Comment on "Nonlinear adiabatic passage from fermion atoms to boson molecules"

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Dynamics of an adiabatic sweep through a Feshbach resonance in a quantum gas of fermionic atoms was considered recently in I.Tikhonenkov, E.Pazy *et al*, Phys. Rev. **A 73**, 043605 (2006). In the mean-field limit, the system is reduced to a classical nonlinear Hamiltonian system with a slowly changing parameter. Analysis of the latter system was done incorrectly; the main results of the commented paper are inconsistent. Here we present accurate study of the mean-field dynamics of the model based on the rigorous separatrix crossing theory. In particular, for the dependence of the remaining atomic fraction Γ on the sweep rate α the abovementioned Refs. predict the power-law $\Gamma \sim \alpha^{1/3}$ (in the case where the initial fraction of molecules is larger than the quantum fluctuations). Instead, Γ is related to a quasi-random jump of an adiabatic invariant which can be calculated using a general method of the separatrix crossing theory and asymptotically $\sim \alpha$.

Dynamics of Bose-Einstein condensates (BEC) introduce paradigm of nonlinearity into quantum systems. Recently, in many models related to BEC physics (like nonlinear Landau-Zener tunneling, macroscopic quantum self-trapping, etc) nonlinear effects that are common for classical nonlinear systems have been revealed. One of the conceptual phenomena of classical adiabatic theory is destruction of the adiabatic invariance at separatrix crossings which is encountered in different fields of physics.

In particular, in a recent Ref. [1], dynamics of an adiabatic sweep through a Feschbach resonance in a quantum gas of fermionic atoms was modeled by a simple mean-field model: a classical Hamiltonian system with a slowly changing parameter. It was argued that for adiabatic Feschbach sweeps that convert degenerate fermionic atoms to diatomic molecules, the Landau-Zener (LZ) behaviour for a single pair of particles is dramatically changed due to many-body effects: the remnant fraction Γ of unconverted atoms is claimed to be a power-law in the sweep rate, rather than being exponentially small. Two main results were obtained: a linear power-law $\Gamma \sim \alpha$ when initial molecular fraction is smaller than the $1/N$ quantum fluctuations, and a cubic power-law $\Gamma \sim \alpha^{1/3}$ when it is larger.

In this Comment we firstly present our analysis of the system and at the end clarify inconsistencies of Ref. [1] that prevent the authors from obtaining adequate results. We note that a general rigorous method exists for treating such kinds of nonlinear systems. It is well-known that action is an adiabatic invariant in a Hamiltonian system that depends on a slowly varying parameter. This result is based on the possibility of averaging over fast motion in the unperturbed [frozen at a certain parameter value] system. The situation is different in case the unperturbed system has separatrices on its phase portrait. As the parameter varies, the separatrices slowly evolve on the phase portrait. A phase trajectory of the exact system may cross the separatrix of the frozen system. On the separatrix, the period of motion tends to infinity. This results in breakdown of the averaging method and in this case more accurate study is necessary to describe behavior of the action. It turns out that at the crossing a quasi-random jump in the value of the adiabatic invariant occurs. The asymptotic formula for this jump in a Hamiltonian system depending on a slowly varying parameter was obtained in [2]. Later, the theory of adiabatic separatrix crossings was also developed for slow-fast Hamiltonian systems [3], volume preserving systems [4], and was applied to certain physical problems (see, for example, Refs. [5–7]). It was also noticed that nonlinear Landau-Zener tunneling models constitute a particular case for which a general theory can be applied [8].

In Ref. [1], the considered two-mode model of the atom-molecular system in the classical field limit leads to the equations of motion (Eq. 6 of Ref. [1])

$$\begin{aligned} \dot u &= \delta(\tau) v, \\ \dot v &= -\delta(\tau) u + \frac{\sqrt{2}}{4}(w-1)(3w+1), \\ \dot w &= \sqrt{2} v, \end{aligned} \tag{1}$$

where u, v, w are real and imaginary parts of atom-molecule coherence, and atom-molecule population imbalance, correspondingly; the dot denotes time derivative, $\tau = \epsilon t$, $0 < \epsilon \ll 1$ is a small parameter, and $\delta(\tau)$ is a slowly varying parameter corresponding to the (scaled) detuning.

Let us firstly consider these equations at frozen values of parameter δ. In this case, equations (1) possess two first integrals:

$$u^2 + v^2 - \frac{1}{2}(w-1)^2(w+1) = 0, \tag{2}$$

$$u - \frac{\delta}{\sqrt{2}} w = \mathrm{const} \tag{3}$$

The first integral defines a surface of rotation around $w-$axis with a singular [conical] point at (0,0,1). Equa-

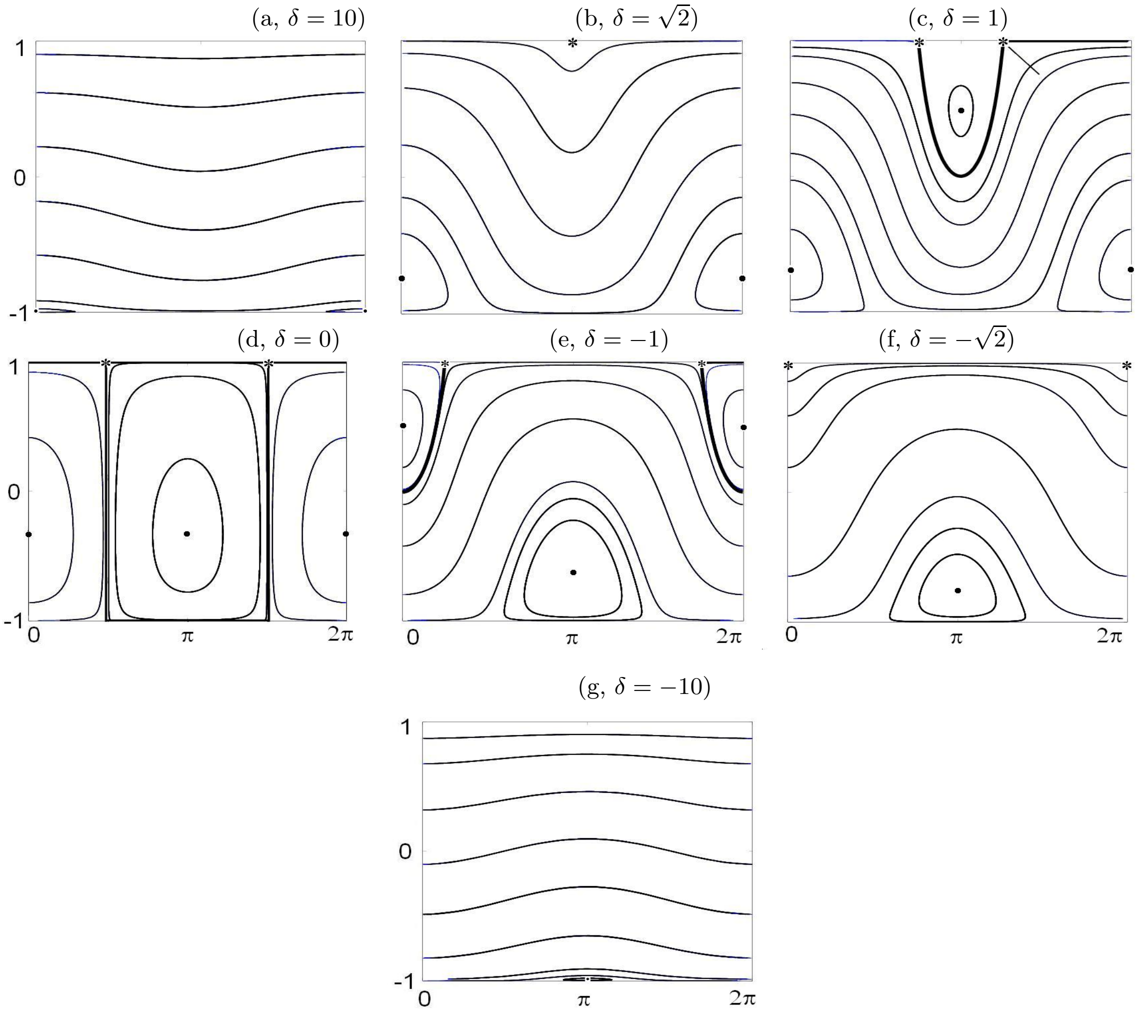

FIG. 1: Phase portraits of the system with Hamiltonian (4) frozen at different values of parameter δ. Stars: unstable fixed points; dotes: elliptical (stable) fixed points. Note the differences with corresponding phase portraits shown in Ref. [1]. While phase trajectories are depicted correctly on that portraits, fixed points are not always determined correctly, and the separatrices are not shown.

tions of motion (1) restricted to surface (2) are equivalent to equations of motion in a Hamiltonian system with Hamiltonian

$$H = -\delta(\tau)w + (1-w)\sqrt{1+w}\cos\phi, \qquad (4)$$

where canonically conjugated variables are w and ϕ, $\phi = \arctan(v/u)$ [see Ref. [1]; there is a misprint in the formula for H there].

Consider phase portraits of the system with Hamiltonian (4) frozen at different values of parameter δ [see Fig. 1]. In these portraits, points (0,0,1) and (0,0,-1) are represented as segments $w = 1$ and $w = -1$ correspondingly. Point (0,0,1) is always a stable point of (1), but formally speaking this is not true for the points of the segment $w = 1$ on phase portraits of the system with Hamiltonian (4). If δ is positive and $|\delta| > \sqrt{2}$, there is only one stable elliptic point on the phase portrait, at $\phi = 0$ and w not far from -1 [see Fig. 1a].

At $\delta = \sqrt{2}$ a bifurcation takes place, and at $\sqrt{2} > \delta > 0$ the phase portrait looks as shown in Fig. 1c. There are two saddle points at $w = 1, \cos\phi = -\delta/\sqrt{2}$ and a newborn elliptic point at $\phi = \pi$. The trajectory connecting these two saddles corresponds to the singular trajectory on surface (2). On the phase portrait of the system with Hamiltonian (4) it separates rotations and oscillating motions and we call it the separatrix of the frozen system. Below, we consider the "shifted" Hamiltonian $E = H + \delta$; E equals zero on the separatrix.

At $\delta = 0$ the singular trajectory on (2) is passing through (0,0,-1), and correspondingly on the phase portrait the segment $w = -1$ belongs to the separatrix [Fig. 1d]. At $0 > \delta > -\sqrt{2}$ the phase portrait looks as shown in Fig. 1e. Finally, at large negative values of δ, again there is only one elliptic stationary point at $\phi = \pi$, and w close to -1.

Consider now a phase trajectory on a phase portrait frozen at a certain value of δ. If the trajectory is closed, the area S inside of it is connected with the action I of the system by a simple relation $S = 2\pi I$. If the trajectory is not closed, we define the action as follows. If the area S bounded by the trajectory and lines $w = 1, \phi = 0, \phi = 2\pi$ is smaller than 2π, we still have $S = 2\pi I$. If S is larger than 2π, we put $2\pi I = 4\pi - S$. Defined in this way, I is a continuous function of the coordinates.

In system with Hamiltonian (4) with $\tau = \epsilon t$, the action I is an adiabatic invariant of motion (see, for example, [3]). Far from the separatrix, it undergoes oscillations of order ϵ. We introduce the so called improved adiabatic invariant $J = I + \epsilon f(w, \phi, \tau)$. The function f is defined as follows [see, for example, [2]]:

$$f = f(w, \phi, \tau) = \frac{1}{2\pi}\int_0^T \left(\frac{T}{2} - t\right)\frac{\partial H}{\partial \tau}dt. \qquad (5)$$

The integral in (5) is computed along the unperturbed trajectory passing through the point (w, ϕ) at the time moment $t = 0$; T is the period of motion on this trajectory. If the trajectory is not closed, T is the time necessary for a phase point on this trajectory to cover the distance of 2π along the ϕ-axis. Far from the separatrices variation of J along a phase trajectory is of order ϵ^2. As the slow time τ grows, the area bounded by the separatrix $S^*(\tau)$ slowly changes. On the other hand, a value of the adiabatic invariant associated with a certain phase trajectory stays well-preserved. Accordingly, phase trajectories of (4) can cross the separatrix. Let initially δ be large in magnitude and positivw, so that the phase portrait is similar to one shown in Fig. 1a. Consider a trajectory rotating close to the singular point on (2). Along the corresponding phase trajectory on the plane (ϕ, w), value of w is close to 1. We assume $1-w$ on this trajectory to be small, yet finite. Hence, the initial value J_- of improved action J is also small. As the time goes, value of parameter δ diminishes, and at $\delta = \sqrt{2}$ the separatrix loop appears. The area $S^*(\tau)$ surrounded by the separatrix grows with time, and the action associated with the phase trajectory stays approximately constant. In the so called adiabatic approximation this action is conserved at $J = J_-$, and at the slow time moment $\tau = \tau_*$ such that $S^*(\tau_*) = 2\pi J_-$ the phase trajectory crosses the separatrix. After the separatrix crossing, the phase point rotates around the elliptic point inside the separatrix loop and slowly drifts with this point to the bottom of the phase portrait. If $\delta(\tau)$ is a monotonous function, the phase trajectory will never again cross the separatrix. Calculating change in the improved action due to the separatrix crossing according to the general method, we find

$$2\pi\Delta J = -2\frac{\epsilon\Theta}{\sqrt{2-\delta_*^2}}\ln(2\sin\pi\xi), \qquad (6)$$

where $\xi = |h_0/(\epsilon\Theta)|$, h_0 is value of the Hamiltonian at the time of the last crossing of the vertex bisecting

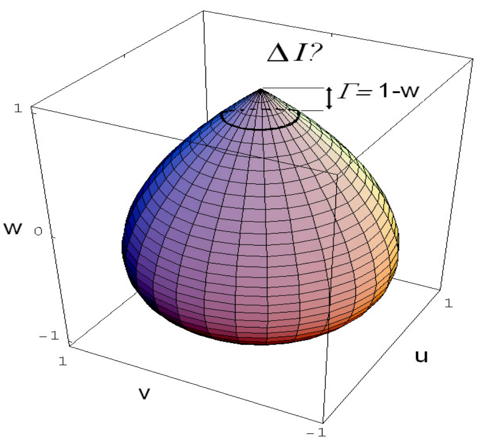

FIG. 2: The surface $u^2+v^2 = (w+1)(w-1)^2/2$. Action variable is related to area within a trajectory on the hamiltonian phase portrait, not on $(u, v)-$ plane. Contrary to Ref. [1], Γ is not proportional to square root of action (i.e., square root of ΔI in the case initial action is zero. Note that we calculate change in action only in the case where the initial action is not very small).

the angle between incoming and outgoing separatrices of the saddle point C outside the separatrix loop (see Fig. 1c) before crossing the separatrix; $\Theta = dS/d\tau|_{\tau=\tau_*}$; $\delta_* = \delta(\tau_*)$. The value ξ strongly depends on initial conditions: a small of order ϵ variation of initial conditions results generally in variation of ξ of order 1. Often, this value can be considered as a random variable (in particular, when one considers a bunch of trajectories with initial conditions distributed in certain volume in phase space). As a result, jump of the adiabatic invariant at separatrix crossing is also quasi-random value (strongly dependent on initial conditions). The formula is valid provided ξ is not too close to 0 or 1 (i.e., not for all trajectories from the bunch, but only for most of them; for details, see Ref. [3]). For large $|\delta|$, the function f tends to zero, and action I coincides with J. Hence, formula (6) gives the variation in action I. As we consider slow change of δ from initial large positive value $\delta(t_i)$ to final large negative value $\delta(t_f)$, in the adiabatic limit $w(t_f) = -w(t_i)$, i.e. sign of w is reversed due to the passage through resonance, but its absolute value is preserved: $\Delta|w| = w(t_f) + w(t_i) = 0$. Small nonadiabatic correction arise due to quasi-random jump of adiabatic invariant at the separatrix crossing as described above, and the resulted $\Delta|w|$ is equal to change in action ΔI.

We present detailed theoretical calculations in a separate paper. Note that similar calculations are done in Ref. [7]. Similar formulas has been used in some problems of plasma physics and hydrodynamics. Although they do not relate explicitly the change in action with initial conditions, they are useful in describing dynamics of the corresponding systems, especially when multiple separatrix crossings occur.

Let us give now several remarks concerning calculations of Ref. [1].

In these papers, it was correctly established that the nonadiabatic correction behaves as a power of the small parameter. However, there are some inconsistencies in derivation of this power law that make the final results of Ref.[1] incorrect. Let us mention some of them.

1) The analysis of phase portraits of the system with Hamiltonian (4) performed in Ref. [1] is inconsistent. The sign of δ is not the most important issue (sweeping from negative to positive δ in the Hamiltonian $\delta w + (1-w)\sqrt{1+w}\cos\phi$ is equivalent to sweeping from positive to negative δ in the Hamiltonian (4) with the $-\delta w$ term instead of $+\delta w$). Main phenomena in this problem occur near the separatrix, and the separatrix is not shown at all in Fig.6 of [1]. There are no stable points on the phase portrait of the Hamiltonian (4) at $w=1$ at $\delta > \sqrt{2}$. At $\delta = \sqrt{2}$, two saddle points and one elliptic point appear. These saddle points are not shown in Fig.6. of Ref. [1] and not accounted for in the analysis (correspondingly, the curve structure of Fig.4 of Ref. [1] seems to be inadequate).

2) The most important issue is calculation of change of the action. Evaluation of integral (14) in Ref. [1] is improper. In that Ref., expression (16) is obtained by differentiation of (11) and describes the rate of motion of elliptic fixed points due to slow variation of δ. However, to obtain their Eq. (18), the authors use this expression near $w=1$, i.e. near a hyperbolic point (actually, even in case the expression were used near an elliptic fixed point as intended, it is not clear why the rate of motion of elliptic fixed points should be related to the change in the action in such a simple way as described in Section 4. Calculation of change of action in this system is a very difficult mathematical problem. In such systems change in action is usually related to motion near unstable fixed points, where manipulations with linearization around elliptic fixed points do not help much).

3) The statement $\Gamma^2 = \Delta I/2$ (where $\Gamma = 1-w$, see Eq.14 of Ref. [1]) is also incorrect. Briefly, for a non-hamiltonian dynamical system, area within a trajectory has nothing to do with the action (note that the expression was related to the case of zero initial action, so it means $\Gamma^2 = I/2$, where I is the action obtained during the motion near $w=1$. We do not consider this case (of zero or very small initial action) in our Comment, still it is important to establish how the action is related to w in the system).

Consider firstly the system in the Hamiltonian form, with Hamiltonian function (4) and canonically conjugated variables w, ϕ. It is a well-known fact that the action is proportional to the area on the phase portrait on plane (ϕ, w). In the case of a large δ and w close to 1 [this is the case discussed above and represented in Fig.3b in Ref. [1]], the area between a phase trajectory and the line $w=1$ is proportional to $(1-w)$, i.e., Γ [see also Fig.2 in our paper]. Considering this question using the system in non-Hamiltonian form (Eqs. (1)) is a little bit more complicated. In Section IV of Ref. [1] the authors consider a closed trajectory with w close to 1 on the surfaces $u^2+v^2 = 1-w^2$ [Bloch sphere of a two-mode Bose-Josephson system] and $u^2+v^2 = (w+1)(w-1)^2/2$ [the corresponding surface in the considered problem]. They state that "the surface area enclosed within this periodic trajectory is just the action ... accumulated during the sweep". However, it is not so. Action variable is the first integral of the unperturbed system. Hence it should be preserved by the flow of system (1) [system (6) in [1]] at a constant value of parameter δ. Consider motion in this system on the level surface $L=0$ of function $L = u^2+v^2-(w+1)(w-1)^2/2$. In order to understand what measure on this surface is preserved, consider a close level surface $L=d$, $|d| \ll 1$. Consider a surface element dS on surface $L=0$ and an elementary cylinder bounded by this surface element, a surface composed of normals to $L=0$ passing through the border of dS, and surface $L=d$. In the limit as $d \to 0$ we find that the volume of this elementary cylinder is $dV = dS/|\nabla L|$. System (1) is volume-preserving. Hence this elementary volume is invariant under the flow of this system. Therefore, the integral of the unperturbed system is not the area on the surface $L=0$ (and, of course, not the area within a trajectory), but the area on the surface divided by magnitude of the gradient $|\nabla L|$. Near the singular point $u=v=0$, $w=1$ the surface is essentially conical and this gradient is proportional to $(1-w)$. Thus, we find that action variable is proportional to $(1-w)$ and not to $(1-w)^2$ as the authors of Ref. [1] state (see also Fig.2).

4) From physical point of view, it should be noted that both Landau-Zener model and the model of Ref. [1] predict 100% transfer efficiency in the adiabatic limit. Nonadiabatic corrections discussed above introduce just small deviations from 100% efficiency. However, the characteristic feature of the experimental data of Ref. [9] is that it has approximately 50% efficiency in the adiabatic limit. The experimental data from Ref. [9] was reproduced in Fig.1 of Ref. [1], but only for the case of not very slow ramp speeds. This makes Fig. 1 of Ref. [1] a little bit confusing. What is shown in the Fig. is related not to regime of exponentially small fraction of remnant atoms in Landau-Zener model (or power-law fraction in the nonlinear model), but to the regime of fast ramping. However, theoretical considerations of Ref. [1] (and our present Comment) are for the case of small nonadiabatic corrections.

To conclude, from our point of view the main result of the commented paper is very interesting analogy between dynamics of Feschbach resonance passage and fundamental issue in classical mechanics, i. e. destruction of adiabatic invariance at separatrix crossings. Unfortunately, the main results declared in this Ref. (Eqs. (26) and (20) of Ref. [1]) seems to be incorrect due to the reasons discussed above. In the first part of the Comment, we presented constructive analysis of the case where initial molecular fraction is not very small $(1-w(t_i) \gg 1/N)$.

We did not consider here the case with very small initial molecular fraction (which lead to inclusion of "quantum-noise" term and Eq. (26) of Ref. [1], $\Gamma \sim \alpha$). However, we noted that statement $\Gamma^2 \sim \Delta I$ is incorrect which would make Eq.(26) also incorrect in case ΔI were calculated correctly. Since this is not the case, the Eq.(26) might be true by coincidence (in case several mistakes cancel each other). This requires more detailed consideration.

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