GEOMETRY OF QUASIMINIMAL PHASE TRANSITIONS

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Abstract. We consider the quasiminima of the energy functional

$$\int_\Omega A(x, \nabla u) + F(x, u)\, dx\,,$$

where $A(x, \nabla u) \sim |\nabla u|^p$ and F is a double-well potential. We show that the Lipschitz quasiminima, which satisfy an equipartition of energy condition, possess density estimates of Caffarelli-Cordoba-type, that is, roughly speaking, the complement of their interfaces occupies a positive density portion of balls of large radii.

From this, it follows that the level sets of the rescaled quasiminima approach locally uniformly hypersurfaces of quasimimal perimeter.

If the quasiminimum is also a solution of the associated PDE, the limit hypersurface is shown to have zero mean curvature and a quantitative viscosity bound on the mean curvature of the level sets is given. In such a case, some Harnack-type inequalities for level sets are obtained and then, if the limit surface if flat, so are the level sets of the solution.

Contents

Keywords: Ginzburg-Landau-Allen-Cahn functionals, quasiminima, geometric and measure theoretic properties.
MS Classification: 35J20, 35J60, 35J70, 49K20.

We thank Berardino Sciunzi for several interesting discussions. This collaboration began during a very pleasant visit of AF to the *Università di Roma La Sapienza* and *Roma Tor Vergata* and it has been completed while the authors were visiting the *Wolfgang Pauli Institute* in Vienna on the occasion of the *Summer School on Gross-Pitaevskii equations for superfluids and Bose-Einstein condensates.* EV has been partially supported by *MIUR Metodi variazionali ed equazioni differenziali nonlineari* and *GNAMPA Principio del massimo e disuguaglianze di Harnack per equazioni ellittiche e sub-ellittiche.*

1. Introduction

We consider here a functional related to the Ginzburg-Landau-Allen-Cahn equation and we prove Caffarelli-Cordoba-type *density estimates* for quasiminima. Namely, we will consider a functional which is the sum of two terms: a double-well potential and a kinetic part. The kinetic part will lead to a (possibly degenerate or singular) elliptic equation, while the effect of the potential is to drive the system towards the two minimal states.

We will prove that all the Lipschitz quasiminima (in the terminology of [GG84]) of our functional, which satisfy an equipartition of energy condition, enjoy suitable density estimates, which, grosso modo, say that the interfaces behave "like codimension 1 sets in a measure theoretic fashion".

From such density estimates, we will derive some connections between the level sets of quasiminima and surfaces of quasiminimal perimeter, or of zero mean curvature. Flatness and symmetry results will also be obtained. These results extend the work of [Sav03, SV05, VSS06], which deal with minimizers, to the quasiminimal setting.

A precise description of the system will be given in § 2 and our result will be described in detail in § 3.

We would like to recall that functionals as the one we consider here have also some physical relevance, since they arise in the theory of superconductor and superfluids (see [GP58, Lan67]), in the study of interfaces in both gasses and solids (see [Row79, AC79]), in questions of fluid dynamics (possibly, with non-Newtonian phenomena, see [Lad67, AC81]) and in cosmology (see [Car95]).

In the purely mathematical setting, functionals as the ones we study here have been the paradigmatic examples for several celebrated Γ-convergence results (see, e.g., [DGF75, Mod87, Ste88, Owe88, Bou90, OS91]) and they are related to a famous conjecture of De Giorgi (see [DG79]).

2. Set-up

Given a domain $\Omega \subset \mathbb{R}^n$, we consider the energy functional

$$\mathcal{F}_\Omega(u) := \int_\Omega A(x, \nabla u) + F(x,u)\, dx\,.$$

Here above, $A(x,\eta)$ is supposed to be of the order of $|\eta|^p$, with $p \in (1,+\infty)$, and F is a double-well potential.

Concrete examples are provided by

$$\mathcal{F}_\Omega(u) = \int_\Omega |\nabla u|^p + K(x)\,\chi_{(-1,1)}(u)\, dx \tag{1}$$

and

$$\mathcal{F}_\Omega(u) = \int_\Omega |\nabla u|^p + K(x)\,(1-u^2)^p\, dx\,, \tag{2}$$

with K positive and bounded from 0 and $+\infty$, and $p \in (1,+\infty)$, but more general cases will also be considered here.

More precisely, we make the following assumptions. We assume that $A \in C^1(\Omega \times \mathbb{R}^n)$ and that

$$a(x,\eta) := D_\eta A(x,\eta)$$

is in $C(\Omega \times \mathbb{R}^n) \cap C^1(\Omega \times \mathbb{R}^n - \{0\})$. We require that

$$A(x,0) = 0, \quad a(x,0) = 0\,, \tag{3}$$

for every $x \in \Omega$ and that there exists $\Lambda > 0$ and $p \in (1, +\infty)$ in such a way that

$$\zeta \cdot D_\eta a(x,\eta)\,\zeta \;\geq\; \Lambda^{-1}|\zeta|^2\,|\eta|^{p-2}\,, \tag{4}$$

$$|D_\eta a(x,\eta)| \;\leq\; \Lambda\,|\eta|^{p-2}\,, \tag{5}$$

$$|D_x a(x,\eta)| \;\leq\; \Lambda\,|\eta|^{p-1}\,, \tag{6}$$

$$\text{and} \quad \eta \cdot a(x,\eta) \;\geq\; \Lambda^{-1}\,|\eta|^p\,, \tag{7}$$

for every $x \in \Omega$, $\eta \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^n$.
Note, in particular, that $A(x,\eta)$ is bounded from above and below by $|\eta|^p$, due to (3), (4) and (5).
We assume that $F : \Omega \times \mathbb{R} \ni (x,u) \longrightarrow \mathbb{R}$ is a Carathéodory function, i.e., continuous in u for a.e. $x \in \Omega$ and measurable in x for every $u \in \mathbb{R}$. We require that

$$0 \leq F \leq M, \qquad F(x,\pm 1) = 0, \qquad \inf_{|u| \leq \theta} F(x,u) \geq \gamma(\theta) \tag{8}$$

for every $0 \leq \theta < 1$, where $\gamma(\theta)$ and M are positive constants. Here and below, all the structural inequalities on F are assumed to be uniform for a.e. $x \in \Omega$. Further, we assume that the partial derivative $F_u(x,u)$ exists for every $u \in (-1,1)$ and for a.e. $x \in \Omega$, and that F is uniformly Lipschitz in $u \in (-1,1)$, that is

$$\sup_{|u|<1} |F_u(x,u)| \leq \Lambda\,. \tag{9}$$

The techniques exploited here would also allow to deal with less regular potentials F, but we will not push ahead with such generality.
We assume the following growth condition near $u = \pm 1$: we suppose that either condition (10) holds or that conditions (11), (12) and (13) hold (see here below). That is, we consider two cases. The first case (which will be dubbed *case* χ, since it is the case in which the potential is just a characteristic function) is the one in which we assume that

$$F(x,\tau) \;\geq\; \frac{1}{\Lambda}\,, \tag{10}$$

for any $\tau \in (-1,1)$ and a.e. $x \in \Omega$.
In the second case (which will be called *case* W, since it is the case in which the potential is a W-shaped function), we take the following assumptions. We suppose that there exists $s_0 > 0$ and $0 < d \leq p$ such that

$$F_u(x,-1+s) \geq \frac{s^{d-1}}{\Lambda}, \qquad F_u(x,1-s) \leq -\frac{s^{d-1}}{\Lambda}\,, \tag{11}$$

for every $s \in (0, s_0)$, that

$$F(x,-1+s) + F(x,1-s) \leq \Lambda s^{d'} \tag{12}$$

for any $s \in [0, s_0)$, for a suitable $d' > 0$, and that

(13) $\quad F_u$ is monotone increasing in u for $u \in (-1, -1+s_0)$ and $u \in (1-s_0, 1)$.

The functionals in (1) and (2) are paradigmatic examples for the cases χ and W, respectively.

We will consider here the quasiminima of $\mathcal{F}$. For this, we recall the following definition (see, e.g., [GG84]). Given $Q \geq 1$, we say that a function $u \in W^{1,p}(\Omega)$ is a *quasiminimum with constant* Q (for short: a Q*-minimum*) of $\mathcal{F}$ in Ω if $\mathcal{F}_\Omega(u) < +\infty$ and

$$\mathcal{F}_\Omega(u) \leq Q\, \mathcal{F}_\Omega(u+\varphi),$$

for any $\varphi \in C_0^\infty(\Omega)$.
The n-dimensional Lebesgue measure will be denoted by $\mathcal{L}^n$.

3. The main result and some consequences

The main result we prove is the following:

Theorem 1. *Fix $\theta \in (0,1)$. Let $|u| \leq 1$ be a Q-minimum for $\mathcal{F}$ in a domain Ω. Assume that*

$$|\nabla u(x)| \leq M_0, \tag{14}$$

for a.e. $x \in \Omega$, and that

$$A(x, \nabla u(x)) \leq M_0\, F(x, u(x)), \tag{15}$$

for a.e. $x \in \Omega \cap \{|u| < 1\}$, for a suitable $M_0 > 0$.
Suppose that there exist two positive real numbers μ_1 and μ_2 in such a way that $B_{\mu_1}(x) \subset \Omega$ and

$$\mathcal{L}^n\Big(B_{\mu_1}(x) \cap \{u > \theta\}\Big) \geq \mu_2. \tag{16}$$

Then, there exist positive r_0 and c, which depend only on n, Q, θ, μ_1, μ_2, M_0 and on the structural constants of the functional $\mathcal{F}$, in such a way that

$$\mathcal{L}^n\Big(B_r(x) \cap \{u > \theta\}\Big) \geq c\, r^n,$$

for any $r \geq r_0$, provided that $B_r(x) \subset \Omega$.

We remark that an analogous result holds with the signs changed, that is: if

$$\mathcal{L}^n\Big(B_{\mu_1}(x) \cap \{u < \theta\}\Big) \geq \mu_2,$$

then

$$\mathcal{L}^n\Big(B_r(x) \cap \{u < \theta\}\Big) \geq c\, r^n,$$

for any $r \geq r_0$ provided that $B_r(x) \subset \Omega$.

We also point out that conditions (14) and (15) are quite natural, since they correspond, respectively, to a Lipschitz condition and to a phenomenon known in the literature as "equipartition of energy". In particular, conditions (14) and (15) are satisfied if u is a solution of the associated PDE (see, e.g., Theorem 1 of [Tol84] and page 1459 of [CGS94]). Also, (14) obviously implies (15) if (10) holds (i.e., if we are in case χ).

As a consequence of Theorem 1, we will show that the (sub)level sets of rescaled Q-minima approach locally uniformly hypersurfaces of Q-minimal weighted perimeter. To this extent, we define the appropriate concept of perimeter by following § 3 of [Bou90], that is, we set

$$h(x,\eta) := \int_{-1}^{1} \left[\inf_{t>0} \left(\frac{A(x,t\eta) + F(x,\tau)}{t}\right)\right] d\tau$$

and

$$\mathrm{Per}\,(E,\Omega;\mathcal{F}) \;:=\; \int_{\partial^\star E\cap\Omega} h(x,\nu_E(x))\,d\mathcal{H}^{n-1}(x)\,.$$

Here above, $E\subseteq\mathbb{R}^n$ is supposed to be a Caccioppoli set with exterior normal ν_E and reduced boundary $\partial^\star E$, and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure (see, e.g., [Giu84] for definitions and basic properties). Note that the above definition of $\mathrm{Per}\,(E,\Omega;\mathcal{F})$ generalizes the standard notion of perimeter of the set E in Ω, which will be denoted by $\mathrm{Per}\,(E,\Omega)$. With this notation, we derive the following result from Theorem 1:

Corollary 2. *Assume that*

$$\sup_{x\in\Omega,\,|\xi|<1}\frac{|F_x(x,\xi)|}{(F(x,\xi))^{1/p}} \;<\; +\infty\,. \tag{17}$$

Consider an infinitesimal sequence of ε's. Let u_ε be a sequence of Q-minima for the functional

$$\mathcal{F}^\varepsilon_\Omega(u) \;:=\; \frac{1}{\varepsilon}\int_\Omega A(x,\varepsilon\nabla u)+F(x,u)\,dx\,.$$

Suppose that

$$|u_\varepsilon| \;\le\; 1\,, \tag{18}$$

that

$$|\nabla u_\varepsilon(x)| \;\le\; M_0/\varepsilon\,, \tag{19}$$

for a.e. $x\in\Omega$, that

$$A(x,\varepsilon\nabla u_\varepsilon(x)) \;\le\; M_0\,F(x,u_\varepsilon(x))\,, \tag{20}$$

for a.e. $x\in\Omega\cap\{|u_\varepsilon|<1\}$, for some $M_0>0$, and that

$$\sup_\varepsilon \mathcal{F}^\varepsilon_\Omega(u_\varepsilon) \;<\; +\infty\,. \tag{21}$$

Then, up to subsequences, u_ε converges in L^1_{loc} to a step function $u_0=\chi_E-\chi_{\Omega-E}$ for a suitable $E\subseteq\Omega$.
Also, given any $\theta\in(0,1)$ the set $\{|u_\varepsilon|<\theta\}$ approaches ∂E locally uniformly.
Moreover, E is a Q-minimum for the weighted perimeter $\mathrm{Per}\,(\cdot,\Omega;\mathcal{F})$.

In case of the spatially homogeneous scaling $u_\varepsilon(x):=u(x/\varepsilon)$, we remark that assumption (21) holds true thanks to Lemma 10 below.

If u is also a solution of the associated PDE, the result of Corollary 2 may be sharpened, by obtaining that the limit interface ∂E has zero mean curvature in the viscosity sense. This fact is indeed a consequence of the density estimates performed here and of the fine viscosity methods invented in [Sav03], as developed in [SV05]. As an example, we provide the following result:

Corollary 3. *Let $A(x,\eta):=|\eta|^p/p$, $F(x,\tau):=(1-\tau^2)^p$. Let*

$$u\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \text{ be a Q-minimum for $\mathcal{F}$ in any domain of } \mathbb{R}^n. \tag{22}$$

Suppose that $|u|\le 1$ and

$$\Delta_p u \;:=\; \mathrm{div}\,(|\nabla u|^{p-2}\nabla u) \;=\; -2p\,u(1-u^2)^{p-1} \tag{23}$$

in $\mathbb{R}^n$.
Let $u_\varepsilon(x):=u(x/\varepsilon)$, for an infinitesimal sequence of ε's.

Then, the claims of Corollary 2 hold true and, if E is as in Corollary 2, we have that ∂E satisfies the zero mean curvature equation in the viscosity sense.
More precisely, let $x^\star \in \partial E$ be so that, for any $r > 0$

$$\mathcal{L}^n\Big(B_r(x^\star) \cap (\mathbb{R}^n - E)\Big) > 0 \ \text{ and } \ \mathcal{L}^n\Big(B_r(x^\star) \cap E\Big) > 0\,,$$

and assume that ∂E admits a tangent hyperplane at $x^\star$. Then, if a paraboloid with vertex in $x^\star$ touches ∂E by below (resp., above) at $x^\star$, then its mean curvature at $x^\star$ must be non-positive (resp., non-negative).
In particular, ∂E satisfies the zero mean curvature equation in the classical sense at any point where it is a C^2 hypersurface.

The connection between Q-minima and solutions of PDEs is a classical topic in the calculus of variations. For instance, solutions of quite general PDEs are known to be Q-minima of *suitable* functionals (see, e.g., § 6.2 of [Giu94]). On the other hand, the meaning of conditions (22) and (23) is that the PDE solved by u turns out to be the Euler-Lagrange equation of a functional of the type "kinetic part plus double-well potential", for which u is a Q-minimum. In particular, the lack of convexity of the potential makes the technique of § 6.2 of [Giu94] not applicable to our case.

In Corollary 3 here above, a quantitative estimate on the mean curvature of the level sets of the solution may also be obtained, as given by the following result, which is related to similar ones in [Sav03, SV05]:

Corollary 4. *Let the assumptions of Corollary 3 hold. Let ℓ, θ, $\delta > 0$, $\xi \in \mathbb{R}^{n-1}$ and $M \in \mathrm{Mat}((n-1) \times (n-1))$.*
Suppose that $u(0) = 0$ and $u(x) < 0$ for any $x = (x', x_n) \in [-\ell, \ell]^N$ so that

$$x_n < \frac{\theta}{2\ell^2} x' \cdot M x' + \frac{\theta}{\ell} \xi \cdot x' \,.$$

Then, there exist a universal constant $\delta_0 > 0$ and a function $\sigma : (0,1) \longrightarrow (0,1)$, so that, if

$$\delta \in (0, \delta_0]\,, \quad \delta \le \theta\,, \quad \frac{\theta}{\ell} \in \Big(0, \sigma(\delta)\Big]\,, \quad \|M\| \le \frac{1}{\delta} \quad \text{and} \quad |\xi| \le \frac{1}{\delta}\,,$$

then $\mathrm{tr} M \le \delta$.

Roughly speaking, Corollary 4 here above states that if the zero level set of u is touched by below by a paraboloid, then the mean curvature (or, equivalently, the trace of the Hessian) of such a paraboloid cannot be too large (and so, the level sets are close to have zero mean curvature, in a weak, quantitative, viscosity sense).

From the results above, some Harnack-type inequalities for level sets of quasiminima follow. Roughly speaking, such results say that, once the zero level set of u is trapped in a rectangle whose height is small enough, then, in a smaller neighborhood, it can be trapped in a rectangle with even smaller height. Accordingly, once such a level set is trapped inside a suitably flat cylinder, then, possibly changing coordinates, it is trapped in an even flatter cylinder in the interior (pictures of these facts are given on pages 3–4 of [VSS06]). More precisely, we have the following results:

Corollary 5. *Let ℓ, $\theta > 0$. Let u, A and F be as in Corollary 3. Assume that $u(0) = 0$ and that*

$$\{u = 0\} \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, |x'| < \ell\,,\ |x_n| < \ell\} \subseteq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, |x_n| < \theta\}\,.$$

Let $u_\varepsilon(x) := u(x/\varepsilon)$ *and suppose that*

$$\lim_{\varepsilon\to 0^+} \mathcal{F}^\varepsilon_\Omega(u_\varepsilon) \le \operatorname{Per}(E,\Omega) \int_{-1}^{1} \Big(\frac{p}{p-1}(1-\tau^2)^p\Big)^{(p-1)/p} d\tau\,, \tag{24}$$

where E *is as in Corollary 2 and* $\operatorname{Per}(E,\Omega)$ *denotes the perimeter of the set* E *in* Ω. *Then, there exists a universal constant* $c \in (0,1)$ *so that, for any* $\theta_0 > 0$, *there exists* $\varepsilon_0(\theta_0) > 0$ *such that, if*

$$\frac{\theta}{\ell} \le \varepsilon_0(\theta_0) \qquad \text{and} \qquad \theta \ge \theta_0\,,$$

then

$$\{u=0\} \cap \{|x'| < c\ell\} \subseteq \{|x_n| < (1-c)\,\theta\}\,.$$

Corollary 6. *Let* ℓ, $\theta > 0$. *Let* u, A *and* F *be as in Corollary 3 and suppose that* (24) *holds. Assume that* $u(0) = 0$ *and that*

$$\{u=0\} \cap \{(x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} \,:\, |x'| < \ell\,,\ |x_n| < \ell\} \subseteq \{(x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} \,:\, |x_n| < \theta\}\,.$$

Then, there exist universal constants η_1, $\eta_2 > 0$, *with* $0 < \eta_1 < \eta_2 < 1$, *such that, for any* $\theta_0 > 0$, *there exists* $\varepsilon_1(\theta_0) > 0$ *such that, if*

$$\frac{\theta}{\ell} \le \varepsilon_1(\theta_0) \qquad \text{and} \qquad \theta \ge \theta_0\,,$$

then

$$\begin{aligned} &\{u=0\} \cap \Big(\{|\pi_\xi\, x| < \eta_2\ell\} \times \{|(x\cdot\xi)| < \eta_2\ell\}\Big) \subseteq \\ \subseteq\ &\Big(\{|\pi_\xi x| < \eta_2\ell\} \times \{|(x\cdot\xi)| < \eta_1\theta\}\Big) \end{aligned}$$

for some unit vector ξ.

Corollaries 5 and 6 are extensions of analogous Harnack-type inequalities for level sets of minimizers, which have been obtained in [Sav03, VSS06].

In the spirit of a famous conjecture by De Giorgi (see [DG79]), we now point out a symmetry feature which extends some results by [BCN97, GG98, Far99, AC00, Sav03, VSS06] to the quasiminima:

Corollary 7. *Let the assumptions and the notations of Corollary 3 hold. Suppose that* (24) *is satisfied and that* ∂E *is a flat hyperplane. Then, the level sets of* u *are flat hyperplanes too.*

The proof of Theorem 1 is the core of the rest of the paper. It is performed in § 5 and it uses some auxiliary results proved in § 4. The proof of Corollaries 2, 3 and 4 will then be dealt with in § 6 and 7. We then devote § 8 and 9 to the proofs of Corollaries 5 and 6. Finally, Corollary 7 will be shown to hold in § 10.
Some results of [GG84, Tol84, Bou90, CGS94, Val04, PV05a, PV05b, SV05, VSS06] will be used in the course of the proofs and some ideas from [Mod87, CC95, Sav03] will also be borrowed.

4. Toolbox

We perform here some technical computations, which will turn out to be useful in the proof of the main result.

Lemma 8. *Let*

$$i := \inf_{x\geq 1} \frac{x^{n-1}}{(x+1)^n - x^n}. \tag{25}$$

Then, $i \in (0, +\infty)$.

Proof. Let

$$f(x) := \frac{x^{n-1}}{(x+1)^n - x^n}.$$

Then,

$$\lim_{x\to+\infty} f(x) = \lim_{y\to 0^+} \frac{y}{(1+y)^n - 1} = \lim_{y\to 0^+} \frac{1}{n(1+y)^{n-1}} = \frac{1}{n}.$$

This and the fact that f is continuous and positive in $[1, +\infty)$ imply the desired result. $\square$

The result to come is a complicated version of an iterative scheme given on page 10 of [CC95] (in our case, the complication will arise from the fact that we deal with quasiminima, instead of minima as in [CC95], thus obtaining additional terms in the computations).

Lemma 9. *Let i be as in (25). Fix $C > 0$ and let $\alpha \in (0, 1)$ in such a way that*

$$\frac{\alpha^{(n-1)/n}}{1-\alpha} = \frac{1}{2C^{1/n}}. \tag{26}$$

Let L and T be positive real numbers and define

$$\gamma := \frac{1}{\alpha}\left(\frac{4C^2Te^{-L}}{1-e^{-L}}\right)^{n/(n-1)}. \tag{27}$$

Suppose that

$$\varepsilon \leq \frac{(\alpha\gamma)^{(n-1)/n}T^{n-1}}{4} \tag{28}$$

and that

$$\gamma \leq \min\left\{\frac{1}{CT^n},\ \alpha^{n-1}\left(\frac{i}{2CT}\right)^n\right\}. \tag{29}$$

Let a_k and $\mathcal{A}_k$ be two sequences of non-negative real numbers, for $k = 1, 2, \ldots,$ and suppose that

$$a_1 \geq 1/C \tag{30}$$

and that

$$a_k \leq CT^nk^{n-1} \qquad \text{for any } k \geq 1. \tag{31}$$

Assume also that the following recursion holds:

$$\mathcal{A}_k + \left(\sum_{j=1}^{k} a_j\right)^{(n-1)/n} \leq \varepsilon k^{n-1} + C\left(\mathcal{A}_{k+1} - \mathcal{A}_k + \sum_{j=1}^{k+1} e^{-L(k+1-j)}a_j\right). \tag{32}$$

Then,

$$\mathcal{A}_k + \sum_{j=1}^{k} a_j \;\geq\; \gamma T^n k^n \qquad \textit{for any } k \geq 1. \tag{33}$$

Proof. The proof is by induction. If $k = 1$, then (33) holds, thanks to (29), (30) and the fact that $\mathcal{A}_k$ is non-negative.
We now suppose that (33) holds for k and we prove it for $k+1$. In this argument, we may also suppose that

$$C\gamma T^n (k+1)^n \geq 1\,, \tag{34}$$

otherwise (33) holds for $k+1$, due to (30) and the fact that both $\mathcal{A}_k$ and a_k are non-negative.
Note that (26) and (34) yield that

$$\gamma^{1/n} Tk \geq \frac{\gamma^{1/n} T\,(k+1)}{2} \geq \frac{1}{2C^{1/n}} = \frac{\alpha^{(n-1)/n}}{1-\alpha}\,,$$

and so

$$(1-\alpha)\,\gamma T^n k^n \;\geq\; (\alpha\gamma)^{(n-1)/n} T^{n-1} k^{n-1}\,. \tag{35}$$

We now prove that

$$\mathcal{A}_k + \left(\sum_{j=1}^{k} a_j\right)^{(n-1)/n} \;\geq\; (\alpha\gamma)^{(n-1)/n}\, T^{n-1} k^{n-1}\,. \tag{36}$$

Indeed, if

$$\sum_{j=1}^{k} a_j \;\geq\; \alpha\gamma T^n k^n\,,$$

then (36) follows since $\mathcal{A}_k$ is non-negative. Therefore, we may suppose that

$$\sum_{j=1}^{k} a_j \;\leq\; \alpha\gamma T^n k^n\,.$$

Then, since (33) is assumed to hold for k, we have that

$$\begin{aligned}
\mathcal{A}_k \;&\geq\; \gamma T^n k^n - \sum_{j=1}^{k} a_j \\
&\geq\; (1-\alpha)\,\gamma T^n k^n \\
&\geq\; (\alpha\gamma)^{(n-1)/n} T^{n-1} k^{n-1}\,,
\end{aligned}$$

thanks to (35), thus proving (36).
Moreover, by using (31) and (27), one obtains that

$$\begin{aligned}
\sum_{j=1}^{k+1} e^{-L(k+1-j)} a_j \;&\leq\; a_{k+1} + CT^n k^{n-1} \sum_{j=1}^{k} e^{-L(k+1-j)} \\
&\leq\; a_{k+1} + CT^n k^{n-1} \frac{e^{-L}}{1-e^{-L}} \\
&=\; a_{k+1} + \frac{(\alpha\gamma)^{(n-1)/n}}{4C}\, T^{n-1}\, k^{n-1}\,.
\end{aligned}$$

The latter estimate, (28), (32) and (36) imply that

$$\mathcal{A}_{k+1} - \mathcal{A}_k + a_{k+1} \,\geq\, \frac{(\alpha\gamma)^{(n-1)/n}}{2C}\, T^{n-1} k^{n-1}\,.$$

Since (33) holds for k, we thus deduce that

$$\begin{aligned} & \mathcal{A}_{k+1} + \sum_{j=1}^{k+1} a_j \\ = \;& \mathcal{A}_{k+1} - \mathcal{A}_k + a_{k+1} + \mathcal{A}_k + \sum_{j=1}^{k} a_j \\ \geq \;& \frac{(\alpha\gamma)^{(n-1)/n}}{2C}\, T^{n-1} k^{n-1} + \gamma T^n k^n\,. \end{aligned}$$

The latter term is bounded from below by $\gamma T^n (k+1)^n$, thanks to (29) and (25), and this gives (33) for $k+1$, as desired. □

The result to come is a preliminary energy estimate, which says that the energy in balls of radius r is controlled by r^{n-1}, as it may be easily heuristically guessed by looking at the energy of "codimension 1 interfaces".

Lemma 10. *Let $r \geq 1$. Let u be as in the statement of Theorem 1. Then,*

$$\mathcal{F}_{B_r}(u) \,\leq\, \hat{C}\, r^{n-1}\,,$$

for a suitable $\hat{C} > 0$, which only depends on n, Q, M_0 and on the structural constants of $\mathcal{F}$, as long as $B_r \subset \Omega$.

Proof. The proof is a variation of analogous estimates performed in [CC95, AAC01, BL03, PV05a, PV05b]. We provide full details for the facility of the reader. Let h be a smooth function so that $h = -1$ in B_{r-1} and $h = 1$ outside B_r. Let $u^* := \min\{u, h\}$ Then, since u is Q-minimal,

$$\begin{aligned} \mathcal{F}_{B_r}(u) &\leq Q\,\mathcal{F}_{B_r}(u^*) \\ &\leq Q \int_{B_r - B_{r-1}} |\nabla u|^p + |\nabla h|^p + F(x,u) + F(x,h)\,dx \\ &\leq CQ\, r^{n-1}\,, \end{aligned}$$

for a suitable $C > 0$, due to (14). □

5. Proof of Theorem 1

The proof of Theorem 1 will use the technique invented in [CC95], as developed in [Val04, PV05a, PV05b]. The arguments performed in the proof will be very technical, and several precise computations will be needed to investigate the compensation of the different quantities involved.

In what follows, we will consider θ and u as in the statement of Theorem 1. With no loss of generality, the point x in the statement of Theorem 1 will be taken to be the origin.

First of all, we point out that it is enough to prove Theorem 1 for θ close to -1:

Lemma 11. *Let us suppose that Theorem 1 holds true for some $\theta_\star \in (-1,1)$. Then, it holds true for any $\theta \in (\theta_\star, 1)$.*

Proof. If $\mathcal{L}^n\left(B_{\mu_1} \cap \{u > \theta\}\right) \geq \mu_2$, then, obviously, $\mathcal{L}^n\left(B_{\mu_1} \cap \{u > \theta_\star\}\right) \geq \mu_2$, and so, since the claim of Theorem 1 holds for $\theta_\star$, we have that $\mathcal{L}^n\left(B_r \cap \{u > \theta_\star\}\right) \geq c\, r^n$, as long as r is large enough.
Accordingly, making use of (8) and Lemma 10, we conclude that

$$\begin{aligned} & \mathcal{L}^n\left(B_r \cap \{u > \theta\}\right) \\ \geq\ & \mathcal{L}^n\left(B_r \cap \{u > \theta_\star\}\right) - \mathcal{L}^n\left(B_r \cap \{\theta_\star < u \leq \theta\}\right) \\ \geq\ & c\, r^n - C\, r^{n-1} \geq \frac{c\, r^n}{2} \end{aligned}$$

if r is sufficiently big. □

In the light of Lemma 11, we will suppose in what follows that θ is close to -1: in particular, we will suppose that $1+\theta \in (0, s_0)$, where s_0 is the structural constant introduced on page 3.
For the proof of Theorem 1, it is convenient to introduce the following quantities:

$$V_r := \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) \qquad \text{and} \qquad A_r = \int_{B_r} F(x, u)\, dx\,.$$

As remarked in [CC95], the above quantities play the rôle of volume and area terms, respectively, in the minimal surface analogous.
The idea of the proof will then be to obtain iterative relations of some quantities that are somewhat reminiscent of similar differential inequalities holding in the minimal surface theory (see again [CC95] for very interesting heuristics on this).
We will take $R > r$ and use a barrier function $h \in C^2(B_R - \{0\}) \cap C^{1,1}(B_R)$ such that $|h| \leq 1$, $h = 1$ outside B_R and

$$|\nabla h| \leq C\,, \tag{37}$$

where $C \geq 1$ denotes (here, and in what follows) a suitable constant (which may depend on n, Q θ, μ_1, μ_2, M_0 and on the structural constants of the functional $\mathcal{F}$ and which may be different at different steps of the computation).
We will set $q := \max\{p, 2\}$ and, as customary, the conjugated exponent of q will be denoted by $q' := q/(q-1)$.
We also set $\epsilon := (1+\theta)/2$ and we define $u^* = \min\{u, h\}$ and $\beta = \min\{u - u^*, \epsilon\}$.
The barrier h will be constructed in such a way that

$$h(x) \leq -1 + \epsilon \quad \text{for any} \quad x \in B_r. \tag{38}$$

As a consequence,

$$(u - u^*)(x) \geq \epsilon \text{ for any } x \in B_r \cap \{u > \theta\}, \tag{39}$$

and therefore

$$\int_{B_R} |\beta|^{qn/(n-1)} \geq \int_{B_r \cap \{u > \theta\}} |\beta|^{qn/(n-1)} \geq \frac{1}{C} V_r\,. \tag{40}$$

We will also make use of a free parameter K that will be chosen conveniently in what follows.

We use (40), the Sobolev inequality applied to β^q and then the Young inequality, to conclude that

$$\begin{aligned} V_r^{(n-1)/n} &\leq C\Big(\int_{B_R}|\beta|^{qn/(n-1)}\Big)^{(n-1)/n} \\ &\leq C\int_{B_R}|\beta|^{q-1}|\nabla\beta| \\ &= C\int_{B_R\cap\{u-u^*<\epsilon\}}|\beta|^{q-1}|\nabla\beta| \\ &\leq C\,K^q\int_{B_R}|\nabla(u-u^*)|^q \\ &\quad+\frac{C}{K^{q'}}\int_{B_R\cap\{u-u^*<\epsilon\}}(u-u^*)^q\,. \end{aligned}\tag{41}$$

If $p\geq 2$, we use the following formula (which is proven on page 1062 of [PV05a]):

$$\frac{1}{C}\,|\xi'-\xi|^p\leq A(x,\xi')-A(x,\xi)-a(x,\xi)\cdot(\xi'-\xi)\,,$$

for every $\xi,\xi'\in\mathbb{R}^n$ and $x\in\Omega$. In particular, taking $\xi:=\nabla u^*$ and $\xi':=\nabla u$, since $q=p$ if $p\geq 2$, we obtain that

$$\frac{1}{C}\,|\nabla(u-u^*)|^q\leq A(x,\nabla u)-A(x,\nabla u^*)-a(x,\nabla u^*)\cdot\nabla(u-u^*)\tag{42}$$

if $p\geq 2$.
On the other hand, if $p\in(1,2)$, we use the following formula (for the proof of which, see again page 1062 of [PV05a]): given $M\geq 0$, there exist $C_M\geq 1$ such that

$$\frac{M^{p-2}}{C_M}\,|\xi'-\xi|^2\leq A(x,\xi')-A(x,\xi)-a(x,\xi)\cdot(\xi'-\xi)\,,$$

for every $\xi,\xi'\in\mathbb{R}^n$ with $|\xi|+|\xi'|\leq M$ and $x\in\Omega$. Since ∇u and ∇u^* are uniformly bounded (because of (14) and (37)), we use the above formula with $\xi:=\nabla u^*$ and $\xi':=\nabla u$ to deduce that (42) holds when $p\in(1,2)$ too.
Then, by (41) and (42),

$$\begin{aligned} V_r^{(n-1)/n} &\leq C\,K^q\int_{B_R}A(x,\nabla u)-A(x,\nabla u^*)\,dx \\ &\quad- C\,K^q\int_{B_R}a(x,\nabla u^*)\cdot\nabla(u-u^*)\,dx \\ &\quad+\frac{C}{K^{q'}}\int_{B_R\cap\{u-u^*<\epsilon\}}(u-u^*)^q\,dx\,. \end{aligned}\tag{43}$$

We integrate by parts in (43), to get that

$$\begin{aligned} V_r^{(n-1)/n} &\leq C\,K^q \int_{B_R} A(x,\nabla u) - A(x,\nabla u^*)\,dx \\ &+ C\,K^q \int_{B_R} \operatorname{div} a(x,\nabla h)\,(u-u^*)\,dx \\ &+ \frac{C}{K^{q'}} \int_{B_R\cap\{u-u^*<\epsilon\}} (u-u^*)^q\,dx\,. \end{aligned}$$

Thus, we use the Q-minimality of u to obtain that

$$\begin{aligned} V_r^{(n-1)/n} &\leq CQ\,K^q \int_{B_R} A(x,\nabla u^*) + F(x,u^*)\,dx \\ &- C\,K^q \int_{B_R} F(x,u)\,dx \\ &+ C\,K^q \int_{B_R} \operatorname{div} a(x,\nabla h)\,(u-u^*)\,dx \\ &+ \frac{C}{K^{q'}} \int_{B_R\cap\{u-u^*<\epsilon\}} (u-u^*)^q\,dx\,. \end{aligned} \tag{44}$$

We now notice that

$$\operatorname{div} a(x,\nabla h) \;\leq\; C\,|\nabla h|^{p-2}(|\nabla h| + |D^2 h|)\,,$$

thanks to (5) and (6).
Such an estimate, (44) and (15) give that

$$\begin{aligned} V_r^{(n-1)/n} &\leq CQ\,K^q \int_{B_R\cap\{h\leq u\}} |\nabla h|^p + F(x,h)\,dx \\ &+ CQ\,K^q \int_{B_R\cap\{u<h\}} F(x,u)\,dx \\ &- C\,K^q \int_{B_R} F(x,u)\,dx \\ &+ C\,K^q \int_{B_R} |\nabla h|^{p-2}(|\nabla h| + |D^2 h|)\,(u-u^*)\,dx \\ &+ \frac{C}{K^{q'}} \int_{B_R\cap\{u-u^*<\epsilon\}} (u-u^*)^q\,dx\,. \end{aligned}$$

This and (39) imply that

$$
\begin{aligned}
V_r^{(n-1)/n} \leq\ & CQ\,K^q \int_{B_R\cap\{h\leq u\}} |\nabla h|^p + F(x,h)\,dx \\
& + CQ\,K^q \int_{B_R\cap\{u<h\}} F(x,u)\,dx \\
& - C\,K^q \int_{B_R} F(x,u)\,dx \\
& + C\,K^q \int_{B_R} |\nabla h|^{p-2}(|\nabla h| + |D^2 h|)\,(u-u^*)\,dx \\
& + \frac{C}{K^{q'}} \int_{(B_R-B_r)\cap\{u-u^*<\epsilon\}\cap\{u>\theta\}} (u-u^*)^q\,dx \\
& + \frac{C}{K^{q'}} \int_{B_R\cap\{u-u^*<\epsilon\}\cap\{u\leq\theta\}} (u-u^*)^q\,dx\,.
\end{aligned}
\tag{45}
$$

We now observe that

$$
CF(x,u) \geq (u-u^*)^q \tag{46}
$$

if $u \leq \theta$. Indeed, if (10) holds (that is, if we are in case χ, according to the notation of page 3), then (46) is obvious; when, on the other hand, we are in case W (according to the notation of page 3), we note that, if $u \leq \theta$,

$$
F(x,u) = \int_0^{u+1} F_u(x,-1+s)\,ds \geq \frac{1}{C}(u+1)^d\,,
$$

thanks to (11), and so (46) follows from the fact that $d \leq p \leq q$.
Therefore, we gather from (45) and (46) that

$$
\begin{aligned}
A_r + V_r^{(n-1)/n} \leq\ & CQ\,K^q \int_{B_R\cap\{h\leq u\}} |\nabla h|^p + F(x,h)\,dx \\
& + CQ\,K^q \int_{B_R\cap\{u<h\}} F(x,u)\,dx \\
& + C\,K^q \int_{B_R} |\nabla h|^{p-2}(|\nabla h| + |D^2 h|)\,(u-u^*)\,dx \\
& + C \int_{(B_R-B_r)\cap\{u-u^*<\epsilon\}\cap\{u>\theta\}} (u-u^*)^q\,dx\,,
\end{aligned}
$$

as long as K is chosen to be suitably large with respect to the structural constants. Such a K will then be fixed once and for all, and it will be absorbed into the constants C from now on. The same will be done for the constant Q. Therefore, we obtain

$$\begin{aligned} A_r + V_r^{(n-1)/n} &\leq C\,(V_R - V_r) \\ &+ C \int_{B_R \cap \{h \leq u\}} |\nabla h|^p + F(x,h)\,dx \\ &+ C \int_{B_R \cap \{u < h\}} F(x,u)\,dx \\ &+ C \int_{B_R} |\nabla h|^{p-2}(|\nabla h| + |D^2 h|)\,(u - u^*)\,dx\,. \end{aligned} \tag{47}$$

We now need to distinguish the cases χ and W in order to properly choose the barrier h. That is, we need to distinguish the case in which condition (10) holds from the case in which conditions (11), (12) and (13) hold.

5.1. **The case** χ**.** In this case, we are assuming condition (10), which implies that

$$A_{r+1} - A_r = \int_{B_{r+1} - B_r} F(x,u)\,dx \geq \frac{1}{C}\,\mathcal{L}^n\Big((B_{r+1} - B_r) \cap \{|u| < 1\}\Big)$$

and therefore

$$\begin{aligned} \mathcal{L}^n\Big((B_{r+1} - B_r) \cap \{u > -1\}\Big) &\leq C\,(A_{r+1} - A_r) + \mathcal{L}^n\Big((B_{r+1} - B_r) \cap \{u = 1\}\Big) \\ &\leq C\,(A_{r+1} - A_r) + (V_{r+1} - V_r)\,. \end{aligned} \tag{48}$$

We now choose $r := k \in \mathbb{N}$, $R := r + 1$ and h to be smooth and so that $h(x) = -1$ for any $x \in B_r$. In particular, h fulfills (37) and (38).
Then, we gather from (47) and (48) that

$$A_k + V_k^{(n-1)/n} \;\leq\; C\,(V_{k+1} + A_{k+1} - V_k - A_k)\,. \tag{49}$$

Also, if k_0 is the smallest integer greater than μ_1, it follows from (16) that

$$A_{k_0} + V_{k_0} \;\geq\; \mu_2\,. \tag{50}$$

We now exploit Lemma 2.1 of [PV05a] (used here with $\alpha := 0$, $v_k := V_{k+k_0-1}$ and $a_k := a_{k+k_0-1}$), to deduce from (49) and (50) that $V_{k+k_0-1} + A_{k+k_0-1} \geq \gamma k^n$ for any $k \in \mathbb{N}$, for a suitable $\gamma > 0$. Thence, by Lemma 10,

$$V_{k+k_0-1} \geq \gamma k^n - \hat{C} k^{n-1} \geq \frac{\gamma}{2} k^n\,,$$

as long as $k \geq 2\hat{C}/\gamma$. Consequently,

$$\mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) \;\geq\; \frac{\gamma}{2^{n+1}}\,r^n\,,$$

for any $r \geq r_0 := 2k_0 + 2\hat{C}/\gamma$, which ends the proof of Theorem 1 in case χ. ◊

5.2. **The case W.** This case is the one in which we assume conditions (11), (12) and (13). By possibly replacing d with $\max\{d,p\}$ in (11), we may suppose that $d = p$. Then, from (11),

$$C\,F(x,\tau) \,\geq\, (\tau+1)^p \tag{51}$$

if $\tau \in [-1,\theta)$.
We let $\Theta > 0$ be suitably small (with respect to the structural constants) and T (and, in fact, ΘT) suitably large (possibly in dependence of θ and Θ). Given $k \in \mathbb{N}$, we take $R := (k+1)T$, $r := kT$ and then we choose h as done on page 183 of [Val04]. Such a barrier satisfies

$$(h+1) + |\nabla h| + |D^2 h| \leq C(h+1) \leq Ce^{-\Theta T(k+1-j)} \tag{52}$$

in $B_{jT} - B_{(j-1)T}$, and

$$|\nabla h| + |D^2 h| \leq C\Theta(h+1) \tag{53}$$

in $B_{(k+1)T}$. Then, the assumptions on h taken on page 11 and, in particular (38), are satisfied provided that ΘT is conveniently large.
Notice also that, since $h \geq -1$,

$$u - h \,\leq\, (u-h) + (h+1) \,=\, u+1\,. \tag{54}$$

We now estimate the contribution of the terms in (47) when $u \leq \theta$. That is, we use (53), (51), (13), (38), (12) and (54) to gather

$$\begin{aligned}
&\int_{B_{(k+1)T}\cap\{h\leq u\}\cap\{u\leq\theta\}} \Big(|\nabla h|^p + F(x,h)\Big) + \int_{B_{(k+1)T}\cap\{u<h\}\cap\{u\leq\theta\}} F(x,u) \\
&\quad + \int_{B_{(k+1)T}\cap\{u\leq\theta\}} |\nabla h|^{p-2}\Big(|\nabla h| + |D^2 h|\Big)(u-u^*) \\
&\leq C\Big[\int_{(B_{(k+1)T}-B_{kT})\cap\{u\leq\theta\}} F(x,u) \\
&\quad + \int_{B_{kT}\cap\{u\leq\theta\}} F(x,h) + \int_{B_{(k+1)T}\cap\{h<u\leq\theta\}} (h+1)^{p-1}(u-h)\Big] \\
&\leq C\Big[A_{(k+1)T} - A_{kT} + k^{n-1}T^n \sum_{j=1}^{k} e^{-c\Theta T(k+1-j)} \\
&\quad + \int_{(B_{(k+1)T}-B_{kT})\cap\{h<u\leq\theta\}} (u+1)^p\Big] \\
&\leq C\Big[A_{(k+1)T} - A_{kT} + k^{n-1}T^n e^{-c\Theta T}\Big],
\end{aligned} \tag{55}$$

where $c := \min\{d', p-1\} > 0$ and d' is the quantity introduced in (12).

We now estimate the contribution of the terms in (47) when $u > \theta$. More precisely, we use (53), (51), (13) and (12) to obtain that

$$\begin{aligned} & \int\limits_{B_{(k+1)T}\cap\{h\le u\}\cap\{u>\theta\}} \Big(|\nabla h|^p + F(x,h)\Big) + \int\limits_{B_{(k+1)T}\cap\{u<h\}\cap\{u>\theta\}} F(x,u) \\ & + \int\limits_{B_{(k+1)T}\cap\{u>\theta\}} |\nabla h|^{p-2}\Big(|\nabla h| + |D^2 h|\Big)(u-u^*) \\ & \le C \sum_{j=1}^{k+1} e^{-c\Theta T(k+1-j)}(V_{jT} - V_{(j-1)T}) \,. \end{aligned} \tag{56}$$

By collecting the results in (47), (55) and (56), we see that

$$\begin{aligned} & A_{kT} + V_{kT}^{(n-1)/n} \\ & \le C \left[A_{(k+1)T} - A_{kT} + k^{n-1}T^n e^{-c\Theta T} + \sum_{j=1}^{k+1} e^{-c\Theta T(k+1-j)}(V_{jT} - V_{(j-1)T}) \right] , \end{aligned} \tag{57}$$

that is, (32) holds, by choosing $\mathcal{A}_k := A_{kT}$, $a_k := V_{kT} - V_{(k-1)T}$, $\varepsilon := CT^n e^{-c\Theta T}$ and $L := c\Theta T$. The constant C in (57) is now fixed once and for all (and the quantity C in the statement of Lemma 9 is assumed to agree with it). Of course, without loss of generality we may suppose that

$$C \ge \max\left\{ \frac{1}{\mu_2},\, 2^n n^3 \Big(\mathcal{L}^n(B_1) + 1 \Big) \right\} , \tag{58}$$

where μ_2 is the quantity introduced in (16).
Note that (29) is satisfied, thanks to (27) and to the assumption that T is suitably large. The facts that $e^{-L} > 0$, $C \ge 1$ and (27) yield (28). Also,

$$a_k = \mathcal{L}^n\Big(\{u > \theta\} \cap (B_{kT} - B_{(k-1)T}) \Big) \le \mathcal{L}^n\Big(B_{kT} - B_{(k-1)T} \Big) \le CT^n k^{n-1} ,$$

due to (58), thus yielding (31). Also, (30) is a consequence of (16) and (58), by taking $T \ge \mu_1$. Consequently, by Lemmata 9 and 10,

$$\begin{aligned} \gamma T^n k^n & \le \int_{B_{kT}} F(x,u)\,dx + \mathcal{L}^n\Big(\{u>\theta\} \cap B_{kT} \Big) \\ & \le \hat{C} T^{n-1} k^{n-1} + \mathcal{L}^n\Big(\{u>\theta\} \cap B_{kT} \Big) , \end{aligned}$$

for a suitable $\gamma > 0$. Thus, if $kT \ge 2\hat{C}/\gamma$,

$$\mathcal{L}^n\Big(\{u>\theta\} \cap B_{kT} \Big) \ge \frac{\gamma T^n}{2} k^n .$$

Consequently, if $r \ge r_0 := T + 2\hat{C}/\gamma$,

$$\mathcal{L}^n\Big(\{u>\theta\} \cap B_r \Big) \ge \frac{\gamma}{2^{n+1}} r^n ,$$

thus completing the proof of Theorem 1 in case W. ◊

6. Proof of Corollary 2

We will need the following auxiliary result, the proof of which follows from Theorem 1, Lemma 10, and the argument on pages 1066–1067 of [PV05a]:

Corollary 12. *Fix $\theta \in (0,1)$. Let u be a Q-minimum for $\mathcal{F}$ in Ω. Suppose that $|u| \leq 1$ and*

$$|\nabla u(x)| \,\leq\, M_0\,, \tag{59}$$

for a.e. $x \in \Omega$, that

$$A(x, \nabla u(x)) \,\leq\, M_0\, F(x, u(x))\,, \tag{60}$$

for a.e. $x \in \Omega \cap \{|u| < 1\}$, for some $M_0 > 0$. Let $x \in \{-\theta < u < \theta\}$ and $y \in \Omega$. Then, there exist positive r_0, c and C, depending only on n, Q, θ, M_0 and the structural constants, such that:

$$\mathcal{L}^n\Big(B_r(x) \cap \{u > \theta\}\Big) \geq c\, r^n\,, \qquad \mathcal{L}^n\Big(B_r(x) \cap \{u < -\theta\}\Big) \geq c\, r^n$$

$$\mathcal{L}^n\Big(B_r(x) \cap \{|u| < \theta\}\Big) \,\geq c\, r^{n-1} \;\; \textit{and} \;\; \mathcal{L}^n\Big(B_r(y) \cap \{|u| < \theta\}\Big) \,\leq C\, r^{n-1},$$

for any $r \geq r_0$, provided that $B_r(x)$, $B_r(y) \subset \Omega$.

We now focus on the proof of Corollary 2. First of all, we deal with the $L^1_{\rm loc}$-convergence of u_ε. The argument we perform for this is, up to now, quite standard (see, e.g., [Mod87]). For $\tau \in [-1,1]$, we define

$$\Phi(\tau, x) \,:=\, \int_0^\tau \Big(F(x,\xi)\Big)^{1/p'}\, d\xi\,,$$

where $p' := p/(p-1)$ is the conjugated exponent of p. Then, for any fixed $x \in \Omega$, the map $[-1,1] \ni \tau \mapsto \Phi(\tau, x)$ is strictly increasing, because of (8), and so we denote by $\Psi(\cdot, x)$ its inverse function.

We define $v_\varepsilon(x) := \Phi(u_\varepsilon(x), x)$. Then,

$$|v_\varepsilon| \,\leq\, 2\left(\sup_{x\in\Omega,\,|\xi|\leq 1} F(x,\xi)\right)^{1/p'} \,<\, +\infty\,, \tag{61}$$

thanks to (18) and (8). Moreover, by the Young inequality, (17) and (21), given a ball $B \subset \Omega$, we have that

$$\int_B |\nabla v_\varepsilon|\, dx \leq \int_B (F(x,u_\varepsilon))^{1/p'} |\nabla u_\varepsilon|\, dx + C\mathcal{L}^n(B) \leq \mathcal{F}^\varepsilon_\Omega(u) + C\mathcal{L}^n(B) < +\infty\,, \tag{62}$$

for a suitable $C > 0$.

The estimates in (61) and (62), together with the Rellich-Kondrashov Theorem, imply that v_ε converges, up to subsequences, in $L^1_{\rm loc}$ and so a.e., to a suitable v_0.

We let $u_0(x) := \Psi(v_0(x), x)$ and we consider a ball $B \subset \Omega$. Note that $u_\varepsilon(x) := \Psi(v_\varepsilon(x), x)$ converges a.e. to v_0. Thus, we use (18) and the Dominated Convergence Theorem to deduce that

$$\lim_{\varepsilon\to 0} \int_B |u_\varepsilon - u_0|\, dx \,=\, 0\,,$$

which gives the $L^1_{\rm loc}$-convergence of u_ε.

We now show that the limit function u_0 is a step function, i.e., that it takes only the values 1 and -1. Indeed, by the Fatou's Lemma and (21),

$$\int_\Omega F(x,u_0)\, dx \leq \liminf_{\varepsilon\to 0} \int_\Omega F(x,u_\varepsilon)\, dx \leq \liminf_{\varepsilon\to 0} \varepsilon \mathcal{F}^\varepsilon_\Omega(u_\varepsilon) = 0\,.$$

Accordingly, $F(x,u_0)=0$ and so $|u_0|=1$, as desired.
The fact that the set $E:=\{u=1\}$ has Q-minimal perimeter follows from Theorem 3.3 of [Bou90] and Theorem 5.1 of [GG84].
The locally uniform convergence of the (sub)level sets of u_ε follows now by a standard argument (see, e.g., page 69 of [PV05b]), which we repeat here for the reader's convenience.
If the locally uniform convergence of the (sub)level sets of u_ε were false, there would exist some $d>0$, an infinitesimal sequence ε_k, and a sequence of points x_k contained in a compact subset of Ω such that $|u_{\varepsilon_k}(x_k)|<\theta$ and in such a way that $B_d(x_k)$ is always contained either in E on in $\Omega-E$. Say, for definiteness, that $B_d(x_k)\subseteq E$. Then,

$$u_0(x)=1 \text{ for any } x\in B_d(x_k). \tag{63}$$

We set $w_\varepsilon(x):=u_\varepsilon(\varepsilon x)$. Note that (59) and (60) for w_ε hold because of (19) and (20), and that $|w_{\varepsilon_k}(x_k/\varepsilon_k)|<\theta$. Also, by a direct computation, one sees that w_ε is a Q-minimum of the functional

$$\mathcal{G}^\varepsilon_{\Omega/\varepsilon}(w) := \int_{\Omega/\varepsilon} A(\varepsilon x,\nabla w)+F(\varepsilon x,w)\,dx\,,$$

where $\Omega/\varepsilon:=\{x/\varepsilon,\ x\in\Omega\}$. We remark that the structural constants of $\mathcal{G}^\varepsilon$ agree with the ones of $\mathcal{F}$, since conditions (3)–(13) are uniform in x.
Accordingly, by Corollary 12,

$$\begin{aligned} c\rho^n &\le \mathcal{L}^n\Big(B_\rho(x_k/\varepsilon_k)\cap\{w_{\varepsilon_k}<-\theta\}\Big)\\ &= \frac{1}{\varepsilon_k^n}\,\mathcal{L}^n\Big(B_{\varepsilon_k\rho}(x_k)\cap\{u_{\varepsilon_k}<-\theta\}\Big)\,, \end{aligned}$$

for some $c>0$, as long as $\rho\ge r_0$, and therefore

$$\mathcal{L}^n\Big(B_d(x_k)\cap\{u_{\varepsilon_k}<-\theta\}\Big)\ \ge\ cd^n\,,$$

if ε_k is small enough.
Consequently, using also (63),

$$\int\limits_{B_d(x_k)} |u_0-u_{\varepsilon_k}| \ge \int\limits_{B_d(x_k)\cap\{u_{\varepsilon_k}<-\theta\}} |u_0-u_{\varepsilon_k}| \ge cd^n\,(1+\theta)$$

in contradiction with the $L^1_{\rm loc}$-convergence of u_ε.
This ends the proof of Corollary 2. ◊

7. Proof of Corollaries 3 and 4

Conditions (17) and (18) in Corollary 2 hold true under the hypotheses of Corollaries 3 and 4. Conditions (19) and (20) are assured by (23) and the results in [Tol84] and [CGS94], respectively. A scaling argument and Lemma 10 imply (21). Therefore, we are able to apply Corollary 2.
Moreover, one easily sees that condition (2.1) in [SV05] is satisfied by u, thanks to Corollary 12 (full details on how to check such a condition are also given in the proof of Corollary 9.2 of [SV05]). Then, the claim of Corollary 3 (resp., Corollary 4) follows from Corollary 2 here and from Theorem 2.1 in [SV05] (resp., Lemma 6.6 in [SV05]). ◊

8. Proof of Corollary 5

The proof is a modification of the one of Theorem 1.1 in [VSS06], according to the following observations. First of all, Lemma 2.23 of [VSS06], which is stated there for minimizers, does hold for quasiminima (indeed, one just uses Corollary 12 of the present paper instead of Theorem 1.1 of [PV05a] to go through the proof of Lemma 2.23 of [VSS06]).
Consequently, Corollary 2.24 of [VSS06] also holds for quasiminima.
Moreover, Lemma 5.1 of [VSS06] holds for quasiminima too (indeed, its proof uses Lemma 2.23 of [VSS06], which we have just shown to hold for quasiminima).
As a consequence, Proposition 5.2 of [VSS06] holds for quasiminima.
The proof of Corollary 5 here is now the same as the one of Theorem 1.1 of [VSS06] (as given on pages 61–67 there). Indeed, the proof of Theorem 1.1 of [VSS06] uses Lemma 5.1, Proposition 5.2 and Corollary 2.24 there (which have just been shown to hold for quasiminima) and the results of [Bou90, PV05a] (which get replaced respectively by (24) and Corollary 12 here). $\diamondsuit$

9. Proof of Corollary 6

The proof is the same as the one of Theorem 1.2 of [VSS06], as given on pages 69–73 there. The only difference is that the use of Theorem 1.1 of [VSS06] (resp., Lemma 9.3 of [SV05]) is replaced by the use of Corollary 5 (resp., Corollary 3) here. $\diamondsuit$

10. Proof of Corollary 7

We note that Lemma 8.1 of [VSS06] holds for quasiminima, since its proof uses Theorem 1.2 of [VSS06] which may be replaced by Corollary 6 here. Then, the desired result follows by repeating verbatim the argument on page 77 of [VSS06]. $\diamondsuit$

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