Stability of Periodic Soliton Equations under Short Range Perturbations

Spyridon Kamvissis*
Department of Applied Mathematics, University of Crete
714 09 Iraklion, Greece

Gerald Teschl†
Faculty of Mathematics, University of Vienna,
Nordbergstrasse 15, 1090 Vienna, Austria
and
International Erwin Schrödinger Institute for Mathematical Physics,
Boltzmanngasse 9, 1090 Vienna, Austria‡
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We consider the stability of (quasi-)periodic solutions of soliton equations under short range perturbations and give a complete description of the long time asymptotics in this situation. We show that, apart from the phenomenon of the solitons travelling on the quasi-periodic background, the perturbed solution asymptotically approaches a modulated solution. We use the Toda lattice as a model but the same methods and ideas are applicable to all soliton equations in one space dimension.

More precisely, let g be the genus of the hyperelliptic Riemann surface associated with the unperturbed solution. We show that the n/t-pane contains $g+2$ areas where the perturbed solution is close to a quasi-periodic solution in the same isospectral torus. In between there are $g+1$ regions where the perturbed solution is asymptotically close to a modulated lattice which undergoes a continuous phase transition (in the Jacobian variety) and which interpolates between these isospectral solutions. In the special case of the free solution ($g=0$) the isospectral torus consists of just one point and we recover the classical result.

Both the solutions in the isospectral torus and the phase transition are explicitly characterized in terms of Abelian integrals on the underlying hyperelliptic Riemann surface.

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I. INTRODUCTION

One of the most important defining properties of solitons is their stability under perturbations. The classical result going back to Zabusky and Kruskal [14] states that a short range perturbation of the constant solution of a soliton equation eventually splits into a number of stable solitons. So the solitons constitute the stable part of arbitrary short range initial conditions. This is the motivation for the result presented here. Our aim is to investigate the case where the constant background solution is replaced by a quasi-periodic one.

Solitons on a (quasi-)periodic background have a long tradition and are used to model localized excitements on a phonon, lattice, or magnetic field background. Of course periodic solutions, as well as solitons travelling on a periodic background, are well understood (see e.g. [8] or [6]) and one might expect that, in analogous fashion, a small initial perturbation of the periodic lattice splits into a number of solitons on a periodic background. The reason why this naive expectation has to be wrong is related to the fact that solitons produce a phase shift; something that was probably first observed in [10] who investigated solitons of the Korteweg-de Vries equation on the background of the two-gap Weierstrass solution. Since the phase shifts of the solitons do not necessarily add up to zero, there must be an additional feature making up for this phase shift. This feature has not been understood so far and will be our main topic here.

To illustrate these facts, let us consider the doubly infinite Toda lattice in Flaschka's variables (see e.g. [12] or [13])

$$\begin{aligned}\dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2),\\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)),\end{aligned}$$

$(n,t) \in \mathbb{Z}\times\mathbb{R}$, where the dot denotes differentiation with respect to time. We will consider a quasi-periodic algebro-geometric background solution (a_q, b_q) (e.g., any periodic solution) plus a short range (in the sense of [5]) perturbation (a,b). The perturbed solution can be computed via the inverse scattering transform. The case where (a_q, b_q) is constant is classical (see again [12] or [13]), but the more general case applicable here has only recently been analyzed in [5] (see also [4]).

In Figure 1 the numerically computed solution corresponding to the initial condition $a(n,0)=1$, $b(n,0) = (-1)^n + 2\delta_{0,n}$ is shown. In this picture the two observed lines express the variables $a(n,t)$ at a frozen time $t=90$.

*Electronic address: spyros@tem.uoc.gr
†Electronic address: Gerald.Teschl@univie.ac.at; URL: http://www.mat.univie.ac.at/~gerald/
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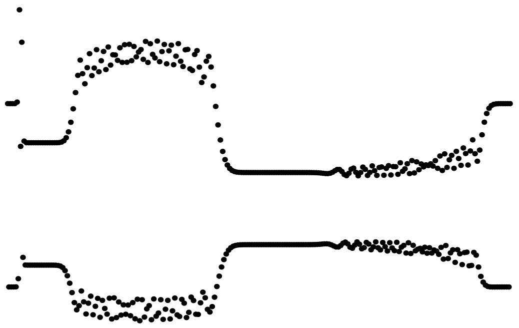

FIG. 1: Numerically computed solution of the Toda lattice, with initial condition a period two solution perturbed at one point in the middle.

In areas where the lines seem to be continuous this is due to the fact that we have plotted a huge number of particles (around 1000) and also due to the 2-periodicity in space. So one can think of the two lines as the even- and odd-numbered particles of the lattice. We first note the single soliton which separates two regions of apparent periodicity on the left. Also, after the soliton, we observe three different areas with apparently periodic solutions of period two. Finally there are some transitional regions in between which interpolate between the different period two regions. It is the purpose of this paper to give a complete explanation of this picture. We provide the details in case of the Toda lattice though it is clear that our arguments apply to other soliton equations as well.

II. QUASIPERIODIC SOLUTIONS IN TERMS OF RIEMANN THETA FUNCTIONS

We begin by recalling that quasi-periodic solutions are most conveniently written in terms of Riemann theta functions ([3]). In case of the Toda lattice the underlying Riemann surface is associated with the square root

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}.$$

We can picture this surface as the end result of cutting and pasting; we consider two "sheets" (copies of the complex plane), we cut them along the segments $[E_0, E_1]$, $[E_2, E_3]$, $\ldots$ and paste the top side of the upper sheet to the bottom side of the lower sheet and the bottom side of the upper sheet to the top side of the lower sheet across every such segment. The numbers $E_j \in \mathbb{R}$ are the band edges of the spectrum $\Sigma = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]$ of the corresponding Jacobi operator in the Lax pair. On this Riemann surface we have a standard basis of normalized holomorphic differentials

$$\zeta_j = \frac{\sum_k c_j(k) z^k}{R_{2g+2}^{1/2}(z)} dz, \quad 1 \le j \le g$$

(the constants $c_j(k)$ have to be determined from the usual normalization with respect to a canoncial homology basis).

Introduce the vector

$$\begin{aligned} z_j(n,t) = {} & \int_{E_0}^{\infty_+} \zeta_j - \sum_{k=1}^{g} \int_{E_0}^{\mu_k} \zeta_j \\ & - n \int_{\infty_-}^{\infty_+} \zeta_j + t 2 c_j(g) - \Xi_{E_0} \in \mathbb{C}^g. \end{aligned}$$

Here $\infty_\pm$ are the points above ∞ on the upper/lower sheet and Ξ_{E_0} is the vector of Riemann constants. The numbers μ_j are some arbitrary points whose images in the complex plane lie in the j-th interior spectral gap. All possible choices form the isospectral class of quasi-periodic Jacobi operators with the given spectral bands. This isospectral class is just a g dimensional torus, since the preimage of each gap with respect to the map that maps the surface to the complex plane (seen as one of the two sheets) consists of two parts, one on the upper and one the lower sheet, which form a circle.

Then the well-known formulas for the solutions read

$$\begin{aligned} a(n,t)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2}, \\ b(n,t) &= \tilde{b} + \frac{1}{2}\frac{d}{dt} \ln\left(\frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\right), \end{aligned}$$

where $\tilde{a}$, $\tilde{b}$ are the averages and

$$\theta(\underline{z}) = \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi \mathrm{i} \left(\langle \underline{m}, \underline{z} \rangle + \frac{\langle \underline{m}, \underline{\tau}\, \underline{m} \rangle}{2} \right), \quad \underline{z} \in \mathbb{C}^g,$$

is the Riemann theta function of our surface. The matrix τ is the matrix of b-periods of the normalized basis of holomorphic differentials ζ_j.

III. THE MAIN RESULT

We will assume for simplicity that no solitons are present. In other words, we assume that the Jacobi operator above has no eigenvalues. The assumption is not crucial, since the solitons can be easily incorporated using a Darboux transform.

To obtain the long time asymptotics one reformulates the problem as a Riemann-Hilbert problem (RHP) on the underlying Riemann surface. This shows that the solution can be read off from the two by two matrix valued function m that is meromorphic off the preimage of the spectrum Σ, with divisor satisfying

$$(m_{j1}) \ge -\sum_k \mathcal{D}_{\mu_k^*(n,t)}, \quad (m_{j2}) \ge -\sum_k \mathcal{D}_{\mu_k(n,t)},$$

$j = 1, 2$, jump matrix given by

$$J(p,n,t) = \begin{pmatrix} 1 - |R(p,n,t)|^2 & -\overline{R(p,n,t)} \\ R(p,n,t) & 1 \end{pmatrix},$$

$p \in \Sigma$, and normalization

$$m(p,n,t) \to \mathbb{I}, \quad \text{as } p \to \infty_+.$$

This is similar to the RHP applicable to the constant background case, with the main difference being that we allow poles at the points $\mu_j(n,t)$, and their flip images $\mu_j^*(n,t)$ on the other sheet. The g points $\mu_j(n,t)$ are uniquely defined by the Jacobi inversion problem

$$\sum_{k=1}^{g} \int_{E_0}^{\mu_k(n,t)} \zeta_j = \sum_{k=1}^{g} \int_{E_0}^{\mu_k} \zeta_j + n \int_{\infty_-}^{\infty_+} \zeta_j - t2c_j(g).$$

This is a crucial point and related to the fact that our RHP is no longer formulated in the complex plane. While a holomorphic RHP with jump of index zero has a solution in the complex plane, this is no longer true on a surface of genus g unless we admit at least g poles ([11]). In particular, the above RHP has no holomorphic solutions except in special cases (e.g. if there is no jump). In fact, a main issue in the mathematical analysis of the RHP ([9]) is to define an appropriate space of solutions which makes the problem well-posed.

The matrix elements of the jump are of the form

$$R(p,n,t) = R(p)\Theta(p,n,t)\exp(t\phi(p)),$$

where $R(p)$ is the reflection coefficient at $t=0$, Θ is a ratio of four theta functions

$$\Theta(p,n,t) = \frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))}\frac{\theta(\underline{z}(p^*,0,0))}{\theta(\underline{z}(p^*,n,t))}$$

(of modulus one), and the phase ϕ is given by

$$\phi(p) = 2\int_{E_0}^{p} \Omega_0 + 2\frac{n}{t}\int_{E_0}^{p} \omega_{\infty_+\infty_-}.$$

Here

$$\omega_{\infty_+\infty_-} = \frac{\prod_{j=1}^{g}(z-\lambda_j)}{R_{2g+2}^{1/2}(z)}dz$$

is the normalized Abelian differential of the third kind with poles at ∞_+ respectively ∞_- and

$$\Omega_0 = \frac{\prod_{j=0}^{g}(z-\tilde\lambda_j)}{R_{2g+2}^{1/2}(z)}dz, \quad \sum_{j=0}^{g}\tilde\lambda_j = \frac{1}{2}\sum_{j=0}^{2g+1}E_j,$$

is the normalized Abelian differential of the second kind with second order poles at ∞_+ respectively ∞_-. The constants λ_j respectively $\tilde\lambda_j$ again have to be determined from the normalization.

There are $g+1$ stationary phase points $z_j(n/t)$ which behave as follows: As $\eta = \frac{n}{t}$ runs from $-\infty$ to $+\infty$ we start with $z_g(\eta)$ moving from ∞ towards E_{2g+1} while the others stay in their spectral gaps until $z_g(\eta)$ has passed the first spectral band. After this has happened, z_{g-1} can leave its gap, while $z_g(\eta)$ remains there, traverses the next spectral band and so on. Until finally $z_0(\eta)$ traverses the last spectral band and escapes to $-\infty$.

Factorizing the jump matrix and using the asymptotic analysis of oscillatory Riemann-Hilbert problems introduced in ([2]), but generalized accordingly in [9], one obtains that for long times the perturbed Toda lattice is asymptotically close to the following limiting lattice defined by

$$\begin{aligned}\prod_{j=n}^{\infty}(\frac{a_l(j,t)}{a_q(j,t)})^2 =& \frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n+1,t))}\frac{\theta(\underline{z}(n+1,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(n,t)+\underline{\delta}(n,t))}\times \\ & \times \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\omega_{\infty_-\infty_+}\right), \\ \delta_\ell(n,t) =& \frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\zeta_\ell,\end{aligned}$$

where R is the associated reflection coefficient, $C(n/t) = \Sigma \cap (-\infty, z_j(n/t))$, and $z_j(n/t)$ is the single stationary phase point lying in the spectrum, if there is such a point, or otherwise, one of the two stationary phase points lying in the same spectral gap.

In summary, for any short range perturbation $a(n,t)$ of a quasi-periodic solution of the Toda lattice one has that

$$\prod_{j=n}^{\infty}\frac{a(j,t)}{a_l(j,t)} \to 1 \tag{1}$$

uniformly in n, as $t \to \infty$.

From (1) one recovers the $a(n,t)$ and by differentiating one recovers the $b(n,t)$. It thus follows that

$$|a(n,t) - a_l(n,t)| + |b(n,t) - b_l(n,t)| \to 0$$

uniformly in n, as $t \to \infty$.

A complete mathematical proof will be given in [9].

If solitons are present we can apply appropriate Darboux transformations to add the effect of such solitons. What we then see asymptotically is additional travelling solitons on a periodic background ([6] or [8]).

We end this note by pointing out that the limiting lattice can be easily computed numerically. For the initial data defined in the introduction the result is shown in Figure 2. The soliton was added using a Darboux transformation.

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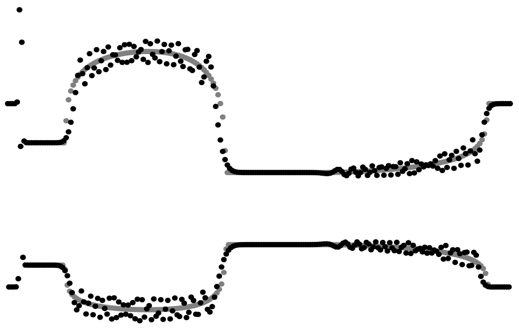

FIG. 2: Solution of the Toda lattice (black) plus corresponding limiting lattice (gray). No gray points visible implies that black points overlap gray points.

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