]_{\tilde{H}} \in \mathcal{S}_{\tilde{H}} \\ U_{\tilde{H}}(B) \cap X \neq \emptyset \\ \delta(B) \leq R}} H_{[B]_{\tilde{H}}} = \sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset \\ \delta(B) \leq R}} H_B \,,$$

and, consequently,

$$S^R_{\sigma,\tilde{H}} = \sum_{\substack{[B]_{\tilde{H}} \in \mathcal{S}_{\tilde{H}} \\ U_{\tilde{H}}(B) \cap X = \emptyset \\ \delta(B) \leq R}} H_{[B]_{\tilde{H}}} + \sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset \\ \delta(B) \leq R}} H_B \,.$$

We now make use of the above equality, (42) and (39) to conclude that

$$\begin{aligned}
\sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset \\ \delta(B) \leq R}} H_B(u^{*,\sigma,R}) &= S^R_{\sigma,\tilde{H}}(u^{*,\sigma,R}) - \sum_{\substack{[B]_{\tilde{H}} \in \mathcal{S}_{\tilde{H}} \\ U_{\tilde{H}}(B) \cap X = \emptyset \\ \delta(B) \leq R}} H_{[B]_{\tilde{H}}}(u^{*,\sigma,R}) \\
&= S^R_{\sigma,\tilde{H}}(u^{*,\sigma,R}) - \sum_{\substack{[B]_{\tilde{H}} \in \mathcal{S}_{\tilde{H}} \\ U_{\tilde{H}}(B) \cap X = \emptyset \\ \delta(B) \leq R}} H_{[B]_{\tilde{H}}}(v) \\
&\leq S^R_{\sigma,\tilde{H}}(v) - \sum_{\substack{[B]_{\tilde{H}} \in \mathcal{S}_{\tilde{H}} \\ U_{\tilde{H}}(B) \cap X = \emptyset \\ \delta(B) \leq R}} H_{[B]_{\tilde{H}}}(v) \\
&= \sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset \\ \delta(B) \leq R}} H_B(v) \,,
\end{aligned}$$

as desired. □

2.6. **The Birkhoff property and a-priori bounds.** We now show that the infimal minimizer is Birkhoff (see Definition 6) and that it belongs to the space $\mathcal{O}_\sigma$ defined on page 6.

Lemma 24. *The infimal minimizer $u^{*,\sigma,R}$ is Birkhoff.*

Proof. Let $\ell \in \mathbb{Z}$ and $g \in G$ be such that $\sigma(g) \geq \ell$ (resp., $\sigma(g) \leq \ell$). By **H3** and **H4**, $\mathcal{R}_{-\ell}\mathcal{T}_g u^{*,\sigma,R}$ (resp., $\mathcal{R}_\ell \mathcal{T}_{g^{-1}} u^{*,\sigma,R}$) is also a minimizer for $S^R_{\sigma,H}$ in $\mathcal{C}_{\sigma,H}$ and it satisfies the constraints in the right hand side of (36). Thus, by the infimal minimizer property, $u^{*,\sigma,R} \leq \mathcal{R}_{-\ell}\mathcal{T}_g u^{*,\sigma,R}$ (resp.,, $u^{*,\sigma,R} \leq \mathcal{R}_\ell \mathcal{T}_{g^{-1}} u^{*,\sigma,R}$). □

Next result points out a useful property of the Birkhoff functions:

Lemma 25. *If v satisfies the Birkhoff property, then $v_{gq} - v_q \in [\sigma(g) - 1, \sigma(g) + 1]$, for any $q \in \Lambda$ and any $g \in G$.*

Proof. Given $g \in G$, let $\ell \in \mathbb{Z}$ be so that $\ell \leq \sigma(g) \leq \ell + 1$. Then, since v is Birkhoff, $\mathcal{T}_g(v) \geq \mathcal{R}_\ell(v)$ and $\mathcal{T}_g(v) \leq \mathcal{R}_{\ell+1}(v)$. That is,

$$v_{gq} - v_q \geq \ell \geq \sigma(g) - 1$$

and

$$v_{gq} - v_q \leq \ell + 1 \leq \sigma(g) + 1\,,$$

for any $q \in \Lambda$, as desired. □

Corollary 26. *If v satisfies the Birkhoff property, then $v_{g_1 p} - v_{g_2 p} \in [\sigma(g_1) - \sigma(g_2) - 1, \sigma(g_1) - \sigma(g_2) + 1]$, for any $p \in \Lambda$ and any $g \in G$.*

Proof. Just apply Lemma 25 to $g := g_1 g_2^{-1}$ and $q := g_2 p$. □

We now perform an easy energy estimate which will imply an a-priori regularity result (that is, Lemma 28).

Lemma 27. *Let K, κ, θ be as in* **H6***. Suppose that v satisfies the Birkhoff property and that*

$$\max_{i,j \in \Lambda/G} |v_i - v_j| \geq 2\,. \tag{44}$$

Then,

$$H_{gF}(v) \;\geq\; \kappa \Big(\max_{i,j\in\Lambda/G} |v_i - v_j| - 2 \Big)^\theta - K\,,$$

for any $g \in G$.

Proof. Let $a, b \in \Lambda/G$ such that

$$v_a - v_b = \max_{i,j\in\Lambda/G} |v_i - v_j|\,.$$

By Lemma 25,

$$v_{ga} - v_{gb} \geq (v_a + \sigma(g) - 1) - (v_b + \sigma(g) + 1) = v_a - v_b - 2\,,$$

which is greater or equal then zero, by (44). Hence, from **H3** and **H6**,

$$\begin{aligned} H_{gF}(v) = H_F(\mathcal{T}_g v) &\geq \kappa \max_{i,j\in(\Lambda/G)} |v_{gi} - v_{gj}|^\theta - K \\ &\geq \kappa \,(v_{ga} - v_{gb})^\theta - K \\ &\geq \kappa \,(v_a - v_b - 2)^\theta - K \,, \end{aligned}$$

as desired. □

We are now in the position of estimating the oscillation of our minimizers:

Lemma 28. *Let $\kappa, K, S, \theta, \vartheta$ be as in* **H6** *and* **H7**. *Let $v \in \mathcal{C}_{\sigma,H}$ be a minimizer for $S^R_{\sigma,H}$ on $\mathcal{C}_{\sigma,H}$. Then,*

$$\max_{i,j\in\Lambda/G} |v_i - v_j| \;\leq\; 2 + \left(\frac{S\,(\|\sigma\|^\vartheta + 1)\,\sharp F + K}{\kappa} \right)^{1/\theta} .$$

Proof. Without loss of generality, we may assume that (44) holds, otherwise we are done. Note also that

$$[iF]_H \neq [jF]_H \text{ for any } i \neq j \in G/H \tag{45}$$

Indeed, if $F = gF$, with $g = i^{-1}hj$ and $h \in H$, then, since F is finite (and not empty), fixed $f \in F$, we have that there exist $n_1 > n_2 \in \mathbb{N}$ in such a way that $g^{n_1} f = g^{n_2} f$. That is, if $n := n_1 - n_2 > 0$, $g^n f = f$ and so, by **G3**, $g \in H$ and so i and j are the same in G/H, yielding (45).
Moreover, by **H10** and (27), $\delta(iF) \leq R$, for any $i \in G/H$. Also, since $F = \Lambda/G$, we have that, for any $i \in G/H$, there exists $h \in H$ in such a way that $(hiF)\cap(\Lambda/H) \neq \emptyset$. Thus, we deduce from **H6** that

$$S^R_{\sigma,H}(v) \;\geq\; \sum_{i\in G/H} H_{iF}(v) \,.$$

Moreover, by **H7**,

$$S^R_{\sigma,H}(\sigma) \leq \sum_{\substack{B\in\mathcal{S}\\ B\cap(\Lambda/H)\neq\emptyset}} \varsigma(B)\,(\|\sigma\|^\vartheta + 1) \leq S\,(\|\sigma\|^\vartheta + 1)\cdot \sharp(\Lambda/H) \,.$$

Hence, we exploit Lemma 27 to conclude that

$$\begin{aligned}
&\left(\kappa\left(\max_{j,h\in\Lambda/G}|v_j-v_h|-2\right)^\theta-K\right)\cdot\sharp(G/H)\\
\le\;& \sum_{i\in G/H}\left(\kappa\left(\max_{j,h\in\Lambda/G}|v_j-v_h|-2\right)^\theta-K\right)\\
\le\;& \sum_{i\in G/H} H_{iF}(v)\\
\le\;& S^R_{\sigma,H}(v)\\
\le\;& S^R_{\sigma,H}(\sigma)\\
\le\;& S\left(\|\sigma\|^\vartheta+1\right)\cdot\sharp(\Lambda/H)\\
=\;& S\left(\|\sigma\|^\vartheta+1\right)\cdot\sharp(\Lambda/G)\cdot\sharp(G/H)\,,
\end{aligned}$$

which yields the desired result. □

Lemma 29. *Let $K^\star_\sigma$ be as in (7). Then,*

$$|u^{*,\sigma,R}_{gi}-\sigma(g)|\le K^\star_\sigma-1\,,$$

for all $i\in\Lambda/G$.

Proof. First, note that, by the constraint in the right hand side of (36), $u^{*,\sigma,R}_{gi}\ge\sigma(g)$, for any $g\in G$ and any $i\in\Lambda/G$.
In fact, we also have that there exist $i^*\in\Lambda/G$ and $g^*\in G$ in such a way that

$$u^{*,\sigma,R}_{g^*i^*}-\sigma(g^*)\in[0,1]\,. \tag{46}$$

To prove this, we argue by contradiction, supposing that $u^{*,\sigma,R}_{gi}-\sigma(g)\ge 1$ for all $i\in\Lambda/G$ and $g\in G$. Then,

$$(\mathcal{R}_{-1}u^{*,\sigma,R})_{gi}=u^{*,\sigma,R}_{gi}-1\ge\sigma(g)\,,$$

that is, $\mathcal{R}_{-1}u^{*,\sigma,R}$ satisfies the constraints in the right hand side of (36) and so, due to **H3** and **H4**, it is also a minimizer. Since $\mathcal{R}_{-1}u^{*,\sigma,R}<u^{*,\sigma,R}$, this would contradict the fact that $u^{*,\sigma,R}$ is the infimal minimizer. This yields the proof of (46).
Now, if $\sharp(\Lambda/G)=1$, that is, if $\Lambda/G=\{i^*\}$, then (46) and Corollary 26 imply that

$$|u^{*,\sigma,R}_{gi^*}-\sigma(g)|\le|u^{*,\sigma,R}_{gi^*}-u^{*,\sigma,R}_{g^*i^*}-\sigma(g)+\sigma(g^*)|+|u^{*,\sigma,R}_{g^*i^*}-\sigma(g^*)|\le 2+1\,,$$

which is the desired result for this case.
If, on the other hand, $\sharp(\Lambda/G)\ge 2$ we exploit (46), Corollary 26 and Lemma 28 to obtain that

$$\begin{aligned}
|u^{*,\sigma,R}_{gi}-\sigma(g)| &\le |u^{*,\sigma,R}_{gi}-u^{*,\sigma,R}_{gi^*}|+|u^{*,\sigma,R}_{gi^*}-u^{*,\sigma,R}_{g^*i^*}-\sigma(g)+\sigma(g^*)|+|u^{*,\sigma,R}_{g^*i^*}-\sigma(g^*)|\\
&\le 2+\left(\frac{S\left(\|\sigma\|^\vartheta+1\right)\sharp F+K}{\kappa}\right)^{1/\theta}+2+1\,,
\end{aligned}$$

as desired. □

2.7. **Approximating irrational cocycles with rational ones.** We now remove the assumption (made on page 16) that the cocycle σ is rational. We keep, for the moment the assumption that the interaction is finite range. The finite range assumption will be removed later.

Lemma 30. *Fix $R > 0$. Let $K_\sigma^\star$ be as in (7). Then, fixed any cocycle σ, there exists a Birkhoff configuration $u^{*,\sigma,R}$ such that*

$$|u_{gi}^{*,\sigma,R} - \sigma(g)| \leq K_\sigma^\star - 1\,, \tag{47}$$

for all $i \in \Lambda/G$.
Also, given any finite set $X \subset \Lambda$ and any configuration β such that $\beta_p = 0$ for any $p \in \Lambda - X$, one has that

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,R}) \;\leq\; \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,R}+\beta)\,. \tag{48}$$

The fact that $u^{*,\sigma,R}$ satisfies (48) means that it is a ground state for the cut–off interaction.
Proof. Proposition 23 and Lemma 29 yields the result for rational cocycle.
Thus, let σ be an irrational cocycle, and let $\sigma^{(n)}$ be as in Proposition 11. Consider the infimal minimizer $u^{*,\sigma^{(n)},R}$ in the light of Proposition 23 and Lemma 29. Since $u^{*,\sigma^{(n)},R} - u_{\sigma^{(n)}}$ belongs to the topological product $\Pi_{p\in\Lambda}[-K^\star_{\sigma^{(n)}},\, K^\star_{\sigma^{(n)}}]$, and so to $\Pi_{p\in\Lambda}[-K_\sigma^\star - 1,\, K_\sigma^\star+1]$ as long as n is large enough, and such a set is compact by Tychonoff Theorem, we deduce that, up to subsequences, $u^{*,\sigma^{(n)},R}$ converges to a suitable $u^{*,\sigma,R}$ pointwise in Λ. Of course, $u^{*,\sigma,R}$ satisfies (47), since so does $u^{*,\sigma^{(n)},R}$.
Also, by **H8**, there exists $C > 0$ in such a way that, for any $B \in \mathcal{S}$,

$$\begin{aligned}|H_B(u^{*,\sigma^{(n)},R}+\beta) - H_B(u^{*,\sigma,R}+\beta)| + |H_B(u^{*,\sigma^{(n)},R}) - H_B(u^{*,\sigma,R})|\\ \leq C\sum_{p\in B} |u_p^{*,\sigma^{(n)},R} - u_p^{*,\sigma,R}|\end{aligned}$$

and so, since B is a finite set,

$$\begin{aligned} &\lim_{n\to+\infty} H_B(u^{*,\sigma^{(n)},R}+\beta) \;=\; H_B(u^{*,\sigma,R}+\beta)\\ \text{and}\qquad &\lim_{n\to+\infty} H_B(u^{*,\sigma^{(n)},R}) \;=\; H_B(u^{*,\sigma,R})\,.\end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{n\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma^{(n)},R}+\beta) \;=\; \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,R}+\beta)\\ \text{and}\qquad &\lim_{n\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma^{(n)},R}) \;=\; \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,R})\,,\end{aligned}$$

since, by Lemma 12, the above series are just finite sums. It follows that $u^{*,\sigma,R}$ satisfies (48), since, by Proposition 23, so does $u^{*,\sigma^{(n)},R}$. $\square$

2.8. **Removing the cut-off.** We now complete the proof of Theorem 8 by taking the limit as $R \to +\infty$, that is, we will remove the conditions that the interaction sets B have to satisfy $\delta(B) \le R$.
To this extent, given a (possibly irrational) cocycle σ, let $u^{*,\sigma,R}$ be as in Lemma 30. By (47) and Tychonoff Theorem, we may suppose that $u^{*,\sigma,R}$ converges to some $u^{*,\sigma}$ pointwise in Λ, possibly up to a subsequence.
We will show that $u^{*,\sigma}$ is the configuration claimed in Theorem 8. Indeed, $u^{*,\sigma}$ satisfies (12), just by taking the limit as $R \to +\infty$ in (47).
Then, the proof of Theorem 8 is completed once we show that $u^{*,\sigma}$ is a ground state.
Proving that $u^{*,\sigma}$ is a minimizer of the energy without the cut–off is somewhat delicate. Of course, the energies considered are different. the reduced energy energy $S^R_{\sigma,H}$ defined in (28) is different. in (1). We also note that we have to deal with the fact that when we define the periodified energy, we are considering only periodified configurations. If we consider increasing the range of the interaction, then the interacting sets will be much larger than the fundamental domain of the periodification. Hence, the interactions could be counted many times.
As we will see, this difficulty is solved thanks to the no-symmetry breaking result of Proposition 22 which makes possible to consider large fundamental domains for the periodification without changing the minimizers. Then, we can increase the range of the perturbation without problems of overcounting.
We now proceed to the formal proof that $u^{*,\sigma}$ is a ground state, and this will end the proof of Theorem 8.
For this, fix a finite set $X \subset \Lambda$ and consider a configuration v in such a way $v_p = u^{*,\sigma}_p$ for any $p \in \Lambda - X$, and let $\beta := v - u^{*,\sigma}$. For a fixed $R > 0$, let Y be the set in (10); then, for any $r \ge R$,

$$
\begin{aligned}
&\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} |H_B(u^{*,\sigma}) - H_B(u^{*,\sigma,r})| \\
\le\;& \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u^{*,\sigma} + (1-\tau) u^{*,\sigma,r}) \right| d\tau \cdot |u^{*,\sigma}_p - u^{*,\sigma,r}_p| \\
\le\;& \sum_{p\in Y} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u^{*,\sigma} + (1-\tau) u^{*,\sigma,r}) \right| d\tau \cdot |u^{*,\sigma}_p - u^{*,\sigma,r}_p| \\
\le\;& C \sum_{p\in Y} |u^{*,\sigma}_p - u^{*,\sigma,r}_p| ,
\end{aligned}
$$

where **H8** has been used. Then, since, by **H10**, Y is a finite set,

$$\lim_{r\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} |H_B(u^{*,\sigma}) - H_B(u^{*,\sigma,r})| \;=\; 0\,,$$

for any fixed $R > 0$.
Analogously, if we set $v^{(r)} := u^{*,\sigma,r} + \beta$,

$$\lim_{r\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} |H_B(v) - H_B(v^{(r)})| \;=\; 0\,,$$

for any fixed $R > 0$.
Fix now $\epsilon > 0$. From **H10**, there exists $R_\epsilon > 0$ in such a way

$$\sup_{u\in\mathcal{O}_\sigma} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)> R}} H_B(u) \;\leq\; \epsilon\,,$$

provided that $R \geq R_\epsilon$.
Therefore, for any $R \geq R_\epsilon$,

$$\begin{aligned}
&\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u^{*,\sigma}) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(v)\\
\leq\;\; & 2\epsilon + \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma}) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(v)\\
=\;\; & 2\epsilon + \lim_{r\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,r}) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(v^{(r)})\\
=\;\; & 2\epsilon + \lim_{r\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,r}) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u^{*,\sigma,r}+\beta)\,.
\end{aligned}$$

Then, by (48),

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u^{*,\sigma}) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(v) \;\leq\; 2\epsilon\,.$$

Since ϵ may be taken arbitrarily small, we have just proved that $u^{*,\sigma}$ is a ground state, thus completing the proof of Theorem 8.

3. Laminations of ground states

We would like now to stress the fact that the ground states constructed in Theorem 8 induce a (possibly singular) lamination of $F \times \mathbb{T}^1$. See Definition 9.
First of all, we prove that the ground states are a closed set under pointwise convergence:

Lemma 31. *Let $u^{(n)} \in \mathcal{O}_\sigma$ be such that*

$$\lim_{n\to+\infty} u_p^{(n)} \,=\, u_p\,,$$

for any $p \in \Lambda$. Suppose that any $u^{(n)}$ is a ground state. Then so is u.

Proof. Let β be a configuration which vanishes outside a finite set X. Then, if

$$K := K_\sigma^\star + \sup_{p\in X} |\beta(p)|\,,$$

we have that

$$u^{(n)} + \beta\,,\ u+\beta \in \mathcal{O}_\sigma^K\,.$$

Thus, fix $\epsilon > 0$ and use **H10** to find $R > 0$ in such a way that

$$\sup_{u\in\mathcal{O}_\sigma^K} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)>R}} H_B(u) \,\le\, \epsilon\,.$$

Observe that

$$\begin{aligned} & |H_B(u^{(n)}) - H_B(u)| \\ \le\ & \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u^{(n)} + (1-\tau)u)\right| d\tau \cdot |u_p^{(n)} - u_p| \\ \le\ & C \sum_{p\in B} |u_p^{(n)} - u_p|,, \end{aligned}$$

for some $C > 0$, thanks to **H8**.
Thus, since B is a finite set,

$$\lim_{n\to+\infty} H_B(u^{(n)}) \,=\, H_B(u)\,.$$

Therefore,

$$\lim_{n\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} H_B(u^{(n)}) \,=\, \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} H_B(u)\,,$$

since the above series are finite sums, due to **H10**. Analogously,

$$\lim_{n\to+\infty} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} H_B(u^{(n)}+\beta) \,=\, \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} H_B(u+\beta)\,.$$

Also, since $u^{(n)}$ is a ground state,

$$\begin{aligned} 0 \ &\le\ \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u^{(n)}+\beta) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u^{(n)}) \\ &\le\ 2\epsilon + \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} H_B(u^{(n)}+\beta) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\le R}} H_B(u^{(n)}) \end{aligned}$$

and so, by passing to the limit,

$$\begin{aligned} 0 \;&\leq\; 2\epsilon + \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u+\beta) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset\\ \delta(B)\leq R}} H_B(u) \\ &\leq\; 4\epsilon + \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u+\beta) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u)\,, \end{aligned}$$

which gives the desired result by taking ϵ as small as we wish. □

We now point out a useful property of Birkhoff functions:

Lemma 32. *If $u \in \mathcal{B}_\sigma$, $g_1, g_2 \in G$ and $\ell_1, \ell_2 \in \mathbb{Z}$, then either $\mathcal{R}_{\ell_1}\mathcal{T}_{g_1}u \geq \mathcal{R}_{\ell_2}\mathcal{T}_{g_2}u$ or $\mathcal{R}_{\ell_1}\mathcal{T}_{g_1}u \leq \mathcal{R}_{\ell_2}\mathcal{T}_{g_2}u$.*

Proof. Let $u \in \mathcal{B}_\sigma$. Since σ is a cocycle, by possibly exchanging the indices, we may suppose that $\sigma(g_1g_2^{-1}) \geq \ell_2 - \ell_1$. Then, by the Birkhoff property, $\mathcal{T}_{g_1g_2^{-1}}u \geq \mathcal{R}_{\ell_2-\ell_1}u$, and so $\mathcal{R}_{\ell_1}\mathcal{T}_{g_1}u \geq \mathcal{R}_{\ell_2}\mathcal{T}_{g_2}u$. □

Corollary 33. *Let u be a Birkhoff function. For any $g \in G$ and $\ell \in \mathbb{Z}$, let*

$$\mathcal{L}_{\ell,g} := \left\{ \Big(p, (\mathcal{R}_\ell\mathcal{T}_g u)_p\Big) \;\Big|\; p \in F \right\}.$$

Let $\mathcal{L}$ be the closure of

$$\bigcup_{\substack{g\in G\\ \ell\in\mathbb{Z}}} \mathcal{L}_{\ell,g}\,.$$

Then, $\mathcal{L}$ is a (possibly singular) lamination on $F \times \mathbb{R}$.
Also, if $(p,r) \in \mathcal{L}$, then $(p, r+s) \in \mathcal{L}$ for any $s \in \mathbb{Z}$, thus $\mathcal{L}$ may be quotiented to a lamination of $F \times \mathbb{T}^1$.

Proof. The first claim follows from Lemma 32. The second one from the observation that if $(p,r) \in \mathcal{L}_{\ell,g}$, then $(p, r+s) \in \mathcal{L}_{\ell+s,g}$. □

As a consequence of Corollary 33 and Lemma 31, we obtain:

Corollary 34. *Let u be a Birkhoff ground state. Then the (possibly singular) lamination of Corollary 33 is made of ground states.*

Let

$$H := \{g \in G \mid \sigma(g) \in \mathbb{Z}\}\,. \tag{49}$$

It is easily seen that, if $u \in \mathcal{B}_\sigma$, then

$$\mathcal{T}_h u = \mathcal{R}_{\sigma(h)}u \tag{50}$$

for any $h \in H$, so all the values of u are determined by the values in Λ/H. This gives another characterization of the group H:

Lemma 35. $H = \{g \in G$ *s.t.* $\exists s \in \mathbb{Z}$ *s.t.* $\mathcal{T}_g u = \mathcal{R}_s u \;\forall\; u \in \mathcal{B}_\sigma\}$.

Proof. Let g be in the set described by Lemma 35. Then, since $u_\sigma \in \mathcal{B}_\sigma$, for any $q \in F$

$$\sigma(g) = u_\sigma(gq) = (\mathcal{T}_g u_\sigma)_q = (\mathcal{R}_s u_\sigma)_q = u_\sigma(q) + s = s \in \mathbb{Z}\,,$$

due to (6), so g belongs to the set in (49).
Viceversa, let g be in set described by (49). Then, it also belongs to the set described by Lemma 35, thanks to (50). □

When the cocycle is rational, H happens to be finite index. Thus, the assumption in Theorem 10 that H is just the identity may be seen as an irrationality condition.
With this notation, we see that the gap is "summable" over the quotient between the crystal and the group H:

Lemma 36. *Let $u^{(0)}$ and $u^{(1)}$ be at the edge of a gap in the minimal (possibly singular) lamination of Corollary 34. Then, the following estimate holds:*

$$\sum_{p \in \Lambda/H} |u^{(1)}(p) - u^{(0)}(p)| \le \sharp F\,.$$

Proof. Given $g \in G/H$ let

$$\Theta_g := \{(p,[t])\,, \quad p \in F\,, \quad u^{(0)}(gp) \le t \le u^{(1)}(gp)\}\,,$$

where $[\cdot]$ here above denotes the integer part. Then, $\Theta_g \subset F \times \mathbb{T}^1$ and $\Theta_g \cap \Theta_{\tilde g} = \emptyset$ if $g \ne \tilde g$, thanks to the Birkhoff property and the fact that $u^{(0)}$ and $u^{(1)}$ are at the edge of a gap in the minimal (possibly singular) lamination.
Let μ be the product measure between the counting measure on F and the Lebesgue measure $\mathcal{L}$ on $\mathbb{T}^1$. Then, the above arguments and Fubini's Theorem yield that

$$\begin{aligned}
\sharp F = \mu(F \times \mathbb{T}^1) &\ge \mu\left(\bigcup_{g \in G/H} \Theta_g\right) = \sum_{g \in G/H} \mu(\Theta_g) \\
&= \sum_{g \in G/H} \sum_{p \in F} |u^{(1)}(gp) - u^{(0)}(gp)| \\
&= \sum_{i \in \Lambda/H} |u^{(1)}(i) - u^{(0)}(i)|\,,
\end{aligned}$$

thanks to Lemma 14, as desired. □

4. Proof of Theorem 10

In order to prove Theorem 10, without loss of generality, we may suppose that $u^{(0)}$ and $u^{(1)}$ are at the edge of a gap in the (possibly singular) lamination induced by the minimizers given in Theorem 8 (see the discussion in Section 3 and, in particular, Corollaries 33 and 34 there). Indeed, if the lamination has no gaps between $u^{(0)}$ and $u^{(1)}$, then infinitely many leaves lie between $u^{(0)}$ and $u^{(1)}$, thus giving the claim of Theorem 10.
Thus, we are now going to construct a new solution inside the gap bordered by $u^{(0)}$ and $u^{(1)}$.

Notice that, due to Lemma 36 and the fact that H only contains the identity, we have that

$$\sum_{p\in\Lambda} |u^{(1)}(p) - u^{(0)}(p)| \le \sharp F\,. \tag{51}$$

This estimate will play a crucial role in the sequel since it will provide some amount of compactness.
The a-priori bound (51) gives us an ℓ^1 a priori bound for the difference of the two solutions. Note that it is a purely geometric bound and that it only depends on the fact that they are at the edge of the gaps of a (possibly singular) lamination. Indeed, the term in the left is just the total volume of the gap. Since we know that all the translates of the gap cannot intersect, we just divide the pieces of the integral so that they fit in the fundamental domain.

4.1. **Set-up and compactness features.** During this proof, we will fix a cocycle σ as in the statement of Theorem 10.
We introduce the sup-norm

$$\|u\| := \sup_{p\in\Lambda} |u(p)|\,.$$

It is easily seen that $\mathcal{O}_\sigma$, endowed with such norm, is a Banach manifold.
We define

$$\mathcal{C}_\sigma(u^{(0)},u^{(1)}) := \Big\{u\in\mathcal{B}_\sigma \quad | \quad u^{(0)}\le u\le u^{(1)}\Big\},$$

where $u^{(0)}$ and $u^{(1)}$ are as in the statement of Theorem 10. To avoid unnecessary complications, we will use $\mathcal{C}$ as a short-hand notation for $\mathcal{C}_\sigma(u^{(0)},u^{(1)})$ in the sequel.
We point out that $\mathcal{C}$ possesses the following useful compactness property:

Lemma 37. *Let $u_n\in\mathcal{C}$ for any $n\in\mathbb{N}$. Then, there exists a subsequence u_{n_k} and a configuration $u\in\mathcal{C}$ such that*

$$\lim_{k\to+\infty} u_{n_k}(p) = u(p)\,, \tag{52}$$

for any $p\in\Lambda$.

Proof. Let u_σ be as defined in (6) and set $v_n := u_n - u_\sigma$. Since $u^{(0)},\ u^{(1)}\in\mathcal{B}_\sigma\subset\mathcal{O}_\sigma$, we deduce from (8) that

$$-K^\star_\sigma \le (u^{(0)}-u_\sigma)(p)\le v_n(p)\le (u^{(1)}-u_\sigma)(p)\le K^\star_\sigma\,.$$

That is, $v_n = \{v_n(p)\,,\quad p\in\Lambda\}$ belongs to the topological product $\Pi_{p\in\Lambda}[-K^\star_\sigma,K^\star_\sigma]$. Since, by Tychonoff Theorem, the latter space is compact, we have that there exists a subsequence v_{n_k} and a configuration v such that

$$\lim_{k\to+\infty} v_{n_k}(p) = v(p)\,,$$

for any $p\in\Lambda$. This proves (52), by setting $u := v+u_\sigma$. Since $u_n\in\mathcal{C}$, it also follows from (52) that $u\in\mathcal{C}$. □

Pointwise and uniform convergence thus agree on $\mathcal{C}$, as next result points out:

Lemma 38. *If $u_n \in \mathcal{C}$ is a sequence such that*

$$\lim_{n\to+\infty} u_n(p) = u(p)\,,$$

for any $p \in \Lambda$, then u_n converges to u in the sup*-norm $\|\cdot\|$.*

Proof. Fix $\epsilon > 0$. By (51), there exists a finite set

$$X_\epsilon = \{p_1, \ldots, p_m\} \subset \Lambda$$

in such a way that

$$|u_n(p) - u^{(0)}(p)| \,\leq\, \epsilon\,,$$

for any $n \in \mathbb{N}$ and any $p \in \Lambda - X_\epsilon$. Since $\mathcal{C}$ is obviously closed under pointwise limit, $u \in \mathcal{C}$ and

$$|u(p) - u^{(0)}(p)| \,\leq\, \epsilon\,,$$

for any $p \in \Lambda - X_\epsilon$.
Also, using again the pointwise convergence of u_n, we have that, for each $j = 1, \ldots, m$, there exists $n(\epsilon, j)$ such that

$$|u_n(p_j) - u(p_j)| \,\leq\, \epsilon\,,$$

for any $n \geq n(\epsilon, j)$. Let

$$n(\epsilon) := \max\{n(\epsilon,1), \ldots, n(\epsilon,m)\}\,.$$

Then,

$$|u_n(p) - u(p)| \,\leq\, \epsilon\,,$$

for any $n \geq n(\epsilon)$ and any $p \in X_\epsilon$.
Thus, if $n \geq n(\epsilon)$,

$$\begin{aligned} \|u_n - u\| &\leq \sup_{p\in X_\epsilon} |u_n(p) - u(p)| \\ &\quad + \sup_{p\in\Lambda - X_\epsilon} \Big(|u_n(p) - u^{(0)}(p)| + |u^{(0)}(p) - u(p)| \Big) \\ &\leq 3\epsilon\,, \end{aligned}$$

proving the desired claim. □

As an easy consequence of Lemmata 37 and 38, we thus deduce that $\mathcal{C}$ is compact in the sup-norm:

Corollary 39. *Let $u_n \in \mathcal{C}$ for any $n \in \mathbb{N}$. Then, there exists a subsequence u_{n_k} and a configuration $u \in \mathcal{C}$ so u_{n_k} converges to u in the* sup*-norm $\|\cdot\|$.*

As in [dlLV05], it is also convenient to introduce a *renormalized energy functional.* Namely, we set

$$S_\sigma(u) \,:=\, \sum_{\substack{B\in\mathcal{S}\\ B\cap\Lambda\neq\emptyset}} \Big(H_B(u) - H_B(u^{(0)}) \Big)\,. \tag{53}$$

The paper [dlLV05] contains a similar result in the case of partial differential equation. In partial differential equations renormalized energies for the study of homoclinic solutions to periodic solutions were also used in [RS03]. We also note that some similar expressions were considered in [Mat86] in the context of the Peierls-Nabarro barrier.
The above renormalized energy (53) provides a "nice" functional in $\mathcal{C}$, in view of the following result:

Lemma 40. *For any $u \in \mathcal{C}$, the series in (53) is absolutely convergent, and*

$$0 \le S_\sigma(u) < +\infty . \tag{54}$$

Moreover, if $S_\sigma(u) = 0$ for some $u \in \mathcal{C}$, then u is a ground state, and, conversely, if $u \in \mathcal{C}$ is a ground state, then $S_\sigma(u) = 0$.

Proof. For any $u \in \mathcal{C}$,

$$\begin{aligned}
& \sum_{B\in\mathcal{S}} |H_B(u) - H_B(u^{(0)})| \\
\le \ & \sum_{B\in\mathcal{S}} \sum_{p\in B} \left| \int_0^1 \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau)u^{(0)})\, d\tau \cdot (u_p - u_p^{(0)}) \right| \\
\le \ & \sum_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau)u^{(0)}) \right| d\tau \cdot |u_p - u_p^{(0)}| \\
\le \ & C \sum_{p\in\Lambda} |u - u^{(0)}| ,
\end{aligned}$$

thanks to **H8**. Thus, by (51), the series in (53) is absolutely convergent, and so $S_\sigma(u) < +\infty$, as desired.
Then, let $u \in \mathcal{C}$ and fix $\epsilon > 0$. We exploit again (51) to construct a set $X_\epsilon \subset \Lambda$ so that

$$\sharp X_\epsilon < +\infty \tag{55}$$

and

$$\sum_{p\in\Lambda - X_\epsilon} |u^{(1)} - u^{(0)}| \le \epsilon . \tag{56}$$

Let now

$$v(p) := \begin{cases} u(p) & \text{if } p \in X_\epsilon, \\ u^{(0)}(p) & \text{if } p \in \Lambda - X_\epsilon. \end{cases}$$

Since the series in (53) is absolutely convergent (as we have just proved), by using **H8** and (56), we see that

$$
\begin{aligned}
|S_\sigma(u)-S_\sigma(v)| &= \left|\sum_{B\in\mathcal{S}} H_B(u)-H_B(v)\right| \\
&\le \sum_{B\in\mathcal{S}}\sum_{p\in B}\int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau u+(1-\tau)v)\right| d\tau\cdot|u_p-v_p| \\
&= \sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau u+(1-\tau)v)\right| d\tau\cdot|u_p-v_p| \\
&\le C\sum_{p\in\Lambda}|u_p-v_p| \\
&= C\sum_{p\in\Lambda-X_\epsilon}|u_p-u_p^{(0)}| \\
&\le C\epsilon\,.
\end{aligned}
\tag{57}
$$

Furthermore, since $u^{(0)}$ is a ground state, we deduce from (55) that

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} \left(H_B(v)-H_B(u^{(0)})\right)\ \ge\ 0\,.$$

This, **H8**, (56) and (57) yield that

$$
\begin{aligned}
2C\epsilon+S_\sigma(u) &\ge C\epsilon+S_\sigma(v)\ge C\epsilon+\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon=\emptyset}}\left(H_B(v)-H_B(u^{(0)})\right) \\
&\ge C\epsilon-\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon=\emptyset}}\sum_{p\in B}\int_0^1\left|\frac{\partial H_B}{\partial u_p}(\tau v+(1-\tau)u^{(0)})\right| d\tau\cdot|v_p-u_p^{(0)}| \\
&\ge C\epsilon-C\sum_{p\in\Lambda-X_\epsilon}|v_p-u_p^{(0)}| \\
&\ge 0\,.
\end{aligned}
$$

Since ϵ may be taken as small as we wish, we have just proved that $S_\sigma(u)\ge 0$, which completes the proof of (54).
Let now $u\in\mathcal{C}$ such that $S_\sigma(u)=0$. Let X be a finite set and let w be so that $w_p=u_p$ for any $p\in\Lambda-X$. Let $X_\epsilon\supset X$ as in (55) and (56) and set

$$z(p):=\begin{cases} w(p) & \text{if } p\in X_\epsilon,\\ u^{(0)}(p) & \text{if } p\in\Lambda-X_\epsilon.\end{cases}$$

Arguing as in the proof of (57), we get that

$$\begin{aligned}\Big| \sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(w)-H_B(u^{(0)})) \ - \sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(z)-H_B(u^{(0)}))\Big| \\ \leq C\sum_{p\in\Lambda}|w_p-z_p| \\ = C\sum_{p\in\Lambda-X_\epsilon}|u^{(0)}_p-z_p| \\ \leq C\epsilon\end{aligned}$$

and that

$$\begin{aligned}\Big|S_\sigma(u) - \sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(u)-H_B(u^{(0)}))\Big| &\leq C\sum_{p\in\Lambda-X_\epsilon}|u_p-u^{(0)}_p| \\ &\leq C\epsilon\,.\end{aligned}$$

Furthermore, since $u^{(0)}$ is a ground state,

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(z)-H_B(u^{(0)})) \ \geq \ 0\,.$$

By collecting the above estimates, and exploiting the fact that $S_\sigma(u)=0=S_\sigma(u^{(0)})$, we conclude that

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(u)-H_B(u^{(0)})) \ - \sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(w)-H_B(u^{(0)})) \ \leq \ 3C\epsilon\,. \tag{58}$$

Moreover, from **H1**,

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X=\emptyset}} |H_B(u)-H_B(w)| \ = \ 0\,.$$

This and (58) imply that

$$\begin{aligned}&\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} (H_B(u)-H_B(w)) \\ =\ &\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} (H_B(u)-H_B(w)) \\ \leq\ & 3C\epsilon\,.\end{aligned}$$

Since ϵ may be taken arbitrarily small, this gives that u is a ground state.
Let us now suppose that $u\in\mathcal{C}$ is a ground state. Fix $\epsilon>0$ and let X_ϵ as in (55) and (56). We set

$$\tilde u(p) := \begin{cases} u^{(0)}(p) & \text{if } p\in X_\epsilon, \\ u(p) & \text{if } p\in\Lambda-X_\epsilon.\end{cases}$$

Then, since u is a ground state,

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} H_B(u) \;\leq\; \sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon\neq\emptyset}} H_B(\tilde u)\,.$$

Moreover, by **H8**, the fact that $u\in\mathcal{C}$ and (56),

$$\begin{aligned}\sum_{B\in\mathcal{S}} |H_B(\tilde u)-H_B(u^{(0)})| \;&\leq\; C\sum_{p\in\Lambda}|\tilde u_p-u_p^{(0)}|\\ &=\; C\sum_{p\in\Lambda-X_\epsilon}|\tilde u_p-u_p^{(0)}|\\ &\leq\; C\epsilon\,,\end{aligned}$$

for some $C>0$, and so

$$\begin{aligned}S_\sigma(u) \;&\leq\; C\epsilon+\sum_{B\in\mathcal{S}}(H_B(u)-H_B(\tilde u))\\ &\leq\; C\epsilon+0+\sum_{\substack{B\in\mathcal{S}\\ B\cap X_\epsilon=\emptyset}}|H_B(u)-H_B(\tilde u)|\,.\end{aligned}$$

Since, if $B\cap X_\epsilon=\emptyset$, we have that $u_p=\tilde u_p$ for any $p\in B$, it thus follows from **H1** that

$$S_\sigma(u)\;\leq\; C\epsilon+0+0\,.$$

Since ϵ may be taken arbitrary small, we obtain that $S_\sigma(u)\leq 0$, and, in fact, $S_\sigma(u)=0$, thanks to (54), which has been already proved. This ends the proof of Lemma 40. □

The renormalized energy "behaves nicely" with respect to pointwise convergence, as we now show:

Lemma 41. *Let $u_n\in\mathcal{C}$ be such that*

$$\lim_{n\to+\infty} u_n(p)\;=\;u(p)\,,$$

for any $p\in\Lambda$. Then,

$$\lim_{n\to+\infty}\sum_{p\in\Lambda}|u_n(p)-u(p)|\;=\;0\,, \tag{59}$$

$$\lim_{n\to+\infty} S_\sigma(u_n)\;=\;S_\sigma(u) \tag{60}$$

and

$$\lim_{n\to+\infty}\sum_{p\in\Lambda}\Big(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(u_n)\Big)^2=\sum_{p\in\Lambda}\Big(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(u)\Big)^2. \tag{61}$$

Proof. By (51), fixed $\epsilon>0$, we can find a finite set $X_\epsilon=\{p_\epsilon^{(1)},\dots,p_\epsilon^{(\ell_\epsilon)}\}$ in such a way that

$$\sum_{p\in\Lambda-X_\epsilon}|u^{(1)}-u^{(0)}|\;\leq\;\epsilon\,.$$

Then,

$$\begin{aligned} & \lim_{n\to+\infty} \sum_{p\in\Lambda} |u_n(p) - u(p)| \\ \leq \; & \lim_{n\to+\infty} \sum_{p\in X_\epsilon} |u_n(p) - u(p)| + \epsilon \\ = \; & \lim_{n\to+\infty} \sum_{k=1}^{\ell_\epsilon} |u_n(p_\epsilon^{(k)}) - u(p_\epsilon^{(k)})| + \epsilon \\ = \; & 0 + \epsilon \,, \end{aligned}$$

due to (38).
By taking ϵ as small as we wish, we thus obtain the proof of (59).
Furthermore, since, by Lemma 40, the series in (53) is absolutely convergent,

$$S_\sigma(u) - S_\sigma(u_n) = \sum_{B\in\mathcal{S}} \Big(H_B(u) - H_B(u_n) \Big) \,.$$

Hence, using **H8**,

$$\begin{aligned} & |S_\sigma(u) - S_\sigma(u_n)| \\ \leq \; & \sum_{B\in\mathcal{S}} \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1 - \tau u_n)) \right| \, d\tau \, |u(p) - u_n(p)| \\ = \; & \sum_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1 - \tau u_n)) \right| \, d\tau \, |u(p) - u_n(p)| \\ \leq \; & C \sum_{p\in\Lambda} |u(p) - u_n(p)| \,, \end{aligned}$$

for a suitable C, which yields (60), due to (59).

Moreover, by using **H8** twice, we have:

$$\begin{aligned}
&\left| \sum_{p\in\Lambda}\left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(u_n)\right)^2 - \sum_{p\in\Lambda}\left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(u)\right)^2\right| \\
\le\ & \sum_{p\in\Lambda}\left[\left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\left|\frac{\partial H_B}{\partial u_p}(u_n)\right| + \left|\frac{\partial H_B}{\partial u_p}(u)\right|\right)\cdot\left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\left|\frac{\partial H_B}{\partial u_p}(u_n) - \frac{\partial H_B}{\partial u_p}(u)\right|\right)\right] \\
\le\ & 2C\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{q\in B}\left(\int_0^1\left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\tau u_n+(1-\tau)u)\right|d\tau\cdot|u_n(q)-u(q)|\right) \\
=\ & 2C\sum_{q\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni q}}\sum_{p\in B}\left(\int_0^1\left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\tau u_n+(1-\tau)u)\right|d\tau\cdot|u_n(q)-u(q)|\right) \\
\le\ & 2C^2\sum_{q\in\Lambda}|u_n(q)-u(q)|\,,
\end{aligned}$$

for some $C>0$. This completes the proof of (61), by exploiting (59) once more. $\square$

4.2. **The heat flow.** We now consider the solution of a "heat flow" problem, which, indeed, is the "gradient flow" of our renormalized energy functional.
Given $u\in\mathcal{C}$ and $t>0$, we denote by $\Phi^t(u)$ the configuration $p\mapsto U(p,t)$ that satisfies

$$\begin{cases} \partial_t U(p,t) = -\displaystyle\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(U(\cdot,t)) & \text{for any } p\in\Lambda \text{ and } t>0\,, \\ U(p,0)=u(p) & \text{for any } p\in\Lambda\,. \end{cases} \tag{62}$$

Then, the following properties hold:

Lemma 42. *If $u\in\mathcal{C}$, then Φ^t is well-defined for any $t>0$. Also, $\Phi^t(\mathcal{C})\subset\mathcal{C}$ and*

$$-\frac{d}{dt}S_\sigma\Big(\Phi^t(u)\Big) = \sum_{p\in\Lambda}\Big(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\Big)^2 \ge 0\,. \tag{63}$$

Furthermore, if $u\in\mathcal{C}$, the map

$$t\mapsto \Phi^t(u)$$

belongs to $C([0,1],\mathcal{C})$. Moreover, fixed $T>0$, the map

$$\mathcal{C}\ni u\mapsto \Phi^t(u)$$

is continuous, with uniform modulus of continuity for $t\in[0,T]$.

Proof. Fixed any $p \in \Lambda$,

$$\begin{aligned}
\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left| \frac{\partial H_B}{\partial u_p}(u) - \frac{\partial H_B}{\partial u_p}(v) \right| &\leq \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{q\in B} \int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau u + (1-\tau)v) \right| d\tau \cdot \|u-v\| \\
&\leq C\,\|u-v\|\,,
\end{aligned}$$

for any u, $v \in \mathcal{C}$, due to **H8**. Accordingly, the map

$$\mathcal{C} \ni u \mapsto \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u)$$

is Lipschitz continuous with respect to the sup-norm. Then, the short time existence of the heat flow with continuous dependence on the initial data follows from the theory of ODE's in Banach spaces (see, e.g., [Hal80]). The short time flow may be continued provided that it does not exit from $\mathcal{C}$ (see, e.g., the Prolongation Lemma in [Hal80]). The comparison principle (see, e.g., Lemma 4.3 in [KdlLR97]) shows that $\Phi^t(\mathcal{C}) \subset \mathcal{C}$ and thus that it is possible to continue Φ^t in $\mathcal{C}$.
It thus remains to prove (63). To this extent, recalling **H8**, we fix $C_1, C_2 > 0$ in such a way

$$\sup_{v\in\mathcal{O}_\sigma} \sup_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left| \frac{\partial H_B}{\partial u_p}(v) \right| \leq C_1\,,$$

$$\sup_{v\in\mathcal{O}_\sigma} \sup_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{q\in B} \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(v) \right| \leq C_2\,.$$

From (51), we also have that there exits a collection of finite sets $X_n \subset \Lambda$, so that $X_n \subset X_{n+1}$ for any $n \in \mathbb{N}$, $X_1 \cup X_2 \cup \cdots = \Lambda$, and

$$\sum_{p\in\Lambda - X_n} |u_p^{(1)} - u_p^{(0)}| \leq \frac{1}{n}\,.$$

For any $t \geq 0$, we define

$$f_n(t) := \sum_{\substack{B\in\mathcal{S}\\ B\cap X_n\neq\emptyset}} \Big(H_B(\Phi^t(u)) - H_B(u^{(0)}) \Big) \tag{64}$$

and we now study the convergence properties of f_n and $\dot{f}_n$. First of all,

$$\begin{aligned}
&|f_n(t) - S_\sigma(\Phi^t(u))| \\
= &\left| \sum_{\substack{B\in\mathcal{S} \\ B\cap X_n=\emptyset}} \Big(H_B(\Phi^t(u)) - H_B(u^{(0)}) \Big) \right| \\
\le &\sum_{\substack{B\in\mathcal{S} \\ B\cap X_n=\emptyset}} \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau\Phi^t(u)) + (1-\tau)u^{(0)}) \right| d\tau \cdot |\Phi^t(u)_p - u_p^{(0)}| \\
\le &\sum_{p\in\Lambda - X_n} \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau\Phi^t(u)) + (1-\tau)u^{(0)}) \right| d\tau \cdot |\Phi^t(u)_p - u_p^{(0)}| \\
\le &\; C_1 \sum_{p\in\Lambda - X_n} |\Phi^t(u)_p - u_p^{(0)}| \\
\le &\; \frac{C_1}{n},
\end{aligned}$$

and so

$$f_n(t) \text{ converges to } S_\sigma(\Phi^t(u)) \text{ uniformly in } t \in (0, +\infty). \tag{65}$$

Let now

$$g(t) := -\sum_{p\in\Lambda} \Big(\sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \frac{\partial H_B}{\partial u_p}(\Phi^t(u)) \Big)^2 . \tag{66}$$

Since $u^{(0)}$ is a ground state, and so it solves the variational equation, estimate (51) yields that, fixed any $p \in \Lambda$,

$$\begin{aligned}
&\left| \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \frac{\partial H_B}{\partial u_p}(w) \right| \\
= &\left| \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \frac{\partial H_B}{\partial u_p}(w) - \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u^{(0)}) \right| \\
\le &\sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \sum_{q\in B} \int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau w + (1-\tau)u^{(0)}) \right| d\tau \, |w_q - u_q^{(0)}| ,
\end{aligned}$$

for any $w \in \mathcal{C}$. And so

$$\begin{aligned}
&\sum_{p\in\Lambda}\left|\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(w)\right| \\
\leq\ & \sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{q\in B}\int_0^1\left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\tau w+(1-\tau)u^{(0)})\right|\,d\tau\,|w_q-u_q^{(0)}| \\
=\ & \sum_{q\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni q}}\sum_{p\in B}\int_0^1\left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\tau w+(1-\tau)u^{(0)})\right|\,d\tau\,|w_q-u_q^{(0)}| \qquad (67)\\
\leq\ & C_2\sum_{q\in\Lambda}|w_q-u_q^{(0)}| \\
\leq\ & C_2\,\sharp F\,,
\end{aligned}$$

for any $w \in \mathcal{C}$, where (51) has been used once more.
As a consequence,

$$\sum_{p\in\Lambda}\Big|\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\Big|^2\ \leq\ C_1C_2\,\sharp F\,, \tag{68}$$

which is a bound for the right hand side of (66). From this and the definition of C_1, we have that all the series defining g converge absolutely.
Hence, by rearranging the terms, it follows that

$$-\sum_{B\in\mathcal{S}}\sum_{p\in B}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\left(\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\,\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\right)\ =\ g(t)\,. \tag{69}$$

We now show that

$$\dot f_n(t)\ =\ -\sum_{\substack{B\in\mathcal{S}\\ B\cap X_n\neq\emptyset}}\ \sum_{p\in B}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\,. \tag{70}$$

For this, we observe that, fixed any $t \in (0, +\infty)$,

$$\lim_{\delta\to 0}\Phi^{t+\delta}(u)_q\ =\ \Phi^t(u)_q\,,$$

for any $q \in \Lambda$, thanks to Lemma 42. Therefore, by Lemma 41,

$$\lim_{\delta\to 0}\sum_{q\in\Lambda}|\Phi^{t+\delta}(u)_q-\Phi^t(u)_q|\ =\ 0\,. \tag{71}$$

Moreover, for any $t > 0$ and any $\delta, \delta' \in \mathbb{R}$ such that $t+\delta$, $t+\delta' > 0$, we have that

$$\begin{aligned}
&\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta\tau}(u)) - \frac{\partial H_B}{\partial u_p}(\Phi^t(u))\right| \cdot \left|\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta'\tau}(u))\right| \, d\tau \\
\le\ & \sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\sum_{q\in B}\int_0^1\int_0^1 \left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\xi\Phi^{t+\delta\tau}(u) + (1-\xi)\Phi^t(u))\right| \\
&\cdot \left|\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta'\tau}(u))\right| \cdot |\Phi^{t+\delta\tau}(u)_q - \Phi^t(u)_q| \, d\xi \, d\tau \\
=\ & \sum_{q\in\Lambda}\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p,q}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1\int_0^1 \left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\xi\Phi^{t+\delta\tau}(u) + (1-\xi)\Phi^t(u))\right| \\
&\cdot \left|\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta'\tau}(u))\right| \cdot |\Phi^{t+\delta\tau}(u)_q - \Phi^t(u)_q| \, d\xi \, d\tau \\
\le\ & C_1 \sum_{q\in\Lambda}\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p,q}}\int_0^1\int_0^1 \left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\xi\Phi^{t+\delta\tau}(u) + (1-\xi)\Phi^t(u))\right| \\
&\cdot|\Phi^{t+\delta\tau}(u)_q - \Phi^t(u)_q| \, d\xi \, d\tau \\
\le\ & C_1 C_2 \sum_{q\in\Lambda} |\Phi^{t+\delta\tau}(u)_q - \Phi^t(u)_q| \,.
\end{aligned}$$

This estimate and (71) give that

$$\begin{aligned}
&\lim_{\delta\to 0}\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta\tau}(u)) - \frac{\partial H_B}{\partial u_p}(\Phi^t(u))\right| \\
&\cdot \left|\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta\tau}(u))\right| \, d\tau \ = \ 0 \,.
\end{aligned} \tag{72}$$

Also,

$$\begin{aligned}
&\frac{f_n(t+\delta) - f_n(t)}{\delta} \\
=\ & \sum_{\substack{B\in\mathcal{S}\\ B\cap X_n\neq\emptyset}} \frac{H_B(\Phi^{t+\delta}(u)) - H_B(\Phi^t(u))}{\delta} \\
=\ & \sum_{\substack{B\in\mathcal{S}\\ B\cap X_n\neq\emptyset}} \sum_{p\in B} \frac{1}{\delta}\int_0^\delta \frac{\partial H_B}{\partial u_p}(\Phi^{t+\tau}(u)) \Big(\frac{\partial}{\partial t}\Phi^{t+\tau}(u)\Big)_p \, d\tau \\
=\ & -\sum_{\substack{B\in\mathcal{S}\\ B\cap X_n\neq\emptyset}} \sum_{p\in B}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1 \frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta\tau}(u))\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta\tau}(u)) \, d\tau \,,
\end{aligned}$$

thanks to (62).

Therefore, by (72),

$$
\begin{aligned}
&\lim_{\delta\to 0}\left|\frac{f_n(t+\delta)-f_n(t)}{\delta}+\sum_{\substack{B\in\mathcal{S}\\ B\cap X_n\neq\emptyset}}\sum_{p\in B}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\right| \\
\le\;& \lim_{\delta\to 0}\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1\left|\frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta\tau}(u))-\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\right|\cdot\left|\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta\tau}(u))\right|\,d\tau \\
&+\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1\left|\frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta\tau}(u))-\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\right|\cdot\left|\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\right|\,d\tau \\
=\;& 0\,,
\end{aligned}
$$

thus yielding the proof of (70).
Thus, from (69) and (70),

$$
\begin{aligned}
&|\dot f_n(t)-g(t)| \\
=\;&\left|\sum_{\substack{B\in\mathcal{S}\\ B\cap X_n=\emptyset}}\sum_{p\in B}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\right| \\
=\;&\left|\sum_{p\in\Lambda}\sum_{\substack{B\in\mathcal{S}\\ B\ni p\\ B\cap X_n=\emptyset}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\right|\,.
\end{aligned}
\tag{73}
$$

Thence, if we set

$$\alpha_p(t)\;:=\;\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_A}{\partial u_p}(\Phi^t(u))\,,$$

we have that

$$\sum_{p\in\Lambda}|\alpha_p(t)|\;\le\;C_2\,\sharp F\,,\tag{74}$$

for any $t\in(0,+\infty)$, due to (67), and so

$$\lim_{n\to+\infty}\sum_{p\in\Lambda-X_n}|\alpha_p(t)|\;=\;0\,,\tag{75}$$

for any $t \in (0, +\infty)$. Also, from (70),

$$\begin{aligned}
|\dot{f}_n(t) - g(t)| &\leq \left| \sum_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S} \\ B\ni p \\ B\cap X_n=\emptyset}} \Big(\alpha_p(t) \frac{\partial H_B}{\partial u_p}(\Phi^t(u)) \Big) \right| \\
&= \left| \sum_{p\in\Lambda - X_n} \sum_{\substack{B\in\mathcal{S} \\ B\ni p \\ B\cap X_n=\emptyset}} \Big(\alpha_p(t) \frac{\partial H_B}{\partial u_p}(\Phi^t(u)) \Big) \right| \\
&\leq \sum_{p\in\Lambda - X_n} \left(|\alpha_p(t)| \cdot \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \left| \frac{\partial H_B}{\partial u_p}(\Phi^t(u)) \right| \right) \\
&\leq C_1 \sum_{p\in\Lambda - X_n} |\alpha_p(t)| \,.
\end{aligned}$$

From this and (75), we conclude that $\dot{f}_n$ converges to g pointwise in $(0, +\infty)$. Moreover, by the latter estimate, (68) and (74), we have that

$$|\dot{f}_n(t)| \leq |g(t)| + C_1 \sum_{p\in\Lambda} |\alpha_p(t)| \leq 2C_1C_2\sharp F \,.$$

Thence, by the Dominated Convergence Theorem,

$$\lim_{n\to+\infty} \int_0^{+\infty} \dot{f}_n(t)\varphi(t)\,dt \;=\; \int_0^{+\infty} g(t)\varphi(t)\,dt\,,$$

for any $\varphi \in C_0^\infty(0, +\infty)$. Then, by (65),

$$\begin{aligned}
&\int_0^{+\infty} S_\sigma(\Phi^t(u))\dot{\varphi}(t)\,dt = \lim_{n\to+\infty} \int_0^{+\infty} f_n(t)\dot{\varphi}(t)\,dt \\
&= -\lim_{n\to+\infty} \int_0^{+\infty} \dot{f}_n(t)\varphi(t)\,dt = -\int_0^{+\infty} g(t)\varphi(t)\,dt\,,
\end{aligned}$$

for any $\varphi \in C_0^\infty(0, +\infty)$.
This implies that $\frac{d}{dt} S_\sigma(\Phi^t(u)) = g$ (in the sense of distributions and thence pointwise), which, together with (66) gives (63). $\square$

Given any $s \in [0, 1]$, we now consider the linear interpolation between $u^{(0)}$ and $u^{(1)}$, by defining

$$u^{(s)} \;:=\; (1-s)\,u^{(0)} \;+\; s\,u^{(1)}\,.$$

It is easily seen that $u^{(s)} \in \mathcal{C}$. Furthermore, the following compactness result holds:

Corollary 43. *Fixed $s \in [0,1]$, there exists a sequence $t_{n,s}$ and a configuration $u_\star^{(s)} \in \mathcal{C}$, which is a solution of our variational equation, so that*

$$\lim_{n\to+\infty} t_{n,s} = +\infty$$

$$\textit{and} \qquad \lim_{n\to+\infty} \left\| \Phi^{t_{n,s}}(u^{(s)}) - u_\star^{(s)} \right\| = 0\,.$$

Proof. Let $f(t) := S_\sigma(\Phi^t(u))$. By Lemmata 40 and 42, we know that f is decreasing and bounded by below. Thus,

$$\lim_{t\to+\infty} f(t) = \inf_{[0,\infty)} f\,.$$

Then, there must exists a divergent sequence of times $t_{n,s}$ so that

$$\lim_{n\to+\infty} f'(t_{n,s}) = 0\,. \tag{76}$$

If not, one would have that $f'(t) \le -c$ for some $c > 0$ and any t large enough, but then

$$0 = \lim_{t\to+\infty} f(t+1) - f(t) \le -c\,,$$

which is a contradiction.
Further, by Lemma 42 and Corollary 39, we may suppose, possibly taking subsequences, that $\Phi^{t_{n,s}}(u^{(s)})$ converges in the sup-norm, say to a suitable $u_\star^{(s)} \in \mathcal{C}$. By Lemma 41, (76), (63) and (11), we have that $u_\star^{(s)}$ is a solution of our variational equation. $\square$

4.3. **The basin of attraction of the heat flow.** We now follow the variational pattern performed in [dlLV05]. For this, we consider the "basin of attraction" of the configurations $u^{(0)}$ and $u^{(1)}$ with respect to the heat flow Φ^t. Namely, for $i \in \{0,1\}$, we define

$$\beta_i := \left\{ s \in [0,1] \quad | \quad \lim_{t\to+\infty} \left\| \Phi^t(u^{(s)}) - u^{(i)} \right\| = 0 \right\}.$$

Note that, since $u^{(i)}$ is a solution of our variational equation, then $\Phi^t(u^{(i)}) = u^{(i)}$, and so $i \in \beta_i$, for $i \in \{0,1\}$. We also consider the configurations in $\mathcal{C}$ with zero renormalized energy, that is the set

$$\mu_\sigma := \left\{ u \in \mathcal{C} \quad | \quad S_\sigma(u) = 0. \right\}.$$

We now introduce the set of the solutions of our variational problem which belong to $\mathcal{C}$ and whose S_σ-value is less than r. More precisely, given $r > 0$, we define

$$\nu_\sigma(r) := \Big\{ u \in \mathcal{C} \quad | \quad u \text{ is a solution of our}$$
$$\text{variational equation and } S_\sigma(u) \le r \Big\}.$$

By Lemma 40, we have that

$$u^{(0)},\ u^{(1)} \in \mu_\sigma \subset \nu_\sigma(r)\,,$$

for any $r > 0$. We now investigate the properties of the basins of attraction when μ_σ and $\nu_\sigma(r)$ are "as small as possible".

Lemma 44. *Assume that*

$$\mu_\sigma - \{u^{(0)}, u^{(1)}\} = \emptyset\,. \tag{77}$$

Let $r > 0$ and suppose that there exists $\gamma \in C([0,1], \mathcal{C})$ such that $\gamma(0) = u^{(0)}$, $\gamma(1) = u^{(1)}$ and $S_\sigma(\gamma(t)) \le r$ for any $t \in [0,1]$. Then, $r \ge r_0$, for some $r_0 > 0$.

Proof. We argue by contradiction. Suppose that, for any $n \in \mathbb{N}$, there exists

$$\gamma_n \in C([0,1], \mathcal{C}) \tag{78}$$

such that

$$\gamma_n(0) = u^{(0)},\ \gamma_n(1) = u^{(1)} \text{ and} \tag{79}$$

$$S_\sigma(\gamma_n(t)) \le 1/n \text{ for any } t \in [0,1]. \tag{80}$$

Let us define

$$g_n(t) := \|\gamma_n(t) - u^{(0)}\|\,.$$

Then, $g_n \in C([0,1])$, thanks to (78). Also, $g_n(0) = 0$ and $g_n(1) = \|u^{(1)} - u^{(0)}\|$, due to (79). Thus, there exists $t_n \in [0,1]$ in such a way

$$g_n(t_n) = \frac{\|u^{(1)} - u^{(0)}\|}{2} =: c\,.$$

That is,

$$\|\gamma_n(t_n) - u^{(0)}\| = c\,. \tag{81}$$

We recall that $c > 0$, since $u^{(0)} \not\equiv u^{(1)}$.
Moreover, by Corollary 39, perhaps taking subsequences, we have that $\gamma_n(t_n)$ converges in the sup-norm to some $\bar{u} \in \mathcal{C}$. By (81), $\|\bar{u} - u^{(0)}\| = c$ and thus $\bar{u} \notin \{u^{(0)}, u^{(1)}\}$. Thus, by (77), $S_\sigma(\bar{u}) > 0$. On the other hand, by (80) and Lemma 41, $S_\sigma(\bar{u}) = 0$. This contradiction yields the desired result. □

Lemma 45. *Assume that*

$$\nu_\sigma(r_0) - \{u^{(0)}, u^{(1)}\} = \emptyset\,, \tag{82}$$

where r_0 is as in Lemma 44. Then, there exists $r_1 > 0$ such that the following holds. If $v \in \mathcal{C}$ and $\|v - u^{(0)}\| \le r_1$, then

$$\lim_{t \to +\infty} \|\Phi^t(v) - u^{(0)}\| = 0\,.$$

Analogously, if $v \in \mathcal{C}$ and $\|v - u^{(1)}\| \le r_1$, then

$$\lim_{t \to +\infty} \|\Phi^t(v) - u^{(1)}\| = 0\,.$$

Proof. We prove the first assertion, the second one following analogously. The proof is by contradiction: assume that there exists $v \in \mathcal{C}$ such that

$$\|v - u^{(0)}\| \le r_1 \tag{83}$$

in such a way

$$\Phi^t(v) \text{ does not converge to } u^{(0)} \text{ uniformly.} \tag{84}$$

Here, r_1 will be a suitably small quantity, defined as follows. By (51), we can find a finite set $X_{r_0} \subset \Lambda$ in such a way that

$$\sum_{p\in\Lambda - X_{r_0}} |u_p^{(1)} - u_p^{(0)}| \le \frac{r_0}{4C}, \tag{85}$$

where C is the quantity in (9). We then set

$$r_1 := \min\left\{ r_0 \, , \, \frac{r_0}{4C\sharp X_{r_0}} \right\} .$$

It follows that

$$\sup_{\substack{w\in\mathcal{C} \\ \|w-u^{(0)}\|\le r_1}} S_\sigma(w) \le \frac{r_0}{2} . \tag{86}$$

Indeed, if w is as here above, we get from **H8** and (85) that

$$\begin{aligned}
S_\sigma(w) &\le \sum_{B\in\mathcal{S}} \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau w + (1-\tau)u^{(0)}) \right| d\tau \cdot |w_p - u_p^{(0)}| \\
&\le \sum_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau w + (1-\tau)u^{(0)}) \right| d\tau \cdot |u_p^{(1)} - u_p^{(0)}| \\
&\le C \sum_{p\in\Lambda} |u_p^{(1)} - u_p^{(0)}| \\
&\le C \sum_{p\in X_{r_0}} |u_p^{(1)} - u_p^{(0)}| + \frac{r_0}{4} \\
&\le C \, \|u^{(1)} - u^{(0)}\| \, \sharp X_{r_0} + \frac{r_0}{4} \\
&\le \frac{r_0}{2} ,
\end{aligned}$$

thus proving (86)
From this and (83),

$$S_\sigma(v) \le \frac{r_0}{2} , \tag{87}$$

provided that r_1 is small enough with respect to r_0.
Also, recalling Corollary 43, possibly extracting a subsequence, we have that $\Phi^{t_n}(v)$ uniformly converges to a suitable configuration $\bar{u} \in \mathcal{C}$, and that

$$\bar{u} \text{ is a solution of our variational equation.} \tag{88}$$

Furthermore, it follows from (87) and Lemma 42 that

$$S_\sigma(\Phi^t(v)) \le \frac{r_0}{2} \text{ for any } t \ge 0. \tag{89}$$

Consequently, by Lemma 41,

$$S_\sigma(\bar{u}) \le \frac{r_0}{2}. \tag{90}$$

Note that $\bar{u}$ cannot be equal to $u^{(0)}$, because of (84); thus, (88), (90) and (82) imply that $\bar{u} = u^{(1)}$.
Therefore, there exists $T > 0$ such that

$$\|\Phi^T(v) - u^{(1)}\| = \|\Phi^T(v) - \bar{u}\| \le r_1 \,. \tag{91}$$

Let now

$$\tilde{\gamma}(t) := \Phi^{2T(t-1/4)}(v)\,, \tag{92}$$

for any $t \in [1/4,\ 3/4]$.
Then, $\tilde{\gamma} \in C([1/4,\ 3/4], \mathcal{C})$, thanks to Lemma 42. Also, $\tilde{\gamma}(1/4) = v$ and $\tilde{\gamma}(3/4) = \Phi^T(v)$. Let

$$\gamma(t) := \begin{cases} u^{(0)} + 4t(v - u^{(0)}) & \text{if } t \in [0,\ 1/4), \\ \tilde{\gamma}(t) & \text{if } t \in [1/4,\ 3/4], \\ \Phi^T(v) + (4t-3)(u^{(1)} - \Phi^T(v)) & \text{if } t \in (3/4,\ 1]. \end{cases}$$

Then,

$$\gamma \in C([0,\ 1], \mathcal{C}),\ \gamma(0) = u^{(0)} \text{ and } \gamma(1) = u^{(1)}. \tag{93}$$

Moreover, if $t \in [0,\ 1/4)$,

$$\|\gamma(t) - u^{(0)}\| \le \|v - u^{(0)}\| \le r_1 \,,$$

due to (83), and, if $t \in (3/4,\ 1]$

$$\|\gamma(t) - u^{(1)}\| \le \|\Phi^T(v) - u^{(1)}\| \le r_1 \,,$$

due to (91). Therefore, making use of (86), we gather that $S_\sigma(\gamma(t)) \le r_0/2$, for $t \in [0,\ 1/4) \cup (3/4,\ 1]$, provided that r_1 is small enough. In fact, recalling (89) and (92),

$$S_\sigma(\gamma(t)) \le r_0/2 \text{ for any } t \in [0,\ 1]. \tag{94}$$

But the existence of a path γ satisfying (93) and (94) is in contradiction with Lemma 44. Notice indeed that the assumptions of Lemma 44 are fulfilled here, since (82) implies (77).
This contradiction ends the proof of Lemma 45. □

Lemma 46. *Assume* (82)*. Then, β_0 and β_1 are open in the relative topology of* $[0, 1]$.

Proof. We prove that β_0 is open, the argument for β_1 being analogous. Let $\bar{s} \in \beta_0$ and take

$$s \in [0,\ 1] \cap [\bar{s} - \epsilon,\ \bar{s} + \epsilon]\,.$$

We show that $s \in \beta_0$, provided that $\epsilon > 0$ is sufficiently small. For this, assume, by contradiction, that $s \notin \beta_0$. Then, there exists a sequence $t_n \to +\infty$ in such a way

$$\Phi^{t_n}(u^{(s)}) \text{ does not converge uniformly to } u^{(0)}. \tag{95}$$

Now, since $\bar{s} \in \beta_0$, there exists $\bar{n} \in \mathbb{N}$ such that

$$\|\Phi^{t_{\bar{n}}}(u^{(\bar{s})}) - u^{(0)}\| \le \frac{r_1}{2}\,,$$

where r_1 is the quantity given in Lemma 45. By the continuity with respect to the initial data (recall Lemma 42), we thus have that

$$\|\Phi^{t_{\bar{n}}}(u^{(s)}) - u^{(0)}\| \leq r_1 \,,$$

provided that ϵ is small enough.
Therefore, by Lemma 45, it follows that $\Phi^t(\Phi^{t_{\bar{n}}}(u^{(s)}))$ uniformly converges to $u^{(0)}$. That is, $\Phi^{t_n}(u^{(s)})$ uniformly converges to $u^{(0)}$. Since the latter assertion is in contradiction with (95), the proof of the desired result is complete. □

Corollary 47. *Suppose that* (82) *holds. Then, there exists* $s \in [0,1] - (\beta_0 \cup \beta_1)$.

Proof. Assume, by contradiction, that $[0,1] = \beta_0 \cup \beta_1$. Let

$$\bar{s} := \sup\{s \in \beta_0\}\,. \tag{96}$$

By Lemma 46, β_0 and β_1 are non-empty, disjoint, relatively open subsets of $[0,1]$. Since $1 \in \beta_1$ and β_1 is open, we have that

$$\bar{s} < 1\,. \tag{97}$$

Analogously, since $0 \in \beta_0$ and β_0 is open, we have that

$$\bar{s} > 0\,. \tag{98}$$

Thus, in the light of (96), there are two cases to distinguish: either $\bar{s} \in \beta_0$ or $\bar{s} \in \beta_1$.
If $\bar{s} \in \beta_0$, then, since β_0 is relatively open, it follows from (97) that $\bar{s} + \epsilon \in \beta_0$, provided that $\epsilon > 0$ is sufficiently small. But this contradicts (96).
If, on the other hand, $\bar{s} \in \beta_1$, the fact that β_1 is open and (98) imply that $\bar{s} - \epsilon \in \beta_1$, for any $\epsilon > 0$ sufficiently small. But this again contradicts (96).
These contradictions give the proof of Corollary 47. □

4.4. **End of the proof of Theorem 10.** We distinguish two alternatives: either (82) is violated or it does hold.
If (82) is violated, take

$$u \in \nu_\sigma(r_0) - \{u^{(0)},\, u^{(1)}\}\,.$$

Then, by construction, $u \in \mathcal{B}_\sigma$, u is a solution of our variational equation, $u^{(0)} \leq u \leq u^{(1)}$ and u does not coincide with either $u^{(0)}$ or $u^{(1)}$. Thus, u fulfills the claim of Theorem 10.
If, on the other hand, (82) holds, then there exists

$$s \in [0,1] - (\beta_0 \cup \beta_1)\,.$$

Therefore, by Corollary 39, there exists a sequence $t_n \to +\infty$ in such a way that $\Phi^{t_n}(u^{(s)})$ uniformly converges to a suitable $u \in \mathcal{C}$, which does not coincide with either $u^{(0)}$ or $u^{(1)}$. Then, the fact that $u \in \mathcal{C}$ gives that $u \in \mathcal{B}_\sigma$, and that $u^{(0)} \leq u \leq u^{(1)}$. Also, Corollary 43 implies that u is a solution of our variational equation. Hence u fulfills the claim of Theorem 10 in this case too. □

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