Limit Laws and Recurrence for the Planar Lorentz Process with Infinite Horizon

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Dedicated to Ya. G. Sinai on the occasion of his seventieth birthday

Abstract

As Bleher, [B 92] observed the free flight vector of the planar, infinite horizon, periodic Lorentz process $\{S_n | n = 0, 1, 2, \dots\}$ belongs to the non-standard domain of attraction of the Gaussian law — actually with the $\sqrt{n \log n}$ scaling. Our first aim is to establish his conjecture that, indeed, $\frac{S_n}{\sqrt{n \log n}}$ converges in distribution to the Gaussian law (a Global Limit Theorem). Here the recent method of Bálint and Gouëzel, [BG 06] helped us to essentially simplify the ideas of our earlier sketchy proof [SzV 04b]. Moreover, we can also derive a.) the local version of the Global Limit Theorem, b.) the recurrence of the planar, infinite horizon, periodic Lorentz process, and finally c.) the ergodicity of its infinite invariant measure.

Key words: Lorentz process, periodic configuration of scatterers, infinite horizon, corridors, non-normal domain of attraction of the gaussian law, local limit law, recurrence, ergodicity

1 Introduction

The Lorentz process is the $\mathbb{Z}^d-$covering of a Sinai billiard — in other words of a dispersing billiard — given on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. If the horizon is finite, i. e. the free flight vector $\psi(x)$ is bounded, then the Lorentz process is a most instructive model of the Brownian motion.

We note here that though this should be true in any dimension $d \geq 2$, mathematical results only exist for $d = 2$. In this paper we will also restrict our attention to this case. As suggested by the beautiful — partially rigorous, partially heuristic — work of Bleher [B 92], in the infinite horizon case ($d = 2$!) the asymptotic behaviour of the displacement S_n, taken in the moment of the n^{th} reflection from a scatterer, is slightly superdiffusive and $\frac{S_n}{\sqrt{n \log n}}$ was expected to possess a limiting Gaussian distribution. The first main aim of the present work is to provide the first rigorous proof of this statement.

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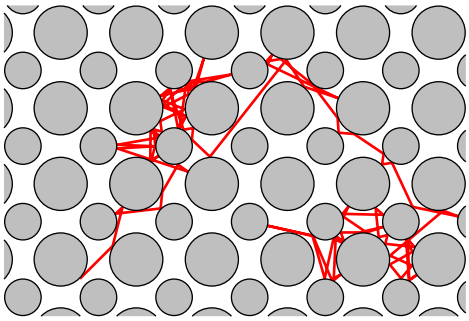

Figure 1: 100 collisions Lorentz trajectory segment with circular scatterers and finite horizon

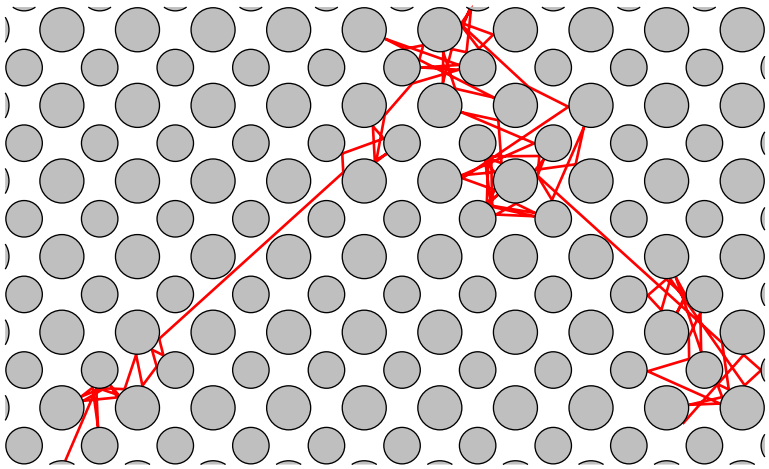

Figure 2: 100 collissions Lorentz trajectory-segment with circular scatterers and infinite horizon

Theorem 1 (Global limit theorem)**.** *Suppose that the direction vectors of infinite, collision-free flights span the plane. Let A be a bounded open set in the plane. Then*

$$\mu\left(\frac{S_n}{\sqrt{n\log n}} \in A\right) \to \int_A \phi_\Sigma(z)dz$$

where ϕ is a nondegenerate gaussian density with zero expectation and covariance matrix Σ.

A mostly geometric proof of this theorem was sketched in [SzV 04b]. However, before we had completed our paper with the technical proof of Theorem 1, there appeared a much interesting work of Bálint and Gouëzel, [BG 06]: for the stadium billiard they gave a quite analytic proof of a global limit theorem which also uses the $\sqrt{n\log n}$ scaling. The coincidence of scalings is explained by the analogous behaviour of long free flights in our model (i. e. in corridors of the Lorentz process) and that of the quasi integrable trajectories between the linear sides of the stadium billiard. The arguments of [BG 06] also helped us to simplify our approach substantially at essentially three points:

1. once one has a tower construction à la Young, their Lemma 3.5 (cf. our Theorem 13) provides a general, concise condition for the validity of a non-standard Gaussian limit law;

2. they reduce the "tower-sums" to a more tractable, still dominant part;

3. for describing excursions from the tail they use a delicate result of Chernov, [Ch 99] (see in our Lemma 16).

It is a question, though, whether Chernov's result, a beautiful but quite strong tool, is indeed, necessary in this proof.

A well-known spectacular property of discrete approximations of the Brownian motion, like of simple symmetric random walks, is Pólya's classical theorem, claiming recurrence in the planar case.

For the diffusive, finite horizon case of the Lorentz process this had been proved by various authors in [Sch 98], [Con 99], [SzV 04a]. (As a matter of fact, the problem was raised by Sinai in 1979, after he, with Bunimovich, had proved its convergence to the Brownian motion. In 1985 — as a partial solution — the first named author jointly with A. Krámli could settle a weaker property: quasi-recurrence of the planar, finite horizon Lorentz process, cf. [KSz 85].)

The arguments of [SzV 04a] relied on a local version, LCLT of the central limit theorem (CLT). Our second main aim here is to first deduce a local limit theorem (Theorem 2, a LLT), a local version of Theorem 1 and then by using it to establish the recurrence of the planar Lorentz process with an infinite horizon (Theorem 3).

Theorem 2 (Local limit theorem)**.** *Suppose that the direction vectors of infinite, collision-free flights span the plane. Let $k_n \in \mathbb{Z}^2$ such that $\frac{k_n}{\sqrt{n\log n}} \to k$. Then for the discretised position vector S_n^κ the following holds:*

$$n\log n \;\; \mu(S_n^\kappa = k_n) \to \phi_\Sigma(k)$$

where ϕ_Σ is a nondegenerate gaussian density with zero expectation from Theorem 1.

Theorem 3 (Recurrence)**.**

$$\mu(\exists n_k \to \infty \quad S^{\kappa}_{n_k} = 0) = 1$$

It is worth mentioning that, for the planar Lorentz process with a finite horizon but with a finite modification of a periodic scatterer configuration, the problem of recurrence and of ergodicity was studied by Lenci [L 03].

The recurrence itself also has an additional interesting conclusion. Let us note first that for the Lorentz process strong stochastic properties, like correlation decay, limit laws, etc. could only be obtained in the case of a periodic configuration of scatterers for then its factor is a Sinai billiard. For this same case, however, it is an interesting question whether the Lorentz dynamics is ergodic without this factorization as well (N. B. in this case the invariant measure is infinite!). The combination of Theorem 3 with an old result of Simányi, [Sim 89] also provides the answer:

Theorem 4. *The planar Lorentz dynamics with a periodic configuration of scatterers is ergodic.*

The paper is organized as follows. Section 2 is devoted to the description of the model and to the study of its important geometric properties. In fact, we recall the notion of corridors also discussed in Bleher's work; they are distinguished parts of the phase space in whose neighbourhood the free flight vector becomes unbounded. Here we also introduce new coordinates to make easier the study of the tail behaviour of the free flight vector and its asymptotic Markov property. In particular, our geometric probabilistic estimates are sharper than those of [B 92]. Finally Section 2 closes with a brief reminder about the non-normal domain attraction of the normal law. Section 3 is preparatory to the technique used: it contains a reminder on Young towers and on the Fourier transform of the Perron-Frobenius operator. Section 4 contains the proof of Theorem 1, while section 5 those of Theorems 2, 3 and 4.

2 The model and its geometry

2.1 The model

Consider finitely many scatterers $\mathcal{O}_i$ on the 2-torus, (also called obstacles) $\mathbb{T}^2 \supset \mathcal{O} = \cup \mathcal{O}_i$ such that each of the scatterers is strictly convex with a $\mathcal{C}^3$-smooth boundary. Let $n(q)$ denote the unit normal vector of the boundary $\partial\mathcal{O}$ at the point q, directed outwards $\mathcal{O}$. The phase space of the system is:

$$X = \{(q \in \partial\mathcal{O}, v \in \mathbb{R}^2) \mid |v| = 1, \langle v, n(q)\rangle \geq 0\}.$$

The dynamics $T : X \to X$ is uniform motion with v velocity vector followed by an elastic collision (v is mirrored to the tangent line at the point of impact). This system has a natural invariant measure: if we denote by l the total length of $\partial\mathcal{O}$, then $d\mu = \frac{1}{2l} \langle v, n(q)\rangle \, dqdv$, is an invariant probability measure, since $\int_X \langle v, n(q)\rangle \, dqdv = 2l$. The normalizing constant $\frac{1}{2l}$ will be denoted by c_μ.

This phase space will be identified with a finite number of cylinders $\partial\mathcal{O} \times \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. So in this paper if v denotes a velocity of a phase point it is meant as $v \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

The boundary of this phase space consists of tangential collisions denoted by S_0. The dynamics resp. the inverse dynamics is non-continuous in backward resp. forward images of this set. We will denote $S_i = T^i S_0,\ i \in \mathbb{Z}$.

The *planar Lorentz process* is the natural $\mathbb{Z}^2$ cover of the above-described toric billiard. More precisely: consider $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$ the factorization by $\mathbb{Z}^2$. Its fundamental domain D is a square (semi-open, semi-closed) in $\mathbb{R}^2$, so $\mathbb{R}^2 = \cup_{z\in\mathbb{Z}^2}(D+z)$, where $D+z$ is the translated fundamental domain.

We lift the obstacles to the plane (i. e. we take $\tilde{\mathcal{O}} = \Pi^{-1}\mathcal{O}$), and define the phase space $\tilde{X}$, and the dynamics $\tilde{T}$ exactly the same way as above. The *free flight function* $\tilde{\psi} : \tilde{X} \to \mathbb{R}^2$ is defined as follows: $\tilde{\psi}(\tilde{x}) = \tilde{q}(\tilde{T}\tilde{x}) - \tilde{q}(\tilde{x})$. The *discrete free flight function* $\tilde{\kappa} : \tilde{X} \to \mathbb{Z}^2$ is defined as follows: $\tilde{\kappa}(\tilde{x}) = \iota(\tilde{T}\tilde{x}) - \iota(\tilde{x})$, where $\iota(\tilde{x}) = z$ if $\tilde{x} \in D + z$. Observe finally that $\tilde{\psi}$ and $\tilde{\kappa}$ are invariant under the $\mathbb{Z}^2$ action, so there are ψ and κ functions defined on X, such that $\tilde{\psi} = \Pi^*\psi$ and $\tilde{\kappa} = \Pi^*\kappa$. Actually for our purposes it will be more convenient to choose the fundamental domain in such a way that $\partial\tilde{\mathcal{O}} \cap \partial D = \emptyset$. In this way κ will be continuous.

Definition 5. *The system is said to have* finite horizon *if the free flight function is bounded. Otherwise the system is said to have* infinite horizon.

2.2 The corridors, their geometry and the tail of the free flight

From now on we will be only considering the *infinite horizon case*. It had been observed in the physical literature (cf. [FM 84], [BD 85], [Bun 85] and [ZGNR 86]) that, in this case, the anomalous diffusive behaviour is intimately related to the unboundedness of the free flight vector. More precisely it is related to its tail behaviour that is to the behavior of the free flight vector in the neighborhood of collision-free orbits. This motivated Bleher [B 92] to introduce corridors and to study probabilistic geometric estimates of the tail behavior of free flight vectors in these corridors. Our method, however, requires more delicate estimates (the results are also sharper), and therefore we are starting to build up our language.

Figure 3: Free-flight crossing a corridor, and some geometric constants

In the infinite horizon case the only reason for the unboundedness of the free flight is the presence of *corridors*. These are bi-infinite strips in the billiard-table $\mathbb{R}^2 \setminus \tilde{\mathcal{O}}$. The strips are tangent to the obstacles, and their slope is necessarily rational, and, moreover, - up to $\mathbb{Z}^2$ translations - there are finitely many of them. We will suppose the - geometrically generic - condition: for each corridor, and each side of the corridor the tangent obstacles are the images of a *single* scatterer under $\mathbb{Z}^2$ translations. Our results are also valid in the excluded cases, but the geometric constants, which we will calculate would have a more complicated form.

For such a corridor there are four corresponding points in the phase-space, as shown on figure 4. These points are on the boundary of X, and are fixed by

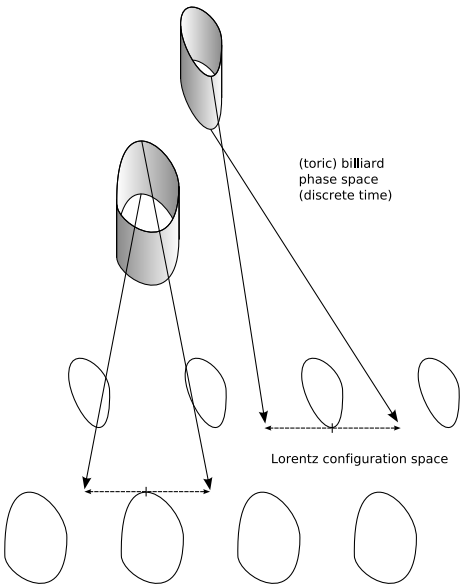

Figure 4: A corridor, and the corresponding four phase points.

the dynamics. (Without the previous condition these would be only periodic points.) Outside of the neigbourhood of these points the free-flight is bounded.

Let us fix such a fixed point on the boundary as $x_0 = (q_0; v_0)$, where v_0 is either $\pi/2$ or $-\pi/2$. Let us denote by $\mathcal{O}_0$ the obstacle on which q_0 is placed. The free flight $\psi(x_0)$ is a lattice vector $\psi(x_0) = \kappa(x_0)$, since $Tx_0 = x_0$. Denote $\kappa(x_0) = w_0$, and the curvature of $\mathcal{O}_0$ at q_0 by ξ_0. Denote the considered (small enough) neighbourhood of x_0 by U_0.

In U_0 (means close enough to x_0) two types of nonsingular collisions can happen. First when the moving particle is "crossing" the corridor (see figure 3). In this case the free-flight is long (actually this is the only case), the closer the phase point lies to x_0 the longer the free-flight can be. The next collision happens on the "other side" of the corridor. To make it precise let us denote by $x_1 = (q_1; v_1)$ the phase point which corresponds to the same corridor as x_0, but $v_1 = -v_0$ and $q_1 \neq q_0$ (see figure 4). Deonte by $\vec{Q}_0$ the planar lattice vector $\iota(\tilde{q}_1) - \iota(\tilde{q}_0)$, where the lifting is such that $\tilde{q}_1 - \tilde{q}_0$ "crosses" the corridor (see figure 3). We also need to define the "width" of the corridor d which is the length of that component of $\tilde{q}_1 - \tilde{q}_0$ which is perpendicular to w_0.

Figure 5: If phase point $T^{-1}x$ is such that $\kappa(x) = w_0$, then if $x \in U_0$ the free flight starting in $q(x)$ "crosses" the corridor.

The other type of nonsingular trajectory in U_0 is when the next collision is on the "same side" of the corridor i. e. $\kappa = w_0$. The phase point Tx is then again close to x_0, but this time it is of the first type, so consecutive "same side" collisions cannot happen. This can be seen on figure 5.

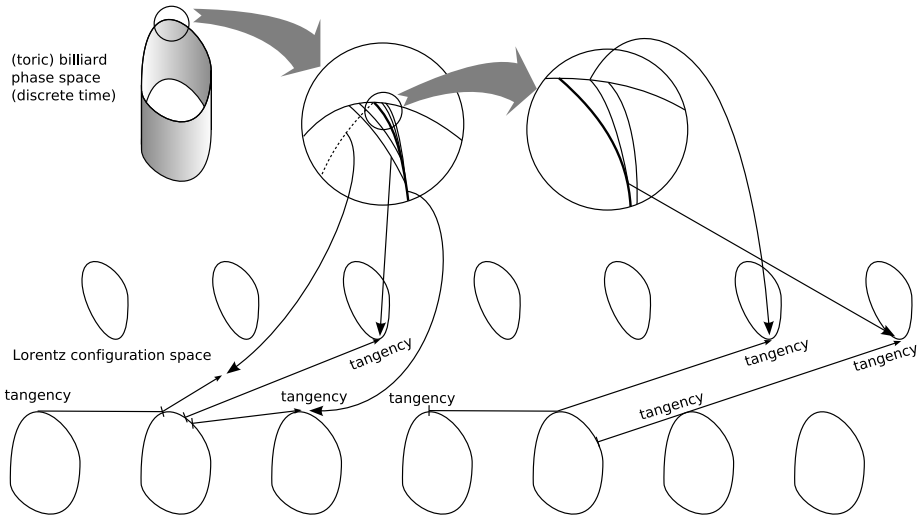

Figure 6: Singularity structure and singular trajectories near x_0

Proposition 6. *Let $U_0 \subset X$ be a sufficiently small neighbourhood of x_0 then*

$$\mu\{x \in U_0 \mid \kappa(x) = Nw_0 + \vec{Q}_0\} \sim c_\mu d^2 |w_0|^{-1} N^{-3}$$

The range of κ in U_0 is $\vec{Q}_0 + w_0\mathbb{Z}^+$ with possibly finitely many exceptions.

We postpone the sketchy proof for the first fact to section 2.4. The second statement was essentially proved in the above text.

2.3 The singularity structure

We are going to describe the singularity structure, and the type of singular trajectories in U_0. The importance of this is that singularities bound the sets for which we want to derive measure estimates. On figure 6 we also plot some trajectory-segments, the configuration component of the corresponding phase point is denoted by a small tick perpendicular to the trajectory.

There is a singularity curve from S_{-1} which is a preimage of tangential collisions, denoted by thick line in figure 6, starting from x_0. This consists of phase points where the next collision will be tangential on $\mathcal{O}_0 + w_0$. This is called the "main" singularity.

There are singularity curves printed on the left of this one, starting on the boundary of the phase space and ending on the main singalarity. These consist of phase points where the free flight "crosses" the corridor, and the next collision is tangential. Therefore this is also a part of S_{-1}. A level-set of the disretized free-flight function κ consists of a curvilinear rectangle bounded by two neighbouring curves from this singularity family, the edge of the phase space and the main singularity.

On the right of the main curve there are some curves from S_{-2}. These also start on the boundary and end on the main curve, but unlike the previous ones, these curves have zero angle with the main line. These consist of phase points, for which the first collision is on $\mathcal{O}_0 + w_0$, and the next one is tangential after

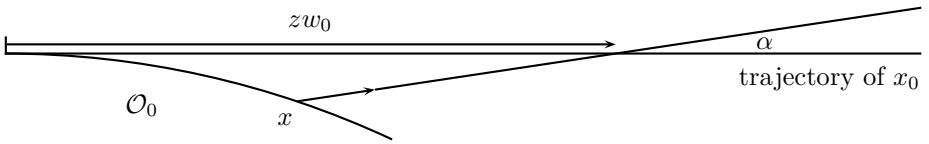

Figure 7: The new coordinates z, α.

crossing the corridor. On the right of the main singularity the next collision for any phase point occurs on $\mathcal{O}_0 + w_0$, therefore κ is constant w_0 in this half of U_0.

These two families of singularity curves have an infinite number of pieces accumulating in x_0. The closer the curve is to x_0 the further the tangential collision occurs, after the moving particle has crossed the corridor. All these singularity curves (including the main one) have $-\xi_0$ as slope on the boundary of the phase space.

The last singularity curve we want to describe is from S_1. It consists of phase points, where the previous collision was tangential on $\mathcal{O}_0 - w_0$. This is the image of the half of $\partial X \cap U_0$, namely that half where $\kappa = w_0$ (on the right of x_0 on figure 6). Consequently (since x_0 is fixed) this curve starts at x_0. This is drawn with a dotted line on figure 6. This curve has slope 2 on the boundary. The sign of the second derivatives of all the singularity curves can be read from the picture.

2.4 New coordinates, and the joint distribution of κ, $\kappa \circ T^{-1}$

In this subsection some proofs will be omitted, some will be sketchy or require further estimates. However, since the missing parts rely on simple but tedious geometrical calculations and the application of these results in later sections does not need sharp estimates, we intended to keep this section not too long.

Instead of using the original (q_0, v_0) coordinates we are going to introdue new (α, z) coordinates in U_0. The *new coordinates* z and α are shown on figure 7 (these coordinates are also different from those of [B 92]). During the free-flight, bouncing off the scatterer $\mathcal{O}_0$, the trajectory of x crosses the trajectory of x_0. This crossing point is therefore $q_0 + zw_0$ for some $z \in \mathbb{R}$. The reader should convince himself that the coordinate z is, in fact, a periodic one with period 1. The other coordinate α is the angle $\measuredangle(w_0, \psi(x))$.

The reason for this change is that these coordinates are more suitable for computations in relation with the free-flight since they are more intrinsically related to the geometry of the model, especially asymptotically (when the free flight goes to infinity). For example, the free flight has the asymptotic form $|\kappa| \sim \frac{d}{\alpha}$ where d is the 'width' of the corridor (cf. fig. 3). Also, the invariant measure is asymptotically equal to $c_\mu |\alpha| |w_0| dz d\alpha$.

We note that the crossing point (which was the base of this coordinatization) does not exist when the next collision occurs on $\mathcal{O}_0 + w_0$. So these new coordinates map only the half of U_0 to the (z, α) plane. Namely that half which is drawn on the left of the main singularity on figure 6. We will denote this part by U_0' (remember that x_0 and U_0 are fixed). However, this restriction does not influence the study of asymptotics, since we miss only the w_0-level set of κ inside U_0.

Proof of Proposition 6. The level-set of κ is a curvilinear rectangle in the (z, α) plane (cf. figure 8). We are going to multiply the height, the width and the density to get the measure. The width is simply 1. The height can be obtained from the formula $\alpha \sim \frac{d}{|\kappa|}$. Writing $\alpha' \sim \frac{d}{|\kappa|+|w_0|}$ we get $\alpha - \alpha' \sim \frac{d|w_0|}{|\kappa|^2}$. The density is $c_\mu|\alpha||w_0|$ and by substituting α we get $c_\mu d\frac{|w_0|}{|\kappa|}$. So the measure is $\sim \frac{c_\mu d^2|w_0|^2}{|\kappa|^3}$. □

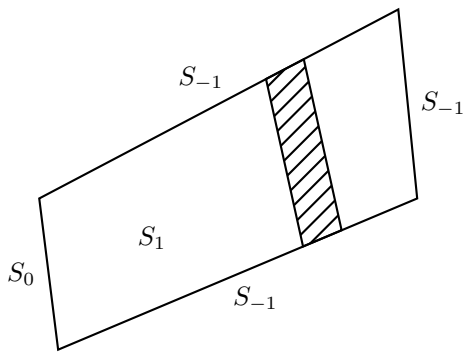

Figure 8: Level set of κ inside U_0, and its intersection with a level set of $\kappa \circ T^{-1}$ (in the (z, α) coordinate plane)

Let us explain how this (z, α) image on figure 8 is related to the phase portrait on figure 6 explained before. The largest curvilinear rectangle on figure 8 is the image of a level-set of κ under the (z, α) coordinate-mapping. This level set is bounded by the boundary of the phase-space on the left, two singularity curves from the first family on the top, and on the bottom, and the main singularity on the right. There is also the dashed line, which is the singularity line from S_1, already explained before, too.

This latter line plays an important role in the joint distribution. On the left of this line $\kappa \circ T^{-1} = w_0$. On the right of this line the mapping T^{-1} takes values in the neighbourhood of another corridor-phase-point x_1 (see subsection 2.2). We are going to use w_1, ξ_1, U_1, U_1' for the point x_1 in the same sense as w_0, ξ_0, U_0, U_0' have been used for the point x_0. Using this notation, on the right of the line we are describing now, there lies TU_1'.

Talking about the joint distribution in terms of our new coordinate functions, note that the α coordinate function is defined in U_0'. Therefore $\alpha \circ T^{-1}$ in the domain $U_0' \cap TU_1'$ has to be meant as applying the same coordinatization rule in U_1. Since $w_1 = w_0$, the α coordinate functions in U_0 and U_1 are comparable also as absolute angles: the observer only has to change signs.

By definition the sign of α is positive, and z is mostly positive. More precisely remember that the asymptotic form of the invariant measure does not depend on z meaning that in the $|\kappa| \to \infty$ limit the distribution of z is uniform. Now consider the range of z in the domain $\kappa = \vec{N}$:

$$z_{\min} \stackrel{\text{def}}{=} \min\{z(x) \mid x \in U_0, \quad \kappa(x) = \vec{N}\}$$

and respectively

$$z_{\max} \stackrel{\text{def}}{=} \max\{z(x) \mid x \in U_0, \quad \kappa(x) = \vec{N}\}$$

Proposition 7. *The asymptotics of the range of z in the domain $\kappa = \vec{N}$ when $\vec{N}$ is in the range of κ (see proposition 6) and $|\vec{N}| \to \infty$ is*

$$z_{\min} \sim 1 - z_{\max} \sim \frac{d}{2|w_0|\xi_0|\vec{N}|}$$

Easy geometrical calculations yield the collision equation:

Proposition 8. *On $U_0' \cap TU_1'$:*

$$\alpha \circ T^{-1} \sim -\alpha + 2\sqrt{\alpha^2 + 2\alpha z|w_0|\xi_0} \qquad (\alpha \to 0)$$

Proposition 9. *In U_0 the following holds:*

$$\min|\kappa| \circ T^{-1} \sim \sqrt{\frac{d|\kappa|}{8\xi_0|w_0|}} \quad \textit{and} \quad \max|\kappa| \circ T^{-1} \sim \frac{8\xi_0|w_0||\kappa|^2}{d} \qquad |\kappa| \to \infty$$

Proof. Substituting the asymptotic maximum of z ($\max z \to 1$ as $\alpha \to 0$) to the collision equation and omitting non-dominant terms we get $\max \alpha \circ T^{-1} \sim 2\sqrt{2\alpha|w_0|\xi_0}$. Since $\alpha \sim \frac{d}{|\kappa|}$ and $\alpha \circ T^{-1} \sim \frac{d}{|\kappa \circ T^{-1}|}$, substituting α and $\alpha \circ T^{-1}$ and then rearranging yields the first statement of the proposition. Using the time-reversion symmetry for this formula we get the second one. □

We can also compute the joint distribution of $(\kappa, \kappa \circ T^{-1})$.

Proposition 10.

$$\mu(\{\kappa = \vec{N}\} \cap \{\kappa \circ T^{-1} = \vec{M}\} \cap U_0) \lesssim \frac{c_\mu d^3|w_0|^2}{4\xi_0} \; \frac{|\vec{N}| + |\vec{M}|}{|\vec{N}|^3|\vec{M}|^3}$$

Proof. For a nonempty intersection we are going to multiply the height the width and the density to get the measure. The density and the height are the same as in the proof of proposition 6. For the width consider the derivative of the collision equation

$$\frac{\partial \alpha \circ T^{-1}}{\partial z} \sim \frac{2\alpha|w_0|\xi_0}{\sqrt{\alpha^2 + 2\alpha z|w_0|\xi_0}}$$

To express the square root in terms of κ and $\kappa \circ T^{-1}$ we can rearrange the collision equation, and substitute α and $\alpha \circ T^{-1}$:

$$\frac{d}{|\kappa \circ T^{-1}|} + \frac{d}{|\kappa|} \sim 2\sqrt{\alpha^2 + 2\alpha z|w_0|\xi_0}$$

We can express the increment of z as the inverse of the derivative multiplied by the increment of $\alpha \circ T^{-1}$. That is

$$\frac{\frac{d}{|\kappa \circ T^{-1}|} + \frac{d}{|\kappa|}}{4\frac{d}{\kappa}|w_0|\xi_0} \; \frac{d|w_0|}{(\kappa \circ T^{-1})^2} \sim d\frac{\kappa + \kappa \circ T^{-1}}{4\xi_0(\kappa \circ T^{-1})^3}$$

Hence the proposition. □

The measure of the set $\{\kappa = \vec{N}\} \cap \{\kappa \circ T^{-1} = \vec{M}\} \cap U_0$ can be zero when $\vec{N}$ or $\vec{M}$ is not in the range of κ inside U_0 (see proposition 6) or they fail the range inequality (we only gave the asymptotics of this in proposition 9 roughly $c_3\sqrt{|\vec{N}|} < |\vec{M}| < c_4|\vec{N}|^2$). It can be also smaller than the expression given in proposition 10 when the pair $(\vec{N}, \vec{M})$ is close to the boundary of the range inequality, but we do not want to formulate the validity precisely, we just mention that in most part of the domain the inequality is sharp. It can also be checked by summing the right hand side, and getting 1 in the limit.

What this essentially means is that the previous free-flight $|\kappa \circ T^{-1}|$ is mostly in the range of $\sqrt{|\kappa|}$. The measure of being in any other range can be estimated from above with $|\kappa|$ powers. To formulate precisely what we will use later in subsection 4.1:

Proposition 11.

$$\mu(\{\kappa = \vec{N}\} \cap \{\kappa \circ T^{-1} > |\kappa|^{\frac{3}{4}}\} \cap U_0) = O(|\vec{N}|^{-3.5})$$

Proof. $\sum_{M=N^{\frac{3}{4}}} \frac{N+M}{N^3M^3} = O(N^{-3.5})$ □

The other level set of $\kappa \circ T^{-1}$ which intersects U_0 is the $\kappa \circ T^{-1} = w_0$ set.

Proposition 12.

$$\mu(\{\kappa = \vec{N}\} \cap \{\kappa \circ T^{-1} = w_0\} \cap U_0) = O(|\vec{N}|^{-4})$$

Proof. As before the measure will be estimated with the product of the height, density and the width. It remained to estimate only the width. The domain is the left-hand side of the dashed line on figure 8. Going from right to left inside the level set in figure 8, the dashed line is reached exactly when $\alpha \circ T^{-1}$ reaches its minimum. According to proposition 9 we have to estimate what is the value of z for which $\alpha \circ T^{-1}$ reaches $\text{const} \cdot \alpha^2$. Exact calculations based on the derivative of the mapping would give $O(|\vec{N}|^{-5})$ in the right hand side of the proposition, but here it is sufficient to observe that this singularity line denoted by the dashed curve is on the left of the $z = 0$ line, where $\alpha = \alpha \circ T^{-1}$. So the width can be estimated by $z_{\min}$ which was given in proposition 7. □

2.5 Non-normal domain of attraction of the normal law

It is well known (cf. [F 66]) that a random variable R with distribution function $\mathbb{P}_R$ belongs to the domain of attraction of a normal distribution if its characteristic function satisfies

$$\log \int e^{itu} d\mathbb{P}_R(u) = it\nu - \frac{1}{2}t^2 L(1/|t|)$$

for some constant $\nu \in \mathbb{R}$ and a slowly varying function $L : \mathbb{R}^+ \to \mathbb{R}^+$ which is bounded below.

The normal (or classical) domain of attraction consists of the class L_2, and is characterized by the boundedness of the slowly varying function L. In the "non-normal" domain of attraction the function L is unbounded and is determined (up to asymptotic equivalence) by the tails of the distribution:

$$\begin{aligned} 1 - \mathbb{P}_R(x) &\sim c_1 x^{-2} l(x) \\ \mathbb{P}_R(x) &\sim c_2 x^{-2} l(x) \quad x \to \infty \end{aligned}$$

for some constants $c_1, c_2 > 0, c_1 + c_2 = 1$ and some slowly vaying function l, which in turn determines L by

$$L(x) = \int_{-x}^{x} u^2 \ d\mathbb{P}_R u.$$

and it follows that $l(x) = o(L(x))$.

For a random variable R in the non-normal domain of attraction of the normal law, the independent sum of $\mathbb{P}_R$-distributed random variables S_n^* satisfies the limit theorem: $\frac{S_n^* - n\nu}{B_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$ as $n \to \infty$ in distribution, where B_n is the normalizing sequence defined by the asymptotics $\frac{nL(B_n)}{B_n^2} \to 1$.

The random variable κ is a vector. It was shown in proposition 6 that its component function belongs to the non-normal domain of attraction of the normal law, if that component is not perpendicular to all of the corridor-free-flights w_i. In this case $l(x)$ is a constant function $l(x) \equiv \mathbf{c}$. Since the free-flight is symmetric, we have $c_1 = c_2 = \frac{1}{2}$, and $\nu = 0$. Consequently $L(x) \sim 2\mathbf{c} \log x$, and $B_n = \sqrt{\mathbf{c} n \log n}$ is a normalizing sequence. The constant $\mathbf{c}$ depends on which component we are looking at. If we choose $\vec{v} \in \mathbb{R}^2$ a unit vector, then the constant of the $\vec{v}$ component has the following expression in the terms of the geometric constants:

$$\mathbf{c} = \sum_{x \in \partial X | Tx = x} c_\mu d_x^2 \frac{\langle \psi(x), \vec{v} \rangle^2}{2|\psi(x)|}. \tag{1}$$

Remember that for such points $\psi(x) = \kappa(x)$, and we used the notation $\psi(x_0) = w_0$ for the fixed corridor we were investigating. Also note, that every term in the above sum appears exactly four times (cf. figure 4 in section 2.2)!

Since the configuration is planar, the following is true: if the corridor free-flight vectors span the plane, then every component of κ, hence the vector itself is in the non-normal domain. If this is not the case, then one has to apply anisotropic scaling to get a nondegenerate limit distribution. Namely it should be $\begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n \log n} \end{pmatrix}$ in a basis, where the first element is perpendicular to the corridor free-flights.

The goal of the forthcoming arguments is to establish Bleher's hypothesis: though the (stationary) process of the free flights of our model is not an independent process, nevertheless in many respects the partial sums behave asymptotically the same way as if the variables were independent.

3 Young towers and the Fourier transform of the Perron-Frobenius operator

3.1 Young towers

According to our recent understanding the most efficient way for constructing Markov partitions for billiards is to use Young towers, cf. [You 98]. We are going to introduce the main concepts without giving a full description.

The presence of singularities prevent stable and unstable curves to possess a lower bound for their size in any part of the phase-space. Therefore the product

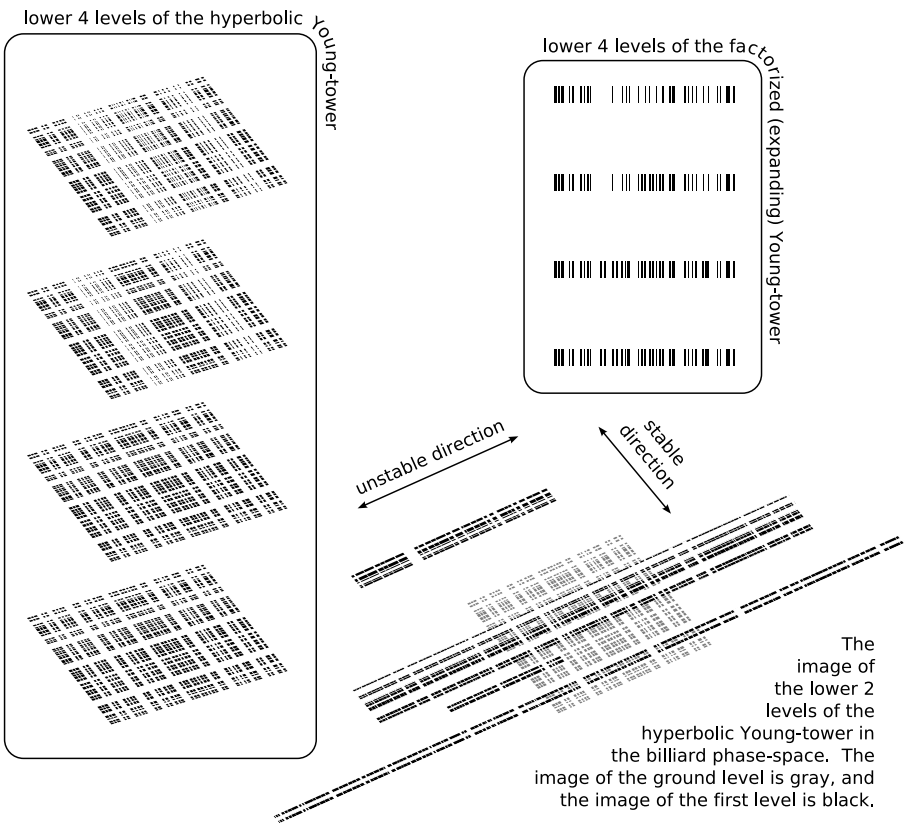

Figure 9: Young-towers, and Markov-return

structure - the key ingredient of several hyperbolic argument - can only be introduced in a complicated set.

First choose an unstable curve W, which is short enough to ensure: a high amount of the points possesses unstable curve of this length. Then define a subset of this curve consisting of points, which remain a certain (exponentially shrinking) distance apart from S_{-1} and other more technical singularities.

$$\Omega_\infty := \{y \in W \mid d(T^n y, S) > \delta_1 \lambda^{-n} \quad \forall n \geq 0\}$$

where λ is the hyperbolicity constant. If δ_1 is chosen small enough this set has positive measure. By construction each point in Ω_∞ possesses a stable curve of length δ_1.

So far we have one unstable curve W, and a family of stable curves $\{\gamma^s\}$. Let us consider all the nearby unstable curves, which are long enough, and intersect all the stable curves in the previous family. These two families of curves $\{\gamma^s\}$ and $\{\gamma^u\}$ define the hyperbolic product-set $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$.

This set is the base of the hyperbolic Young-tower. To continue the construction of the tower we are going to focus on recurring subsets of Λ. On figure 9 we can see that some parts of Λ are mapped to Λ. However we are only interested in those retunrs, which respect the product structure. A subset of Λ is said to be an *s-subset* if it is the product of the full family of stable curves and some part of the unstable family. The notion of *u-subset* is defined -mutatis mutandis- in the same way.

We can see three intersections on the figure, the lower and upper ones are u-subsets. Talking about these intersections black covers gray in the unstable direction (when the reader sees black in these intersections, then on that unstable curve black covers gray). On the contrary gray covers black in the stable direction (on each stable line black can appear only where gray is already there). The inverse image of each of these two intersections is an s-subset.

A Markov-return is an event when some $T^n\Lambda \cap \Lambda$ is a u-subset, and it's inverse image under T^{-n} is an s-subset. The possible non-Markov returns are when the intersection is not a u-subset (this is printed as the middle intersection), or when the inverse image is not an s-subset. This latter event occurs when a recurring part goes over the edge of Λ in the stable direction.

The inverse image of the Markov-recurring part is not necessarily a solid rectangle intersected with Λ. It can have infinitely many "holes" in it, as demonstrated on figure 9.

The tower is built using these Markov-type returns. The basic Λ set is divided into s-subsets according to Markov-returns, and each subset is marked by the return time R. In this way R will be a function on Λ which is constant on these s-subsets. Not all Markov-returns are considered, for sophisticated details please consult [You 98]! The tower itself

$$\Delta \stackrel{\text{def}}{=} \{(x, \omega) : x \in \Lambda;\ \ \omega = 0, 1, \ldots, R(x) - 1\}$$

and the dynamics on the tower is

$$F(x, \omega) = \begin{cases} (x, \omega + 1) & \text{if} \quad \omega + 1 < R(x) \\ (T^R x, 0) & \text{if} \quad \omega + 1 = R(x) \end{cases}$$

Note that we have a decomposition into s-subsets which give rise to a Markov partition on the tower. This tower is only hyperbolic, and as a usual tool in this

field Young has also introduced a factorized version of it $\bar{\Delta}$. Simply collapse the stable direction! This is also demonstrated on figure 9. We have the following commutative diagram of measure preserving transformations:

$$\begin{array}{ccccc} (\bar{\Delta}, \bar{\mu}_\Delta) & \xleftarrow{\pi_{\bar{\Delta}}} & (\Delta, \mu_\Delta) & \xrightarrow{\pi_X} & (X, \mu) \\ \bar{F}\uparrow & & F\uparrow & & T\uparrow \\ (\bar{\Delta}, \bar{\mu}_\Delta) & \xleftarrow{\pi_{\bar{\Delta}}} & (\Delta, \mu_\Delta) & \xrightarrow{\pi_X} & (X, \mu) \end{array} \tag{2}$$

The projection to the original phase-space is not 1-1. On figure 9 the intersection in the middle has at least two inverse images. One of them is in the ground floor, and the other is on the first floor. Since the return is not Markovian these point are to be considered as different points on the tower.

Functions on the original phase space X can be lifted to Δ. Functions on Δ which are constant along stable directions can be considered as functions on $\bar{\Delta}$. For any function ψ on Δ there exists functions h and φ, such that $\varphi - \psi = h - h \circ F$, and φ is constant along stable directions. In this equation the regularity of the functions can be examined, but we will skip the details, and only introduce distance, and function norms on the factorised tower $\bar{\Delta}$.

Remember that the factorised tower $\bar{\Delta}$ has a Cantor-structure, and a Markov-partition. The Cantor-hierarchy can be redefined with the separation time $s(x, y) = \min\{k \geq 0 \mid \bar{F}^k x \text{ and } \bar{F}^k y \text{ lie in different elements of the Markov-partition}\}$. With any $0 < \beta < 1$ the function β^s is a metric providing the original Cantor topology.

On $\bar{\Delta}$ Young uses two kind of norms: the $\mathcal{C}$ norm is

$$\|\varphi\|_{\mathcal{C}} \stackrel{\text{def}}{=} \sup_{l,j} \left\|\varphi|_{\Delta_{l,j}}\right\|_\infty e^{-l\epsilon}$$

where $\|\,.\,\|_\infty$ is the essential supremum wrt μ, and the indices (l, j) refer to the elements of the Markov-partition. The $\mathcal{L}$ norm is a sum of this, and the h-norm:

$$\|\varphi\|_h \stackrel{\text{def}}{=} \sup_{l,j} \left(\sup_{x,y \in \Delta_{l,j}} \frac{|\varphi(x) - \varphi(y)|}{\beta^{s(x,y)}} \right) e^{-l\epsilon};$$

where the inner sup is again essential supremum wrt $\mu \times \mu$. For a Hölder function f on the original billiard phase-space, we can associate a function on $\bar{\Delta}$ as described above, such that for any β smaller than a certain number (computed from the original Hölder exponent of f) the resulting φ has a finite h-norm.

In these definitions the role of ϵ is the following: without ϵ the Jacobian of the mapping would be 1 except when recurring to the base of the tower. However estimates expressed in the terms of this norm see a uniform expansion. Not to do harm the mapping should be expanding, when recurring to the base, so we have to choose ϵ smaller then the Lyapunov-exponent.

The Perron-Frobenius operator P is defined on functions on $\bar{\Delta}$ with finite $\mathcal{L}$-norm as the adjoint of $\bar{F}$ wrt the measure $\bar{\mu}_\Delta$.

$$\int f \cdot g \circ \bar{F} \; d\bar{\mu}_\Delta = \int P(f) \cdot g \; d\bar{\mu}_\Delta \qquad f, g \in \mathcal{L}$$

It also has an algebraic form

$$P(f)(x) = \sum_{y|\bar{F}y=x} \frac{f(y)}{J(y)} \qquad f \in \mathcal{L}$$

where J in the denominator is the Jacobian $\frac{\partial \bar{\mu}^*_\Delta}{\partial \bar{\mu}_\Delta}$ the Radon-Nikodym derivative of the pullback measure wrt the original measure. On the top of the tower the number of inverse images is infinite, otherwise there is a single preimage, and the Jaocbian is 1.

The main issue here is that the transfer operator P possesses a spectral gap. This is the main reason for introducing this rather complicated and implicit symbolic dynamics. This gives those analytical tools in our hands which we describe now. Although κ is defined in the original phase-space X, we will use the same notation for the function lifted to the tower. Since κ is locally constant it can also be considered as a function on $\bar{\Delta}$.

3.2 Fourier transform of the Perron-Frobenius operator

In our case the Birkhoff sum $S_n = \sum_k \kappa \circ T^k$ is not an independent sum. Since Nagaev's 1957 work, [Nag 57], for concluding a limit theorem, one traditionally uses the Fourier-transform of the transfer operator. The Fourier transform operator is defined on the Young-tower, since that is the symbolic space, on which we have the full power of analytical tools built on the transfer operator:

$$P_t(h) \stackrel{\text{def}}{=} P(e^{i\langle t,\kappa\rangle}h) \qquad (h \in \mathcal{L}, t \in \mathbb{R}^2)$$

It has the following simple connection with the characteristic function of the Birkhoff-sum:

$$\int \exp(i\,\langle t, S_n\rangle)d\mu = \int P_t^n(1)d\bar{\mu}_\Delta.$$

For small values of t, P_t can be considered as a perturbation of P, since $P_0 = P$. Then one, in general, proves that P_t possesses a gap between the leading simple eigenvalue λ_t and the rest of the spectrum and the gap is uniformly bounded away from zero.

For large values of t one needs to know exactly for which t values will the unit circle intersect the spectrum. In our case for the continuous free flight function this occurs when $t \in 2\pi\mathbb{Z}^2$. For that reason we swiched to the discretized function κ, since the latter one is an integer valued vector function, and thus $P_t = P$ when $t \in 2\pi\mathbb{Z}^2$, so we can factorize, and consider $t \in 2\pi\mathbb{T}^2$. This is the question of minimality which is completely described in [SzV 04a].

Moreover, if our interest is a global limit law, then it is sufficient to derive an appropriate expansion for λ_t for small values of t.

Indeed, this is also contained in the statement of Theorem 13 from [BG 06]. As a matter of fact, our main task will be to check the key condition of that theorem and thus we will actually derive a simple asymptotics $P_t^n = \lambda_t^n + O(\vartheta^n)$, for a suitable $\vartheta < 1$. Consequently, the characteristic function of the dependent sum can be approximated with a power. Thus it is only the asymptotics of λ_t that has remained to be investigated, and we expect the same behaviour what we have seen in the single term characteristic function. This will mean that the effect of dependence is negligible: both the independent and the dependent

sums have the same asymptotic expansion for the Fourier transform, namely $1 + \mathbf{c}n|t|^2 \log |t|$. (In this formula $\mathbf{c}$ describes the direction-dependence, and $|t|$ the length, remember the definition of $\mathbf{c}$) in 1! Further, relying upon the method of [SzV 04a], we will also be able to handle the local version of the global limit law.

4 Proof of Theorem 1: The asymptotics of λ_t

In the forthcoming section we are going to establish the global limit law of Theorem 1 via an asymptotic expression at $t \to 0$ of λ_t, the leading eigenvalue of the Fourier-transform of the Perron-Frobenius operator. We will rely upon ideas of [BG 06] and of our work [SzV 04b].

The following theorem of [BG 06] provides a condition for a limit law in the non-standard domain of attraction of the gaussian law for models possessing a Young tower with an exponential tail bound for the height function, in general.

Theorem 13. *([BG 06], Theorem 3.5) Assume that the distribution of $g : \bar{\Delta} \to \mathbb{R}$ is in the nonstandard domain of attraction of the normal law. Remember that ω was the level function of the tower. Denote the "tower-sum" by $G = \sum_{k=1}^{\omega} g \circ \bar{F}^{-k}$. Let L, l be as in subsection 2.5. Assume, moreover, that $l(x \log x)/l(x) \to 1$, $L(x \log x)/L(x) \to 1$ when $x \to \infty$. Finally, assume that there exists a real number $a \neq -1/2$ such that*

$$\int g(e^{itG} - 1) = (a + o(1))itL(1/|t|). \tag{3}$$

Write $L_1(x) = (2a+1)L(x)$, and choose a sequence $B_n \to \infty$ such that $\frac{n}{B_n^2} L_1(B_n) \to 1$. Then for λ_t the leading eigenvalue of the $P_t(h) = P(e^{itg}h)$ Fourier-transform-operator:

$$\lambda_t = 1 - \frac{t^2}{2} L_1(1/|t|)(1 + o(1))$$

and consequently for the $S_n = \sum_{k=0}^{n-1} g \circ \bar{F}^k$ Birkhoff-sum:

$$\frac{S_n - n \int g}{B_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1).$$

We will indeed show that in our case the integral condition (3) of the above theorem holds for $g = \kappa$ with $a = 0$. (This is actually the case discussed by [AD 01b].) We note that this case will allow a simpler treatment than that of the stadium because here the expansion rate is much larger (though the logarithm of the free flight function in a corridor does not form a random walk as it is the case in the stadium).

Our arguments for the local limit theorem will use again the asymptotics of the main eigenvalue λ_t, which is expressed for our case by the next Corollary.

Corollary 14. *The Fourier transform operator of the discretized free-flight function has the following expansion for its leading eigenvalue: $\lambda_t \sim 1 + \mathbf{c}n|t|^2 \log |t|$.*

Turn to the proof of the integral condition (3). In our case the function κ is locally constant, so we can pass the integral immediately to Δ. Denote the "tower-sum" by $K = \sum_{k=1}^{\omega} \kappa \circ F^{-k}$. It is easy to see that the dominating terms

of the integral in (3) are those corresponding to parts of the phase space when the process is close to a singular point say x_0 (discussed in 2.2).

The estimate of this integral is based on the following fact. We have already observed that high values of κ are typically reached rapidly: $\kappa \circ T^{-1}$ is in the order of $\sqrt{|\kappa|}$. During this fast trajectory segment the tower-sum can be estimated with the last term, hence it is also of order $\sqrt{|\kappa|}$. Since $|e^{it\bar{K}}-1| \leq t\bar{K}$, the integral can be estimated by $t\sum_n n\sqrt{n}\cdot\mu\{|\kappa| = n\} = O(t)$. The trajectories which do not provide fast reach have polinomially small (in $|\kappa|$) relative measure in the level-sets. These domains can be discarded due to the following lemma:

Lemma 15. *Any part of the integration domain A with measure $\mu(A \cap \{|\kappa| = N\}) = O(N^{-3-\alpha})$ with any $\alpha > 0$ can be thrown away:*

$$\int_\Delta \kappa(e^{i\langle t,K\rangle} - 1) = \int_{\Delta\setminus A} \kappa(e^{i\langle t,K\rangle} - 1) + O(|t|)$$

This is proved in proposition (4.17) of [BG 06]. Though the integration domains are in the hyperbolic Young-tower, if we identify the sets to be disregarded on the original phase-space X, then by measure preservation their pullbacks will satisfy the lemma.

4.1 Reduction of the integration domain

First we are going to discard that part A_1 of the neighbourhood of x_0 where the last step was not fast enough, i. e. $A_1 = \{x \in U_0 \mid |\kappa \circ T^{-1}| > |\kappa|^{\frac{3}{4}}\} \subset X$. We also discard the set A_2 of those points where $\kappa \circ T^{-1} = w_0$, i. e. $A_2 = U_0 \setminus U_0'$.

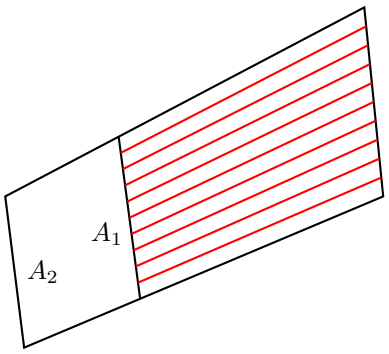

Figure 10: The discarded sets A_1 and A_2 in the level set of κ and the foliation with nearly unstable curves

We already know that the relative measure of $(A_1 \cup A_2)$ inside $\{|\kappa| = N\}$ is $O(N^{-1/2})$ (cf. propositions 11 and 12).

Let us foliate the remaining part with curves whose direction is nearly unstable (the derivative is in the unstable cone for T^{-1}).

The third discarded set A_3 will consist of points, for which the backward $Z \log N$-step trajectory meets the $\{|\kappa| > N^{\frac{4}{5}}\}$ set, where Z is a large number to be chosen later. To estimate the relative measure of A_3 we prove the following lemma:

Lemma 16. *There exists a constant C such that, for any large enough integer Z it is true: for any large enough N, given any unstable curve (for the mapping T^{-1}) in the set $\{|\kappa| = N\} \setminus \bigcup_{i=1,2} A_i$, the points for which $|\kappa|$ increases above $N^{4/5}$ within $Z \log N$ iterations of T^{-1} occupy a subset whose relative measure is less than $CN^{-1/11}$ in that curve.*

Proof. The proof is a suitable modification of Lemma 4.14 of [BG 06]. Note first that in the set $\{|\kappa| = N\} \setminus A_1$ one has $|\kappa \circ T^{-1}| < N^{\frac{3}{4}}$. The points which reach $|\kappa| > N^{4/5}$ get closer to the singularity than const $\cdot N^{-8/5}$. We are going to measure the set of points of the latter type.

The map T^{-1} satisfies Chernov's axioms [Ch 99] as this was checked for our infinite horizon case by Chernov in section 8 of the same article. Using exactly the same ideas as in [BG 06] we choose $\delta = (Z[\mathcal{D}, \mathcal{D}, 0]n)^{-1/\sigma}$ and apply Theorem 3.1 in [Ch 99].

We obtain a decreasing sequence $W_0^1 \supset W_1^1 \supset \cdots \supset W_{[Z \log N]}^1$ of subsets of the given LUM such that

$$\forall c > 0, \forall 0 \le p \le Z \log N$$
$$\mathrm{Leb}\{x \in W_p^1 \mid \mathrm{dist}(T^{-p}x, S_0) \le cN^{-8/5}\} \le CcN^{-8/5}$$

(by equation (3.3) in [Ch 99]), and

$$\forall 0 \le p \le Z \log N, \quad \mathrm{Leb}(W_p^1 \setminus W_{p+1}^1) \le \frac{C}{N}\mathrm{Leb(LUM)}$$

(By (iv), (3.5) in [Ch 99] and our choice of δ).

The second estimate is itself a relative measure estimate. For the first one we observe that $N^{-8/5} = N^{-1/10}N^{-6/4} \le N^{-1/10}\mathrm{const} \cdot \mathrm{Leb(LUM)}$. This gives a measure at most $C \log N \ N^{-1/10}$, hence the lemma. □

The last discarded set will be defined on the tower: We are going to throw away that part A_4 of the integration domain, which is too high on the tower $\omega > Z \log |\kappa|$. Since the tower has exponentially small tails, if Z was chosen large enough, then the discarded set has measure $\mu_\Delta(A_4 \cap \{|\kappa| = N\}) = O(N^{-4})$, and thus by lemma 15 the integral can be restricted to its complement.

Now it remained to estimate the integral on the non-discarded set.

Lemma 17.
$$\int_{\pi_X^{-1}(X \setminus \bigcup_{i=1}^3 A_i) \setminus A_4} \kappa(e^{i\langle t, K\rangle} - 1) = O(|t|)$$

Proof. Consider the integral on the left. By the definition of the discarded sets we have:

$$\left| \int_{\pi_X^{-1}(X \setminus \bigcup_{i=1}^3 A_i) \setminus A_4} \kappa(e^{i\langle t, K\rangle} - 1) \right| \le$$
$$\le |t| \int_{\pi_X^{-1}(X \setminus \bigcup_{i=1}^3 A_i) \setminus A_4} |\kappa||K| \le C|t| \sum_n \mu\{|\kappa| = n\} n \log n \ n^{4/5} \le C|t|$$

□

5 Proof of Theorems 2, 3 and 4

The proof of the local limit theorems (Theorems 19 and 20 below) and that of the recurrence is -mutatis mutandis- the same as for the finite horizon case [SzV 04a]. Here we are going to state the theorems only, and comment just the beginning, where the different norms used by the different approaches ([BG 06] and [SzV 04a]) could lead to a confusion.

Theorem 18 (Nagaev-type theorem, modification of [SzV 04a] Theorem 3.3)**.** *There are constants $\epsilon > 0$, $K > 0$ and $\theta < 1$ and a function $\rho : (-\epsilon, \epsilon) \to \mathcal{L}$ such that*

$$\left\| P_t^n h - \lambda_t^n \rho_t \int_{\bar{\Delta}} h d\bar{\mu}_\Delta \right\|_{L^1} \leq K\theta^n \|h\|_{\mathcal{L}} \quad \forall |t| < \epsilon, n \geq 1, h \in \mathcal{L} \tag{4}$$

and

$$\|\rho_t - 1\|_{L^1} \to 0 \ \ (t \to 0), \ \ \lambda_t = 1 + (1 + o(1))\mathbf{c}|t|^2 \log|t|$$

The statement in [SzV 04a] was stronger in the sense of norms, but inequality (4) is completely enough for the proof of our local limit theorems. The proof of Theorem 18 can also be straightforwardly derived from considerations in [BG 06], namely the proof of their theorem 3.5.

Theorem 19 (Local limit theorem)**.** *Suppose that the corridor free flights $\{\kappa(x) \mid x \in \partial X, \ Tx = x\}$ span the plane. Let $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{\sqrt{n \log n}} \to k \in \mathbb{R}^2$. Then*

$$n \log n \ \ \mu\{S_n = k_n\} \to \varphi(k)$$

where φ is a non-degenerate normal density function with zero expectation and covariance matrix

$$\sum_{x \in \partial X | Tx = x} \frac{c_\mu d_x^2}{2|\kappa(x)|} \begin{pmatrix} \kappa_1^2(x) & \kappa_1(x)\kappa_2(x) \\ \kappa_1(x)\kappa_2(x) & \kappa_2^2(x) \end{pmatrix}$$

where $\kappa = (\kappa_1, \kappa_2)$ is the notation for the component functions.

The proof of recurrence follows a version of the Borel-Cantelli lemma due to Lamperti. See [SzV 04a] for the statement, conditions and the details of application. For this part of the Borel-Cantelli lemma one needs some kind of independence. Lamperti's condition can be fulfilled when one proves an asymptotic independence statement:

Theorem 20. *Let $j_n \in \mathbb{Z}^2$ be such that $\frac{j_n}{\sqrt{n \log n}} \to j \in \mathbb{R}^2$, and $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{\sqrt{n \log n}} \to k \in \mathbb{R}^2$. If the corridor free flights span the plane, then*

$$\lim_{m, n-m \to \infty} m \log m \ \ (n-m) \log(n-m) \ \ \mu\{S_m = j_m, S_n = j_m + k_{n-m}\} = \varphi(j)\varphi(k).$$

The proof consists of repeating twice the same integral transformations as in the proof of the local limit theorem (again see the proof of Theorem 4.2 in [SzV 04a] for details; we note, however, that the claim of Theorem 4.2 in [SzV 04a] contains a misprint, in fact it ought to be analogous with that of Theorem 20).

Theorem 21. *The invariant (infinite) measure of the Lorentz process* $d\tilde{\mu} = \langle v, n(q)\rangle\, dqdv$ *(where* dq *is the arclength measure on the boundary of infinitely many scatterers) is ergodic i. e. for any invariant set* A *either* $\tilde{\mu}(A) = 0$ *or* $\tilde{\mu}(\tilde{X} \setminus A) = 0$.

This follows from recurrence and [Sim 89], where the author showed the equivalence between recurrence and the ergodicity of the infinite measure (see also [Pen 00]).

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